Lord Brouncker's forgotten sequence of continued fractions for pi

Article *in* International Journal of Mathematical Education · January 2010

DOI: 10.1080/00207390903189195

1 author:

Thomas Osler
Rowan University

LORD BROUNCKER'S FORGOTTEN SEQUENCE OF CONTINUED FRACTIONS FOR PI

Thomas J. Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

osler@rowan.edu

Abstract

"Lord Brouncker's continued fraction for π" is a well known result. In this paper we show that Brouncker found not only this one continued fraction, but an entire infinite sequence of related continued fractions for π. These were recorded in the *Arithmetica Infinitorum* by John Wallis, but appear to have been ignored and forgotten by modern mathematicians. We identify these continued fractions and give a modern derivation.

Key words

Continued fractions, Lord Brouncker, John Wallis, pi.

The forgotten continued fractions

Ask a mathematician for "Lord Brouncker's continued fraction for π" and you will likely be shown

(1) $$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}.$$

{In this paper we will use the more convenient notation $\frac{4}{\pi} = 1 + \frac{1^2}{2+}\ \frac{3^2}{2+}\ \frac{5^2}{2+}\ \cdots$.) This fraction first appeared in the *Arithmetica Infinitorum* [11] by John Wallis. In this book, along with topics that lead to Newton's calculus, Wallis derives his famous product

(2) $$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots.$$

Wallis tells us that he showed this product to Lord Brouncker, and in turn Brouncker converted it ***not only*** into (1), but into an ***infinite sequence*** of continued fractions which we begin listing as:

$$1 + \frac{1^2}{2+}\ \frac{3^2}{2+}\ \frac{5^2}{2+}\ \cdots = \frac{4}{\pi}$$ (familiar Lord Brouncker's fraction.)

$$3 + \frac{1^2}{6+}\ \frac{3^2}{6+}\ \frac{5^2}{6+}\ \cdots = \pi$$. (Lange's continued fraction.)

$$5 + \frac{1^2}{10+}\ \frac{3^2}{10+}\ \frac{5^2}{10+}\ \cdots = 3\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4}{\pi}$$

$$7 + \frac{1^2}{14+}\ \frac{3^2}{14+}\ \frac{5^2}{14+}\ \cdots = 3\cdot\frac{1\cdot 3}{2\cdot 2}\pi$$

$$9 + \frac{1^2}{18+}\ \frac{3^2}{18+}\ \frac{5^2}{18+}\ \cdots = 5\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{4}{\pi}$$

$$11 + \frac{1^2}{22+}\ \frac{3^2}{22+}\ \frac{5^2}{22+}\ \cdots = 5\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\pi$$

$$13 + \frac{1^2}{26+}\ \frac{3^2}{26+}\ \frac{5^2}{26+}\ \cdots = 7\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 5}\cdot\frac{4}{\pi}$$

$$15 + \frac{1^2}{30+}\ \frac{3^2}{30+}\ \frac{5^2}{30+}\ \cdots = 7\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\pi$$

…

Apparently, all but the first of these fractions seems to have been, for the most part, forgotten in recent times. As just one example, Lange [6] in 1999 called attention to the second fraction in this list without recognizing its appearance almost 350 years earlier in Wallis [11]. Also Brezinski's history [2] discusses only (1) and ignores the other fractions.

Stedall, who recently translated Wallis's book, called attention to the forgotten nature of this sequence of continued fractions in the year 2000 in her paper [9]. She wrote [9, p. 307]. "To credit Brouncker only with a single fraction is to miss the true significance of what he achieved".

The above list of continued fractions shows that they are all of the form

$$CF(x) = x + \frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\cdots,$$

where x is an odd integer. Notice also that π appears on the RHS in the denominator when x is 1, 5, 9, 13, … and in the numerator when x is 3, 7, 11, 15, … . Thus it is convenient to describe the above list in two equations. With

$$W(n) = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots\frac{(2n-1)(2n+1)}{2n\cdot 2n}$$

defined as the partial Wallis product, we write the two general formulas

(3) $$(4n+1) + \frac{1^2}{2(4n+1)+}\;\frac{3^2}{2(4n+1)+}\;\frac{5^2}{2(4n+1)+}\cdots = (2n+1)\frac{1}{W(n)}\cdot\frac{4}{\pi}$$, and

(4) $$(4n+3) + \frac{1^2}{2(4n+3)+}\;\frac{3^2}{2(4n+3)+}\;\frac{5^2}{2(4n+3)+}\cdots = (2n+1)W(n)\pi\,.$$

Notice also that $CF(2n-1)CF(2n+1) = (2n)^2$.

Derivation of the results

The formulas (3) and (4) are special cases of the known formula [8, page 35]

(5) $$\frac{4\Gamma\left(\frac{x+y+3}{4}\right)\Gamma\left(\frac{x-y+3}{4}\right)}{\Gamma\left(\frac{x+y+1}{4}\right)\Gamma\left(\frac{x-y+1}{4}\right)} = x + \frac{1^2-y^2}{2x\,+}\,\frac{3^2-y^2}{2x\,+}\,\frac{5^2-y^2}{2x\,+}\cdots,$$

valid for either *y* an odd integer and *x* any complex number or *y* any complex number and $\mathrm{Re}(x) > 0$. The names of Euler, Stieltjes, and Ramanujan [1, page 140] have been associated with this result. Using the very well known formulas $\Gamma(n+1) = n!$, $\Gamma(x)x = \Gamma(x+1)$ and $\Gamma(1/2) = \sqrt{\pi}$ we have $\Gamma\left(\frac{2k+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k}\sqrt{\pi}$, valid for $k = 1, 2, 3, \cdots$. With this last result and appropriate values of *x* and *y,* the left hand side of (5) can be expressed in terms of rational numbers and π. For example, if we set $y = 0$ and $x = 4n+1$ in (5) we get our general formula (3) and setting $y = 0$ and $x = 4n+3$ we get our general formula (4). The manipulations are simple and the reader will have no difficulty verifying our formulas.

The author obtained the idea for the above proof from reading Lange's paper [6].

Approximating pi

We can use (3) and (4) to make numerical estimates of π. Let us use the notation

$$CF_m(x) = x + \frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\cdots\frac{(2m-1)^2}{+\quad 2x}$$

for the finite continued fraction with *m* denominators. Then our equations (3) and (4) can be solved for the approximations to π

(6) $$pi1_m(n) = \frac{4(2n+1)}{W(n)\; CF_m(4n+1)}, \text{ and}$$

(7) $$pi2_m(n) = \frac{CF_m(2n+3)}{(2n+1)W(n)}.$$

In these formulas *m* is the number of denominators in the terminated continued fraction and *n* is the number of factors in the partial Wallis product as well as the essential number determining the continued fractions.

Both the original Wallis product (1), ($m = 0$ and $n = \infty$) and the first Brouncker fraction (1), ($m = \infty$ and $n = 0$) are very poor for numerical approximation. For example, using 10,000 factors in the Wallis product (2) gives us π to only four decimal places. However, both (6) and (7) for small values of *m* and *n* give useful results. The following table shows the number of accurate digits obtained by using (6) to estimate π.

Number of accurate decimal digits of pi obtained from (6)

	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$
$n = 1$	2	3	3	4	4	4	5	5	5	5
$n = 2$	3	4	5	6	6	7	7	8	8	9
$n = 3$	3	5	6	7	8	9	9	10	11	11
$n = 4$	4	6	7	8	9	10	11	12	12	13
$n = 5$	4	6	8	9	10	11	12	13	14	15
$n = 6$	5	7	8	10	11	12	13	15	15	16
$n = 7$	5	7	9	10	12	13	14	16	17	18
$n = 8$	5	7	9	11	13	14	15	17	18	19
$n = 9$	5	8	10	11	13	15	16	17	19	20
$n = 10$	5	8	10	12	14	15	17	18	20	21

Brouncker gave the estimates

$$3.14159\ 25535\ 69 < \pi < 3.14159\ 25536\ 96.$$

It is easy to see how Brouncker could have made these estimates of only ten accurate decimal digits from his sequence of continued fractions. In a recent paper [3], Dutka conjectured how the above estimates could have been obtained without the knowledge of Brouncker's full sequence.

Final remarks

Wallis described the ingenious way in which he obtained his product (2) in [11]. He states that he showed his product to Lord Brouncker who then obtained the continued fractions listed in this paper. It appears that Brouncker never published his method of finding these continued fractions and only partially explained his reasoning to Wallis. Wallis gives some hints in [11, pages 167 - 178] as to how Brouncker proceeded but the explanation is incomplete. Euler took keen interest in Brouncker's continued fractions and gave his own derivations and generalizations in [4]. See also [3] and [5] for recent discussions of how Brouncker might have derived the one fraction (1) from the product (2).. Stedall in [9, pages 300-310] made her own conjecture as to how Brouncker might have reasoned to obtain the entire sequence of continued fractions. Her explanation is the most convincing that this author has seen.

Wallis published a table in [11, page 172] which we reproduce here.

Q2	Q4	Q6	Q8	Q10	Q12	Q14	Q16	
$\square$	B	C	D	E	F	G	H	I
$1\ \frac{1}{2}\ \frac{9}{2}+$	$3\ \frac{1}{6}\ \frac{9}{6}+$	$5\ \frac{1}{10}\ \frac{9}{10}+$	$7\ \frac{1}{14}\ \frac{9}{14}+$	$9\ \frac{1}{18}\ \frac{9}{18}+$	$11\ \frac{1}{22}\ \frac{9}{22}+$	$13\ \frac{1}{26}\ \frac{9}{26}+$	$15\ \frac{1}{30}\ \frac{9}{30}+$	$17\ \frac{1}{34}\ \frac{9}{34}+$
		$\frac{4}{1}\square$	$\frac{9}{4}B$	$\frac{16}{9}C$	$\frac{25}{16}D$	$\frac{36}{25}E$	$\frac{49}{36}F$	$\frac{64}{49}G$
				$\frac{4}{1}\times\frac{16}{9}\square$	$\frac{9}{4}\times\frac{25}{16}B$	$\frac{16}{9}\times\frac{36}{25}C$	$\frac{25}{16}\times\frac{49}{36}D$	$\frac{36}{25}\times\frac{64}{49}E$
						$\frac{4}{1}\times\frac{16}{9}\times\frac{36}{25}\square$	$\frac{9}{4}\times\frac{25}{16}\times\frac{49}{36}B$	$\frac{16}{9}\times\frac{36}{25}\times\frac{64}{49}C$
								$\frac{4}{1}\times\frac{16}{9}\times\frac{36}{25}\times\frac{64}{49}\square$

Look carefully at the third row of this table. We see the continued fractions obtained in our list. Wallis writes them in the form

$$1\;\frac{1}{2}\;\frac{9}{2}\;+\;,\quad 3\;\frac{1}{6}\;\frac{9}{6}\;+\;,\quad 5\;\frac{1}{10}\;\frac{9}{10}\;+\;,\quad 7\;\frac{1}{14}\;\frac{9}{14}\;+\;,\ \text{etc.}$$

In the second row Wallis uses a box Π to stand for our $\frac{4}{\pi}$ and the remaining letters B, C, D, etc., to stand for fractions beneath them. Continuing down the columns we find values for the fractions. For example, at the bottom of the third column we find $\frac{4}{1}\Pi$ which is $\frac{4}{1}\cdot\frac{4}{\pi}$ the correct value for the third fraction in Brouncker's list. We see in this table that Wallis and Brouncker have written the equivalent of the value of these fractions in terms of rational numbers and pi.

Acknowledgement

The author wishes to thank James Smoak for his generous assistance with the historical items.

References

[1] Berndt, B. C., *Ramanujan's Notebooks, Part II,* Springer-Verlag, New York, 1989.

[2] Brezinski, Claude, *History of Continued Fractions and Pade Approximations,* Springer-Verlag, Berlin, 1991.

[3] Dutka, J., *Wallis's product, Brouncker's continued fraction, and Leibniz'sseries*, Arch. History Exact Sciences 26 (1982), 115–126.

[4] Euler, L., *De fractionibus continuis Wallisii,* (On the continued fractions of Wallis) , Originally published in *Memoires de l'academie des sciences de St.-Petersbourg* 5, 1815,

pp. 24-44 . Also see *Opera Omnia*: Series 1, Volume 16, pp. 178 – 199. On the web at the Euler Archive http://www.math.dartmouth.edu/~euler/.

[5] Khrushchev, S., *A recovery of Brouncker's proof for the quadrature continued fraction*, Publicacions matematiques, 50(2006), pp. 3-42.

[6] Lange, L. J., *An Elegant Continued Fraction for* π *,* The American Mathematical Monthly, 106 (1999), pp. 456-458.

[7] Osler, T. J., *The united Vieta's and Wallis's products for pi,* American Mathematical Monthly, 106 (1999), pp. 774-776.

[8] Perron, O., *Die Lehre von den Kettenbruchen*, Band II, Teubner, Stuttgart, 1957.

[9] Stedall, Jacqueline A., *Catching Proteus: The Collaborations of Wallis and Brouncker. I. Squaring the Circle*, Notes and Records of the Royal Society of London, Vol. 54, No. 3, (Sep., 2000), pp. 293 -316

[10] Vieta, F., *Variorum de Rebus Mathematicis Reponsorum Liber VII,* (1593) in: *Opera Mathematica, (reprinted)* Georg Olms Verlag, Hildesheim, New York, 1970, pp. 398-400 and 436-446.

[11] Wallis, John, *The Arithmetic of Infinitesimals*, (Translated from Latin by Jacqueline A. Stedall), Springer Verlag, New York, 2004.