EUCLID'S ELEMENTS OF GEOMETRY

The Greek text of J.L. Heiberg (1883–1885)

from Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883–1885

edited, and provided with a modern English translation, by

Richard Fitzpatrick

First edition - 2007 Revised and corrected - 2008

ISBN 978-0-6151-7984-1

Contents

Introduction	4
Book 1	5
Book 2	49
Book 3	69
Book 4	109
Book 5	129
Book 6	155
Book 7	193
Book 8	227
Book 9	253
Book 10	281
Book 11	423
Book 12	471
Book 13	505
Greek-English Lexicon	539

Introduction

Euclid's Elements is by far the most famous mathematical work of classical antiquity, and also has the distinction of being the world's oldest continuously used mathematical textbook. Little is known about the author, beyond the fact that he lived in Alexandria around 300 BCE. The main subjects of the work are geometry, proportion, and number theory.

Most of the theorems appearing in the Elements were not discovered by Euclid himself, but were the work of earlier Greek mathematicians such as Pythagoras (and his school), Hippocrates of Chios, Theaetetus of Athens, and Eudoxus of Cnidos. However, Euclid is generally credited with arranging these theorems in a logical manner, so as to demonstrate (admittedly, not always with the rigour demanded by modern mathematics) that they necessarily follow from five simple axioms. Euclid is also credited with devising a number of particularly ingenious proofs of previously discovered theorems: *e.g.*, Theorem 48 in Book 1.

The geometrical constructions employed in the Elements are restricted to those which can be achieved using a straight-rule and a compass. Furthermore, empirical proofs by means of measurement are strictly forbidden: *i.e.*, any comparison of two magnitudes is restricted to saying that the magnitudes are either equal, or that one is greater than the other.

The Elements consists of thirteen books. Book 1 outlines the fundamental propositions of plane geometry, including the three cases in which triangles are congruent, various theorems involving parallel lines, the theorem regarding the sum of the angles in a triangle, and the Pythagorean theorem. Book 2 is commonly said to deal with "geometric algebra", since most of the theorems contained within it have simple algebraic interpretations. Book 3 investigates circles and their properties, and includes theorems on tangents and inscribed angles. Book 4 is concerned with regular polygons inscribed in, and circumscribed around, circles. Book 5 develops the arithmetic theory of proportion. Book 6 applies the theory of proportion to plane geometry, and contains theorems on similar figures. Book 7 deals with elementary number theory: *e.g.*, prime numbers, greatest common denominators, *etc.* Book 8 is concerned with geometric series. Book 9 contains various applications of results in the previous two books, and includes theorems on the infinitude of prime numbers, as well as the sum of a geometric series. Book 10 attempts to classify incommensurable (*i.e.*, irrational) magnitudes using the so-called "method of exhaustion", an ancient precursor to integration. Book 11 deals with the fundamental propositions of three-dimensional geometry. Book 12 calculates the relative volumes of cones, pyramids, cylinders, and spheres using the method of exhaustion. Finally, Book 13 investigates the five so-called Platonic solids.

This edition of Euclid's Elements presents the definitive Greek text—*i.e.*, that edited by J.L. Heiberg (1883–1885)—accompanied by a modern English translation, as well as a Greek-English lexicon. Neither the spurious books 14 and 15, nor the extensive scholia which have been added to the Elements over the centuries, are included. The aim of the translation is to make the mathematical argument as clear and unambiguous as possible, whilst still adhering closely to the meaning of the original Greek. Text within square parenthesis (in both Greek and English) indicates material identified by Heiberg as being later interpolations to the original text (some particularly obvious or unhelpful interpolations have been omitted altogether). Text within round parenthesis (in English) indicates material which is implied, but not actually present, in the Greek text.

My thanks to Mariusz Wodzicki (Berkeley) for typesetting advice, and to Sam Watson & Jonathan Fenno (U. Mississippi), and Gregory Wong (UCSD) for pointing out a number of errors in Book 1.

ELEMENTS BOOK 1

Fundamentals of Plane Geometry Involving Straight-Lines

Ὅροι.

α΄. Σημεϊόν ἐστιν, οὕ μέρος οὐθέν.

β΄. Γραμμή δὲ μῆχος ἀπλατές.

γ΄. Γραμμῆς δὲ πέρατα σημεῖα.

δ'. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις χεῖται.

ε΄. Ἐπιφάνεια δέ ἐστιν, ὃ μῆχος χαὶ πλάτος μόνον ἔχει.

γ΄. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις χεῖται.

η'. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῷ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ' εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ'. Όταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὥσιν, εὐθύγραμμος χαλεῖται ἡ γωνία.

ι΄. Όταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἶσας ἀλλήλαις ποιῆ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ' ἡν ἐφέστηκεν.

ια'. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

ιβ΄. Όξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

ιγ΄. Όρος ἐστίν, ὅ τινός ἐστι πέρας.

ιδ'. Σχημά έστι τὸ ὑπό τινος ἤ τινων ὄρων περιεχόμενον.

ιε΄. Κύχλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἢ καλεῖται περιφέρεια], πρὸς ἢν ἀφ᾽ ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἰ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

ις'. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

ιζ΄. Διάμετρος δὲ τοῦ χύχλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύχλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύχλον.

ιη'. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

ιθ΄. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

κ΄. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

κα΄ Έτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

Definitions

1. A point is that of which there is no part.

2. And a line is a length without breadth.

3. And the extremities of a line are points.

4. A straight-line is (any) one which lies evenly with points on itself.

5. And a surface is that which has length and breadth only.

6. And the extremities of a surface are lines.

7. A plane surface is (any) one which lies evenly with the straight-lines on itself.

8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.

11. An obtuse angle is one greater than a right-angle.

12. And an acute angle (is) one less than a right-angle.

13. A boundary is that which is the extremity of something.

14. A figure is that which is contained by some boundary or boundaries.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the center of the circle.

17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.[†]

18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.

19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides. ×β΄. Τών δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρϑογώνιον, ἑτερόμηκες δέ, ὃ ὀρϑογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρϑογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οῦτε ἰσόπλευρόν ἐστιν οὕτε ὀρϑογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσϑω.

κγ΄. Παράλληλοί εἰσιν εὐθεῖαι, αἴτινες ἐν τῷ αὐτῷ ἐπιπέδῷ οῦσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις. 21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

[†] This should really be counted as a postulate, rather than as part of a definition.

Αἰτήματα.

α'. Ἡιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πῶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β'. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

γ΄. Καὶ παντὶ κέντρω καὶ διαστήματι κύκλον γράφεσθαι.

δ'. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε'. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῆ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἂ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

 Let it have been postulated[†] to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).[‡]

[†] The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative $H_{\rm trtf}\sigma\vartheta\omega$ could be translated as "let it be postulated", in the sense "let it stand as postulated", but not "let the postulate be now brought forward". The literal translation "let it have been postulated" sounds awkward in English, but more accurately captures the meaning of the Greek.

[‡] This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

Κοιναὶ ἔννοιαι.

α'. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β'. Καὶ ἐὰν ἴσοις ἴσα προστεθῆ, τὰ ὅλα ἐστὶν ἴσα.

γ΄. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεϑῆ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ'. Καὶ τὰ ἐφαρμόζοντα ἐπ' ἀλλήλα ἴσα ἀλλήλοις ἐστίν.

ε΄. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].

Common Notions

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

3. And if equal things are subtracted from equal things then the remainders are equal.^{\dagger}

4. And things coinciding with one another are equal to one another.

5. And the whole [is] greater than the part.

[†] As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains

an inequality of the same type.

α΄.

Έπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.



Έστω ή δοθεῖσα εὐθεῖα πεπερασμένη ή AB.

Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.

Κέντρω μὲν τῷ Α διαστήματι δὲ τῷ AB κύκλος γεγράφθω ὁ BΓΔ, καὶ πάλιν κέντρω μὲν τῷ B διαστήματι δὲ τῷ BA κύκλος γεγράφθω ὁ AΓE, καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ A, B σημεῖα ἐπεζεύγθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῆ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῆ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῆ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῆ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῆ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ισόπλευρον ἄρα ἐστὶ τὸ ABΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς AB. ὅπερ ἔδει ποιῆσαι.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line *AB*.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,[†] to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

[†] The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

β΄.

Πρὸς τῷ δοθέντι σημείῳ τῆ δοθείσῃ εὐθεία ἴσην εὐθεῖαν θέσθαι.

Έστω τὸ μὲν δοθὲν σημεῖον τὸ A, ἡ δὲ δοθεῖσα εὐθεῖα
ἡ BΓ· δεῖ δỳ πρὸς τῷ A σημείω τῆ δοθείσῃ εὐθεία τῆ BΓ
ἴσην εὐθεῖαν θέσθαι.

Ἐπεζεύχθω γὰρ ἀπὸ τοῦ Α σημείου ἐπί τὸ Β σημεῖον εὐθεῖα ἡ ΑΒ, καὶ συνεστάτω ἐπ' αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΔΑΒ, καὶ ἐκβεβλήσθωσαν ἐπ' εὐθείας ταῖς ΔΑ, ΔΒ

Proposition 2[†]

To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let A be the given point, and BC the given straightline. So it is required to place a straight-line at point Aequal to the given straight-line BC.

For let the straight-line AB have been joined from point A to point B [Post. 1], and let the equilateral triangle DAB have been been constructed upon it [Prop. 1.1]. εὐθεῖαι αἱ ΑΕ, ΒΖ, καὶ κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΓ κύκλος γεγράφθω ὁ ΓΗΘ, καὶ πάλιν κέντρῳ τῷ Δ καὶ διαστήματι τῷ ΔΗ κύκλος γεγράφθω ὁ ΗΚΛ.



Έπεὶ οὖν τὸ B σημεῖον κέντρον ἐστὶ τοῦ ΓΗΘ, ἴση ἐστὶν $\mathring{\eta}$ BΓ τῆ BH. πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΗΚΛ κύκλου, ἴση ἐστὶν ἡ ΔΛ τῆ ΔΗ, ῶν ἡ ΔΑ τῆ ΔB ἴση ἐστίν. λοιπὴ ἄρα ἡ ΑΛ λοιπῆ τῆ BH ἐστιν ἴση. ἐδείχϑη δὲ καὶ ἡ BΓ τῆ BH ἴση· ἑκατέρα ἄρα τῶν ΑΛ, BΓ τῆ BH ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΑΛ ἄρα τῆ BΓ ἐστιν ἴση.

Πρὸς ἄρα τῷ δοθέντι σημεί
ώ τῷ Α τῆ δοθείση εὐθεία τῆ ΒΓ ἴση εὐθεία κεῖται
ἡ ΑΛ· ὅπερ ἔδει ποιῆσαι.

And let the straight-lines AE and BF have been produced in a straight-line with DA and DB (respectively) [Post. 2]. And let the circle CGH with center B and radius BC have been drawn [Post. 3], and again let the circle GKL with center D and radius DG have been drawn [Post. 3].



Therefore, since the point B is the center of (the circle) CGH, BC is equal to BG [Def. 1.15]. Again, since the point D is the center of the circle GKL, DL is equal to DG [Def. 1.15]. And within these, DA is equal to DB. Thus, the remainder AL is equal to the remainder BG [C.N. 3]. But BC was also shown (to be) equal to BG. Thus, AL and BC are each equal to BG. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, AL is also equal to BC.

Thus, the straight-line AL, equal to the given straight-line BC, has been placed at the given point A. (Which is) the very thing it was required to do.

[†] This proposition admits of a number of different cases, depending on the relative positions of the point A and the line BC. In such situations, Euclid invariably only considers one particular case—usually, the most difficult—and leaves the remaining cases as exercises for the reader.

γ'.

Δύο δοθεισῶν εὐθειῶν ἀνίσων ἀπὸ τῆς μείζονος τῆ ἐλάσσονι ἴσην εὐθεῖαν ἀφελεῖν.

Έστωσαν αί δοθεϊσαι δύο εὐθεῖαι ἄνισοι αί AB, Γ, ῶν μείζων ἔστω ἡ AB· δεῖ δὴ ἀπὸ τῆς μείζονος τῆς AB τῆ ἐλάσσονι τῆ Γ ἴσην εὐθεῖαν ἀφελεῖν.

Κείσθω πρὸς τῷ Α σημείῳ τῆ Γ εὐθεία ἴση ἡ ΑΔ· καὶ κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΔ κύκλος γεγράφθω ὁ ΔΕΖ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΔΕΖ κύκλου,

Proposition 3

For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

Let AB and C be the two given unequal straight-lines, of which let the greater be AB. So it is required to cut off a straight-line equal to the lesser C from the greater AB.

Let the line AD, equal to the straight-line C, have been placed at point A [Prop. 1.2]. And let the circle DEF have been drawn with center A and radius AD [Post. 3].

ἴση ἐστιν ἡ AE τῆ AΔ· ἀλλὰ καὶ ἡ Γ τῆ AΔ ἐστιν ἴση. ἑκατέρα ἄρα τῶν AE, Γ τῆ AΔ ἐστιν ἴση· ὥστε καὶ ἡ AE τῆ Γ ἐστιν ἴση.



Δύο άρα δοθεισῶν εὐθειῶν ἀνίσων τῶν AB, Γ ἀπὸ τῆς μείζονος τῆς AB τῆ ἐλάσσονι τῆ Γ ἴση ἀφήρηται ἡ AE· ὅπερ ἔδει ποιῆσαι.

And since point A is the center of circle DEF, AE is equal to AD [Def. 1.15]. But, C is also equal to AD. Thus, AE and C are each equal to AD. So AE is also equal to C [C.N. 1].



Thus, for two given unequal straight-lines, AB and C, the (straight-line) AE, equal to the lesser C, has been cut off from the greater AB. (Which is) the very thing it was required to do.

Proposition 4

If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. And (let) the angle BAC (be) equal to the angle EDF. I say that the base BC is also equal to the base EF, and triangle ABC will be equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) ABC to DEF, and ACB to DFE.

For if triangle ABC is applied to triangle DEF,[†] the point *A* being placed on the point *D*, and the straight-line

δ'.

Έὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δυσὶ πλευραῖς ἴσας ἔχῃ ἐκατέραν ἐκατέρα καὶ τὴν γωνίαν τῆ γωνία ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τὴ βάσει ἴσην ἔξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρα, ὑφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς AB, AΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρα τὴν μὲν AB τῆ ΔΕ τὴν δὲ AΓ τῆ ΔΖ καὶ γωνίαν τὴν ὑπὸ BAΓ γωνία τῆ ὑπὸ EΔΖ ἴσην. λέγω, ὅτι καὶ βάσις ἡ BΓ βάσει τῆ ΕΖ ἴση ἐστίν, καὶ τὸ ABΓ τρίγωνον τῷ ΔΕΖ τριγώνῷ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρα, ὑφ' ἀς aἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ABΓ τῆ ὑπὸ ΔΕΖ, ἡ δὲ ὑπὸ AΓΒ τῆ ὑπὸ ΔΖΕ.

Έφαρμοζομένου γὰρ τοῦ ABΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Δ σημεῖον

τῆς δὲ AB εὐθείας ἐπὶ τὴν ΔΕ, ἐφαρμόσει καὶ τὸ B σημεῖον ἐπὶ τὸ E διὰ τὸ ἴσην εἶναι τὴν AB τῆ ΔΕ· ἐφαρμοσάσης δὴ τῆς AB ἐπὶ τὴν ΔΕ ἑφαρμόσει καὶ ἡ AΓ εὐθεῖα ἐπὶ τὴν ΔΖ διὰ τὸ ἴσην εἶναι τὴν ὑπὸ BAΓ γωνίαν τῆ ὑπὸ EΔΖ· ὥστε καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ σημεῖον ἐφαρμόσει διὰ τὸ ἴσην πάλιν εἴναι τὴν AΓ τῆ ΔΖ. ἀλλὰ μὴν καὶ τὸ B ἐπὶ τὸ E ἐφηρμόκει· ὥστε βάσις ἡ BΓ ἐπὶ βάσιν τὴν EZ ἐφαρμόσει. εἰ γὰρ τοῦ μὲν B ἐπὶ τὸ E ἐφαρμόσαντος τοῦ δὲ Γ ἐπὶ τὸ Ζ ἡ BΓ βάσις ἐπὶ τὴν EZ οὐκ ἐφαρμόσει, δύο εὐθεῖαι χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. ἐφαρμόσει ἄρα ἡ BΓ βάσις ἐπὶ τὴν EZ καὶ ἴση αὐτῆ ἔσται· ὥστε καὶ ὅλον τὸ ABΓ τρίγωνον ἐπὶ ὅλον τὸ ΔΕΖ τρίγωνον ἐφαρμόσει καὶ ἴσον αὐτῷ ἕσται, καὶ αἱ λοιπαὶ γωνίαι ἐπὶ τὰς λοιπὰς γωνίας ἐφαρμόσουσι καὶ ἴσαι αὐταῖς ἔσονται, ἡ μὲν ὑπὸ ABΓ τῆ ὑπὸ ΔΕΖ ἡ δὲ ὑπὸ AΓΒ τῆ ὑπὸ ΔΖΕ.

Έὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἐκατέραν ἑκατέρα καὶ τὴν γωνίαν τῆ γωνία ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τὴ βάσει ἴσην ἔξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρα, ὑφ' ὰς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ὅπερ ἔδει δεῖξαι.

AB on DE, then the point B will also coincide with E, on account of AB being equal to DE. So (because of) AB coinciding with DE, the straight-line AC will also coincide with DF, on account of the angle BAC being equal to EDF. So the point C will also coincide with the point F, again on account of AC being equal to DF. But, point B certainly also coincided with point E, so that the base BC will coincide with the base EF. For if B coincides with E, and C with F, and the base BC does not coincide with EF, then two straight-lines will encompass an area. The very thing is impossible [Post. 1].[‡] Thus, the base BC will coincide with EF, and will be equal to it [C.N. 4]. So the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. (That is) ABC to DEF, and ACB to DFE [C.N. 4].

Thus, if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-line equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (Which is) the very thing it was required to show.

 † The application of one figure to another should be counted as an additional postulate.

[‡] Since Post. 1 implicitly assumes that the straight-line joining two given points is unique.

ε΄.

Τῶν ἰσοσχελῶν τριγώνων αἱ τρὸς τῆ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεχβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται.



Έστω τρίγωνον ἰσοσχελὲς τὸ ΑΒΓ ἴσην ἔχον τὴν ΑΒ πλευρὰν τῆ ΑΓ πλευρῷ, καὶ προσεκβεβλήσθωσαν ἐπ' εὐθείας ταῖς ΑΒ, ΑΓ εὐθεῖαι αἱ ΒΔ, ΓΕ· λέγω, ὅτι ἡ μὲν ὑπὸ ΑΒΓ γωνία τῆ ὑπὸ ΑΓΒ ἴση ἐστίν, ἡ δὲ ὑπὸ ΓΒΔ τῆ ὑπὸ ΒΓΕ.

Εἰλήφθω γὰρ ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, καὶ ἀφηρήσθω ἀπὸ τῆς μείζονος τῆς ΑΕ τῆ ἐλάσσονι τῆ ΑΖ

Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.



Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively) [Post. 2]. I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

For let the point F have been taken at random on BD, and let AG have been cut off from the greater AE, equal

ἴση ἡ ΑΗ, καὶ ἐπεζεύχθωσαν αἱ ΖΓ, ΗΒ εὐθεῖαι.

Έπεὶ οῦν ἴση ἐστὶν ἡ μὲν ΑΖ τῆ ΑΗ ἡ δὲ ΑΒ τῆ ΑΓ, δύο δη αί ΖΑ, ΑΓ δυσι ταῖς ΗΑ, ΑΒ ἴσαι εἰσιν ἑκατέρα έκατέρα και γωνίαν κοινήν περιέχουσι τήν ὑπὸ ΖΑΗ βάσις ἄρα ἡ ΖΓ βάσει τῆ HB ἴση ἐστίν, καὶ τὸ ΑΖΓ τρίγωνον τῷ ΑΗΒ τριγώνω ίσον έσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρα, ὑφ' ς αἱ ἴσαι πλευραὶ ύποτείνουσιν, ή μέν ύπὸ ΑΓΖ τῆ ὑπὸ ΑΒΗ, ή δὲ ὑπὸ ΑΖΓ τῆ ὑπὸ ΑΗΒ. καὶ ἐπεὶ ὅλη ἡ ΑΖ ὅλῃ τῆ ΑΗ ἐστιν ἴση, ῶν ή ΑΒ τῆ ΑΓ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΒΖ λοιπῆ τῆ ΓΗ ἐστιν ἴση. ἐδείχϑη δὲ ϰαὶ ἡ ΖΓ τῆ ΗΒ ἴση· δύο δὴ αἱ ΒΖ, ΖΓ δυσὶ ταῖς ΓΗ, ΗΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ γωνία ἡ ὑπὸ BZΓ γωνία τη ὑπὸ ΓΗΒ ἴση, καὶ βάσις αὐτῶν κοινὴ ἡ ΒΓ· καὶ τὸ ΒΖΓ ἄρα τρίγωνον τῷ ΓΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα έκατέρα, ὑφ' ὡς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ή μέν ὑπὸ ΖΒΓ τῆ ὑπὸ ΗΓΒ ή δὲ ὑπὸ ΒΓΖ τῆ ὑπὸ ΓΒΗ. έπεὶ οῦν ὅλη ἡ ὑπὸ ΑΒΗ γωνία ὅλῃ τῆ ὑπὸ ΑΓΖ γωνία έδείχθη ἴση, ῶν ἡ ὑπὸ ΓΒΗ τῆ ὑπὸ ΒΓΖ ἴση, λοιπὴ ἄρα ἡ ύπὸ ΑΒΓ λοιπῆ τῆ ὑπὸ ΑΓΒ ἐστιν ἴση· καί εἰσι πρὸς τῆ βάσει τοῦ ΑΒΓ τριγώνου. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΒΓ τῆ ύπὸ ΗΓΒ ἴση· καί εἰσιν ὑπὸ τὴν βάσιν.

Τῶν ἄρα ἰσοσκελῶν τριγώνων αἰ τρὸς τῆ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι. to the lesser AF [Prop. 1.3]. Also, let the straight-lines FC and GB have been joined [Post. 1].

In fact, since AF is equal to AG, and AB to AC, the two (straight-lines) FA, AC are equal to the two (straight-lines) GA, AB, respectively. They also encompass a common angle, FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtendend by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFCto AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [C.N. 3]. But FCwas also shown (to be) equal to GB. So the two (straightlines) BF, FC are equal to the two (straight-lines) CG, GB, respectively, and the angle BFC (is) equal to the angle CGB, and the base BC is common to them. Thus, the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [C.N. 3]. And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Proposition 6

If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.



Let ABC be a triangle having the angle ABC equal to the angle ACB. I say that side AB is also equal to side AC.



Έὰν τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ῶσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις ἔσονται.



Έστω τρίγωνον τὸ ΑΒΓ ἴσην ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῆ ὑπὸ ΑΓΒ γωνία· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΒ πλευρᾶ τῆ ΑΓ ἐστιν ἴση.

El γὰρ ἄνισός ἐστιν ἡ AB τῆ AΓ, ἡ ἑτέρα αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ AB, καὶ ἀφηρήσθω ἀπὸ τῆς μείζονος τῆς AB τῆ ἐλάττονι τῆ AΓ ἴση ἡ ΔB, καὶ ἐπεζεύχθω ἡ ΔΓ.

Έπεὶ οῦν ἴση ἐστὶν ἡ ΔΒ τῆ ΑΓ κοινὴ δὲ ἡ ΒΓ, δύο δὴ aì ΔΒ, ΒΓ δύο ταῖς ΑΓ, ΓΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα, καὶ γωνία ἡ ὑπὸ ΔΒΓ γωνία τῆ ὑπὸ ΑΓΒ ἐστιν ἴση· βάσις ἄρα ἡ ΔΓ βάσει τῆ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΒΓ τρίγωνον τῷ ΑΓΒ τριγώνῳ ἴσον ἔσται, τὸ ἔλασσον τῷ μείζονι· ὅπερ ἄτοπον· οὐκ ἄρα ἄνισός ἐστιν ἡ ΑΒ τῆ ΑΓ· ἴση ἄρα.

Έὰν ἄρα τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ὥσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι. For if AB is unequal to AC then one of them is greater. Let AB be greater. And let DB, equal to the lesser AC, have been cut off from the greater AB [Prop. 1.3]. And let DC have been joined [Post. 1].

Therefore, since DB is equal to AC, and BC (is) common, the two sides DB, BC are equal to the two sides AC, CB, respectively, and the angle DBC is equal to the angle ACB. Thus, the base DC is equal to the base AB, and the triangle DBC will be equal to the triangle ACB [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, AB is not unequal to AC. Thus, (it is) equal.[†]

Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show.

[†] Here, use is made of the previously unmentioned common notion that if two quantities are not unequal then they must be equal. Later on, use is made of the closely related common notion that if two quantities are not greater than or less than one another, respectively, then they must be equal to one another.

ζ.

Έπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρα οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις.



Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς AB δύο ταῖς αὐταῖς εὐθείαις ταῖς AΓ, ΓΒ ἄλλαι δύο εὐθεῖαι αἱ AΔ, ΔΒ ἴσαι ἑκατέρα ἑκατέρα συνεστάτωσαν πρὸς ἄλλφ καὶ ἄλλφ σημείφ τῷ τε Γ καὶ Δ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι, ὥστε ἴσην εἶναι τὴν μὲν ΓΑ τῆ ΔΑ τὸ αὐτὸ πέρας ἔχουσαν αὐτῆ τὸ Α, τὴν δὲ ΓΒ τῆ ΔΒ τὸ αὐτὸ πέρας ἔχουσαν αὐτῆ τὸ Β, καὶ ἐπεζεύχθω ἡ ΓΔ.

Έπεὶ οῦν ἴση ἐστὶν ἡ ΑΓ τῆ ΑΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΓΔ τῆ ὑπὸ ΑΔΓ· μείζων ἄρα ἡ ὑπὸ ΑΔΓ τῆς ὑπὸ ΔΓΒ· πολλῷ ἄρα ἡ ὑπὸ ΓΔΒ μείζων ἐστί τῆς ὑπὸ ΔΓΒ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῆ ΔΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΓΔΒ γωνία τῆ ὑπὸ ΔΓΒ. ἐδείχθη δὲ αὐτῆς καὶ πολλῷ μείζων· ὅπερ ἐστὶν ἀδύνατον.

Ούκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις

Proposition 7

On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines.



For, if possible, let the two straight-lines AC, CB, equal to two other straight-lines AD, DB, respectively, have been constructed on the same straight-line AB, meeting at different points, C and D, on the same side (of AB), and having the same ends (on AB). So CA is equal to DA, having the same end A as it, and CB is equal to DB, having the same end B as it. And let CD have been joined [Post. 1].

Therefore, since AC is equal to AD, the angle ACD is also equal to angle ADC [Prop. 1.5]. Thus, ADC (is) greater than DCB [C.N. 5]. Thus, CDB is much greater than DCB [C.N. 5]. Again, since CB is equal to DB, the angle CDB is also equal to angle DCB [Prop. 1.5]. But it was shown that the former (angle) is also much greater

άλλαι δύο εύθεῖαι ἴσαι ἑκατέρα ἑκατέρα συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις. ὅπερ ἔδει δεῖξαι.

η'.

Έὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἐκατέραν ἐκατέρα, ἔχῃ δὲ καὶ τὴν βάσιν τῆ βάσει ἴσην, καὶ τὴν γωνίαν τῆ γωνία ἴσην ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς AB, AΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρα, τὴν μὲν AB τῆ ΔΕ τὴν δὲ AΓ τῆ ΔΖ· ἐχέτω δὲ καὶ βάσιν τὴν BΓ βάσει τῆ ΕΖ ἴσην· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ BAΓ γωνία τῆ ὑπὸ ΕΔΖ ἐστιν ἴση.

Έφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Β σημείου ἐπὶ τὸ Ε σημεῖον τῆς δὲ ΒΓ εὐθείας ἐπὶ τὴν ΕΖ ἐφαρμόσει καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ διὰ τὸ ἴσην εἶναι τὴν ΒΓ τῆ ΕΖ· ἐφαρμοσάσης δὴ τῆς ΒΓ ἐπὶ τὴν ΕΖ ἐφαρμόσουσι καὶ αἱ ΒΑ, ΓΑ ἐπὶ τὰς ΕΔ, ΔΖ. εἰ γὰρ βάσις μὲν ἡ ΒΓ ἐπὶ βάσιν τὴν ΕΖ ἐφαρμόσει, αἰ δὲ ΒΑ, ΑΓ πλευραὶ ἐπὶ τὰς ΕΔ, ΔΖ οὐκ ἐφαρμόσουσιν ἀλλὰ παραλλάξουσιν ὡς αἱ ΕΗ, ΗΖ, συσταθήσονται ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρα πρὸς ἄλλῳ καὶ ἄλλῳ σημείω ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι. οὐ συνίστανται δέ· οὐκ ἄρα ἐφαρμοζομένης τῆς ΒΓ βάσεως ἐπὶ τὴν ΕΖ βάσιν οὐκ ἐφαρμόσουσι καὶ αἱ ΒΑ, ΑΓ πλευραὶ ἐπὶ τὰς ΕΔ, ΔΖ. ἐφαρμόσουσιν ἄρα· ὥστε καὶ γωνία ἡ ὑπὸ ΒΑΓ ἐπὶ γωνίαν τὴν ὑπὸ ΕΔΖ ἑφαρμόσει καὶ ἴση αὐτῆ ἔσται.

Ἐἀν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρα καὶ τὴν βάσιν τῇ βάσει ἴσην ἔχῃ, καὶ τὴν γωνίαν τῇ γωνία ἴσην ἔξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην. ὅπερ ἔδει δεῖξαι. (than the latter). The very thing is impossible.

Thus, on the same straight-line, two other straightlines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

Proposition 8

If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines.



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. Let them also have the base BC equal to the base EF. I say that the angle BAC is also equal to the angle EDF.

For if triangle ABC is applied to triangle DEF, the point B being placed on point E, and the straight-line BC on EF, then point C will also coincide with F, on account of BC being equal to EF. So (because of) BCcoinciding with EF, (the sides) BA and CA will also coincide with ED and DF (respectively). For if base BCcoincides with base EF, but the sides AB and AC do not coincide with ED and DF (respectively), but miss like EG and GF (in the above figure), then we will have constructed upon the same straight-line, two other straightlines equal, respectively, to two (given) straight-lines, and (meeting) at a different point on the same side (of the straight-line), but having the same ends. But (such straight-lines) cannot be constructed [Prop. 1.7]. Thus, the base BC being applied to the base EF, the sides BAand AC cannot not coincide with ED and DF (respectively). Thus, they will coincide. So the angle BAC will also coincide with angle EDF, and will be equal to it [C.N. 4].

Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base, θ'.

Τήν δοθεῖσαν γωνίαν εὐθύγραμμον δίχα τεμεῖν.



Έστω ή δοθεῖσα γωνία εὐθύγραμμος ή ὑπὸ ΒΑΓ. δεῖ δὴ αὐτὴν δίχα τεμεῖν.

Εἰλήφθω ἐπὶ τῆς AB τυχὸν σημεῖον τὸ Δ, καὶ ἀφηρήσθω ἀπὸ τῆς AΓ τῆ AΔ ἴση ἡ AE, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΔΕΖ, καὶ ἐπεζεύχθω ἡ AZ· λέγω, ὅτι ἡ ὑπὸ BAΓ γωνία δίχα τέτμηται ὑπὸ τῆς AZ εὐθείας.

Έπεὶ γὰρ ἴση ἐστὶν ἡ ΑΔ τῆ ΑΕ, κοινὴ δὲ ἡ ΑΖ, δύο δὴ αἱ ΔΑ, ΑΖ δυσὶ ταῖς ΕΑ, ΑΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα. καὶ βάσις ἡ ΔΖ βάσει τῆ ΕΖ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΑΖ γωνία τῆ ὑπὸ ΕΑΖ ἴση ἐστίν.

Ή ἄρα δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας· ὅπερ ἔδει ποιῆσαι.

ι΄.

Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τεμεῖν.

Έστω ή δοθεΐσα εὐθεῖα πεπερασμένη ή AB· δεῖ δὴ τὴν AB εὐθεῖαν πεπερασμένην δίχα τεμεῖν.

Συνεστάτω ἐπ' αὐτῆς τρίγωνον ἰσόπλευρον τὸ ABΓ, καὶ τετμήσθω ἡ ὑπὸ AΓΒ γωνία δίχα τῆ ΓΔ εὐθεία. λέγω, ὅτι ἡ AB εὐθεῖα δίχα τέτμηται κατὰ τὸ Δ σημεῖον.

Έπεὶ γὰρ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΒ, κοινὴ δὲ ἡ ΓΔ, δύο δὴ aἱ ΑΓ, ΓΔ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ γωνία ἡ ὑπὸ ΑΓΔ γωνία τῆ ὑπὸ ΒΓΔ ἴση ἐστίν· βάσις ἄρα then they will also have equal the angles encompassed by the equal straight-lines. (Which is) the very thing it was required to show.

Proposition 9

To cut a given rectilinear angle in half.



Let BAC be the given rectilinear angle. So it is required to cut it in half.

Let the point D have been taken at random on AB, and let AE, equal to AD, have been cut off from AC [Prop. 1.3], and let DE have been joined. And let the equilateral triangle DEF have been constructed upon DE [Prop. 1.1], and let AF have been joined. I say that the angle BAC has been cut in half by the straight-line AF.

For since AD is equal to AE, and AF is common, the two (straight-lines) DA, AF are equal to the two (straight-lines) EA, AF, respectively. And the base DFis equal to the base EF. Thus, angle DAF is equal to angle EAF [Prop. 1.8].

Thus, the given rectilinear angle BAC has been cut in half by the straight-line AF. (Which is) the very thing it was required to do.

Proposition 10

To cut a given finite straight-line in half.

Let AB be the given finite straight-line. So it is required to cut the finite straight-line AB in half.

Let the equilateral triangle ABC have been constructed upon (AB) [Prop. 1.1], and let the angle ACBhave been cut in half by the straight-line CD [Prop. 1.9]. I say that the straight-line AB has been cut in half at point D.

For since AC is equal to CB, and CD (is) common,

ή Α
Δ βάσει τῆ ΒΔ ἴση ἐστίν.



Ή ἄρα δοθεῖσα εὐθεῖα πεπερασμένη ἡ AB δίχα τέτμηται κατὰ τὸ Δ· ὅπερ ἔδει ποιῆσαι.

ια΄.

Τῆ δοθείση εὐθεία ἀπὸ τοῦ πρὸς αὐτῆ δοθέντος σημείου πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.



Έστω ή μὲν δοθεῖσα εὐθεῖα ή AB τὸ δὲ δοθὲν σημεῖον ἐπ' αὐτῆς τὸ Γ· δεῖ δὴ ἀπὸ τοῦ Γ σημείου τῆ AB εὐθεία πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω ἐπὶ τῆς ΑΓ τυχὸν σημεῖον τὸ Δ, καὶ κείσθω τῆ ΓΔ ἴση ἡ ΓΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΖΔΕ, καὶ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι τῆ δοθείση εὐθεία τῆ ΑΒ ἀπὸ τοῦ πρὸς αὐτῆ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἦκται ἡ ΖΓ.

Έπεὶ γὰρ ἴση ἐστὶν ἡ ΔΓ τῆ ΓΕ, κοινὴ δὲ ἡ ΓΖ, δύο δὴ αἱ ΔΓ, ΓΖ δυσὶ ταῖς ΕΓ, ΓΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ βάσις ἡ ΔΖ βάσει τῆ ΖΕ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΓΖ γωνία τῆ ὑπὸ ΕΓΖ ἴση ἐστίν· καί εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῆ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΔΓΖ, ΖΓΕ. the two (straight-lines) AC, CD are equal to the two (straight-lines) BC, CD, respectively. And the angle ACD is equal to the angle BCD. Thus, the base AD is equal to the base BD [Prop. 1.4].



Thus, the given finite straight-line AB has been cut in half at (point) D. (Which is) the very thing it was required to do.

Proposition 11

To draw a straight-line at right-angles to a given straight-line from a given point on it.



Let AB be the given straight-line, and C the given point on it. So it is required to draw a straight-line from the point C at right-angles to the straight-line AB.

Let the point D be have been taken at random on AC, and let CE be made equal to CD [Prop. 1.3], and let the equilateral triangle FDE have been constructed on DE[Prop. 1.1], and let FC have been joined. I say that the straight-line FC has been drawn at right-angles to the given straight-line AB from the given point C on it.

For since DC is equal to CE, and CF is common, the two (straight-lines) DC, CF are equal to the two (straight-lines), EC, CF, respectively. And the base DFis equal to the base FE. Thus, the angle DCF is equal to the angle ECF [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line Τῆ ἄρα δοθείση εὐθεία τῆ AB ἀπὸ τοῦ πρὸς αὐτῆ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ῆχται ἡ ΓΖ· ὅπερ ἔδει ποιῆσαι.

ıβ'.

Έπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον ἀπὸ τοῦ δοθέντος σημείου, ὃ μή ἐστιν ἐπ' αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.



Έστω ή μὲν δοθεῖσα εὐθεῖα ἄπειρος ή AB τὸ δὲ δοθὲν σημεῖον, ὃ μή ἐστιν ἐπ' αὐτῆς, τὸ Γ· δεῖ δὴ ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν AB ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ' αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω γὰρ ἐπὶ τὰ ἕτερα μέρη τῆς AB εὐθείας τυχὸν σημεῖον τὸ Δ, καὶ κέντρῳ μὲν τῷ Γ διαστήματι δὲ τῷ ΓΔ κύκλος γεγράφθω ὁ ΕΖΗ, καὶ τετμήσθω ἡ ΕΗ εὐθεῖα δίχα κατὰ τὸ Θ, καὶ ἐπεζεύχθωσαν αἱ ΓΗ, ΓΘ, ΓΕ εὐθεῖαι· λέγω, ὅτι ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν AB ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ² αὐτῆς, κάθετος ἦκται ἡ ΓΘ.

Έπεὶ γὰρ ἴση ἐστὶν ἡ ΗΘ τῆ ΘΕ, κοινὴ δὲ ἡ ΘΓ, δύο δὴ ai HΘ, ΘΓ δύο ταῖς ΕΘ, ΘΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ βάσις ἡ ΓΗ βάσει τῆ ΓΕ ἐστιν ἴση· γωνία ἄρα ἡ ὑπὸ ΓΘΗ γωνία τῆ ὑπὸ ΕΘΓ ἐστιν ἴση. καί εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ[°] εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῆ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται ἐφ[°] ῆν ἐφέστηκεν.

Έπὶ τὴν δοθεῖσαν ắρα εὐθεῖαν ắπειρον τὴν AB ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ' αὐτῆς, κάθετος ῆκται ἡ ΓΘ· ὅπερ ἔδει ποιῆσαι. makes the adjacent angles equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, each of the (angles) DCF and FCE is a right-angle.

Thus, the straight-line CF has been drawn at rightangles to the given straight-line AB from the given point C on it. (Which is) the very thing it was required to do.

Proposition 12

To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.



Let AB be the given infinite straight-line and C the given point, which is not on (AB). So it is required to draw a straight-line perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB).

For let point D have been taken at random on the other side (to C) of the straight-line AB, and let the circle EFG have been drawn with center C and radius CD [Post. 3], and let the straight-line EG have been cut in half at (point) H [Prop. 1.10], and let the straight-lines CG, CH, and CE have been joined. I say that the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB).

For since GH is equal to HE, and HC (is) common, the two (straight-lines) GH, HC are equal to the two (straight-lines) EH, HC, respectively, and the base CGis equal to the base CE. Thus, the angle CHG is equal to the angle EHC [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straightline is called a perpendicular to that upon which it stands [Def. 1.10].

Thus, the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the

ιγ'**.**

Έὰν εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα γωνίας ποιῆ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσει.



Εὐθεῖα γάρ τις ἡ AB ἐπ' εὐθεῖαν τὴν ΓΔ σταθεῖσα γωνίας ποιείτω τὰς ὑπὸ ΓΒΑ, ABΔ· λὲγω, ὅτι αἱ ὑπὸ ΓΒΑ, ABΔ γωνίαι ἤτοι δύο ὀρθαί εἰσιν ἢ δυσὶν ὀρθαῖς ἴσαι.

Εἰ μὲν οὕν ἴση ἐστὶν ἡ ὑπὸ ΓΒΑ τῆ ὑπὸ ΑΒΔ, δύο ὀρθαί εἰσιν. εἰ δὲ οὕ, ἦχθω ἀπὸ τοῦ Β σημείου τῆ ΓΔ [εὐθεία] πρὸς ὀρθὰς ἡ ΒΕ· αἱ ἄρα ὑπὸ ΓΒΕ, ΕΒΔ δύο ὀρθαί εἰσιν· καὶ ἐπεὶ ἡ ὑπὸ ΓΒΕ δυσὶ ταῖς ὑπὸ ΓΒΑ, ΑΒΕ ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ ΕΒΔ· αἱ ἄρα ὑπὸ ΓΒΕ, ΕΒΔ τρισὶ ταῖς ὑπὸ ΓΒΑ, ΑΒΕ, ΕΒΔ ἴσαι εἰσίν. πάλιν, ἐπεὶ ἡ ὑπὸ ΔΒΑ δυσὶ ταῖς ὑπὸ ΔΒΕ, ΕΒΑ ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· αἱ ἄρα ὑπὸ ΔΒΑ, ΑΒΓ τρισὶ ταῖς ὑπὸ ΔΒΑ, ΑΒΓ ἴσαι εἰσίν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓΒΕ, ΕΒΔ τρισὶ ταῖς αὐταῖς ἴσαι· τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἀ ὑπὸ ΓΒΕ, ΕΒΔ ἄρα ταῖς ὑπὸ ΔΒΑ, ΑΒΓ ἴσαι εἰσίν· ἀλλὰ αἱ ὑπὸ ΓΒΕ, ΕΒΔ ὄύο ὀρθαί εἰσιν· καὶ αἱ ὑπὸ ΔΒΑ, ΑΒΓ ἅρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Έὰν ἄρα εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα γωνίας ποιῆ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσει· ὅπερ ἔδει δεῖξαι. given point C, which is not on (AB). (Which is) the very thing it was required to do.

Proposition 13

If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two rightangles, or (angles whose sum is) equal to two rightangles.



For let some straight-line AB stood on the straightline CD make the angles CBA and ABD. I say that the angles CBA and ABD are certainly either two rightangles, or (have a sum) equal to two right-angles.

In fact, if CBA is equal to ABD then they are two right-angles [Def. 1.10]. But, if not, let BE have been drawn from the point B at right-angles to [the straightline] CD [Prop. 1.11]. Thus, CBE and EBD are two right-angles. And since CBE is equal to the two (angles) *CBA* and *ABE*, let *EBD* have been added to both. Thus, the (sum of the angles) CBE and EBD is equal to the (sum of the) three (angles) CBA, ABE, and EBD [C.N. 2]. Again, since DBA is equal to the two (angles) *DBE* and *EBA*, let *ABC* have been added to both. Thus, the (sum of the angles) DBA and ABC is equal to the (sum of the) three (angles) DBE, EBA, and ABC[C.N. 2]. But (the sum of) CBE and EBD was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) CBE and EBD is also equal to (the sum of) DBA and ABC. But, (the sum of) *CBE* and *EBD* is two right-angles. Thus, (the sum of) ABD and ABC is also equal to two right-angles.

Thus, if a straight-line stood on a(nother) straightline makes angles, it will certainly either make two rightangles, or (angles whose sum is) equal to two rightangles. (Which is) the very thing it was required to show. ιδ'.

Έὰν πρός τινι εὐθεία καὶ τῷ πρὸς αὐτῆ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ' εὐθείας ἔσονται ἀλλήλαις αἱ εὐθεῖαι.



Πρὸς γάρ τινι εὐθεία τῆ AB καὶ τῷ πρὸς αὐτῆ σημείῳ τῷ B δύο εὐθεῖαι aἱ BΓ, BΔ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ABΓ, ABΔ δύο ὀρθαῖς ἴσας ποιείτωσαν· λέγω, ὅτι ἐπ᾽ εὐθείας ἐστὶ τῆ ΓΒ ἡ BΔ.

Eỉ γὰρ μή ἐστι τ
ῆ BΓ ἐπ' εὐθείας ἡ BΔ, ἔστω τῆ ΓΒ ἐπ' εὐθείας ἡ BE.

Έπεὶ οὖν εὐθεῖα ἡ AB ἐπ' εὐθεῖαν τὴν ΓΒΕ ἐφέστηκεν, αί ἄρα ὑπὸ ABΓ, ABΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσίν· εἰσὶ δὲ καὶ αἱ ὑπὸ ABΓ, ABΔ δύο ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓΒΑ, ABE ταῖς ὑπὸ ΓΒΑ, ABΔ ἴσαι εἰσίν. κοινὴ ἀφηρήσθω ἡ ὑπὸ ΓΒΑ· λοιπὴ ἄρα ἡ ὑπὸ ABE λοιπῆ τῆ ὑπὸ ABΔ ἐστιν ἴση, ἡ ἐλάσσων τῆ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐπ' εὐθείας ἐστὶν ἡ BE τῆ ΓΒ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις πλὴν τῆς BΔ· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΓΒ τῆ BΔ.

Έλν ἄρα πρός τινι εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω δύο εὐθεῖαι μὴ ἐπὶ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ' εὐθείας ἔσονται ἀλλήλαις αἱ εὐθεῖαι. ὅπερ ἔδει δεῖζαι.

Proposition 14

If two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another.



For let two straight-lines BC and BD, not lying on the same side, make adjacent angles ABC and ABD (whose sum is) equal to two right-angles with some straight-line AB, at the point B on it. I say that BD is straight-on with respect to CB.

For if BD is not straight-on to BC then let BE be straight-on to CB.

Therefore, since the straight-line AB stands on the straight-line CBE, the (sum of the) angles ABC and ABE is thus equal to two right-angles [Prop. 1.13]. But (the sum of) ABC and ABD is also equal to two right-angles. Thus, (the sum of angles) CBA and ABE is equal to (the sum of angles) CBA and ABD [C.N. 1]. Let (angle) CBA have been subtracted from both. Thus, the remainder ABE is equal to the remainder ABD [C.N. 3], the lesser to the greater. The very thing is impossible. Thus, BE is not straight-on with respect to CB. Similarly, we can show that neither (is) any other (straight-line) than BD. Thus, CB is straight-on with respect to BD.

Thus, if two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two rightangles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another. (Which is) the very thing it was required to show.

Proposition 15

Έὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν.

ιε΄.

If two straight-lines cut one another then they make the vertically opposite angles equal to one another.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΕΓ γωνία τῆ ύπὸ ΔΕΒ, ἡ δὲ ὑπὸ ΓΕΒ τῆ ὑπὸ ΑΕΔ.



Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΕ ἐπ' εὐθεῖαν τὴν ΓΔ ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΓΕΑ, ΑΕΔ, αἱ ἄρα ὑπὸ ΓΕΑ, ΑΕΔ γωνίαι δυσιν όρθαῖς ἴσαι εἰσίν. πάλιν, ἐπεὶ εὐθεῖα ἡ ΔΕ ἐπ εὐθεῖαν τὴν AB ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΑΕΔ, ΔEB , at ắpa theorem AE Δ , ΔEB ywhich duoth ophaic to at eloin. έδείχθησαν δὲ καὶ αἱ ὑπὸ ΓΕΑ, ΑΕΔ δυσὶν ὀρθαῖς ἴσαι· αἱ άρα ὑπὸ ΓΕΑ, ΑΕΔ ταῖς ὑπὸ ΑΕΔ, ΔΕΒ ἴσαι εἰσίν. χοινὴ άφηρήσθω ή ὑπὸ ΑΕΔ· λοιπὴ ἄρα ή ὑπὸ ΓΕΑ λοιπῆ τῆ ὑπὸ BEΔ ἴση ἐστίν· ὑμοίως δὴ δειχθήσεται, ὅτι καὶ αἱ ὑπὸ ΓΕΒ, ΔEA ἴσαι εἰσίν.

Ἐὰν ἄρα δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφήν γωνίας ἴσας ἀλλήλαις ποιοῦσιν. ὅπερ ἔδει δεῖξαι.

เร'.

Παντός τριγώνου μιας τῶν πλευρῶν προσεκβληθείσης ή ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν.

Έστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρά ή ΒΓ ἐπὶ τὸ Δ· λὲγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ μείζων ἐστὶν ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον τῶν ύπὸ ΓΒΑ, ΒΑΓ γωνιῶν.

Τετμήσθω ή ΑΓ δίχα κατὰ τὸ Ε, καὶ ἐπιζευχθεῖσα ή ΒΕ έκβεβλήσθω ἐπ' εὐθείας ἐπὶ τὸ Ζ, καὶ κείσθω τῆ ΒΕ ἴση ἡ ΕΖ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω ἡ ΑΓ ἐπὶ τὸ Η.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΕ τῆ ΕΓ, ἡ δὲ ΒΕ τῆ ΕΖ, δύο δή αί ΑΕ, ΕΒ δυσί ταῖς ΓΕ, ΕΖ ἴσαι εἰσὶν ἑχατέρα ἑχατέρα. καὶ γωνία ἡ ὑπὸ ΑΕΒ γωνία τῆ ὑπὸ ΖΕΓ ἴση ἐστίν· κατὰ κορυφήν γάρ· βάσις ἄρα ή ΑΒ βάσει τῆ ΖΓ ἴση ἐστίν, καὶ τὸ ABE τρίγωνον τῷ ZEΓ τριγώνω ἐστιν ἴσον, και αί λοιπαι the two (straight-lines) AE, EB are equal to the two

For let the two straight-lines AB and CD cut one another at the point E. I say that angle AEC is equal to (angle) DEB, and (angle) CEB to (angle) AED.



For since the straight-line AE stands on the straightline CD, making the angles CEA and AED, the (sum of the) angles CEA and AED is thus equal to two rightangles [Prop. 1.13]. Again, since the straight-line DE stands on the straight-line AB, making the angles AEDand DEB, the (sum of the) angles AED and DEB is thus equal to two right-angles [Prop. 1.13]. But (the sum of) CEA and AED was also shown (to be) equal to two right-angles. Thus, (the sum of) CEA and AED is equal to (the sum of) AED and DEB [C.N. 1]. Let AED have been subtracted from both. Thus, the remainder CEA is equal to the remainder BED [C.N. 3]. Similarly, it can be shown that CEB and DEA are also equal.

Thus, if two straight-lines cut one another then they make the vertically opposite angles equal to one another. (Which is) the very thing it was required to show.

Proposition 16

For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let ABC be a triangle, and let one of its sides BChave been produced to D. I say that the external angle ACD is greater than each of the internal and opposite angles, CBA and BAC.

Let the (straight-line) AC have been cut in half at (point) E [Prop. 1.10]. And BE being joined, let it have been produced in a straight-line to (point) $F.^{\dagger}$ And let EF be made equal to BE [Prop. 1.3], and let FC have been joined, and let AC have been drawn through to (point) G.

Therefore, since AE is equal to EC, and BE to EF,

γωνίαι ταῖς λοιπαῖς γωνίαις ἶσαι εἰσὶν ἑκατέρα ἑκατέρα, ὑφ' ἂς αἱ ἶσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΕ τῆ ὑπὸ ΕΓΖ. μείζων δέ ἐστιν ἡ ὑπὸ ΕΓΔ τῆς ὑπὸ ΕΓΖ· μείζων ἄρα ἡ ὑπὸ ΑΓΔ τῆς ὑπὸ ΒΑΕ. Ὁμοίως δὴ τῆς ΒΓ τετμημένης δίχα δειχθήσεται καὶ ἡ ὑπὸ ΒΓΗ, τουτέστιν ἡ ὑπὸ ΑΓΔ, μείζων καὶ τῆς ὑπὸ ΑΒΓ.



Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

(straight-lines) CE, EF, respectively. Also, angle AEB is equal to angle FEC, for (they are) vertically opposite [Prop. 1.15]. Thus, the base AB is equal to the base FC, and the triangle ABE is equal to the triangle FEC, and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 1.4]. Thus, BAE is equal to ECF. But ECD is greater than ECF. Thus, ACD is greater than BAE. Similarly, by having cut BC in half, it can be shown (that) BCG—that is to say, ACD—(is) also greater than ABC.



Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. (Which is) the very thing it was required to show.

[†] The implicit assumption that the point *F* lies in the interior of the angle *ABC* should be counted as an additional postulate.

ιζ΄.

Παντὸς τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονές εἰσι πάντῆ μεταλαμβανόμεναι.



Έστω τρίγωνον τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάττονές εἰσι πάντῃ μεταλαμβανόμεναι.

Proposition 17

For any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles.



Let ABC be a triangle. I say that (the sum of) two angles of triangle ABC taken together in any (possible way) is less than two right-angles.

Έκβεβλήσθω γὰρ ἡ ΒΓ ἐπὶ τὸ Δ.

Καὶ ἐπεὶ τριγώνου τοῦ ΑΒΓ ἐχτός ἐστι γωνία ἡ ὑπὸ ΑΓΔ, μείζων ἐστὶ τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ ΑΒΓ. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τῶν ὑπὸ ΑΒΓ, ΒΓΑ μείζονές εἰσιν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δύο ὀρθαῖς ἴσαι εἰσίν· αἱ ἄρα ὑπὸ ΑΒΓ, ΒΓΑ δύο ὀρθῶν ἐλάσσονές εἰσιν. ὁμοίως δὴ δείζομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΓ, ΑΓΒ δύο ὀρθῶν ἐλάσσονές εἰσι καὶ ἔτι αἱ ὑπὸ ΓΑΒ, ΑΒΓ.

Παντὸς ἄρα τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσςονές εἰσι πάντῆ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

*ι*η'.

Παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει.



Έστω γὰρ τρίγωνον τὸ ABΓ μείζονα ἔχον τὴν AΓ πλευρὰν τῆς AB· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ABΓ μείζων ἐστὶ τῆς ὑπὸ BΓA·

Έπεὶ γὰρ μείζων ἐστὶν ἡ ΑΓ τῆς ΑΒ, κείσθω τῆ ΑΒ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΒΔ.

Καὶ ἐπεὶ τριγώνου τοῦ ΒΓΔ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΔΒ, μείζων ἐστὶ τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ ΔΓΒ· ἴση δὲ ἡ ὑπὸ ΑΔΒ τῆ ὑπὸ ΑΒΔ, ἐπεὶ καὶ πλευρὰ ἡ ΑΒ τῆ ΑΔ ἐστιν ἴση· μείζων ἄρα καὶ ἡ ὑπὸ ΑΒΔ τῆς ὑπὸ ΑΓΒ· πολλῷ ἄρα ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΑΓΒ.

Παντὸς ἄρα τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει· ὅπερ ἔδει δεῖξαι.

ເປ'.

Παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει.

Έστω τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῆς ὑπὸ ΒΓΑ· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΓ πλευρᾶς τῆς ΑΒ μείζων ἐστίν. For let *BC* have been produced to *D*.

And since the angle ACD is external to triangle ABC, it is greater than the internal and opposite angle ABC[Prop. 1.16]. Let ACB have been added to both. Thus, the (sum of the angles) ACD and ACB is greater than the (sum of the angles) ABC and BCA. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ABC and BCA is less than two rightangles. Similarly, we can show that (the sum of) BACand ACB is also less than two right-angles, and further (that the sum of) CAB and ABC (is less than two rightangles).

Thus, for any triangle, (the sum of) two angles taken together in any (possible way) is less than two rightangles. (Which is) the very thing it was required to show.

Proposition 18

In any triangle, the greater side subtends the greater angle.



For let ABC be a triangle having side AC greater than AB. I say that angle ABC is also greater than BCA.

For since AC is greater than AB, let AD be made equal to AB [Prop. 1.3], and let BD have been joined.

And since angle ADB is external to triangle BCD, it is greater than the internal and opposite (angle) DCB[Prop. 1.16]. But ADB (is) equal to ABD, since side AB is also equal to side AD [Prop. 1.5]. Thus, ABD is also greater than ACB. Thus, ABC is much greater than ACB.

Thus, in any triangle, the greater side subtends the greater angle. (Which is) the very thing it was required to show.

Proposition 19

In any triangle, the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than BCA. I say that side AC is also greater than side AB.

Eἰ γὰρ μή, ἤτοι ἴση ἐστὶν ἡ ΑΓ τῆ ΑΒ ἢ ἐλάσσων· ἴση μὲν οῦν οὐχ ἔστιν ἡ ΑΓ τῆ ΑΒ· ἴση γὰρ ἂν ῆν xaì γωνία ἡ ὑπὸ ΑΒΓ τῆ ὑπὸ ΑΓΒ· οὐχ ἔστι δέ· οὐχ ἄρα ἴση ἐστὶν ἡ ΑΓ τῆ ΑΒ. οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ΑΓ τῆς ΑΒ· ἐλάσσων γὰρ ἂν ῆν xaì γωνία ἡ ὑπὸ ΑΒΓ τῆς ὑπὸ ΑΓΒ· οὐχ ἔστι δέ· οὐχ ἄρα ἐλάσσων ἐστὶν ἡ ΑΓ τῆς ΑΒ. ἑδείχθη δέ, ὅτι οὐδὲ ἴση ἐστίν. μείζων ἄρα ἐστὶν ἡ ΑΓ τῆς ΑΒ.



Παντὸς ἄρα τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· ὅπερ ἔδει δεῖξαι.

κ΄.

Παντός τριγώνου αί δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.



Έστω γὰρ τρίγωνον τὸ ABΓ· λέγω, ὅτι τοῦ ABΓ τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν BA, AΓ τῆς BΓ, αἱ δὲ AB, BΓ τῆς AΓ, αἱ δὲ BΓ, ΓΑ τῆς AB. For if not, AC is certainly either equal to, or less than, AB. In fact, AC is not equal to AB. For then angle ABC would also have been equal to ACB [Prop. 1.5]. But it is not. Thus, AC is not equal to AB. Neither, indeed, is AC less than AB. For then angle ABC would also have been less than ACB [Prop. 1.18]. But it is not. Thus, AC is not less than AB. But it was shown that (AC) is not equal (to AB) either. Thus, AC is greater than AB.



Thus, in any triangle, the greater angle is subtended by the greater side. (Which is) the very thing it was required to show.

Proposition 20

In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side).



For let ABC be a triangle. I say that in triangle ABC (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (So), (the sum of) BA and AC (is greater) than BC, (the sum of) AB

Διήχθω γὰρ ἡ BA ἐπὶ τὸ Δ σημεῖον, καὶ κείσθω τῆ ΓΑ ἶση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ.

Έπει οὖν ἴση ἐστὶν ἡ ΔA τῆ $A\Gamma$, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ $A\Delta\Gamma$ τῆ ὑπὸ $A\Gamma\Delta$ · μείζων ἄρα ἡ ὑπὸ $B\Gamma\Delta$ τῆς ὑπὸ $A\Delta\Gamma$ · καὶ ἐπεὶ τρίγωνόν ἐστι τὸ $\Delta\Gamma B$ μείζονα ἔχον τὴν ὑπὸ $B\Gamma\Delta$ γωνίαν τῆς ὑπὸ $B\Delta\Gamma$, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, ἡ ΔB ἄρα τῆς $B\Gamma$ ἐστι μείζων. ἴση δὲ ἡ ΔA τῆ $A\Gamma$ · μείζονες ἄρα αἱ BA, $A\Gamma$ τῆς $B\Gamma$ · ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν AB, $B\Gamma$ τῆς ΓA μείζονές εἰσιν, αἱ δὲ $B\Gamma$, ΓA τῆς AB.

Παντὸς ἄρα τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

κα'.

Έὰν τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μὲν ἔσονται, μείζονα δὲ γωνίαν περιέξουσιν.



Τριγώνου γὰρ τοῦ ABΓ ἐπὶ μιᾶς τῶν πλευρῶν τῆς BΓ ἀπὸ τῶν περάτων τῶν Β, Γ δύο εὐθεῖαι ἐντὸς συνεστάτωσαν αἱ BΔ, ΔΓ· λέγω, ὅτι αἱ BΔ, ΔΓ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τῶν BA, ΑΓ ἐλάσσονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσι τὴν ὑπὸ BΔΓ τῆς ὑπὸ BAΓ.

Διήχθω γὰρ ἡ BΔ ἐπὶ τὸ Ε. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσιν, τοῦ ABE ἄρα τριγώνου αἱ δύο πλευραὶ αἱ AB, AE τῆς BE μείζονές εἰσιν· κοινὴ προσκείσθω ἡ ΕΓ· αἱ ἄρα BA, AΓ τῶν BE, ΕΓ μείζονές εἰσιν. πάλιν, ἐπεὶ τοῦ ΓΕΔ τριγώνου αἱ δύο πλευραὶ αἱ ΓΕ, ΕΔ τῆς ΓΔ μείζονές εἰσιν, κοινὴ προσκείσθω ἡ ΔB· αἱ ΓΕ, ΕΒ ἄρα τῶν ΓΔ, ΔΒ μείζονές εἰσιν. ἀλλὰ τῶν BE, ΕΓ μείζονες ἐδείχθησαν αἱ BA, ΑΓ· πολλῷ ἄρα αἱ BA, ΑΓ τῶν BΔ, ΔΓ μείζονές εἰσιν.

Πάλιν, ἐπεὶ παντὸς τριγώνου ἡ ἐκτὸς γωνία τῆς ἐντὸς καὶ ἀπεναντίον μείζων ἐστίν, τοῦ ΓΔΕ ἄρα τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΓΕΔ. διὰ ταὐτὰ τοίνυν καὶ τοῦ ΑΒΕ τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ

and BC than AC, and (the sum of) BC and CA than AB.

For let BA have been drawn through to point D, and let AD be made equal to CA [Prop. 1.3], and let DC have been joined.

Therefore, since DA is equal to AC, the angle ADC is also equal to ACD [Prop. 1.5]. Thus, BCD is greater than ADC. And since DCB is a triangle having the angle BCD greater than BDC, and the greater angle subtends the greater side [Prop. 1.19], DB is thus greater than BC. But DA is equal to AC. Thus, (the sum of) BA and AC is greater than BC. Similarly, we can show that (the sum of) AB and BC is also greater than CA, and (the sum of) BC and CA than AB.

Thus, in any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (Which is) the very thing it was required to show.

Proposition 21

If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.



For let the two internal straight-lines BD and DC have been constructed on one of the sides BC of the triangle ABC, from its ends B and C (respectively). I say that BD and DC are less than the (sum of the) two remaining sides of the triangle BA and AC, but encompass an angle BDC greater than BAC.

For let BD have been drawn through to E. And since in any triangle (the sum of any) two sides is greater than the remaining (side) [Prop. 1.20], in triangle ABE the (sum of the) two sides AB and AE is thus greater than BE. Let EC have been added to both. Thus, (the sum of) BA and AC is greater than (the sum of) BE and EC. Again, since in triangle CED the (sum of the) two sides CE and ED is greater than CD, let DB have been added to both. Thus, (the sum of) CE and EB is greater than (the sum of) CD and DB. But, (the sum of) BA and AC was shown (to be) greater than (the sum of) BE and EC. Thus, (the sum of) BA and AC is much greater than ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ· πολλῷ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ.

Έλν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν. ὅπερ ἔδει δεῖζαι.

хβ′.

Έχ τριῶν εὐθειῶν, αἴ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].



Έστωσαν αί δοθεῖσαι τρεῖς εὐθεῖαι αί A, B, Γ, ῶν αἰ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν A, B τῆς Γ, αἱ δὲ A, Γ τῆς B, καὶ ἔτι αἱ B, Γ τῆς A[.] δεῖ δὴ ἐκ τῶν ἴσων ταῖς A, B, Γ τρίγωνον συστήσασθαι.

Έκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῆ μὲν Α ἴση ἡ ΔΖ, τῆ δὲ Β ἴση ἡ ΖΗ, τῆ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Z σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστιν ἡ ZΔ τῃ ΖΚ· ἀλλὰ ἡ ZΔ τῃ Α ἐστιν ἴση. καὶ ἡ

(the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].



Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KLH have been drawn with center G and radius GH. And let KF and KG have been joined. I say that the triangle KFG has

KZ ἄρα τῆ A ἐστιν ἴση. πάλιν, ἐπεὶ τὸ H σημεῖον κέντρον ἐστὶ τοῦ ΛΚΘ κύκλου, ἴση ἐστὶν ἡ HΘ τῆ HK· ἀλλὰ ἡ HΘ τῆ Γ ἐστιν ἴση· καὶ ἡ KH ἄρα τῆ Γ ἐστιν ἴση. ἐστὶ δὲ καὶ ἡ ZH τῆ B ἴση· αἱ τρεῖς ἄρα εὐθεῖαι αἱ KZ, ZH, HK τρισὶ ταῖς A, B, Γ ἴσαι εἰσίν.

Έχ τριῶν ἄρα εὐθειῶν τῶν KZ, ZH, HK, αἴ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις εὐθείαις ταῖς Α, Β, Γ, τρίγωνον συνέσταται τὸ KZH· ὅπερ ἔδει ποιῆσαι.

χγ'.

Πρός τῆ δοθείση εὐθεία καὶ τῷ πρός αὐτῆ σημείω τῆ δοθείση γωνία εὐθυγράμμω ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.



Έστω ή μὲν δοθεῖσα εὐθεῖα ή AB, τὸ δὲ πρὸς αὐτῆ σημεῖον τὸ A, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΔΓΕ· δεῖ δὴ πρὸς τῆ δοθείση εὐθεία τῆ AB καὶ τῷ πρὸς αὐτῆ σημείω τῷ A τῆ δοθείση γωνία εὐθυγράμμω τῆ ὑπὸ ΔΓΕ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.

Εἰλήφθω ἐφ' ἑκατέρας τῶν ΓΔ, ΓΕ τυχόντα σημεῖα τὰ Δ, Ε, καὶ ἐπεζεύχθω ἡ ΔΕ· καὶ ἐκ τριῶν εὐθειῶν, αἴ εἰσιν ἴσαι τρισὶ ταῖς ΓΔ, ΔΕ, ΓΕ, τρίγωνον συνεστάτω τὸ ΑΖΗ, ὥστε ἴσην εἶναι τὴν μὲν ΓΔ τῆ ΑΖ, τὴν δὲ ΓΕ τῆ ΑΗ, καὶ ἔτι τὴν ΔΕ τῆ ΖΗ.

Έπει οὖν δύο αί ΔΓ, ΓΕ δύο ταῖς ΖΑ, ΑΗ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα, καὶ βάσις ἡ ΔΕ βάσει τῆ ΖΗ ἴση, γωνία ἄρα ἡ ὑπὸ ΔΓΕ γωνία τῆ ὑπὸ ΖΑΗ ἐστιν ἴση.

Πρὸς ἄρα τῆ δοθείση εὐθεία τῆ AB καὶ τῷ πρὸς αὐτῆ σημείω τῷ A τῆ δοθείση γωνία εὐθυγράμμω τῆ ὑπὸ ΔΓΕ ἴση γωνία εὐθύγραμμος συνέσταται ἡ ὑπὸ ZAH· ὅπερ ἔδει ποιῆσαι. been constructed from three straight-lines equal to A, B, and C.

For since point F is the center of the circle DKL, FD is equal to FK. But, FD is equal to A. Thus, KF is also equal to A. Again, since point G is the center of the circle LKH, GH is equal to GK. But, GH is equal to C. Thus, KG is also equal to C. And FG is also equal to B. Thus, the three straight-lines KF, FG, and GK are equal to A, B, and C (respectively).

Thus, the triangle KFG has been constructed from the three straight-lines KF, FG, and GK, which are equal to the three given straight-lines A, B, and C (respectively). (Which is) the very thing it was required to do.

Proposition 23

To construct a rectilinear angle equal to a given rectilinear angle at a (given) point on a given straight-line.



Let AB be the given straight-line, A the (given) point on it, and DCE the given rectilinear angle. So it is required to construct a rectilinear angle equal to the given rectilinear angle DCE at the (given) point A on the given straight-line AB.

Let the points D and E have been taken at random on each of the (straight-lines) CD and CE (respectively), and let DE have been joined. And let the triangle AFGhave been constructed from three straight-lines which are equal to CD, DE, and CE, such that CD is equal to AF, CE to AG, and further DE to FG [Prop. 1.22].

Therefore, since the two (straight-lines) DC, CE are equal to the two (straight-lines) FA, AG, respectively, and the base DE is equal to the base FG, the angle DCE is thus equal to the angle FAG [Prop. 1.8].

Thus, the rectilinear angle FAG, equal to the given rectilinear angle DCE, has been constructed at the (given) point A on the given straight-line AB. (Which

χδ'.

Έὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρα, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἔξει.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς AB, AΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρα, τὴν μὲν AB τῆ ΔΕ τὴν δὲ AΓ τῆ ΔΖ, ἡ δὲ πρὸς τῷ A γωνία τῆς πρὸς τῷ Δ γωνίας μείζων ἔστω· λέγω, ὅτι καὶ βάσις ἡ BΓ βάσεως τῆς ΕΖ μείζων ἐστίν.

Έπει γὰρ μείζων ἡ ὑπὸ ΒΑΓ γωνία τῆς ὑπὸ ΕΔΖ γωνίας, συνεστάτω πρὸς τῆ ΔΕ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Δ τῆ ὑπὸ ΒΑΓ γωνία ἴση ἡ ὑπὸ ΕΔΗ, καὶ κείσθω ὁποτέρα τῶν ΑΓ, ΔΖ ἴση ἡ ΔΗ, καὶ ἐπεζεύχθωσαν αἱ ΕΗ, ZH.

Έπει οὕν ἴση ἐστιν ἡ μὲν ΑΒ τῆ ΔΕ, ἡ δὲ ΑΓ τῆ ΔΗ, δύο δὴ ai BA, ΑΓ δυσι ταῖς ΕΔ, ΔΗ ἴσαι εἰσιν ἑκατέρα ἑκατέρα καὶ γωνία ἡ ὑπὸ BAΓ γωνία τῆ ὑπὸ EΔΗ ἴση· βάσις ἄρα ἡ BΓ βάσει τῆ EH ἐστιν ἴση. πάλιν, ἐπεὶ ἴση ἐστιν ἡ ΔΖ τῆ ΔΗ, ἴση ἐστι καὶ ἡ ὑπὸ ΔΗΖ γωνία τῆ ὑπὸ ΔΖΗ· μείζων ἄρα ἡ ὑπὸ ΔΖΗ τῆς ὑπὸ EHΖ· πολλῷ ἄρα μείζων ἐστιν ἡ ὑπὸ EZH τῆς ὑπὸ EHZ. καὶ ἐπεὶ τρίγωνόν ἐστι τὸ EZH μείζονα ἔχον τὴν ὑπὸ EZH γωνίαν τῆς ὑπὸ EHZ, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, μείζων ἄρα καὶ πλευρὰ ἡ EH τῆς EZ. ἴση δὲ ἡ EH τῆ BΓ· μείζων ἄρα καὶ ἡ BΓ τῆς EZ.

Έὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρα, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἕξει· ὅπερ ἔδει δείζαι. is) the very thing it was required to do.

Proposition 24

If two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is), AB (equal) to DE, and AC to DF. Let them also have the angle at A greater than the angle at D. I say that the base BC is also greater than the base EF.

For since angle BAC is greater than angle EDF, let (angle) EDG, equal to angle BAC, have been constructed at the point D on the straight-line DE [Prop. 1.23]. And let DG be made equal to either of AC or DF [Prop. 1.3], and let EG and FG have been joined.

Therefore, since AB is equal to DE and AC to DG, the two (straight-lines) BA, AC are equal to the two (straight-lines) ED, DG, respectively. Also the angle BAC is equal to the angle EDG. Thus, the base BCis equal to the base EG [Prop. 1.4]. Again, since DFis equal to DG, angle DGF is also equal to angle DFG[Prop. 1.5]. Thus, DFG (is) greater than EGF. Thus, EFG is much greater than EGF. And since triangle EFG has angle EFG greater than EGF, and the greater angle is subtended by the greater side [Prop. 1.19], side EG (is) thus also greater than EF. But EG (is) equal to BC. Thus, BC (is) also greater than EF.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter). ×ε′.

Έὰν δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρα, τὴν δὲ βάσιν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἔξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς AB, AΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρα, τὴν μὲν AB τỹ ΔΕ, τὴν δὲ AΓ τỹ ΔΖ· βάσις δὲ ἡ BΓ βάσεως τῆς ΕΖ μείζων ἔστω· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ BAΓ γωνίας τῆς ὑπὸ ΕΔΖ μείζων ἐστίν.

Εἰ γὰρ μή, ἤτοι ἴση ἐστὶν αὐτῆ ἢ ἐλάσσων· ἴση μὲν οῦν οὐκ ἔστιν ἡ ὑπὸ ΒΑΓ τῆ ὑπὸ ΕΔΖ· ἴση γὰρ ἂν ἤν καὶ βάσις ἡ ΒΓ βάσει τῆ ΕΖ· οὐκ ἔστι δέ. οὐκ ἄρα ἴση ἐστὶ γωνία ἡ ὑπὸ ΒΑΓ τῆ ὑπὸ ΕΔΖ· οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ὑπὸ ΒΑΓ τῆς ὑπὸ ΕΔΖ· ἐλάσσων γὰρ ἂν ῆν καὶ βάσις ἡ ΒΓ βάσεως τῆς ΕΖ· οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῆς ὑπὸ ΕΔΖ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ τῆς ὑπὸ ΕΔΖ.

Έλν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἐκατέραν ἑκάτερα, τὴν δὲ βασίν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην. ὅπερ ἔδει δεῖξαι.

หร'.

Έὰν δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρα καὶ μίαν πλευρὰν μιῷ πλευρῷ ἴσην ἤτοι τὴν πρὸς ταῖς ἴσαις γωνίαις ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει [ἑκατέραν ἑκατέρα] καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίạ.

Έστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο γωνίας τὰς

(Which is) the very thing it was required to show.

Proposition 25

If two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter).



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively (That is), AB (equal) to DE, and AC to DF. And let the base BC be greater than the base EF. I say that angle BAC is also greater than EDF.

For if not, (BAC) is certainly either equal to, or less than, (EDF). In fact, BAC is not equal to EDF. For then the base BC would also have been equal to the base EF [Prop. 1.4]. But it is not. Thus, angle BAC is not equal to EDF. Neither, indeed, is BAC less than EDF. For then the base BC would also have been less than the base EF [Prop. 1.24]. But it is not. Thus, angle BAC is not less than EDF. But it was shown that (BAC is) not equal (to EDF) either. Thus, BAC is greater than EDF.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter). (Which is) the very thing it was required to show.

Proposition 26

If two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle (equal) to the remaining angle.

ύπὸ ABΓ, BΓA δυσὶ ταῖς ὑπὸ ΔΕΖ, ΕΖΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρα, τὴν μὲν ὑπὸ ABΓ τῆ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ BΓA τῆ ὑπὸ ΕΖΔ· ἐχέτω δὲ καὶ μίαν πλευρὰν μιῷ πλευρῷ ἴσην, πρότερον τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν BΓ τῆ ΕΖ· λέγω, ὅτι καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει ἑκατέραν ἑκατέρα, τὴν μὲν AB τῆ ΔΕ τὴν δὲ AΓ τῆ ΔΖ, καὶ τὴν λοιπὴν γωνίαν τῆ λοιπῆ γωνία, τὴν ὑπὸ BAΓ τῆ ὑπὸ ΕΔΖ.



Eἰ γὰρ ἄνισός ἐστιν ἡ AB τῆ ΔE, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ AB, καὶ κείσθω τῆ ΔE ἴση ἡ BH, καὶ ἐπεζεύχθω ἡ HΓ.

Έπεὶ οὖν ἴση ἐστὶν ἡ μὲν BH τῆ ΔΕ, ἡ δὲ BΓ τῆ EZ, δύο δὴ ai BH, BΓ δυσὶ ταῖς ΔΕ, EZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα: καὶ γωνία ἡ ὑπὸ HBΓ γωνία τῆ ὑπὸ ΔΕΖ ἴση ἐστίν· βάσις ἄρα ἡ HΓ βάσει τῆ ΔΖ ἴση ἐστίν, καὶ τὸ HBΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ HΓΒ γωνία τῆ ὑπὸ ΔΖΕ. ἀλλὰ ἡ ὑπὸ ΔΖΕ τῆ ὑπὸ BΓΑ ὑπόκειται ἴση· καὶ ἡ ὑπὸ BΓΗ ἄρα τῆ ὑπὸ ΔΖΕ τῆ ὑπὸ BΓΑ ὑπόκειται ἴση· καὶ ἡ ὑπὸ BΓΗ ἄρα τῆ ὑπὸ BΓΑ ἴση ἐστίν, ἡ ἐλάσσων τῆ μείζονι· ὅπερ ἀδύνατον. οὐκ ἄρα ἀνισός ἐστιν ἡ AB τῆ ΔΕ. ἴση ἄρα. ἔστι δὲ καὶ ἡ BΓ τῆ EZ ἴση· δύο δὴ ai AB, BΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα· καὶ γωνία ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ ἐστιν ἴση· βάσις ἄρα ἡ ΑΓ βάσει τῆ ΔΖ ἴση ἐστίν, καὶ λοιπὴ γωνία ἡ ὑπὸ BΑΓ τῆ λοιπῆ γωνία τῆ ὑπὸ ΕΔΖ ἴση ἐστίν.

Ἀλλὰ δὴ πάλιν ἔστωσαν αί ὑπὸ τὰς ἴσας γωνίας πλευραὶ ὑποτείνουσαι ἴσαι, ὡς ἡ AB τῆ ΔΕ· λέγω πάλιν, ὅτι καὶ αἰ λοιπαὶ πλευραὶ ταῖς λοιπαῖς πλευραῖς ἴσαι ἔσονται, ἡ μὲν AΓ τῆ ΔΖ, ἡ δὲ BΓ τῆ ΕΖ καὶ ἔτι ἡ λοιπὴ γωνία ἡ ὑπὸ BAΓ τῆ λοιπῆ γωνία τῆ ὑπὸ EΔΖ ἴση ἐστίν.

Εἰ γὰρ ἄνισός ἐστιν ἡ ΒΓ τῆ ΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων, εἰ δυνατόν, ἡ ΒΓ, καὶ κείσθω τῆ ΕΖ ἴση ἡ ΒΘ, καὶ ἐπεζεύχθω ἡ ΑΘ. καὶ ἐπὲι ἴση ἐστὶν ἡ μὲν ΒΘ τῆ ΕΖ ἡ δὲ ΑΒ τῆ ΔΕ, δύο δὴ αἱ ΑΒ, ΒΘ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκαρέρα· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ ΑΘ βάσει τῆ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΕΖ τριγώνῷ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἂς αἱ ἴσας πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΘΑ γωνία τῆ ὑπὸ ΕΖΔ. ἀλλὰ ἡ ὑπὸ ELEMENTS BOOK 1

Let ABC and DEF be two triangles having the two angles ABC and BCA equal to the two (angles) DEFand EFD, respectively. (That is) ABC (equal) to DEF, and BCA to EFD. And let them also have one side equal to one side. First of all, the (side) by the equal angles. (That is) BC (equal) to EF. I say that they will have the remaining sides equal to the corresponding remaining sides. (That is) AB (equal) to DE, and AC to DF. And (they will have) the remaining angle (equal) to the remaining angle. (That is) BAC (equal) to EDF.



For if AB is unequal to DE then one of them is greater. Let AB be greater, and let BG be made equal to DE [Prop. 1.3], and let GC have been joined.

Therefore, since BG is equal to DE, and BC to EF, the two (straight-lines) GB, BC^{\dagger} are equal to the two (straight-lines) DE, EF, respectively. And angle GBC is equal to angle DEF. Thus, the base GC is equal to the base DF, and triangle GBC is equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, GCB (is equal) to DFE. But, DFEwas assumed (to be) equal to BCA. Thus, BCG is also equal to BCA, the lesser to the greater. The very thing (is) impossible. Thus, AB is not unequal to DE. Thus, (it is) equal. And BC is also equal to EF. So the two (straight-lines) AB, BC are equal to the two (straightlines) DE, EF, respectively. And angle ABC is equal to angle DEF. Thus, the base AC is equal to the base DF, and the remaining angle BAC is equal to the remaining angle EDF [Prop. 1.4].

But, again, let the sides subtending the equal angles be equal: for instance, (let) AB (be equal) to DE. Again, I say that the remaining sides will be equal to the remaining sides. (That is) AC (equal) to DF, and BC to EF. Furthermore, the remaining angle BAC is equal to the remaining angle EDF.

For if BC is unequal to EF then one of them is greater. If possible, let BC be greater. And let BH be made equal to EF [Prop. 1.3], and let AH have been joined. And since BH is equal to EF, and AB to DE, the two (straight-lines) AB, BH are equal to the two ΕΖΔ τῆ ὑπὸ ΒΓΑ ἐστιν ἴση· τριγώνου δὴ τοῦ ΑΘΓ ἡ ἐπτὸς γωνία ἡ ὑπὸ ΒΘΑ ἴση ἐστὶ τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ ΒΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ΒΓ τῆ ΕΖ· ἴση ἄρα. ἐστὶ δὲ καὶ ἡ ΑΒ τῆ ΔΕ ἴση. δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσιν ἑκατέρα ἑκατέρα· καὶ γωνίας ἴσας περιέχουσι· βάσις ἄρα ἡ ΑΓ βάσει τῆ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῷ ἴσον καὶ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῆ λοιπὴ γωνία τῆ ὑπὸ ΕΔΖ ἴση.

Έὰν ἄρα δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρα καὶ μίαν πλευρὰν μιῷ πλευρῷ ἴσην ἤτοι τὴν πρὸς ταῖς ἴσαις γωνίαις, ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει καὶ τὴν λοιπὴν γωνίαν τῆ λοιπῆ γωνία. ὅπερ ἔδει δεῖξαι.

^{\dagger} The Greek text has "BG, BC", which is obviously a mistake.

хζ'.

Έὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῆ, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.



Εἰς γὰρ δύο εὐθείας τὰς AB, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΕΖ, ΕΖΔ ἴσας ἀλλήλαις ποιείτω· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῆ ΓΔ.

Εἰ γὰρ μή, ἐκβαλλόμεναι αἰ ΑΒ, ΓΔ συμπεσοῦνται ἤτοι ἐπὶ τὰ Β, Δ μέρη ἢ ἐπὶ τὰ Α, Γ. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν ἐπὶ τὰ Β, Δ μέρη κατὰ τὸ Η. τριγώνου δὴ τοῦ ΗΕΖ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΕΖ ἴση ἐστὶ τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ ΕΖΗ· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα αἱ ΑΒ, ΔΓ ἐκβαλλόμεναι συμπεσοῦνται ἐπὶ τὰ Β, Δ μέρη. ὁμοίως

(straight-lines) DE, EF, respectively. And the angles they encompass (are also equal). Thus, the base AH is equal to the base DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, angle BHA is equal to EFD. But, EFD is equal to BCA. So, in triangle AHC, the external angle BHA is equal to the internal and opposite angle BCA. The very thing (is) impossible [Prop. 1.16]. Thus, BC is not unequal to EF. Thus, (it is) equal. And AB is also equal to DE. So the two (straight-lines) AB, BC are equal to the two (straightlines) DE, EF, respectively. And they encompass equal angles. Thus, the base AC is equal to the base DF, and triangle ABC (is) equal to triangle DEF, and the remaining angle BAC (is) equal to the remaining angle *EDF* [Prop. 1.4].

Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle (equal) to the remaining angle. (Which is) the very thing it was required to show.

Proposition 27

If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.



For let the straight-line EF, falling across the two straight-lines AB and CD, make the alternate angles AEF and EFD equal to one another. I say that AB and CD are parallel.

For if not, being produced, AB and CD will certainly meet together: either in the direction of B and D, or (in the direction) of A and C [Def. 1.23]. Let them have been produced, and let them meet together in the direction of B and D at (point) G. So, for the triangle δη δειχθήσεται, ὄτι οὐδὲ ἐπὶ τὰ Α, Γ· αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῆ ΓΔ.

Έὰν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῆ, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

×η'.

Έὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῆ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην ποιῆ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.



Εἰς γὰρ δύο εὐθείας τὰς AB, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ ΕΗΒ τῆ ἐντὸς καὶ ἀπεναντίον γωνία τῆ ὑπὸ ΗΘΔ ἴσην ποιείτω ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ ΒΗΘ, ΗΘΔ δυσὶν ὀρθαῖς ἴσας· λέγω, ὅτι παράλληλός ἐστιν ἡ AB τῆ ΓΔ.

Έπει γὰρ ἴση ἐστιν ἡ ὑπὸ ΕΗΒ τῆ ὑπὸ ΗΘΔ, ἀλλὰ ἡ ὑπὸ ΕΗΒ τῆ ὑπὸ ΑΗΘ ἐστιν ἴση, καὶ ἡ ὑπὸ ΑΗΘ ἄρα τῆ ὑπὸ ΗΘΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστιν ἡ ΑΒ τῆ ΓΔ.

Πάλιν, ἐπεὶ αἱ ὑπὸ ΒΗΘ, ΗΘΔ δύο ὀρθαῖς ἴσαι εἰσίν, εἰσὶ δὲ xaì αἱ ὑπὸ ΑΗΘ, ΒΗΘ δυσὶν ὀρθαῖς ἴσαι, αἱ ἄρα ὑπὸ ΑΗΘ, ΒΗΘ ταῖς ὑπὸ ΒΗΘ, ΗΘΔ ἴσαι εἰσίν· xοινὴ ἀφηρήσθω ἡ ὑπὸ ΒΗΘ· λοιπὴ ἄρα ἡ ὑπὸ ΑΗΘ λοιπῆ τῆ ὑπὸ ΗΘΔ ἐστιν ἴση· xaí εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῆ ΓΔ.

Έὰν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῆ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην *GEF*, the external angle *AEF* is equal to the interior and opposite (angle) *EFG*. The very thing is impossible [Prop. 1.16]. Thus, being produced, *AB* and *CD* will not meet together in the direction of *B* and *D*. Similarly, it can be shown that neither (will they meet together) in (the direction of) *A* and *C*. But (straight-lines) meeting in neither direction are parallel [Def. 1.23]. Thus, *AB* and *CD* are parallel.

Thus, if a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Proposition 28

If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two rightangles, then the (two) straight-lines will be parallel to one another.



For let EF, falling across the two straight-lines AB and CD, make the external angle EGB equal to the internal and opposite angle GHD, or the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles. I say that AB is parallel to CD.

For since (in the first case) EGB is equal to GHD, but EGB is equal to AGH [Prop. 1.15], AGH is thus also equal to GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

Again, since (in the second case, the sum of) BGH and GHD is equal to two right-angles, and (the sum of) AGH and BGH is also equal to two right-angles [Prop. 1.13], (the sum of) AGH and BGH is thus equal to (the sum of) BGH and GHD. Let BGH have been subtracted from both. Thus, the remainder AGH is equal to the remainder GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

ποιῆ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

хθ'.

Ή εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τάς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῆ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας.



Eἰς γὰρ παραλλήλους εὐθείας τὰς AB, ΓΔ εὐθεῖα ἑμπιπτέτω ἡ EZ· λέγω, ὅτι τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ AHΘ, HΘΔ ἴσας ποιεῖ καὶ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ EHB τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ HΘΔ ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ BHΘ, HΘΔ δυσὶν ὀρθαῖς ἴσας.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΑΗΘ τῆ ὑπὸ ΗΘΔ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΑΗΘ· ×οινὴ προσχείσθω ἡ ὑπὸ ΒΗΘ· αἰ ἄρα ὑπὸ ΑΗΘ, ΒΗΘ τῶν ὑπὸ ΒΗΘ, ΗΘΔ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ ΑΗΘ, ΒΗΘ δυσὶν ὀρθαῖς ἴσαι εἰσίν. [xαl] αἱ ἄρα ὑπὸ ΒΗΘ, ΗΘΔ δύο ὀρθῶν ἐλάσσονές εἰσιν. αἱ δὲ ἀπ² ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν· αἱ ἄρα ΑΒ, ΓΔ ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν· αἱ ἄρα ΑΒ, ΓΔ ἐκβαλλόμεναι εἰς ἄπειρον συμπέσοῦνται· οὐ συμπίπτουσι δὲ διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι· οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΑΗΘ τῆ ὑπὸ ΗΘΔ· ἴση ἄρα. ἀλλὰ ἡ ὑπὸ ΑΗΘ τῆ ὑπὸ ΕΗΒ ἐστιν ἴση· ×αὶ ἡ ὑπὸ ΕΗΒ ἄρα τῆ ὑπὸ ΗΘΔ ἐστιν ἴση· ×οινὴ προσχείσθω ἡ ὑπὸ ΒΗΘ· αἱ ἄρα ὑπὸ ΕΗΒ, ΒΗΘ τοῖς ὑπὸ ΒΗΘ, ΗΘΔ ἴσαι εἰσίν. ἀλλὰ αἱ ὑπὸ ΕΗΒ, ΒΗΘ δύο ὀρθαῖς ἴσαι εἰσίν· ×αὶ αἱ ὑπὸ ΒΗΘ, ΗΘΔ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν.

Ή ἄρα εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τάς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῆ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ

Thus, if a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two rightangles, then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Proposition 29

A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.



For let the straight-line EF fall across the parallel straight-lines AB and CD. I say that it makes the alternate angles, AGH and GHD, equal, the external angle EGB equal to the internal and opposite (angle) GHD, and the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles.

For if AGH is unequal to GHD then one of them is greater. Let AGH be greater. Let BGH have been added to both. Thus, (the sum of) AGH and BGH is greater than (the sum of) BGH and GHD. But, (the sum of) AGH and BGH is equal to two right-angles [Prop 1.13]. Thus, (the sum of) BGH and GHD is [also] less than two right-angles. But (straight-lines) being produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, AB and CD, being produced to infinity, will meet together. But they do not meet, on account of them (initially) being assumed parallel (to one another) [Def. 1.23]. Thus, AGH is not unequal to GHD. Thus, (it is) equal. But, AGH is equal to EGB [Prop. 1.15]. And EGB is thus also equal to GHD. Let BGH be added to both. Thus, (the sum of) EGB and BGH is equal to (the sum of) BGH and GHD. But, (the sum of) EGB and BGH is equal to two rightμέρη δυσίν ὀρθαῖς ἴσας. ὅπερ ἔδει δεῖξαι.

λ'.

Αἱ τῆ αὐτῆ εὐθεία παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι.



Έστω ἑκατέρα τῶν AB, ΓΔ τῆ ΕΖ παράλληλος· λέγω, ὅτι καὶ ἡ AB τῆ ΓΔ ἐστι παράλληλος.

Ἐμπιπτέτω γὰρ εἰς αὐτὰς εὐθεῖα ἡ ΗΚ.

Καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς AB, EZ εὐθεῖα ἐμπέπτωκεν ἡ HK, ἴση ἄρα ἡ ὑπὸ AHK τῆ ὑπὸ HΘΖ. πάλιν, ἐπεὶ εἰς παραλλήλους εὐθείας τὰς EZ, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ HK, ἴση ἐστὶν ἡ ὑπὸ HΘΖ τῆ ὑπὸ HKΔ. ἐδείχθη δὲ καὶ ἡ ὑπὸ AHK τῆ ὑπὸ HΘΖ ἴση. καὶ ἡ ὑπὸ AHK ἄρα τῆ ὑπὸ HKΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ. παράλληλος ἄρα ἐστὶν ἡ AB τῆ ΓΔ.

[Αἱ ἄρα τῆ αὐτῆ εὐθεία παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι·] ὅπερ ἔδει δεῖξαι.

λα΄.

Διὰ τοῦ δοθέντος σημείου τῆ δοθείσῃ εὐθεία παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Έστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ διὰ τοῦ Α σημείου τῆ ΒΓ εὐθεία παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω ἐπὶ τῆς ΒΓ τυχὸν σημεῖον τὸ Δ, καὶ ἐπεζεύχθω ἡ ΑΔ· καὶ συνεστάτω πρὸς τῆ ΔΑ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Α τῆ ὑπὸ ΑΔΓ γωνία ἴση ἡ ὑπὸ ΔΑΕ· καὶ

angles [Prop. 1.13]. Thus, (the sum of) BGH and GHD is also equal to two right-angles.

Thus, a straight-line falling across parallel straightlines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 30

(Straight-lines) parallel to the same straight-line are also parallel to one another.



Let each of the (straight-lines) AB and CD be parallel to EF. I say that AB is also parallel to CD.

For let the straight-line GK fall across (AB, CD, and EF).

And since the straight-line GK has fallen across the parallel straight-lines AB and EF, (angle) AGK (is) thus equal to GHF [Prop. 1.29]. Again, since the straight-line GK has fallen across the parallel straight-lines EF and CD, (angle) GHF is equal to GKD [Prop. 1.29]. But AGK was also shown (to be) equal to GHF. Thus, AGK is also equal to GKD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

[Thus, (straight-lines) parallel to the same straightline are also parallel to one another.] (Which is) the very thing it was required to show.

Proposition 31

To draw a straight-line parallel to a given straight-line, through a given point.

Let A be the given point, and BC the given straightline. So it is required to draw a straight-line parallel to the straight-line BC, through the point A.

Let the point D have been taken a random on BC, and let AD have been joined. And let (angle) DAE, equal to angle ADC, have been constructed on the straight-line

ἐκβεβλήσθω ἐπ' εὐθείας τῆ ΕΑ εὐθεῖα ἡ ΑΖ.



Καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΒΓ, ΕΖ εὐθεῖα ἐμπίπτουσα ἡ ΑΔ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΕΑΔ, ΑΔΓ ἴσας ἀλλήλαις πεποίηχεν, παράλληλος ἄρα ἐστὶν ἡ ΕΑΖ τῇ ΒΓ.

Διὰ τοῦ δοθέντος ἄρα σημείου τοῦ Α τῆ δοθείση εὐθεία τῆ ΒΓ παράλληλος εὐθεῖα γραμμὴ ῆκται ἡ ΕΑΖ· ὅπερ ἔδει ποιῆσαι.

λβ'.

Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.



Έστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΓΑΒ, ΑΒΓ, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ήχθω γὰρ διὰ τοῦ Γ σημείου τῆ ΑΒ εὐθεία παράλληλος ή ΓΕ.

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῆ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΕ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῆ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΕΓΔ ἴση ἐστὶ τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ ΑΒΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῆ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΔ γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΒΑΓ, ΑΒΓ. DA at the point A on it [Prop. 1.23]. And let the straightline AF have been produced in a straight-line with EA.



And since the straight-line AD, (in) falling across the two straight-lines BC and EF, has made the alternate angles EAD and ADC equal to one another, EAF is thus parallel to BC [Prop. 1.27].

Thus, the straight-line EAF has been drawn parallel to the given straight-line BC, through the given point A. (Which is) the very thing it was required to do.

Proposition 32

In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.



Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is equal to the (sum of the) two internal and opposite angles CAB and ABC, and the (sum of the) three internal angles of the triangle—ABC, BCA, and CAB—is equal to two right-angles.

For let CE have been drawn through point C parallel to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE, and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE, and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC. Thus, the whole anΚοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσίν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεχβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὅπερ ἔδει δεῖξαι.

λγ΄.

Αἱ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν.



Έστωσαν ίσαι τε καὶ παράλληλοι αἱ AB, ΓΔ, καὶ ἐπιζευγνύτωσαν αὐτὰς ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι αἱ AΓ, ΒΔ· λέγω, ὅτι καὶ αἱ AΓ, ΒΔ ἴσαι τε καὶ παράλληλοί εἰσιν.

Έπεζεύχθω ή ΒΓ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῆ ΓΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἰ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῆ ΓΔ κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνία τῆ ὑπὸ ΒΓΔ ἴση· βάσις ἄρα ἡ ΑΓ βάσει τῆ ΒΔ ἐστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρα, ὑφʾ ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῆ ὑπὸ ΓΒΔ. καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΑΓ, ΒΔ εὐθεῖα ἐμπίπτουσα ἡ ΒΓ τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ ΑΓ τῆ ΒΔ. ἐδείχθη δὲ αὐτῆ καὶ ἴση.

Αἱ ἄρα τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν. ὅπερ ἕδει δεῖξαι. gle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC.

Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles) ABC, BCA, and CAB. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ACB, CBA, and CAB is also equal to two right-angles.

Thus, in any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 33

Straight-lines joining equal and parallel (straightlines) on the same sides are themselves also equal and parallel.



Let AB and CD be equal and parallel (straight-lines), and let the straight-lines AC and BD join them on the same sides. I say that AC and BD are also equal and parallel.

Let BC have been joined. And since AB is parallel to CD, and BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. And since AB is equal to CD, and BCis common, the two (straight-lines) AB, BC are equal to the two (straight-lines) DC, CB.[†]And the angle ABCis equal to the angle BCD. Thus, the base AC is equal to the base BD, and triangle ABC is equal to triangle DCB^{\ddagger} , and the remaining angles will be equal to the corresponding remaining angles subtended by the equal sides [Prop. 1.4]. Thus, angle ACB is equal to CBD. Also, since the straight-line BC, (in) falling across the two straight-lines AC and BD, has made the alternate angles (ACB and CBD) equal to one another, AC is thus parallel to BD [Prop. 1.27]. And (AC) was also shown (to be) equal to (BD).

Thus, straight-lines joining equal and parallel (straight-

lines) on the same sides are themselves also equal and parallel. (Which is) the very thing it was required to show.

[†] The Greek text has "BC, CD", which is obviously a mistake. ^{\ddagger} The Greek text has "*DCB*", which is obviously a mistake.

λδ'.

τέμνει.



Έστω παραλληλόγραμμον χωρίον τὸ ΑΓΔΒ, διάμετρος δὲ αὐτοῦ ἡ ΒΓ· λέγω, ὅτι τοῦ ΑΓΔΒ παραλληλογράμμου αἱ άπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ ΒΓ διάμετρος αὐτὸ δίχα τέμνει.

Έπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῆ ΓΔ, καὶ εἰς αὐτὰς έμπέπτωκεν εύθεῖα ή ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. πάλιν ἐπεὶ παράλληλός ἐστιν ἡ ΑΓ τῆ ΒΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΓΒ, ΓΒΔ ἴσαι ἀλλήλαις εἰσίν. δύο δὴ τρίγωνά ἐστι τὰ ΑΒΓ, ΒΓΔ τὰς δύο γωνίας τὰς ὑπὸ ΑΒΓ, ΒΓΑ δυσὶ ταῖς ὑπὸ ΒΓΔ, ΓΒΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρα καὶ μίαν πλευράν μια πλευρα ίσην την πρός ταις ίσαις γωνίαις χοινην αὐτῶν τὴν ΒΓ· καὶ τὰς λοιπὰς ẳρα πλευρὰς ταῖς λοιπαῖς ίσας ἕξει ἑκατέραν ἑκατέρα καὶ τὴν λοιπὴν γωνίαν τῆ λοιπῆ γωνία ἴση ἄρα ἡ μὲν ΑΒ πλευρὰ τῆ ΓΔ, ἡ δὲ ΑΓ τῆ ΒΔ, καὶ ἔτι ἴση ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΓΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ABF γωνία τῃ ὑπὸ BFΔ, ἡ δὲ ὑπὸ FBΔ τῆ ὑπὸ ΑΓΒ, ὅλη ἄρα ἡ ὑπὸ ΑΒΔ ὅλῃ τῆ ὑπὸ ΑΓΔ ἐστιν ίση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΒ ἴση.

Τῶν ἄρα παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Λέγω δή, ὅτι καὶ ἡ διάμετρος αὐτὰ δίχα τέμνει. ἐπεὶ γὰρ ἴση ἐστὶν ἡ AB τ
ỹ ΓΔ, χοινὴ δὲ ἡ BΓ, δύο δὴ αἰ AB, BΓ δυσὶ ταῖς ΓΔ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ γωνία ἡ ύπὸ ΑΒΓ γωνία τῆ ὑπὸ ΒΓΔ ἴση. καὶ βάσις ἄρα ἡ ΑΓ τῆ ΔB ἴση. καὶ τὸ $AB\Gamma$ [ắρα] τρίγωνον τῷ $B\Gamma\Delta$ τριγώνῳ ἴσον ἐστίν.

Η άρα ΒΓ διάμετρος δίγα τέμνει τὸ ΑΒΓΔ παραλληλόγραμμον. ὅπερ ἔδει δεῖξαι.

Proposition 34

Tῶν παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί In parallelogrammic figures the opposite sides and angles τε και γωνίαι ίσαι άλλήλαις είσίν, και ή διάμετρος αύτα δίχα are equal to one another, and a diagonal cuts them in half.



Let ACDB be a parallelogrammic figure, and BC its diagonal. I say that for parallelogram ACDB, the opposite sides and angles are equal to one another, and the diagonal BC cuts it in half.

For since AB is parallel to CD, and the straight-line BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. Again, since AC is parallel to BD, and BC has fallen across them, the alternate angles ACB and CBD are equal to one another [Prop. 1.29]. So ABC and BCD are two triangles having the two angles ABC and BCA equal to the two (angles) BCD and CBD, respectively, and one side equal to one side-the (one) by the equal angles and common to them, (namely) BC. Thus, they will also have the remaining sides equal to the corresponding remaining (sides), and the remaining angle (equal) to the remaining angle [Prop. 1.26]. Thus, side AB is equal to CD, and AC to BD. Furthermore, angle BAC is equal to CDB. And since angle ABC is equal to BCD, and CBD to ACB, the whole (angle) ABD is thus equal to the whole (angle) ACD. And BAC was also shown (to be) equal to CDB.

Thus, in parallelogrammic figures the opposite sides and angles are equal to one another.

And, I also say that a diagonal cuts them in half. For since AB is equal to CD, and BC (is) common, the two (straight-lines) AB, BC are equal to the two (straightlines) DC, CB^{\dagger} , respectively. And angle ABC is equal to angle BCD. Thus, the base AC (is) also equal to DB,
and triangle ABC is equal to triangle BCD [Prop. 1.4]. Thus, the diagonal BC cuts the parallelogram $ACDB^{\ddagger}$ in half. (Which is) the very thing it was required to show.

[†] The Greek text has "*CD*, *BC*", which is obviously a mistake. [‡] The Greek text has "*ABCD*", which is obviously a mistake.

λε΄.

Τὰ παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.



Έστω παραλληλόγραμμα τὰ ΑΒΓΔ, ΕΒΓΖ ἐπὶ τῆς αὐτῆς βάσεως τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΖ, ΒΓ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔ τῷ ΕΒΓΖ παραλληλογράμμῳ.

Έπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΑΒΓΔ, ἴση ἐστὶν ἡ ΑΔ τῆ ΒΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΕΖ τῆ ΒΓ ἐστιν ἴση· ὥστε καὶ ἡ ΑΔ τῆ ΕΖ ἐστιν ἴση· καὶ κοινὴ ἡ ΔΕ· ὅλη ἄρα ἡ ΑΕ ὅλη τῆ ΔΖ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΑΒ τῆ ΔΓ ἴση· δύο δὴ αἱ ΕΑ, ΑΒ δύο ταῖς ΖΔ, ΔΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα· καὶ γωνία ἡ ὑπὸ ΖΔΓ γωνία τῆ ὑπὸ ΕΑΒ ἐστιν ἴση ἡ ἐκτὸς τῆ ἐντός· βάσις ἄρα ἡ ΕΒ βάσει τῆ ΖΓ ἴση ἐστίν, καὶ τὸ ΕΑΒ τρίγωνον τῷ ΔΖΓ τριγώνῳ ἴσον ἔσται· κοινὸν ἀφηρήσϑω τὸ ΔΗΕ· λοιπὸν ἄρα τὸ ΑΒΗΔ τραπέζιον λοιπῷ τῷ ΕΗΓΖ τραπεζίῳ ἐστὶν ἴσον· κοινὸν προσκείσϑω τὸ ΗΒΓ τρίγωνον· ὅλον ἄρα τὸ ΑΒΓΔ παραλληλόγραμμον ὅλῳ τῷ ΕΒΓΖ παραλληλογράμμῳ ἴσον ἐστίν.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 35

Parallelograms which are on the same base and between the same parallels are equal[†] to one another.



Let ABCD and EBCF be parallelograms on the same base BC, and between the same parallels AF and BC. I say that ABCD is equal to parallelogram EBCF.

For since ABCD is a parallelogram, AD is equal to BC [Prop. 1.34]. So, for the same (reasons), EF is also equal to BC. So AD is also equal to EF. And DE is common. Thus, the whole (straight-line) AE is equal to the whole (straight-line) DF. And AB is also equal to DC. So the two (straight-lines) EA, AB are equal to the two (straight-lines) FD, DC, respectively. And angle FDC is equal to angle EAB, the external to the internal [Prop. 1.29]. Thus, the base EB is equal to the base FC, and triangle EAB will be equal to triangle DFC [Prop. 1.4]. Let DGE have been taken away from both. Thus, the remaining trapezium ABGD is equal to the remaining trapezium EGCF. Let triangle GBC have been added to both. Thus, the whole parallelogram ABCD is equal to the whole parallelogram EBCF.

Thus, parallelograms which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

 † Here, for the first time, "equal" means "equal in area", rather than "congruent".

אק'.

Τὰ παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Έστω παραλληλόγραμμα τὰ ΑΒΓΔ, ΕΖΗΘ ἐπὶ ἴσων βάσεων ὄντα τῶν ΒΓ, ΖΗ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΘ, ΒΗ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔ παραλ-

Proposition 36

Parallelograms which are on equal bases and between the same parallels are equal to one another.

Let ABCD and EFGH be parallelograms which are on the equal bases BC and FG, and (are) between the same parallels AH and BG. I say that the parallelogram



Έπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῆ ΖΗ, ἀλλὰ ἡ ΖΗ τῆ ΕΘ ἐστιν ἴση, καὶ ἡ ΒΓ ἄρα τῆ ΕΘ ἐστιν ἴση. εἰσὶ δὲ καὶ παράλληλοι. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΕΒ, ΘΓ· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι ἴσαι τε καὶ παράλληλοί εἰσι [καὶ αἱ ΕΒ, ΘΓ ἄρα ἴσαι τέ εἰσι καὶ παράλληλοι]. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΕΒΓΘ. καί ἐστιν ἴσον τῷ ΑΒΓΔ· βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει τὴν ΒΓ, καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστὶν αὐτῷ ταῖς ΒΓ, ΑΘ. δὶα τὰ αὐτὰ δὴ καὶ τὸ ΕΖΗΘ τῷ αὐτῷ τῷ ΕΒΓΘ ἐστιν ἴσον. ὥστε καὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΖΗΘ ἐστιν ἴσον.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖζαι.

λζ'.



Έστω τρίγωνα τὰ ABΓ, Δ BΓ ἐπὶ τῆς αὐτῆς βάσεως τῆς BΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς AΔ, BΓ· λέγω, ὅτι ἴσον ἐστὶ τὸ ABΓ τρίγωνον τῷ Δ BΓ τριγώνῳ.

Έκβεβλήσθω ή ΑΔ ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Ε, Ζ, καὶ διὰ μὲν τοῦ Β τῆ ΓΑ παράλληλος ἤχθω ή ΒΕ, δὶα δὲ τοῦ Γ τῆ ΒΔ παράλληλος ἤχθω ή ΓΖ. παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΕΒΓΑ, ΔΒΓΖ· καί εἰσιν ἴσα· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΕΖ· καί ἐστι τοῦ μὲν ΕΒΓΑ παραλληλογράμμου ἤμισυ τὸ ΑΒΓ τρίγωνον· ἡ γὰρ ΑΒ διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ ΔΒΓΖ παραλληλογράμμου ἤμισυ τὸ ΔΒΓ τρίγωνον· ἡ γὰρ ΔΓ διάμετρος αὐτὸ δίχα τέμνει. [τὰ δὲ



For let BE and CH have been joined. And since BC is equal to FG, but FG is equal to EH [Prop. 1.34], BC is thus equal to EH. And they are also parallel, and EB and HC join them. But (straight-lines) joining equal and parallel (straight-lines) on the same sides are (themselves) equal and parallel [Prop. 1.33] [thus, EB and HC are also equal and parallel]. Thus, EBCH is a parallelogram [Prop. 1.34], and is equal to ABCD. For it has the same base, BC, as (ABCD), and is between the same parallels, BC and AH, as (ABCD) [Prop. 1.35]. So, for the same (reasons), EFGH is also equal to the same (parallelogram) EBCH [Prop. 1.34]. So that the parallelogram ABCD is also equal to EFGH.

Thus, parallelograms which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 37

Triangles which are on the same base and between the same parallels are equal to one another.



Let ABC and DBC be triangles on the same base BC, and between the same parallels AD and BC. I say that triangle ABC is equal to triangle DBC.

Let AD have been produced in both directions to E and F, and let the (straight-line) BE have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) CF have been drawn through C parallel to BD [Prop. 1.31]. Thus, EBCA and DBCF are both parallelograms, and are equal. For they are on the same base BC, and between the same parallels BC and EF [Prop. 1.35]. And the triangle ABC is half of the parallelogram EBCA. For the diagonal AB cuts the latter in

τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΒΓ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

[†] This is an additional common notion.

λη'.

Τὰ τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.



 Ἐστω τρίγωνα τὰ ABΓ, ΔΕΖ ἐπὶ ἴσων βάσεων τῶν BΓ,
 ΕΖ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς BΖ, ΑΔ· λέγω, ὅτι
 ἴσον ἐστὶ τὸ ABΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Έκβεβλήσθω γὰρ ἡ ΑΔ ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Η, Θ, καὶ διὰ μὲν τοῦ Β τῆ ΓΑ παράλληλος ἤχθω ἡ ΒΗ, δὶα δὲ τοῦ Ζ τῆ ΔΕ παράλληλος ἤχθω ἡ ΖΘ. παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΗΒΓΑ, ΔΕΖΘ· καὶ ἴσον τὸ ΗΒΓΑ τῷ ΔΕΖΘ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΓ, ΕΖ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΖ, ΗΘ· καί ἐστι τοῦ μὲν ΗΒΓΑ παραλληλογράμμου ἤμισυ τὸ ΑΒΓ τρίγωνον. ἡ γὰρ ΑΒ διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ ΔΕΖΘ παραλληλογράμμου ἤμισυ τὸ ΖΕΔ τρίγωνον· ἡ γὰρ ΔΖ δίαμετρος αὐτὸ δίχα τέμνει [τὰ δὲ τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λθ'.

Τὰ ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Έστω ίσα τρίγωνα τὰ ABΓ, Δ BΓ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη τῆς BΓ· λέγω, ὅτι καὶ ἐν ταῖς

half [Prop. 1.34]. And the triangle DBC (is) half of the parallelogram DBCF. For the diagonal DC cuts the latter in half [Prop. 1.34]. [And the halves of equal things are equal to one another.][†] Thus, triangle ABC is equal to triangle DBC.

Thus, triangles which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 38

Triangles which are on equal bases and between the same parallels are equal to one another.



Let ABC and DEF be triangles on the equal bases BC and EF, and between the same parallels BF and AD. I say that triangle ABC is equal to triangle DEF.

For let AD have been produced in both directions to G and H, and let the (straight-line) BG have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) FH have been drawn through F parallel to DE [Prop. 1.31]. Thus, GBCA and DEFH are each parallelograms. And GBCA is equal to DEFH. For they are on the equal bases BC and EF, and between the same parallels BF and GH [Prop. 1.36]. And triangle ABC is half of the parallelogram GBCA. For the diagonal AB cuts the latter in half [Prop. 1.34]. And triangle FED (is) half of parallelogram DEFH. For the diagonal DF cuts the latter in half. [And the halves of equal things are equal to one another.] Thus, triangle ABC is equal to triangle DEF.

Thus, triangles which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 39

Equal triangles which are on the same base, and on the same side, are also between the same parallels.

Let ABC and DBC be equal triangles which are on the same base BC, and on the same side (of it). I say that αὐταῖς παραλλήλοις ἐστίν.



Έπεζεύχθω γὰρ ἡ ΑΔ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΔ τῆ ΒΓ.

Εἰ γὰρ μή, ἤχθω διὰ τοῦ Α σημείου τῆ ΒΓ εὐθεία παράλληλος ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΕΓ. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις. ἀλλὰ τὸ ΑΒΓ τῷ ΔΒΓ ἐστιν ἴσον· καὶ τὸ ΔΒΓ ἄρα τῷ ΕΒΓ ἴσον ἐστὶ τὸ μεῖζον τῷ ἐλάσσονι· ὅπερ ἐστιν ἀδύνατον· οὐκ ἄρα παράλληλός ἐστιν ἡ ΑΕ τῆ ΒΓ. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς ΑΔ· ἡ ΑΔ ἄρα τῆ ΒΓ ἐστι παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μ΄.

Τὰ ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.



Έστω ίσα τρίγωνα τὰ ΑΒΓ, ΓΔΕ ἐπὶ ίσων βάσεων τῶν ΒΓ, ΓΕ καὶ ἐπὶ τὰ αὐτὰ μέρη. λέγω, ὅτι καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Έπεζεύχθω γὰρ ἡ ΑΔ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΔ τῆ ΒΕ.

Εἰ γὰρ μή, ἤχθω διὰ τοῦ Α τῆ ΒΕ παράλληλος ἡ ΑΖ, καὶ ἐπεζεύχθω ἡ ΖΕ. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΖΓΕ τριγώνῳ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΓ, ΓΕ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΕ, ΑΖ. ἀλλὰ τὸ ΑΒΓ τρίγωνον ἴσον ἐστὶ τῷ ΔΓΕ [τρίγωνῳ]· καὶ τὸ ΔΓΕ ἄρα [τρίγωνον] ἴσον ἐστὶ τῷ ΖΓΕ τριγώνῳ τὸ μεῖζον τῷ they are also between the same parallels.



For let AD have been joined. I say that AD and BC are parallel.

For, if not, let AE have been drawn through point A parallel to the straight-line BC [Prop. 1.31], and let EC have been joined. Thus, triangle ABC is equal to triangle EBC. For it is on the same base as it, BC, and between the same parallels [Prop. 1.37]. But ABC is equal to DBC. Thus, DBC is also equal to EBC, the greater to the lesser. The very thing is impossible. Thus, AE is not parallel to BC. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BC.

Thus, equal triangles which are on the same base, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

Proposition 40[†]

Equal triangles which are on equal bases, and on the same side, are also between the same parallels.



Let ABC and CDE be equal triangles on the equal bases BC and CE (respectively), and on the same side (of BE). I say that they are also between the same parallels.

For let *AD* have been joined. I say that *AD* is parallel to *BE*.

For if not, let AF have been drawn through A parallel to BE [Prop. 1.31], and let FE have been joined. Thus, triangle ABC is equal to triangle FCE. For they are on equal bases, BC and CE, and between the same parallels, BE and AF [Prop. 1.38]. But, triangle ABC is equal ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλος ἡ ΑΖ τῆ ΒΕ. ὑμοίως δὴ δείξομεν, ὅτι οὐδ² ἄλλη τις πλὴν τῆς ΑΔ· ἡ ΑΔ ἄρα τῆ ΒΕ ἐστι παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι. to [triangle] DCE. Thus, [triangle] DCE is also equal to triangle FCE, the greater to the lesser. The very thing is impossible. Thus, AF is not parallel to BE. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BE.

Thus, equal triangles which are on equal bases, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

[†] This whole proposition is regarded by Heiberg as a relatively early interpolation to the original text.

μα΄.

Έὰν παραλληλόγραμμον τριγώνω βάσιν τε ἔχη τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ῆ, διπλάσιόν ἐστί τὸ παραλληλόγραμμον τοῦ τριγώνου.



Παραλληλόγραμμον γὰρ τὸ ΑΒΓΔ τριγώνω τῷ ΕΒΓ βάσιν τε ἐχέτω τὴν αὐτὴν τὴν ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἔστω ταῖς ΒΓ, ΑΕ· λέγω, ὅτι διπλάσιόν ἐστι τὸ ΑΒΓΔ παραλληλόγραμμον τοῦ ΒΕΓ τριγώνου.

Έπεζεύχθω γὰρ ἡ ΑΓ. ἴσον δή ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνω· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΕ. ἀλλὰ τὸ ΑΒΓΔ παραλληλόγραμμον διπλάσιόν ἐστι τοῦ ΑΒΓ τριγώνου· ἡ γὰρ ΑΓ διάμετρος αὐτὸ δίχα τέμνει· ὥστε τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τοῦ ΕΒΓ τριγώνου ἐστὶ διπλάσιον.

Έλν ἄρα παραλληλόγραμμον τριγώνω βάσιν τε ἕχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἢ, διπλάσιόν ἐστί τὸ παραλληλόγραμμον τοῦ τριγώνου· ὅπερ ἔδει δεῖξαι.

μβ΄.

Τῷ δοθέντι τριγώνω ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῆ δοθείση γωνία εὐθυγράμμω.

Έστω τὸ μὲν δοθὲν τρίγωνον τὸ ABΓ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ τῷ ABΓ τριγώνῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῆ Δ γωνία εὐθυγράμμῳ.

Proposition 41

If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.



For let parallelogram ABCD have the same base BC as triangle EBC, and let it be between the same parallels, BC and AE. I say that parallelogram ABCD is double (the area) of triangle BEC.

For let AC have been joined. So triangle ABC is equal to triangle EBC. For it is on the same base, BC, as (EBC), and between the same parallels, BC and AE[Prop. 1.37]. But, parallelogram ABCD is double (the area) of triangle ABC. For the diagonal AC cuts the former in half [Prop. 1.34]. So parallelogram ABCD is also double (the area) of triangle EBC.

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

Proposition 42

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let ABC be the given triangle, and D the given rectilinear angle. So it is required to construct a parallelogram equal to triangle ABC in the rectilinear angle D.



Τετμήσθω ή BΓ δίχα κατὰ τὸ Ε, καὶ ἐπεζεύχθω ή ΑΕ, καὶ συνεστάτω πρὸς τῆ ΕΓ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Ε τῆ Δ γωνία ἴση ἡ ὑπὸ ΓΕΖ, καὶ διὰ μὲν τοῦ Α τῆ ΕΓ παράλληλος ἤχθω ἡ ΑΗ, διὰ δὲ τοῦ Γ τῆ ΕΖ παράλληλος ἤχθω ἡ ΓΗ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΕΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ BE τῆ ΕΓ, ἴσον ἐστὶ καὶ τὸ ΑBE τρίγωνον τῷ ΑΕΓ τριγώνω· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν BE, ΕΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς BΓ, ΑΗ· διπλάσιον ἄρα ἐστὶ τὸ ABΓ τρίγωνον τοῦ ΑΕΓ τριγώνου. ἔστι δὲ καὶ τὸ ΖΕΓΗ παραλληλόγραμμον διπλάσιον τοῦ ΑΕΓ τριγώνου· βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει καὶ ἐν ταῖς αὐταῖς ἐστιν αὐτῷ παραλλήλοις· ἴσον ἄρα ἑστὶ τὸ ΖΕΓΗ παραλληλόγραμμον τῷ ΑΒΓ τριγώνω. καὶ ἔχει τὴν ὑπὸ ΓΕΖ γωνίαν ἴσην τῆ δοθείσῃ τῆ Δ.

Τῷ ἄρα δοθέντι τριγώνω τῷ ΑΒΓ ἴσον παραλληλόγραμμον συνέσταται τὸ ΖΕΓΗ ἐν γωνία τῆ ὑπὸ ΓΕΖ, ἥτις ἐστὶν ἴση τῆ Δ· ὅπερ ἔδει ποιῆσαι.

μγ΄.

Παντὸς παραλληλογράμμου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν.

Έστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα μὲν ἔστω τὰ ΕΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΒΚ παραπλήρωμα τῷ ΚΔ παραπληρώματι.

Έπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΓΔ τριγώνῳ. πάλιν, ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΕΘ, διάμετρος δὲ αὐτοῦ ἐστιν ἡ ΑΚ, ἴσον ἐστὶ τὸ ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΚΖΓ τρίγωνον τῷ ΚΗΓ ἐστιν ἴσον. ἐπεὶ οῦν τὸ μὲν ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ ἐστὶν ἴσον, τὸ δὲ ΚΖΓ τῷ ΚΗΓ, τὸ ΑΕΚ τρίγωνον μετὰ τοῦ ΚΗΓ ἴσον ἐστὶ τῷ ΑΘΚ τριγώνῳ μετὰ τοῦ ΚΖΓ. ἔστι δὲ καὶ ὅλον τὸ ΑΒΓ τρίγωνον ὅλῳ τῷ ΑΔΓ ἴσον. λοιπὸν ἄρα τὸ ΒΚ παραπλήρωμα λοιπῷ τῷ ΚΔ παρα-



Let BC have been cut in half at E [Prop. 1.10], and let AE have been joined. And let (angle) CEF, equal to angle D, have been constructed at the point E on the straight-line EC [Prop. 1.23]. And let AG have been drawn through A parallel to EC [Prop. 1.31], and let CGhave been drawn through C parallel to EF [Prop. 1.31]. Thus, FECG is a parallelogram. And since BE is equal to EC, triangle ABE is also equal to triangle AEC. For they are on the equal bases, BE and EC, and between the same parallels, BC and AG [Prop. 1.38]. Thus, triangle ABC is double (the area) of triangle AEC. And parallelogram *FECG* is also double (the area) of triangle AEC. For it has the same base as (AEC), and is between the same parallels as (AEC) [Prop. 1.41]. Thus, parallelogram FECG is equal to triangle ABC. (FECG) also has the angle CEF equal to the given (angle) D.

Thus, parallelogram FECG, equal to the given triangle ABC, has been constructed in the angle CEF, which is equal to D. (Which is) the very thing it was required to do.

Proposition 43

For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

Let ABCD be a parallelogram, and AC its diagonal. And let EH and FG be the parallelograms about AC, and BK and KD the so-called complements (about AC). I say that the complement BK is equal to the complement KD.

For since ABCD is a parallelogram, and AC its diagonal, triangle ABC is equal to triangle ACD [Prop. 1.34]. Again, since EH is a parallelogram, and AK is its diagonal, triangle AEK is equal to triangle AHK [Prop. 1.34]. So, for the same (reasons), triangle KFC is also equal to (triangle) KGC. Therefore, since triangle AEK is equal to triangle AHK, and KFC to KGC, triangle AEK plus KGC is equal to triangle AHK plus KFC. And the whole triangle ABC is also equal to the whole (triangle) ADC. Thus, the remaining complement BK is equal to

πληρώματί ἐστιν ἴσον.



Παντὸς ἄρα παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μδ'.

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι τριγώνῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν τῆ δοθείσῃ γωνία εὐθυγράμ- a given straight-line in a given rectilinear angle. μω.



Έστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν τρίγωνον τὸ Γ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνω τῷ Γ ίσον παραλληλόγραμμον παραβαλεῖν ἐν ἴσῃ τῇ Δ γωνία.

Συνεστάτω τῷ Γ τριγώνω ἴσον παραλληλόγραμμον τὸ BEZH ἐν γωνία τῆ ὑπὸ EBH,
 ἤ ἐστιν ἴση τῆ Δ· καὶ κείσθω ώστε ἐπ' εὐθείας εἶναι τὴν ΒΕ τῆ ΑΒ, καὶ διήχθω ἡ ΖΗ έπὶ τὸ Θ, καὶ διὰ τοῦ Α ὁποτέρα τῶν ΒΗ, ΕΖ παράλληλος ήχθω ή ΑΘ, καὶ ἐπεζεύχθω ή ΘΒ. καὶ ἐπεὶ εἰς παραλλήλους τὰς AΘ, EZ εὐθεῖα ἐνέπεσεν ἡ ΘΖ, αἱ ἄρα ὑπὸ AΘΖ, ΘΖΕ γωνίαι δυσὶν ὀρθαῖς εἰσιν ἴσαι. αἱ ἄρα ὑπὸ ΒΘΗ, ΗΖΕ δύο όρθῶν ἐλάσσονές εἰσιν· αί δὲ ἀπὸ ἐλασσόνων ἢ δύο όρθῶν εἰς ἄπειρον ἐκβαλλόμεναι συμπίπτουσιν αί ΘΒ, ΖΕ



Thus, for any parallelogramic figure, the complements of the parallelograms about the diagonal are equal to one another. (Which is) the very thing it was required to show.

Proposition 44

To apply a parallelogram equal to a given triangle to



Let AB be the given straight-line, C the given triangle, and D the given rectilinear angle. So it is required to apply a parallelogram equal to the given triangle C to the given straight-line AB in an angle equal to (angle) D.

Let the parallelogram BEFG, equal to the triangle C, have been constructed in the angle EBG, which is equal to D [Prop. 1.42]. And let it have been placed so that BE is straight-on to AB.[†] And let FG have been drawn through to H, and let AH have been drawn through A parallel to either of BG or EF [Prop. 1.31], and let HBhave been joined. And since the straight-line HF falls across the parallels AH and EF, the (sum of the) angles AHF and HFE is thus equal to two right-angles άρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Κ, καὶ διὰ τοῦ Κ σημείου ὁποτέρα τῶν ΕΑ, ΖΘ παράλληλος ἦχθω ἡ ΚΛ, καὶ ἐκβεβλήσθωσαν αἱ ΘΑ, ΗΒ ἐπὶ τὰ Λ, Μ σημεῖα. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΛΚΖ, διάμετρος δὲ αὐτοῦ ἡ ΘΚ, περὶ δὲ τὴν ΘΚ παραλληλόγραμμα μὲν τὰ ΑΗ, ΜΕ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΛΒ, ΒΖ· ἴσον ἄρα ἐστὶ τὸ ΛΒ τῷ ΒΖ. ἀλλὰ τὸ ΒΖ τῷ Γ τριγώνῳ ἐστὶν ἴσον· καὶ τὸ ΛΒ ἄρα τῷ Γ ἐστιν ἴσον. καὶ ἐπεὶ ἴση ἐστιν ἤ ὑπὸ ΗΒΕ γωνία τῆ ὑπὸ ABM, ἀλλὰ ἡ ὑπὸ ΗΒΕ τῆ Δ ἐστιν ἴση, καὶ ἡ ὑπὸ ABM ἄρα τῆ Δ γωνία ἐστὶν ἴση.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν AB τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ AB ἐν γωνία τῇ ὑπὸ ABM, ἦ ἐστιν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

[†] This can be achieved using Props. 1.3, 1.23, and 1.31.

με΄.

Τῷ δοθέντι εὐθυγράμμῷ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῆ δοθείση γωνία εὐθυγράμμῳ.

Έστω τὸ μὲν δοθὲν εὐθύγραμμον τὸ ΑΒΓΔ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Ε· δεῖ δὴ τῷ ΑΒΓΔ εὐθυγράμμω ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῆ δοθείσῃ γωνία τῆ Ε.

Ἐπεζεύχθω ἡ ΔΒ, καὶ συνεστάτω τῷ ΑΒΔ τριγώνῷ ἴσον παραλληλόγραμμον τὸ ΖΘ ἐν τῆ ὑπὸ ΘΚΖ γωνία, ἥ έστιν ἴση τῆ Ε· καὶ παραβεβλήσθω παρὰ τὴν ΗΘ εὐθεῖαν τῷ ΔΒΓ τριγώνω ίσον παραλληλόγραμμον τὸ ΗΜ ἐν τῆ ὑπὸ ΗΘΜ γωνία, ή ἐστιν ἴση τῆ Ε. καὶ ἐπεὶ ἡ Ε γωνία ἑκατέρα τῶν ὑπὸ ΘΚΖ, ΗΘΜ ἐστιν ἴση, καὶ ἡ ὑπὸ ΘΚΖ ἄρα τῇ ὑπὸ ΗΘΜ ἐστιν ἴση. κοινὴ προσκείσθω ἡ ὑπὸ ΚΘΗ· αἱ ἄρα ύπὸ ΖΚΘ, ΚΘΗ ταῖς ὑπὸ ΚΘΗ, ΗΘΜ ἴσαι εἰσίν. ἀλλ' αἱ ύπὸ ΖΚΘ, ΚΘΗ δυσὶν ὀρθαῖς ἴσαι εἰσίν καὶ ἀί ὑπὸ ΚΘΗ, ΗΘΜ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν. πρὸς δή τινι εὐθεῖα τῆ ΗΘ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ δύο εὐθεῖαι αἱ ΚΘ, ΘΜ μὴ έπὶ τὰ αὐτὰ μέρη χείμεναι τὰς ἐφεξῆς γωνίας δύο ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΚΘ τῆ ΘΜ· καὶ ἐπεὶ εἰς παραλλήλους τὰς KM, ZH εὐθεῖα ἐνέπεσεν ἡ ΘΗ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΜΘΗ, ΘΗΖ ἴσαι ἀλλήλαις εἰσίν. κοινὴ προσκείσθω ἡ ὑπὸ ΘΗΛ· αἰ ἄρα ὑπὸ ΜΘΗ, ΘΗΛ ταῖς ύπὸ ΘΗΖ, ΘΗΛ ἴσαι εἰσιν. ἀλλ' αι ὑπὸ ΜΘΗ, ΘΗΛ δύο όρθαῖς ἴσαι εἰσίν καὶ αἱ ὑπὸ ΘΗΖ, ΘΗΛ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΖΗ τῆ ΗΛ. ϰαὶ ἐπεὶ ἡ ΖΚ τῆ ΘΗ ἴση τε καὶ παράλληλός ἐστιν, ἀλλὰ καὶ ἡ ΘΗ τῆ ΜΛ, καὶ ἡ ΚΖ ἄρα τῆ ΜΛ ἴση τε καὶ παράλληλός ἐστιν· καὶ

[Prop. 1.29]. Thus, (the sum of) *BHG* and *GFE* is less than two right-angles. And (straight-lines) produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced, HB and FE will meet together. Let them have been produced, and let them meet together at K. And let KL have been drawn through point K parallel to either of EA or FH [Prop. 1.31]. And let HA and GB have been produced to points L and M (respectively). Thus, HLKF is a parallelogram, and HK its diagonal. And AG and ME (are) parallelograms, and LB and BF the so-called complements, about HK. Thus, LB is equal to BF [Prop. 1.43]. But, BF is equal to triangle C. Thus, LB is also equal to C. Also, since angle GBE is equal to ABM [Prop. 1.15], but GBE is equal to D, ABM is thus also equal to angle D.

Thus, the parallelogram LB, equal to the given triangle C, has been applied to the given straight-line AB in the angle ABM, which is equal to D. (Which is) the very thing it was required to do.

Proposition 45

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Let ABCD be the given rectilinear figure,[†] and E the given rectilinear angle. So it is required to construct a parallelogram equal to the rectilinear figure ABCD in the given angle E.

Let DB have been joined, and let the parallelogram FH, equal to the triangle ABD, have been constructed in the angle HKF, which is equal to E [Prop. 1.42]. And let the parallelogram GM, equal to the triangle DBC, have been applied to the straight-line GH in the angle GHM, which is equal to E [Prop. 1.44]. And since angle E is equal to each of (angles) HKF and GHM, (angle) HKF is thus also equal to GHM. Let KHG have been added to both. Thus, (the sum of) FKH and KHGis equal to (the sum of) KHG and GHM. But, (the sum of) FKH and KHG is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) KHG and GHM is also equal to two right-angles. So two straight-lines, KHand HM, not lying on the same side, make adjacent angles with some straight-line GH, at the point H on it, (whose sum is) equal to two right-angles. Thus, KH is straight-on to HM [Prop. 1.14]. And since the straightline HG falls across the parallels KM and FG, the alternate angles MHG and HGF are equal to one another [Prop. 1.29]. Let HGL have been added to both. Thus, (the sum of) MHG and HGL is equal to (the sum of) ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ KM, ΖΛ· καὶ αἱ KM, ΖΛ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΚΖΛΜ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΒΔ τρίγωνον τῷ ΖΘ παραλληλογράμμῳ, τὸ δὲ ΔΒΓ τῷ HM, ὅλον ἄρα τὸ ΑΒΓΔ εὐθύγραμμον ὅλῳ τῷ ΚΖΛΜ παραλληλογράμμῳ ἐστὶν ἴσον.



Τῷ ἄρα δοθέντι εὐθυγράμμω τῷ ΑΒΓΔ ἴσον παραλληλόγραμμον συνέσταται τὸ ΚΖΛΜ ἐν γωνία τῆ ὑπὸ ΖΚΜ, ἤ ἐστιν ἴση τῆ δοθείσῃ τῆ Ε· ὅπερ ἔδει ποιῆσαι.

HGF and HGL. But, (the sum of) MHG and HGL is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) HGF and HGL is also equal to two right-angles. Thus, FG is straight-on to GL [Prop. 1.14]. And since FK is equal and parallel to HG [Prop. 1.34], but also HG to ML [Prop. 1.34], KF is thus also equal and parallel to ML [Prop. 1.30]. And the straight-lines KM and FLjoin them. Thus, KM and FL are equal and parallel as well [Prop. 1.33]. Thus, KFLM is a parallelogram. And since triangle ABD is equal to parallelogram FH, and DBC to GM, the whole rectilinear figure ABCD is thus equal to the whole parallelogram KFLM.



Thus, the parallelogram KFLM, equal to the given rectilinear figure ABCD, has been constructed in the angle FKM, which is equal to the given (angle) E. (Which is) the very thing it was required to do.

[†] The proof is only given for a four-sided figure. However, the extension to many-sided figures is trivial.

Άπὸ τῆς δοθείσης εὐθείας τετράγωνον ἀναγράψαι.

Έστω ή δοθεϊσα εὐθεῖα ή AB· δεῖ δὴ ἀπὸ τῆς AB εὐθείας τετράγωνον ἀναγράψαι.

Ήχθω τῆ AB εὐθεία ἀπὸ τοῦ πρὸς αὐτῆ σημείου τοῦ A πρὸς ὀρθὰς ἡ AΓ, καὶ κείσθω τῆ AB ἴση ἡ AΔ· καὶ διὰ μὲν τοῦ Δ σημείου τῆ AB παράλληλος ἤχθω ἡ ΔΕ, διὰ δὲ τοῦ B σημείου τῆ AΔ παράλληλος ἤχθω ἡ BE. παραλληλόγραμμον ἄρα ἐστὶ τὸ AΔEB· ἴση ἄρα ἐστὶν ἡ μὲν AB τῆ ΔΕ, ἡ δὲ AΔ τῆ BE. ἀλλὰ ἡ AB τῆ AΔ ἐστιν ἴση· αἱ τέσσαρες ἄρα αἱ BA, AΔ, ΔΕ, EB ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ τὸ AΔEB παραλληλόγραμμον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ εἰς παραλλήλους τὰς AB, ΔΕ εὐθεῖα ἐνέπεσεν ἡ AΔ, αἱ ἄρα ὑπὸ BAΔ, ΑΔΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ BAΔ· ὀρθὴ ἄρα καὶ

Proposition 46

To describe a square on a given straight-line.

Let AB be the given straight-line. So it is required to describe a square on the straight-line AB.

Let AC have been drawn at right-angles to the straight-line AB from the point A on it [Prop. 1.11], and let AD have been made equal to AB [Prop. 1.3]. And let DE have been drawn through point D parallel to AB [Prop. 1.31], and let BE have been drawn through point B parallel to AD [Prop. 1.31]. Thus, ADEB is a parallelogram. Therefore, AB is equal to DE, and AD to BE [Prop. 1.34]. But, AB is equal to AD. Thus, the four (sides) BA, AD, DE, and EB are equal to one another. Thus, the parallelogram ADEB is equilateral. So I say that (it is) also right-angled. For since the straight-line

ή ὑπὸ ΑΔΕ. τῶν δὲ παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὀρθὴ ἄρα καὶ ἑκατέρα τῶν ἀπεναντίον τῶν ὑπὸ ΑΒΕ, ΒΕΔ γωνιῶν· ὀρθογώνιον ἄρα ἐστὶ τὸ ΑΔΕΒ. ἐδείχθη δὲ καὶ ἰσόπλευρον.



Τετράγωνον ἄρα ἐστίν· καί ἐστιν ἀπὸ τῆς AB εὐθείας ἀναγεγραμμένον· ὅπερ ἔδει ποιῆσαι.

μζ΄.

Έν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

Έστω τρίγωνον ὀρθογώνιον τὸ ABΓ ὀρθὴν ἔχον τὴν ὑπὸ BAΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς BΓ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν BA, ΑΓ τετραγώνοις.

Άναγεγράφθω γὰρ ἀπὸ μὲν τῆς ΒΓ τετράγωνον τὸ ΒΔΕΓ, ἀπὸ δὲ τῶν ΒΑ, ΑΓ τὰ ΗΒ, ΘΓ, καὶ διὰ τοῦ Α ὁποτέρα τῶν ΒΔ, ΓΕ παράλληλος ἤχθω ἡ ΑΛ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΖΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΒΑΓ, ΒΑΗ γωνιῶν, πρὸς δή τινι εὐθεία τῆ ΒΑ καὶ τῷ πρὸς αὐτῆ σημείω τῷ Α δύο εὐθεῖαι αἱ ΑΓ, ΑΗ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΓΑ τῆ ΑΗ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΒΑ τῆ ΑΘ ἐστιν ἐπ' εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῆ ὑπὸ ΖΒΑ· ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· ὅλη ἄρα ἡ ὑπὸ ΔΒΑ ὅλῃ τῆ ὑπὸ ΖΒΓ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῆ ΒΓ, ἡ δὲ ΖΒ τῆ ΒΑ, δύο δὴ αἱ ΔΒ, ΒΑ δύο ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα· καὶ γωνία ἡ ὑπὸ ΔΒΑ γωνία τῆ ὑπὸ ΖΒΓ AD falls across the parallels AB and DE, the (sum of the) angles BAD and ADE is equal to two right-angles [Prop. 1.29]. But BAD (is a) right-angle. Thus, ADE (is) also a right-angle. And for parallelogrammic figures, the opposite sides and angles are equal to one another [Prop. 1.34]. Thus, each of the opposite angles ABE and BED (are) also right-angles. Thus, ADEB is right-angled. And it was also shown (to be) equilateral.



Thus, (ADEB) is a square [Def. 1.22]. And it is described on the straight-line AB. (Which is) the very thing it was required to do.

Proposition 47

In right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides containing the right-angle.

Let ABC be a right-angled triangle having the angle BAC a right-angle. I say that the square on BC is equal to the (sum of the) squares on BA and AC.

For let the square *BDEC* have been described on BC, and (the squares) GB and HC on AB and AC(respectively) [Prop. 1.46]. And let AL have been drawn through point A parallel to either of BD or CE[Prop. 1.31]. And let AD and FC have been joined. And since angles BAC and BAG are each right-angles, then two straight-lines AC and AG, not lying on the same side, make the adjacent angles with some straight-line BA, at the point A on it, (whose sum is) equal to two right-angles. Thus, CA is straight-on to AG [Prop. 1.14]. So, for the same (reasons), BA is also straight-on to AH. And since angle DBC is equal to FBA, for (they are) both right-angles, let ABC have been added to both. Thus, the whole (angle) DBA is equal to the whole (angle) FBC. And since DB is equal to BC, and FB to BA, the two (straight-lines) DB, BA are equal to the τρίγωνον τῷ ΖΒΓ τριγώνω ἐστὶν ἴσον· καί [ἐστι] τοῦ μὲν ABΔ τριγώνου διπλάσιον τὸ BΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν BΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς BΔ, AΛ· τοῦ δὲ ΖΒΓ τριγώνου διπλάσιον τὸ HB τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ZB καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ZB, HΓ. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἑστίν·] ἴσον ἄρα ἐστὶ καὶ τὸ BΛ παραλληλόγραμμον τῷ HB τετραγώνω. ὁμοίως δὴ ἐπιζευγνυμένων τῶν AE, BK δειχθήσεται καὶ τὸ ΓΛ παραλληλόγραμμον ἴῷ ΘΓ τετραγώνω· ὅλον ἄρα τὸ BΔΕΓ τετράγωνον δυσὶ τοῖς HB, ΘΓ τετραγώνοις ἴσον ἐστίν. καί ἐστι τὸ μὲν BΔΕΓ τετράγωνον ἀπὸ τῆς BΓ ἀναγραφέν, τὰ δὲ HB, ΘΓ ἀπὸ τῶν BA, AΓ. τὸ ἄρα ἀπὸ τῆς BΓ πλευρῶς τετράγωνοις.



Έν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

two (straight-lines) CB, BF,[†] respectively. And angle DBA (is) equal to angle FBC. Thus, the base AD [is] equal to the base FC, and the triangle ABD is equal to the triangle FBC [Prop. 1.4]. And parallelogram BL [is] double (the area) of triangle ABD. For they have the same base, BD, and are between the same parallels, BD and AL [Prop. 1.41]. And square GB is double (the area) of triangle FBC. For again they have the same base, FB, and are between the same parallels, FB and GC [Prop. 1.41]. [And the doubles of equal things are equal to one another.]^{\ddagger} Thus, the parallelogram *BL* is also equal to the square GB. So, similarly, AE and BKbeing joined, the parallelogram CL can be shown (to be) equal to the square HC. Thus, the whole square BDEC is equal to the (sum of the) two squares GB and HC. And the square BDEC is described on BC, and the (squares) GB and HC on BA and AC (respectively). Thus, the square on the side BC is equal to the (sum of the) squares on the sides BA and AC.



Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show.

 $^{^\}dagger$ The Greek text has "*FB*, *BC*", which is obviously a mistake.

[‡] This is an additional common notion.

μη΄.

Έὰν τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ἢ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν.



Τριγώνου γὰρ τοῦ ΑΒΓ τὸ ἀπὸ μιᾶς τῆς ΒΓ πλευρᾶς τετράγωνον ἶσον ἔστω τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις· λέγω, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΒΑΓ γωνία.

Ήχθω γὰρ ἀπὸ τοῦ Α σημείου τῆ ΑΓ εὐθεία πρὸς ὀρθὰς ἡ ΑΔ καὶ κείσθω τῆ ΒΑ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ. ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῆ ΑΒ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΔΑ τετράγωνον τῷ ἀπὸ τῆς ΑΒ τετραγώνω. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΑΓ τετράγωνον· τὰ ἄρα ἀπὸ τῶν ΔΑ, ΑΓ τετράγωνα ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΔΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΓ· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΔΑΓ γωνία· τοῖς δὲ ἀπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΓ· ὑπόκειται γάρ· τὸ ἄρα ἀπὸ τῆς ΔΓ τετράγωνον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετραγώνω· ὥστε καὶ πλευρὰ ἡ ΔΓ τῆ ΒΓ ἐστιν ἴση· καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῆ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δύο ταῖς ΒΑ, ΑΓ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΓ βάσει τῆ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνία τῆ ὑΒΑΓ [ἐστιν] ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΔΑΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΒΑΓ.

Έὰν ἀρὰ τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ἢ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν. ὅπερ ἔδει δεῖξαι.

Proposition 48

If the square on one of the sides of a triangle is equal to the (sum of the) squares on the two remaining sides of the triangle then the angle contained by the two remaining sides of the triangle is a right-angle.



For let the square on one of the sides, BC, of triangle ABC be equal to the (sum of the) squares on the sides BA and AC. I say that angle BAC is a right-angle.

For let AD have been drawn from point A at rightangles to the straight-line AC [Prop. 1.11], and let AD have been made equal to BA [Prop. 1.3], and let DChave been joined. Since DA is equal to AB, the square on DA is thus also equal to the square on AB^{\dagger} . Let the square on AC have been added to both. Thus, the (sum of the) squares on DA and AC is equal to the (sum of the) squares on BA and AC. But, the (square) on DC is equal to the (sum of the squares) on DA and AC. For angle *DAC* is a right-angle [Prop. 1.47]. But, the (square) on BC is equal to (sum of the squares) on BA and AC. For (that) was assumed. Thus, the square on DC is equal to the square on BC. So side DC is also equal to (side) BC. And since DA is equal to AB, and AC (is) common, the two (straight-lines) DA, AC are equal to the two (straight-lines) BA, AC. And the base DC is equal to the base BC. Thus, angle DAC [is] equal to angle BAC [Prop. 1.8]. But DAC is a right-angle. Thus, BAC is also a right-angle.

Thus, if the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining two sides of the triangle then the angle contained by the remaining two sides of the triangle is a right-angle. (Which is) the very thing it was required to show.

[†] Here, use is made of the additional common notion that the squares of equal things are themselves equal. Later on, the inverse notion is used.

ELEMENTS BOOK 2

Fundamentals of Geometric Algebra

Οροι.

α'. Πᾶν παραλληλόγραμμον ὀρθογώνιον περιέχεσθαι λέγεται ὑπὸ δύο τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν εὐθειῶν.

β'. Παντὸς δὲ παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον αὐτοῦ παραλληλογράμμων ἒν ὁποιονοῦν σὺν τοῖς δυσὶ παραπληρώμασι γνώμων καλείσθω.

α΄.

Έλν ῶσι δύο εὐθεῖαι, τμηθῆ δὲ ἡ ἑτέρα αὐτῶν εἰς ὁσαδηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπό τε τῆς ἀτμήτου καὶ ἑκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις.



Έστωσαν δύο εὐθεῖαι αἰ Α, ΒΓ, καὶ τετμήσθω ἡ ΒΓ, ὡς ἔτυχεν, κατὰ τὰ Δ, Ε σημεῖα λέγω, ὅτι τὸ ὑπὸ τῶν Α, ΒΓ περιεχομένον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν Α, ΒΔ περιεχομένω ὀρθογωνίω καὶ τῷ ὑπὸ τῶν Α, ΔΕ καὶ ἔτι τῷ ὑπὸ τῶν Α, ΕΓ.

Ήχθω γὰρ ἀπὸ τοῦ Β τῆ ΒΓ πρὸς ὀρθὰς ἡ ΒΖ, καὶ κείσθω τῆ Α ἴση ἡ ΒΗ, καὶ διὰ μὲν τοῦ Η τῆ ΒΓ παράλληλος ἤχθω ἡ ΗΘ, διὰ δὲ τῶν Δ, Ε, Γ τῆ ΒΗ παράλληλοι ἤχθωσαν αἱ ΔΚ, ΕΛ, ΓΘ.

Τσον δή ἑστι τὸ BΘ τοῖς BK, $\Delta\Lambda$, EΘ. καί ἑστι τὸ μὲν BΘ τὸ ὑπὸ τῶν A, BΓ· περιέχεται μὲν γὰρ ὑπὸ τῶν HB, BΓ, ἴση δὲ ἡ BH τῆ A· τὸ δὲ BK τὸ ὑπὸ τῶν A, BΔ· περιέχεται μὲν γὰρ ὑπὸ τῶν HB, BΔ, ἴση δὲ ἡ BH τῆ A. τὸ δὲ $\Delta\Lambda$ τὸ ὑπὸ τῶν A, Δ E· ἴση γὰρ ἡ Δ K, τουτέστιν ἡ BH, τῆ A. καὶ ἔτι ὑμοίως τὸ EΘ τὸ ὑπὸ τῶν A, EΓ· τὸ ἄρα ὑπὸ τῶν A, BΓ ἴσον ἐστὶ τῷ τε ὑπὸ A, BΔ καὶ τῷ ὑπὸ A, ΔΕ καὶ ἔτι τῷ ὑπὸ A, EΓ.

Έλν ἄρα ῶσι δύο εὐθεῖαι, τμηθῆ δὲ ἡ ἑτέρα αὐτῶν εἰς ὑσαδηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπό τε τῆς ἀτμήτου καὶ ἑκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις· ὅπερ

Definitions

1. Any rectangular parallelogram is said to be contained by the two straight-lines containing the rightangle.

2. And in any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a gnomon.

Proposition 1^{\dagger}

If there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line).



Let A and BC be the two straight-lines, and let BC be cut, at random, at points D and E. I say that the rectangle contained by A and BC is equal to the rectangle(s) contained by A and BD, by A and DE, and, finally, by A and EC.

For let BF have been drawn from point B, at rightangles to BC [Prop. 1.11], and let BG be made equal to A [Prop. 1.3], and let GH have been drawn through (point) G, parallel to BC [Prop. 1.31], and let DK, EL, and CH have been drawn through (points) D, E, and C(respectively), parallel to BG [Prop. 1.31].

So the (rectangle) BH is equal to the (rectangles) BK, DL, and EH. And BH is the (rectangle contained) by A and BC. For it is contained by GB and BC, and BG (is) equal to A. And BK (is) the (rectangle contained) by A and BD. For it is contained by GB and BD, and BG (is) equal to A. And DL (is) the (rectangle contained) by A and DE. For DK, that is to say BG [Prop. 1.34], (is) equal to A. Similarly, EH (is) also the (rectangle contained) by A and BC is equal to the (rectangle contained) by A and BC.

and BD, by A and DE, and, finally, by A and EC.

Thus, if there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line). (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $a(b + c + d + \cdots) = ab + ac + ad + \cdots$.

β'.

Έὰν εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ.



Εὐθεῖα γὰρ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ὑπὸ ΒΑ, ΑΓ περιεχομένου ὀρθογωνίου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνω.

Άναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ AΔEB, καὶ ἦχθω διὰ τοῦ Γ ὁποτέρ
ợ τῶν AΔ, BE παράλληλος ἡ ΓΖ.

Ίσον δή ἐστὶ τὸ ΑΕ τοῖς ΑΖ, ΓΕ. καί ἐστι τὸ μὲν ΑΕ τὸ ἀπὸ τῆς ΑΒ τετράγωνον, τὸ δὲ ΑΖ τὸ ὑπὸ τῶν ΒΑ, ΑΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΔΑ, ΑΓ, ἴση δὲ ἡ ΑΔ τῆ ΑΒ· τὸ δὲ ΓΕ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΕ τῆ ΑΒ. τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ μετὰ τοῦ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ.

Έὰν ἄρα εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ· ὅπερ ἔδει δεῖζαι.

Proposition 2^{\dagger}

If a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole.



For let the straight-line AB have been cut, at random, at point C. I say that the rectangle contained by AB and BC, plus the rectangle contained by BA and AC, is equal to the square on AB.

For let the square ADEB have been described on AB [Prop. 1.46], and let CF have been drawn through C, parallel to either of AD or BE [Prop. 1.31].

So the (square) AE is equal to the (rectangles) AF and CE. And AE is the square on AB. And AF (is) the rectangle contained by the (straight-lines) BA and AC. For it is contained by DA and AC, and AD (is) equal to AB. And CE (is) the (rectangle contained) by AB and BC. For BE (is) equal to AB. Thus, the (rectangle contained) by BA and AC, plus the (rectangle contained) by AB and BC, is equal to the square on AB.

Thus, if a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straightline), and each of the pieces (of the straight-line), is equal to the square on the whole. (Which is) the very thing it was required to show. [†] This proposition is a geometric version of the algebraic identity: $a b + a c = a^2$ if a = b + c.

γ΄.

Έὰν εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένω ὀρθογωνίω καὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνω.



Εὐθεῖα γὰρ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΒΓ τετραγώνου.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΔΕΒ, καὶ διήχθω ἡ ΕΔ ἐπὶ τὸ Ζ, καὶ διὰ τοῦ Α ὁποτέρα τῶν ΓΔ, ΒΕ παράλληλος ἤχθω ἡ ΑΖ. ἴσον δή ἐστι τὸ ΑΕ τοῖς ΑΔ, ΓΕ· καί ἐστι τὸ μὲν ΑΕ τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΑΒ, ΒΕ, ἴση δὲ ἡ ΒΕ τῆ ΒΓ· τὸ δὲ ΑΔ τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴση γὰρ ἡ ΔΓ τῆ ΓΒ· τὸ δὲ ΔΒ τὸ ἀπὸ τῆς ΓΒ τετράγωνον· τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΒΓ τετραγώνου.

Έὰν ἄρα εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης xαὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ xαὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνῳ. ὅπερ ἔδει δεῖξαι.

Proposition 3[†]

If a straight-line is cut at random then the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece.



For let the straight-line AB have been cut, at random, at (point) C. I say that the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

For let the square CDEB have been described on CB[Prop. 1.46], and let ED have been drawn through to F, and let AF have been drawn through A, parallel to either of CD or BE [Prop. 1.31]. So the (rectangle) AEis equal to the (rectangle) AD and the (square) CE. And AE is the rectangle contained by AB and BC. For it is contained by AB and BE, and BE (is) equal to BC. And AD (is) the (rectangle contained) by AC and CB. For DC (is) equal to CB. And DB (is) the square on CB. Thus, the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

Thus, if a straight-line is cut at random then the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece. (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $(a + b) a = a b + a^2$.

δ'.

Έαν εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δἰς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθο-

Proposition 4[†]

If a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the



Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ. λέγω, ὅτι τὸ ἀπὸ τῆς ΑΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένω ὀρθογωνίω.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ ἐπεζεύχθω ἡ ΒΔ, καὶ διὰ μὲν τοῦ Γ ὑποτέρα τῶν ΑΔ, ΕΒ παράλληλος ἤχθω ἡ ΓΖ, διὰ δὲ τοῦ Η ὑποτέρα τῶν ΑΒ, ΔΕ παράλληλος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΓΖ τῆ ΑΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΔ, ἡ ἐκτὸς γωνία ή ὑπὸ ΓΗΒ ἴση ἐστὶ τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ ΑΔΒ. άλλ' ή ὑπὸ ΑΔΒ τῆ ὑπὸ ΑΒΔ ἐστιν ἴση, ἐπεὶ καὶ πλευρὰ ἡ BA τῆ AΔ ἐστιν ἴση· καὶ ἡ ὑπὸ ΓΗΒ ẳρα γωνιά τῆ ὑπὸ ΗΒΓ έστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΒΓ πλευρῷ τῆ ΓΗ ἐστιν ἴση· άλλ' ἡ μὲν ΓΒ τῆ ΗΚ ἐστιν ἴση. ἡ δὲ ΓΗ τῆ ΚΒ· ϰαὶ ἡ ΗΚ άρα τῆ KB ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΓΗKB. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΓΗ τῆ ΒΚ [καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΓΒ], αἱ ἄρα ὑπὸ ΚΒΓ, ΗΓΒ γωνίαι δύο ὀρθαῖς εἰσιν ἴσαι. ὀρθὴ δὲ ἡ ὑπὸ ΚΒΓ· ὀρθή ἄρα καὶ ἡ ὑπὸ ΒΓΗ· ὥστε καὶ αἱ ἀπεναντίον αί ὑπὸ ΓΗΚ, ΗΚΒ ὀρθαί εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ ΓΗΚΒ· ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν· καί ἐστιν ἀπὸ τῆς ΓΒ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΘΖ τετράγωνόν έστιν καί έστιν άπὸ τῆς ΘΗ, τουτέστιν [ἀπὸ] τῆς ΑΓ· τὰ άρα ΘΖ, ΚΓ τετράγωνα ἀπὸ τῶν ΑΓ, ΓΒ εἰσιν. καὶ ἐπεὶ ίσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, καί ἐστι τὸ ΑΗ τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴση γὰρ ἡ ΗΓ τῆ ΓΒ· καὶ τὸ ΗΕ ἄρα ἴσον ἐστὶ τῷ ύπὸ ΑΓ, ΓΒ· τὰ ἄρα ΑΗ, ΗΕ ἴσα ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. ἔστι δὲ καὶ τὰ ΘΖ, ΓΚ τετράγωνα ἀπὸ τῶν ΑΓ, ΓB · τὰ ἄρα τέσσαρα τὰ ΘZ , ΓK , AH, HE ἴσα ἐστὶ τοῖς τε άπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένω όρθογωνίω. ἀλλὰ τὰ ΘΖ, ΓΚ, ΑΗ, ΗΕ ὅλον έστι τὸ ΑΔΕΒ, ὅ ἐστιν ἀπὸ τῆς ΑΒ τετράγωνον τὸ ἄρα άπὸ τῆς ΑΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένω όρθογωνίω.

Έὰν ἄρα εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς





For let the straight-line AB have been cut, at random, at (point) C. I say that the square on AB is equal to the (sum of the) squares on AC and CB, and twice the rectangle contained by AC and CB.

For let the square ADEB have been described on AB[Prop. 1.46], and let BD have been joined, and let CFhave been drawn through C, parallel to either of AD or EB [Prop. 1.31], and let HK have been drawn through G, parallel to either of AB or DE [Prop. 1.31]. And since CF is parallel to AD, and BD has fallen across them, the external angle CGB is equal to the internal and opposite (angle) ADB [Prop. 1.29]. But, ADB is equal to ABD, since the side BA is also equal to AD [Prop. 1.5]. Thus, angle CGB is also equal to GBC. So the side BC is equal to the side CG [Prop. 1.6]. But, CB is equal to GK, and CG to KB [Prop. 1.34]. Thus, GK is also equal to KB. Thus, CGKB is equilateral. So I say that (it is) also right-angled. For since CG is parallel to BK [and the straight-line CB has fallen across them], the angles KBCand GCB are thus equal to two right-angles [Prop. 1.29]. But KBC (is) a right-angle. Thus, BCG (is) also a rightangle. So the opposite (angles) CGK and GKB are also right-angles [Prop. 1.34]. Thus, CGKB is right-angled. And it was also shown (to be) equilateral. Thus, it is a square. And it is on CB. So, for the same (reasons), *HF* is also a square. And it is on *HG*, that is to say [on] AC [Prop. 1.34]. Thus, the squares HF and KC are on AC and CB (respectively). And the (rectangle) AGis equal to the (rectangle) GE [Prop. 1.43]. And AG is the (rectangle contained) by AC and CB. For GC (is) equal to CB. Thus, GE is also equal to the (rectangle contained) by AC and CB. Thus, the (rectangles) AGand GE are equal to twice the (rectangle contained) by AC and CB. And HF and CK are the squares on ACand CB (respectively). Thus, the four (figures) HF, CK, AG, and GE are equal to the (sum of the) squares on όλης τετράγωνον ίσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ· ὅπερ ἔδει δεῖξαι. AC and BC, and twice the rectangle contained by AC and CB. But, the (figures) HF, CK, AG, and GE are (equivalent to) the whole of ADEB, which is the square on AB. Thus, the square on AB is equal to the (sum of the) squares on AC and CB, and twice the rectangle contained by AC and CB.

Thus, if a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces. (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $(a + b)^2 = a^2 + b^2 + 2 a b$.

ε΄.

Έὰν εὐθεῖα γραμμὴ τμηθῆ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ.



Εὐθεĩα γάρ τις ή AB τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὸ ὑπὸ τῶν AΔ, ΔB περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΕΖΒ, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διὰ μὲν τοῦ Δ ὑποτέρα τῶν ΓΕ, ΒΖ παράλληλος ἦχθω ἡ ΔΗ, διὰ δὲ τοῦ Θ ὑποτέρα τῶν AB, ΕΖ παράλληλος πάλιν ἦχθω ἡ KM, καὶ πάλιν διὰ τοῦ A ὑποτέρα τῶν ΓΛ, BM παράλληλος ἦχθω ἡ AK. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΓΘ παραπλήρωμα τῷ ΘΖ παραπληρώματι, κοινὸν προσχείσθω τὸ ΔΜ· ὅλον ἄρα τὸ ΓΜ ὅλω τῷ ΔΖ ἴσον ἐστίν. ἀλλὰ τὸ ΓΜ τῷ ΑΛ ἴσον ἐστίν, ἐπεὶ καὶ ἡ ΑΓ τῆ ΓΒ ἐστιν ἴση· καὶ τὸ ΑΛ ἄρα τῷ ΔΖ ἴσον ἐστίν. κοινὸν προσχείσθω τὸ ΓΘ· ὅλον ἄρα τὸ ΑΘ τῷ MNΞ[†] γνώμονι ἴσον ἐστίν. ἀλλὰ τὸ ΑΘ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ἐστιν· ἴση γὰρ ἡ ΔΘ τῆ ΔΒ· καὶ ὁ MNΞ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ ΑΔ, ΔΒ. κοινὸν προσχείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΓΔ· ὁ ὅρα MNΞ γνώμων καὶ τῷ ΑΠ τῆ

Proposition 5[‡]

If a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line).



For let any straight-line AB have been cut—equally at C, and unequally at D. I say that the rectangle contained by AD and DB, plus the square on CD, is equal to the square on CB.

For let the square CEFB have been described on CB[Prop. 1.46], and let BE have been joined, and let DGhave been drawn through D, parallel to either of CE or BF [Prop. 1.31], and again let KM have been drawn through H, parallel to either of AB or EF [Prop. 1.31], and again let AK have been drawn through A, parallel to either of CL or BM [Prop. 1.31]. And since the complement CH is equal to the complement HF [Prop. 1.43], let the (square) DM have been added to both. Thus, the whole (rectangle) CM is equal to the whole (rectangle) DF. But, (rectangle) CM is equal to (rectangle) AL, since AC is also equal to CB [Prop. 1.36]. Thus, (rectangle) AL is also equal to (rectangle) DF. Let (rectangle) CH have been added to both. Thus, the whole (rectangle) AH is equal to the gnomon NOP. But, AH ΓΔ τετραγώνω. ἀλλὰ ὁ ΜΝΞ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΒ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνω.

Έὰν ἄρα εὐθεῖα γραμμὴ τμηθῆ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ. ὅπερ ἔδει δεῖξαι. is the (rectangle contained) by AD and DB. For DH(is) equal to DB. Thus, the gnomon NOP is also equal to the (rectangle contained) by AD and DB. Let LG, which is equal to the (square) on CD, have been added to both. Thus, the gnomon NOP and the (square) LG are equal to the rectangle contained by AD and DB, and the square on CD. But, the gnomon NOP and the (square) LG is (equivalent to) the whole square CEFB, which is on CB. Thus, the rectangle contained by AD and DB, plus the square on CD, is equal to the square on CB.

Thus, if a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line). (Which is) the very thing it was required to show.

 † Note the (presumably mistaken) double use of the label M in the Greek text.

[‡] This proposition is a geometric version of the algebraic identity: $a b + [(a + b)/2 - b]^2 = [(a + b)/2]^2$.

ኖ'.

Έαν εὐθεῖα γραμμὴ τμηθῆ δίχα, προστεθῆ δέ τις αὐτῆ εὐθεῖα ἐπ' εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῆ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθόγώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ.



Εὐθεῖα γάρ τις ή AB τετμήσθω δίχα κατὰ τὸ Γ σημεῖον, προσκείσθω δέ τις αὐτῆ εὐθεῖα ἐπ' εὐθείας ή BΔ· λέγω, ὅτι τὸ ὑπὸ τῶν AΔ, ΔB περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓB τετραγώνου ἶσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνω.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΓΔ τετράγωνον τὸ ΓΕΖΔ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ διὰ μὲν τοῦ Β σημείου ὁποτέρα τῶν ΕΓ, ΔΖ παράλληλος ἤχθω ἡ BH, διὰ δὲ τοῦ Θ σημείου ὁποτέρα τῶν AB, ΕΖ παράλληλος ἤχθω ἡ KM, καὶ ἔτι διὰ τοῦ Α ὁποτέρα τῶν ΓΛ, ΔΜ παράλληλος ἤχθω ἡ AK.

Έπεὶ οῦν ἴση ἐστὶν ἡ ΑΓ τῆ ΓΒ, ἴσον ἐστὶ καὶ τὸ ΑΛ

Proposition 6[†]

If a straight-line is cut in half, and any straight-line added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added.



For let any straight-line AB have been cut in half at point C, and let any straight-line BD have been added to it straight-on. I say that the rectangle contained by AD and DB, plus the square on CB, is equal to the square on CD.

For let the square CEFD have been described on CD [Prop. 1.46], and let DE have been joined, and let BG have been drawn through point B, parallel to either of EC or DF [Prop. 1.31], and let KM have been drawn through point H, parallel to either of AB or EF [Prop. 1.31], and finally let AK have been drawn

τῷ ΓΘ. ἀλλὰ τὸ ΓΘ τῷ ΘΖ ἴσον ἐστίν. καὶ τὸ ΑΛ ἄρα τῷ ΘΖ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΜ· ὅλον ἄρα τὸ ΑΜ τῷ ΝΞΟ γνώμονί ἐστιν ἴσον. ἀλλὰ τὸ ΑΜ ἐστι τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἴση γάρ ἐστιν ἡ ΔΜ τῆ ΔΒ· καὶ ὁ ΝΞΟ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ [περιεχομένῳ ὀρθογωνίῳ]. κοινὸν προσκείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ΝΞΟ γνώμονι καὶ τῷ ΛΗ. ἀλλὰ ὁ ΝΞΟ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΔ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΓΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ψετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς

Έαν ἄρα εὐθεῖα γραμμὴ τμηθῆ δίχα, προστεθῆ δέ τις αὐτῆ εὐθεῖα ἐπ' εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῆ προσχειμένῃ καὶ τῆς προσχειμένης περιεχόμενον ὀρθόγώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγχειμένῃς ἔχ τε τῆς ἡμισείας καὶ τῆς προσχειμένῃς τετραγώνῳ. ὅπερ ἔδει δεῖξαι. through A, parallel to either of CL or DM [Prop. 1.31].

Therefore, since AC is equal to CB, (rectangle) AL is also equal to (rectangle) CH [Prop. 1.36]. But, (rectangle) CH is equal to (rectangle) HF [Prop. 1.43]. Thus, (rectangle) AL is also equal to (rectangle) HF. Let (rectangle) CM have been added to both. Thus, the whole (rectangle) AM is equal to the gnomon NOP. But, AMis the (rectangle contained) by AD and DB. For DM is equal to DB. Thus, gnomon NOP is also equal to the [rectangle contained] by AD and DB. Let LG, which is equal to the square on BC, have been added to both. Thus, the rectangle contained by AD and DB, plus the square on CB, is equal to the gnomon NOP and the (square) LG. But the gnomon NOP and the (square) LG is (equivalent to) the whole square CEFD, which is on CD. Thus, the rectangle contained by AD and DB, plus the square on CB, is equal to the square on CD.

Thus, if a straight-line is cut in half, and any straightline added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straightline), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added. (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $(2 a + b) b + a^2 = (a + b)^2$.

ζ.

Έαν εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ᾽ ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.



Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένω ὀρθογωνίω καὶ τῷ

Proposition 7[†]

If a straight-line is cut at random then the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece.



For let any straight-line AB have been cut, at random, at point C. I say that the (sum of the) squares on AB and BC is equal to twice the rectangle contained by AB and

ἀπὸ τῆς ΓΑ τετραγώνω.

Άναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ AΔEB· καὶ καταγεγράφθω τὸ σχῆμα.

Έπεὶ οὖν ἴσον ἐστὶ τὸ AH τῷ HE, κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ AZ ὅλῳ τῷ ΓΕ ἴσον ἐστίν· τὰ ἄρα AZ, ΓΕ διπλάσιά ἐστι τοῦ AZ. ἀλλὰ τὰ AZ, ΓΕ ὁ ΚΛΜ ἐστι γνώμων καὶ τὸ ΓΖ τετράγωνον· ὁ ΚΛΜ ἄρα γνώμων καὶ τὸ ΓΖ διπλάσιά ἐστι τοῦ AZ. ἔστι δὲ τοῦ AZ διπλάσιον καὶ τὸ δἰς ὑπὸ τῶν AB, BΓ· ἴση γὰρ ἡ BZ τῆ BΓ· ὁ ἄρα ΚΛΜ γνώμων καὶ τὸ ΓΖ τετράγωνον ἴσον ἐστὶ τῷ δἰς ὑπὸ τῶν AB, BΓ. κοινὸν προσκείσθω τὸ ΔΗ, ὅ ἐστιν ἀπὸ τῆς ΑΓ τετράγωνον· ὁ ἄρα ΚΛΜ γνώμων καὶ τὰ BH, HΔ τετράγωνα ἴσα ἐστὶ τῷ τε δἰς ὑπὸ τῶν AB, BΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ. ἀλλὰ ὁ ΚΛΜ γνώμων καὶ τὰ BH, HΔ τετράγωνα ὅλον ἐστὶ τὸ ΑΔΕΒ καὶ τὸ ΓΖ, ἅ ἐστιν ἀπὸ τῶν AB, BΓ τετράγωνα· τὰ ἄρα ἀπὸ τῶν AB, BΓ τετράγωνα ἴσα ἐστὶ τῷ [τε] δἰς ὑπὸ τῶν AB, BΓ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου.

Έαν ἄρα εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ᾽ ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δἰς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ. ὅπερ ἔδει δεῖξαι. BC, and the square on CA.

For let the square ADEB have been described on AB [Prop. 1.46], and let the (rest of) the figure have been drawn.

Therefore, since (rectangle) AG is equal to (rectangle) GE [Prop. 1.43], let the (square) CF have been added to both. Thus, the whole (rectangle) AF is equal to the whole (rectangle) CE. Thus, (rectangle) AF plus (rectangle) CE is double (rectangle) AF. But, (rectangle) AF plus (rectangle) CE is the gnomon KLM, and the square CF. Thus, the gnomon KLM, and the square CF, is double the (rectangle) AF. But double the (rectangle) AF is also twice the (rectangle contained) by ABand BC. For BF (is) equal to BC. Thus, the gnomon KLM, and the square CF, are equal to twice the (rectangle contained) by AB and BC. Let DG, which is the square on AC, have been added to both. Thus, the gnomon KLM, and the squares BG and GD, are equal to twice the rectangle contained by AB and BC, and the square on AC. But, the gnomon KLM and the squares BG and GD is (equivalent to) the whole of ADEB and *CF*, which are the squares on *AB* and *BC* (respectively). Thus, the (sum of the) squares on AB and BC is equal to twice the rectangle contained by AB and BC, and the square on AC.

Thus, if a straight-line is cut at random then the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece. (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $(a + b)^2 + a^2 = 2 (a + b) a + b^2$.

η΄.

Έὰν εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ τετράχις ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσον ἐστὶ τῷ ἀπό τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ AB τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ τετράχις ὑπὸ τῶν AB, BΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς AΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς AB, BΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνω.

Έκβεβλήσθω γὰρ ἐπ' εὐθείας [τῆ AB εὐθεῖα] ἡ BΔ, καὶ κείσθω τῆ ΓΒ ἴση ἡ BΔ, καὶ ἀναγεγράφθω ἀπὸ τῆς AΔ τετράγωνον τὸ AEZΔ, καὶ καταγεγράφθω διπλοῦν τὸ σχῆμα.

Proposition 8[†]

If a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line).

For let any straight-line AB have been cut, at random, at point C. I say that four times the rectangle contained by AB and BC, plus the square on AC, is equal to the square described on AB and BC, as on one (complete straight-line).

For let BD have been produced in a straight-line [with the straight-line AB], and let BD be made equal to CB [Prop. 1.3], and let the square AEFD have been described on AD [Prop. 1.46], and let the (rest of the) figure have been drawn double.



Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΓΒ τῆ ΒΔ, ἀλλὰ ἡ μὲν ΓΒ τῆ ΗΚ ἐστιν ἴση, ἡ δὲ B Δ τῆ KN, καὶ ἡ HK ἄρα τῆ KN ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΠΡ τῆ ΡΟ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ή ΒΓ τὴ ΒΔ, ή δὲ ΗΚ τῆ ΚΝ, ἴσον ἄρα ἐστὶ καὶ τὸ μὲν ΓΚ τῷ ΚΔ, τὸ δὲ ΗΡ τῷ ΡΝ. ἀλλὰ τὸ ΓΚ τῷ ΡΝ ἐστιν ίσον· παραπληρώματα γάρ τοῦ ΓΟ παραλληλογράμμου· καὶ τὸ $K\Delta$ ắρα τῷ HP ἴσον ἐστίν· τὰ τέσσαρα ắρα τὰ ΔK , ΓK , ΗΡ, ΡΝ ἴσα ἀλλήλοις ἐστίν. τὰ τέσσαρα ἄρα τετραπλάσιά έστι τοῦ ΓΚ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῆ ΒΔ, ἀλλὰ ἡ μὲν $B\Delta$ τ
 \breve{n} BK, τουτέστι τῆ ΓΗ ἴση, ἡ δὲ ΓΒ τῆ ΗΚ, τουτέστι τῆ ΗΠ, ἐστιν ἴση, καὶ ἡ ΓΗ ἄρα τῆ ΗΠ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΓΗ τῆ ΗΠ, ἡ δὲ ΠΡ τῆ PO, ἴσον ἐστὶ xαὶ τὸ μέν ΑΗ τῷ ΜΠ, τὸ δὲ ΠΛ τῷ ΡΖ. ἀλλὰ τὸ ΜΠ τῷ ΠΛ ἐστιν ίσον· παραπληρώματα γὰρ τοῦ ΜΛ παραλληλογράμμου· καὶ τὸ ΑΗ ἄρα τῷ ΡΖ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΑΗ, ΜΠ, ΠΛ, ΡΖ ίσα ἀλλήλοις ἐστίν· τὰ τέσσαρα ἄρα τοῦ ΑΗ ἐστι τετραπλάσια. ἐδείχθη δὲ καὶ τὰ τέσσαρα τὰ ΓΚ, ΚΔ, ΗΡ, ΡΝ τοῦ ΓΚ τετραπλάσια· τὰ ἄρα ὀχτώ, য় περιέχει τὸν ΣΤΥ γνώμονα, τετραπλάσιά ἐστι τοῦ ΑΚ. καὶ ἐπεὶ τὸ ΑΚ τὸ ὑπὸ τῶν AB, B Δ ἐστιν· ἴση γὰρ ἡ BK τῆ B Δ · τὸ ẳρα τετράχις ύπὸ τῶν ΑΒ, ΒΔ τετραπλάσιόν ἐστι τοῦ ΑΚ. ἐδείχθη δὲ τοῦ ΑΚ τετραπλάσιος καὶ ὁ ΣΤΥ γνώμων· τὸ ẳρα τετράκις ύπὸ τῶν ΑΒ, ΒΔ ἴσον ἐστὶ τῷ ΣΤΥ γνώμονι. χοινὸν προσκείσθω τὸ $\Xi\Theta$, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς $A\Gamma$ τετραγών ω · τὸ άρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ΣΤΥ γνώμονι καὶ τῷ ΞΘ. ἀλλὰ ὁ ΣΤΥ γνώμων καὶ τὸ ΞΘ ὅλον ἐστὶ τὸ $AEZ\Delta$ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς $A\Delta$ · τὸ ἄρα τετράχις ύπὸ τῶν AB, BΔ μετὰ τοῦ ἀπὸ AΓ ἴσον ἐστὶ τῷ ἀπὸ AΔ τετραγώνω· ἴση δὲ ἡ ΒΔ τῆ ΒΓ. τὸ ἄρα τετράχις ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ίσον έστι τῷ ἀπὸ τῆς ΑΔ, τουτέστι τῷ ἀπὸ τῆς ΑΒ καὶ ΒΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνω.

Ἐἀν ἄρα εὐθεῖα γραμμὴ τμηθῆ, ὡς ἔτυχεν, τὸ τετράχις ὑπὸ τῆς ὅλης xαὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσου



Therefore, since CB is equal to BD, but CB is equal to GK [Prop. 1.34], and BD to KN [Prop. 1.34], GK is thus also equal to KN. So, for the same (reasons), QR is equal to RP. And since BC is equal to BD, and GK to KN, (square) CK is thus also equal to (square) KD, and (square) GR to (square) RN [Prop. 1.36]. But, (square) CK is equal to (square) RN. For (they are) complements in the parallelogram CP [Prop. 1.43]. Thus, (square) KD is also equal to (square) GR. Thus, the four (squares) DK, CK, GR, and RN are equal to one another. Thus, the four (taken together) are quadruple (square) CK. Again, since CB is equal to BD, but BD(is) equal to BK—that is to say, CG—and CB is equal to GK—that is to say, GQ—CG is thus also equal to GQ. And since CG is equal to GQ, and QR to RP, (rectangle) AG is also equal to (rectangle) MQ, and (rectangle) QL to (rectangle) RF [Prop. 1.36]. But, (rectangle) MQis equal to (rectangle) QL. For (they are) complements in the parallelogram *ML* [Prop. 1.43]. Thus, (rectangle) AG is also equal to (rectangle) RF. Thus, the four (rectangles) AG, MQ, QL, and RF are equal to one another. Thus, the four (taken together) are quadruple (rectangle) AG. And it was also shown that the four (squares) CK, KD, GR, and RN (taken together are) quadruple (square) CK. Thus, the eight (figures taken together), which comprise the gnomon STU, are quadruple (rectangle) AK. And since AK is the (rectangle contained) by AB and BD, for BK (is) equal to BD, four times the (rectangle contained) by AB and BD is quadruple (rectangle) AK. But the gnomon STU was also shown (to be equal to) quadruple (rectangle) AK. Thus, four times the (rectangle contained) by AB and BD is equal to the gnomon STU. Let OH, which is equal to the square on AC, have been added to both. Thus, four times the rectangle contained by AB and BD, plus the square on AC, is equal to the gnomon STU, and the (square) OH. But,

έστὶ τῷ ἀπό τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῷ. ὅπερ ἔδει δεῖξαι.

the gnomon STU and the (square) OH is (equivalent to) the whole square AEFD, which is on AD. Thus, four times the (rectangle contained) by AB and BD, plus the (square) on AC, is equal to the square on AD. And BD(is) equal to BC. Thus, four times the rectangle contained by AB and BC, plus the square on AC, is equal to the (square) on AD, that is to say the square described on AB and BC, as on one (complete straight-line).

Thus, if a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line). (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $4(a + b) a + b^2 = [(a + b) + a]^2$.

θ'.

Έαν εὐθεῖα γραμμὴ τμηθῆ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου.



Eùdeĩa γάρ τις ή AB τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ · λέγω, ὅτι τὰ ἀπὸ τῶν A Δ , Δ B τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν AΓ, Γ Δ τετραγώνων.

Ήχθω γὰρ ἀπὸ τοῦ Γ τῆ ΑΒ πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρα τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ, καὶ διὰ μὲν τοῦ Δ τῆ ΕΓ παράλληλος ἤχθω ἡ ΔΖ, διὰ δὲ τοῦ Ζ τῆ ΑΒ ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΑΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΕ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΕΑΓ γωνία τῆ ὑπὸ ΑΕΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ πρὸς τῷ Γ, λοιπαὶ ἄρα αἱ ὑπὸ ΕΑΓ, ΑΕΓ μιῷ ὀρθῆ ἴσαι εἰσίν· καί εἰσιν ἴσαι· ἡμίσεια ἄρα ὀρθῆς ἐστιν ἑκατέρα τῶν ὑπὸ ΓΕΑ, ΓΑΕ. δὶα τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν ὀρθῆς· ὅλη ἄρα ἡ ὑπὸ ΑΕΒ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ὑπὸ ΗΕΖ ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΕΗΖ· ἴση γάρ ἐστι τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΕΖΗ ἡμίσειά ἐστιν

Proposition 9[†]

If a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces.



For let any straight-line AB have been cut—equally at C, and unequally at D. I say that the (sum of the) squares on AD and DB is double the (sum of the squares) on AC and CD.

For let CE have been drawn from (point) C, at rightangles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EBhave been joined. And let DF have been drawn through (point) D, parallel to EC [Prop. 1.31], and (let) FG(have been drawn) through (point) F, (parallel) to AB[Prop. 1.31]. And let AF have been joined. And since AC is equal to CE, the angle EAC is also equal to the (angle) AEC [Prop. 1.5]. And since the (angle) at C is a right-angle, the (sum of the) remaining angles (of triangle AEC), EAC and AEC, is thus equal to one right-

όρθῆς· ἴση ἄρα [ἐστὶν] ἡ ὑπὸ ΗΕΖ γωνία τῇ ὑπὸ ΕΖΗ· ὥστε καὶ πλευρὰ ἡ ΕΗ τῆ ΗΖ ἐστιν ἴση. πάλιν ἐπεὶ ἡ πρὸς τῷ Β γωνία ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ $Z\Delta B^{\cdot}$ ἴση γὰρ πάλιν ἐστὶ τῆ ἐντὸς καὶ ἀπεναντίον τῆ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ή ὑπὸ $\mathrm{BZ}\Delta$ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ πρὸς τῷ B γωνία τῆ ὑπὸ ΔZB · ὥστε καὶ πλευρὰ ἡ Z Δ πλευρᾶ τῆ ΔB ἐστιν ίση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΕ, ἴσον ἐστὶ καὶ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΓΕ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΕ τετράγωνα διπλάσιά έστι τοῦ ἀπὸ ΑΓ. τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ τετράγωνον· ὀρθὴ γὰρ ἡ ὑπὸ ΑΓΕ γωνία· τὸ ἄρα άπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ. πάλιν, ἐπεὶ ἴση έστιν ή EH τῆ HZ, ἴσον και τὸ ἀπὸ τῆς EH τῷ ἀπὸ τῆς HZ· τὰ ἄρα ἀπὸ τῶν ΕΗ, ΗΖ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΗΖ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΗ, ΗΖ τετραγώνοις ίσον ἐστὶ τὸ ἀπὸ τῆς ΕΖ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΕΖ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΗΖ. ἴση δὲ ἡ ΗΖ τῆ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΖ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. έστι δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΖ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΖ τετράγωνον· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΑΕΖ γωνία· τὸ ἄρα ἀπὸ τῆς ΑΖ τετράγωνον διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΖ ἴσα τὰ ἀπὸ τῶν ΑΔ, ΔZ · $d\rho \vartheta \eta$ yàp η προς τῶ Δ γωνία· τὰ ἄρα ἀπὸ τῶν $A\Delta$, ΔZ διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. ἴση δὲ ἡ ΔZ τῆ ΔB · τὰ ἄρα ἀπὸ τῶν $A\Delta$, ΔB τετράγωνα διπλάσιά έστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετράγώνων.

Έὰν ἄρα εὐθεῖα γραμμὴ τμηθῆ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου. ὅπερ ἔδει δεῖξαι.

angle [Prop. 1.32]. And they are equal. Thus, (angles) CEA and CAE are each half a right-angle. So, for the same (reasons), (angles) CEB and EBC are also each half a right-angle. Thus, the whole (angle) AEB is a right-angle. And since GEF is half a right-angle, and EGF (is) a right-angle—for it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) *EFG* is thus half a right-angle [Prop. 1.32]. Thus, angle GEF [is] equal to EFG. So the side EG is also equal to the (side) GF [Prop. 1.6]. Again, since the angle at B is half a right-angle, and (angle) FDB (is) a right-angle-for again it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) BFD is half a right-angle [Prop. 1.32]. Thus, the angle at B (is) equal to DFB. So the side FD is also equal to the side DB [Prop. 1.6]. And since AC is equal to CE, the (square) on AC (is) also equal to the (square) on CE. Thus, the (sum of the) squares on AC and CE is double the (square) on AC. And the square on EA is equal to the (sum of the) squares on AC and CE. For angle ACE (is) a right-angle [Prop. 1.47]. Thus, the (square) on EA is double the (square) on AC. Again, since EGis equal to GF, the (square) on EG (is) also equal to the (square) on GF. Thus, the (sum of the squares) on EG and GF is double the square on GF. And the square on EF is equal to the (sum of the) squares on EG and GF [Prop. 1.47]. Thus, the square on EF is double the (square) on GF. And GF (is) equal to CD [Prop. 1.34]. Thus, the (square) on EF is double the (square) on CD. And the (square) on EA is also double the (square) on AC. Thus, the (sum of the) squares on AE and EF is double the (sum of the) squares on AC and CD. And the square on AF is equal to the (sum of the squares) on AE and EF. For the angle AEF is a right-angle [Prop. 1.47]. Thus, the square on AF is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DF (is) equal to the (square) on AF. For the angle at D is a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AD and DF is double the (sum of the) squares on AC and CD. And DF (is) equal to DB. Thus, the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD.

Thus, if a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces. (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $a^2 + b^2 = 2[([a+b]/2)^2 + ([a+b]/2 - b)^2].$

ι′**.**

Έαν εὐθεῖα γραμμὴ τμηθῆ δίχα, προστεθῆ δέ τις αὐτῆ εὐθεῖα ἐπ' εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῆ προσκειμένῃ καὶ τὸ ἀπὸ τῆς προσκειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου.



Εὐθεῖα γάρ τις ἡ AB τετμήσθω δίχα κατὰ τὸ Γ, προσκείσθω δέ τις αὐτῆ εὐθεῖα ἐπ' εὐθείας ἡ BΔ· λέγω, ὅτι τὰ ἀπὸ τῶν AΔ, ΔB τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν AΓ, ΓΔ τετραγώνων.

"Ηχθω γὰρ ἀπὸ τοῦ Γ σημείου τῆ AB πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρα τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αί ΕΑ, ΕΒ΄ καὶ διὰ μὲν τοῦ Ε τῆ ΑΔ παράλληλος ἤχθω ἡ ΕΖ, διὰ δὲ τοῦ Δ τῆ ΓΕ παράλληλος ἤχθω ἡ ΖΔ. καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς $\mathrm{E}\Gamma,\,\mathrm{Z}\Delta$ εὐθεῖά τις ἐνέπεσεν ή EZ, αί ὑπὸ ΓΕΖ, ΕΖΔ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν· αί άρα ὑπὸ ΖΕΒ, ΕΖΔ δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ άπ' έλασσόνων ή δύο όρθῶν ἐκβαλλόμεναι συμπίπτουσιν. αί ἄρα ΕΒ, ΖΔ ἐκβαλλόμεναι ἐπὶ τὰ Β, Δ μέρη συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΑΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΕ, ἴση έστι και γωνία ή ὑπὸ ΕΑΓ τῆ ὑπὸ ΑΕΓ· και ὀρθή ή πρὸς τῷ Γ· ἡμίσεια ἄρα ὀρθῆς [ἐστιν] ἑχατέρα τῶν ὑπὸ ΕΑΓ, ΑΕΓ. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν όρθῆς ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ. καὶ ἐπεὶ ἡμίσεια ὀρθῆς έστιν ή ὑπὸ ΕΒΓ, ἡμίσεια ἄρα ὀρθῆς καὶ ἡ ὑπὸ ΔΒΗ. ἔστι δὲ καὶ ἡ ὑπὸ ΒΔΗ ὀρθή· ἴση γάρ ἐστι τῇ ὑπὸ ΔΓΕ· ἐναλλὰξ γάρ· λοιπή ἄρα ή ὑπὸ ΔΗΒ ἡμίσειά ἐστιν ὀρθῆς· ἡ ἄρα ὑπὸ ΔHB τῆ ὑπὸ ΔBH ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ $B\Delta$ πλευρᾶ τῆ ΗΔ ἐστιν ἴση. πάλιν, ἐπεὶ ἡ ὑπὸ ΕΗΖ ἡμίσειά ἐστιν όρθῆς, ὀρθὴ δὲ ἡ πρὸς τῷ Ζ΄ ἴση γάρ ἐστι τῆ ἀπεναντίον τῆ πρός τῷ Γ· λοιπὴ ἄρα ἡ ὑπὸ ΖΕΗ ἡμίσειά ἐστιν ὀρθῆς. ἴση ἄρα ἡ ὑπὸ ΕΗΖ γωνία τῆ ὑπὸ ΖΕΗ· ὥστε καὶ πλευρὰ ἡ ΗΖ πλευρα τη ΕΖ έστιν ίση. και έπει ίση έστιν ή ΕΓ τη ΓΑ], ἴσον ἐστὶ [καὶ] τὸ ἀπὸ τῆς ΕΓ τετράγωνον τῷ ἀπὸ τῆς ΓΑ

Proposition 10^{\dagger}

If a straight-line is cut in half, and any straight-line added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line).



For let any straight-line AB have been cut in half at (point) C, and let any straight-line BD have been added to it straight-on. I say that the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD.

For let CE have been drawn from point C, at rightangles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EB have been joined. And let EF have been drawn through E, parallel to AD [Prop. 1.31], and let FD have been drawn through D, parallel to CE [Prop. 1.31]. And since some straight-line EF falls across the parallel straight-lines ECand FD, the (internal angles) CEF and EFD are thus equal to two right-angles [Prop. 1.29]. Thus, FEB and EFD are less than two right-angles. And (straight-lines) produced from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced in the direction of *B* and *D*, the (straight-lines) EB and FD will meet. Let them have been produced, and let them meet together at G, and let AG have been joined. And since AC is equal to CE, angle EAC is also equal to (angle) AEC [Prop. 1.5]. And the (angle) at C (is) a right-angle. Thus, EAC and AEC [are] each half a right-angle [Prop. 1.32]. So, for the same (reasons), *CEB* and *EBC* are also each half a right-angle. Thus, (angle) AEB is a right-angle. And since EBCis half a right-angle, DBG (is) thus also half a rightangle [Prop. 1.15]. And *BDG* is also a right-angle. For it is equal to DCE. For (they are) alternate (angles)

τετραγώνω· τὰ ἄρα ἀπὸ τῶν ΕΓ, ΓΑ τετράγωνα διπλάσιά έστι τοῦ ἀπὸ τῆς ΓΑ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΓ, ΓΑ ίσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ· τὸ ἄρα ἀπὸ τῆς ΕΑ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου. πάλιν, ἐπεὶ ἴση έστιν ή ΖΗ τῆ ΕΖ, ἴσον ἐστι και τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΖΕ΄ τὰ ἄρα ἀπὸ τῶν ΗΖ, ΖΕ διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΕΖ. τοῖς δὲ ἀπὸ τῶν ΗΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΗ· τὸ ẳρα ἀπὸ τῆς EH διπλάσιόν ἐστι τοῦ ἀπὸ τῆς EZ. ἴση δὲ ἡ EZ τῆ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΗ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΗ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΗ τετραγώνοις ίσον έστι τὸ ἀπὸ τῆς ΑΗ τετράγωνον· τὸ άρα ἀπὸ τῆς ΑΗ διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς AH ἴσα ἐστὶ τὰ ἀπὸ τῶν AΔ, ΔH^{\cdot} τὰ ẳρα ἀπὸ τῶν ΑΔ, ΔΗ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, $\Gamma\Delta$ [τετραγώνων]. ἴση δὲ ἡ ΔH τῆ ΔB · τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Έαν ἄρα εὐθεῖα γραμμὴ τμηθῆ δίχα, προστεθῆ δέ τις αὐτῆ εὐθεῖα ἐπ' εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῆ προσχειμένῃ καὶ τὸ ἀπὸ τῆς προσχειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγχειμένης ἕχ τε τῆς ἡμισείας καὶ τῆς προσχειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου. ὅπερ ἕδει δεῖξαι. [Prop. 1.29]. Thus, the remaining (angle) DGB is half a right-angle. Thus, DGB is equal to DBG. So side BDis also equal to side GD [Prop. 1.6]. Again, since EGF is half a right-angle, and the (angle) at F (is) a right-angle, for it is equal to the opposite (angle) at C [Prop. 1.34], the remaining (angle) *FEG* is thus half a right-angle. Thus, angle EGF (is) equal to FEG. So the side GFis also equal to the side EF [Prop. 1.6]. And since [ECis equal to CA the square on EC is [also] equal to the square on CA. Thus, the (sum of the) squares on ECand CA is double the square on CA. And the (square) on EA is equal to the (sum of the squares) on EC and CA [Prop. 1.47]. Thus, the square on EA is double the square on AC. Again, since FG is equal to EF, the (square) on FG is also equal to the (square) on FE. Thus, the (sum of the squares) on GF and FE is double the (square) on EF. And the (square) on EG is equal to the (sum of the squares) on GF and FE [Prop. 1.47]. Thus, the (square) on EG is double the (square) on EF. And EF (is) equal to CD [Prop. 1.34]. Thus, the square on EG is double the (square) on CD. But it was also shown that the (square) on EA (is) double the (square) on AC. Thus, the (sum of the) squares on AE and EG is double the (sum of the) squares on AC and CD. And the square on AG is equal to the (sum of the) squares on AEand EG [Prop. 1.47]. Thus, the (square) on AG is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DG is equal to the (square) on AG [Prop. 1.47]. Thus, the (sum of the) [squares] on AD and DG is double the (sum of the) [squares] on ACand CD. And DG (is) equal to DB. Thus, the (sum of the) [squares] on AD and DB is double the (sum of the) squares on AC and CD.

Thus, if a straight-line is cut in half, and any straightline added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line). (Which is) the very thing it was required to show.

[†] This proposition is a geometric version of the algebraic identity: $(2 a + b)^2 + b^2 = 2 [a^2 + (a + b)^2]$.

ıα'.

Τὴν δοθεῖσαν εὐθεῖαν τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἶσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Proposition 11[†]

To cut a given straight-line such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.



Έστω ή δοθεϊσα εὐθεῖα ή AB· δεῖ δὴ τὴν AB τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνω.

Άναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ AB $\Delta\Gamma$, καὶ τετμήσθω ἡ AΓ δίχα κατὰ τὸ E σημεῖον, καὶ ἐπεζεύχθω ἡ BE, καὶ διήχθω ἡ ΓA ἐπὶ τὸ Z, καὶ κείσθω τῆ BE ἴση ἡ EZ, καὶ ἀναγεγράφθω ἀπὸ τῆς AZ τετράγωνον τὸ ZΘ, καὶ διήχθω ἡ HΘ ἐπὶ τὸ K· λέγω, ὅτι ἡ AB τέτμηται κατὰ τὸ Θ, ὥστε τὸ ὑπὸ τῶν AB, BΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς AΘ τετραγώνω.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ε, πρόσκειται δὲ αὐτῆ ἡ ΖΑ, τὸ ẳρα ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΕ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ τετραγώνῳ. ἴση δὲ ἡ ΕΖ τῆ ΕΒ· τὸ ẳρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τῷ ἀπὸ ΕΒ. ἀλλὰ τῷ ἀπὸ ΕΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΒΑ, ΑΕ· ὀρθή γὰρ ή πρὸς τῷ Α γωνία· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΕ. κοινόν ἀφηρήσθω τὸ ἀπὸ τῆς ΑΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνω. καί έστι τὸ μὲν ὑπὸ τῶν ΓΖ, ΖΑ τὸ ΖΚ· ἴση γὰρ
ἡ AZ τỹ ZH· τὸ δὲ ἀπὸ τῆς AB τὸ A
Δ· τὸ ẳρα ZK ἴσον έστι τῷ ΑΔ. κοινὸν ἀρηρήσθω τὸ ΑΚ· λοιπὸν ἄρα τὸ ΖΘ τῷ ΘΔ ἴσον ἐστίν. χαί ἐστι τὸ μὲν ΘΔ τὸ ὑπὸ τῶν ΑΒ, $B\Theta$ · ἴση γὰρ ἡ AB τῆ $B\Delta$ · τὸ δὲ $Z\Theta$ τὸ ἀπὸ τῆς $A\Theta$ · τὸ άρα ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ ΘΑ τετραγώνω.

Ή ἄρα δοθεῖσα εὐθεῖα ἡ ΑΒ τέτμηται κατὰ τὸ Θ ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς ΘΑ τετραγώνῳ· ὅπερ ἔδει ποιῆσαι.



Let AB be the given straight-line. So it is required to cut AB such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straightline), is equal to the square on the remaining piece.

For let the square ABDC have been described on AB [Prop. 1.46], and let AC have been cut in half at point E [Prop. 1.10], and let BE have been joined. And let CA have been drawn through to (point) F, and let EF be made equal to BE [Prop. 1.3]. And let the square FH have been described on AF [Prop. 1.46], and let GH have been drawn through to (point) K. I say that AB has been cut at H such as to make the rectangle contained by AB and BH equal to the square on AH.

For since the straight-line AC has been cut in half at E, and FA has been added to it, the rectangle contained by CF and FA, plus the square on AE, is thus equal to the square on EF [Prop. 2.6]. And EF (is) equal to EB. Thus, the (rectangle contained) by CF and FA, plus the (square) on AE, is equal to the (square) on EB. But, the (sum of the squares) on BA and AE is equal to the (square) on EB. For the angle at A (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by CF and FA, plus the (square) on AE, is equal to the (sum of the squares) on BA and AE. Let the square on AE have been subtracted from both. Thus, the remaining rectangle contained by CF and FA is equal to the square on AB. And FK is the (rectangle contained) by CF and FA. For AF (is) equal to FG. And AD (is) the (square) on AB. Thus, the (rectangle) FK is equal to the (square) AD. Let (rectangle) AK have been subtracted from both. Thus, the remaining (square) FH is equal to the (rectangle) HD. And HD is the (rectangle contained) by ABand BH. For AB (is) equal to BD. And FH (is) the (square) on AH. Thus, the rectangle contained by AB

and BH is equal to the square on HA.

Thus, the given straight-line AB has been cut at (point) H such as to make the rectangle contained by AB and BH equal to the square on HA. (Which is) the very thing it was required to do.

[†] This manner of cutting a straight-line—so that the ratio of the whole to the larger piece is equal to the ratio of the larger to the smaller piece—is sometimes called the "Golden Section".

ıβ'.

Έν τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δἰς ὑπὸ τε μιᾶς τῶν περὶ τὴν ἀμβλεῖαν γωνίαν, ἐφ' ἢν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῆ ἀμβλεία γωνία.



Έστω ἀμβλυγώνιον τρίγωνον τὸ ABΓ ἀμβλεῖαν ἔχον τὴν ὑπὸ BAΓ, καὶ ἦχθω ἀπὸ τοῦ B σημείου ἐπὶ τὴν ΓΑ ἐκβληθεῖσαν κάθετος ἡ BΔ. λέγω, ὅτι τὸ ἀπὸ τῆς BΓ τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν BA, AΓ τετραγώνων τῷ δἰς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ.

Έπει γὰρ εὐθεῖα ἡ ΓΔ τέτμηται, ὡς ἔτυχεν, κατὰ τὸ Α σημεῖον, τὸ ἄρα ἀπὸ τῆς ΔΓ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓΑ, ΑΔ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔΒ· τὰ ἄρα ἀπὸ τῶν ΓΔ, ΔΒ ἴσα ἐστὶ τοῖς τε ἀπὸ τῶν ΓΑ, ΑΔ, ΔΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ [περιεχομένῳ ὀρθογωνίῳ]. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΓΔ, ΔΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΓΒ· ὀρθὴ γὰρ ἡ προς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν ΑΔ, ΔΒ ἴσον τὸ ἀπὸ τῆς ΑΒ· τὸ ἄρα ἀπὸ τῆς ΓΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΓΑ, ΑΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ· ὥστε τὸ ἀπὸ τῆς ΓΒ τετράγωνον τῶν ἀπὸ τῶν ΓΑ, ΑΒ τετραγώνων μεῖζόν ἐστι τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ.

Έν ἄρα τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλεῖαν γωνίαν περιεχουσῶν

Proposition 12[†]

In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle.



Let ABC be an obtuse-angled triangle, having the angle BAC obtuse. And let BD be drawn from point B, perpendicular to CA produced [Prop. 1.12]. I say that the square on BC is greater than the (sum of the) squares on BA and AC, by twice the rectangle contained by CA and AD.

For since the straight-line CD has been cut, at random, at point A, the (square) on DC is thus equal to the (sum of the) squares on CA and AD, and twice the rectangle contained by CA and AD [Prop. 2.4]. Let the (square) on DB have been added to both. Thus, the (sum of the squares) on CD and DB is equal to the (sum of the) squares on CA, AD, and DB, and twice the [rectangle contained] by CA and AD. But, the (square) on CB is equal to the (sum of the squares) on CD and DB. For the angle at D (is) a right-angle [Prop. 1.47]. And the (square) on AB (is) equal to the (sum of the squares) on AD and DB [Prop. 1.47]. Thus, the square on CBis equal to the (sum of the) squares on CA and AB, and twice the rectangle contained by CA and AD. So the square on CB is greater than the (sum of the) squares on πλευρῶν τετραγώνων τῷ περιχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ἀμβλεῖαν γωνίαν, ἐφ᾽ ῆν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῆ ἀμβλεία γωνία. ὅπερ ἔδει δεῖξαι. CA and AB by twice the rectangle contained by CA and AD.

Thus, in obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle. (Which is) the very thing it was required to show.

[†] This proposition is equivalent to the well-known cosine formula: $BC^2 = AB^2 + AC^2 - 2ABAC \cos BAC$, since $\cos BAC = -AD/AB$.

ιγ΄.

Έν τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἐλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῷ δἰς ὑπό τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ' ῆν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῆ ὀξεία γωνία.



Έστω ὀξυγώνιον τρίγωνον τὸ ABΓ ὀξεῖαν ἔχον τὴν πρὸς τῷ B γωνίαν, καὶ ἦχθω ἀπὸ τοῦ A σημείου ἐπὶ τὴν BΓ κάθετος ἡ AΔ· λέγω, ὅτι τὸ ἀπὸ τῆς AΓ τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν ΓΒ, BA τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΒ, BΔ περιεχομένῳ ὀρθογωνίῳ.

Έπεὶ γὰρ εὐθεῖα ἡ ΓΒ τέτμηται, ὡς ἔτυχεν, κατὰ τὸ Δ, τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΔ τετράγωνα ἴσα ἐστὶ τῷ τε δἰς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῷ ὀρθογωνίῷ καὶ τῷ ἀπὸ τῆς ΔΓ τετραγώνῷ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔΑ τετράγωνον· τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΔ, ΔΑ τετράγωνα ἴσα ἐστὶ τῷ τε δἰς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῷ ὀρθογωνίῷ καὶ τοῖς ἀπὸ τῶν ΑΔ, ΔΓ τετραγώνιος. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΒΔ, ΔΑ ἴσον τὸ ἀπὸ τῆς ΑΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν ΑΔ, ΔΓ ἴσον τὸ ἀπὸ τῆς ΑΓ· τὰ ἀρα ἀπὸ τῶν ΓΒ, ΒΑ ἴσα ἑστὶ τῷ τε ἀπὸ τῆς ΑΓ καὶ τῷ δἰς ὑπὸ τῶν ΓΒ, ΒΔ· ὥστε μόνον τὸ ἀπὸ τῆς ΑΓ ἕλαττόν ἐστι

Proposition 13[†]

In acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle.



Let ABC be an acute-angled triangle, having the angle at (point) B acute. And let AD have been drawn from point A, perpendicular to BC [Prop. 1.12]. I say that the square on AC is less than the (sum of the) squares on CB and BA, by twice the rectangle contained by CB and BD.

For since the straight-line CB has been cut, at random, at (point) D, the (sum of the) squares on CB and BD is thus equal to twice the rectangle contained by CBand BD, and the square on DC [Prop. 2.7]. Let the square on DA have been added to both. Thus, the (sum of the) squares on CB, BD, and DA is equal to twice the rectangle contained by CB and BD, and the (sum of the) squares on AD and DC. But, the (square) on AB(is) equal to the (sum of the squares) on BD and DA. For the angle at (point) D is a right-angle [Prop. 1.47]. τῶν ἀπὸ τῶν ΓΒ, ΒΑ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένω ὀρθογωνίω.

Έν ἄρα τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἐλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δἰς ὑπό τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ' ῆν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῆ ὀξεία γωνία. ὅπερ ἔδει δεῖξαι. And the (square) on AC (is) equal to the (sum of the squares) on AD and DC [Prop. 1.47]. Thus, the (sum of the squares) on CB and BA is equal to the (square) on AC, and twice the (rectangle contained) by CB and BD. So the (square) on AC alone is less than the (sum of the) squares on CB and BA by twice the rectangle contained by CB and BD.

Thus, in acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle. (Which is) the very thing it was required to show.

[†] This proposition is equivalent to the well-known cosine formula: $AC^2 = AB^2 + BC^2 - 2ABBC \cos ABC$, since $\cos ABC = BD/AB$.



Τῷ δοθέντι εὐθυγράμμῷ ἴσον τετράγωνον συστήσαςθαι.



Έστω τὸ δοθὲν εὐθύγραμμον τὸ Α· δεῖ δὴ τῷ Α εὐθυγράμμῷ ἴσον τετράγωνον συστήσασθαι.

Συνεστάτω γὰρ τῷ Α ἑυθυγράμμῳ ἴσον παραλληλόγραμμον ὀρθογώνιον τὸ ΒΔ· εἰ μὲν οὖν ἴση ἐστὶν ἡ ΒΕ τῆ ΕΔ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. συνέσταται γὰρ τῷ Α εὐθυγράμμῳ ἴσον τετράγωνον τὸ ΒΔ· εἰ δὲ οὖ, μία τῶν ΒΕ, ΕΔ μείζων ἐστίν. ἔστω μείζων ἡ ΒΕ, καὶ ἐκβεβλήσθω ἐπὶ τὸ Ζ, καὶ κείσθω τῆ ΕΔ ἴση ἡ ΕΖ, καὶ τετμήσθω ἡ ΒΖ δίχα κατὰ τὸ Η, καὶ κέντρῳ τῷ Η, διαστήματι δὲ ἐνὶ τῶν ΗΒ, ΗΖ ἡμικύκλιον γεγράφθω τὸ ΒΘΖ, καὶ ἐκβεβλήσθω ἡ ΔΕ ἐπὶ τὸ Θ, καὶ ἐπεζεύχθω ἡ ΗΘ.

Έπει οῦν εὐθεĩα ἡ BZ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ ἄρα ὑπὸ τῶν BE, EZ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς HZ τετραγώνω. ἴση δὲ ἡ HZ τῆ HΘ· τὸ ἄρα ὑπὸ τῶν BE, EZ μετὰ τοῦ ἀπὸ τῆς HE ἴσον ἐστὶ τῷ ἀπὸ τῆς HΘ. τῷ δὲ ἀπὸ τῆς HΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΘΕ, EH

Proposition 14

To construct a square equal to a given rectilinear figure.



Let *A* be the given rectilinear figure. So it is required to construct a square equal to the rectilinear figure *A*.

For let the right-angled parallelogram BD, equal to the rectilinear figure A, have been constructed [Prop. 1.45]. Therefore, if BE is equal to ED then that (which) was prescribed has taken place. For the square BD, equal to the rectilinear figure A, has been constructed. And if not, then one of the (straight-lines) BE or ED is greater (than the other). Let BE be greater, and let it have been produced to F, and let EF be made equal to ED[Prop. 1.3]. And let BF have been cut in half at (point) G [Prop. 1.10]. And, with center G, and radius one of the (straight-lines) GB or GF, let the semi-circle BHFhave been drawn. And let DE have been produced to H, and let GH have been joined.

ὑπὸ τῶν BE, EZ μετὰ τοῦ ἀπὸ τῆς HE ἴσον ἐστὶ τῷ ἀπὸ Therefore, since the straight-line BF has been cut τῆς HΘ. τῷ δὲ ἀπὸ τῆς HΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΘΕ, EH equally at G, and unequally at E—the rectangle conτετράγωνα· τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ μετὰ τοῦ ἀπὸ ΗΕ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΘΕ, ΕΗ. Χοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς ΗΕ τετράγωνον· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΒΕ, ΕΖ περιεχόμενον ὄρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ τετραγώνω. ἀλλὰ τὸ ὑπὸ τῶν ΒΕ, ΕΖ τὸ ΒΔ ἐστιν· ἴση γὰρ ἡ ΕΖ τῆ ΕΔ· τὸ ἄρα ΒΔ παραλληλόγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘΕ τετραγώνω. ἴσον δὲ τὸ ΒΔ τῷ Α εὐθυγράμμω. Χαὶ τὸ Α ἄρα εὐθύγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ ἀναγραφησομένω τετραγώνω.

Τῷ ἄρα δοθέντι εὐθυγράμμω τῷ Α ἴσον τετράγωνον συνέσταται τὸ ἀπὸ τῆς ΕΘ ἀναγραφησόμενον· ὅπερ ἔδει ποιῆσαι.

tained by BE and EF, plus the square on EG, is thus equal to the square on GF [Prop. 2.5]. And GF (is) equal to GH. Thus, the (rectangle contained) by BE and EF, plus the (square) on GE, is equal to the (square) on GH. And the (sum of the) squares on HE and EG is equal to the (square) on GH [Prop. 1.47]. Thus, the (rectangle contained) by BE and EF, plus the (square) on GE, is equal to the (sum of the squares) on HE and EG. Let the square on GE have been taken from both. Thus, the remaining rectangle contained by BE and EF is equal to the square on EH. But, BD is the (rectangle contained) by BE and EF. For EF (is) equal to ED. Thus, the parallelogram BD is equal to the square on HE. And BD(is) equal to the rectilinear figure A. Thus, the rectilinear figure A is also equal to the square (which) can be described on EH.

Thus, a square—(namely), that (which) can be described on EH—has been constructed, equal to the given rectilinear figure A. (Which is) the very thing it was required to do.

ELEMENTS BOOK 3

Fundamentals of Plane Geometry Involving Circles

Οροι.

α'. Ίσοι κύκλοι εἰσίν, ῶν αἱ διάμετροι ἴσαι εἰσίν, ἢ ῶν αἱ ἐκ τῶν κέντρων ἴσαι εἰσίν.

β'. Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἁπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

γ'. Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἴτινες ἁπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

δ'. Έν κύκλω ΐσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ' αὐτὰς κάθετοι ἀγόμεναι ἴσαι ῶσιν.

ε΄. Μεΐζον δὲ ἀπέχειν λέγεται, ἐφ' ἢν ἡ μείζων κάθετος πίπτει.

τ΄. Τμήμα κύκλου ἐστὶ τὸ περιεχόμενον σχήμα ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

ζ΄. Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

η'. Έν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῆ τι σημεῖον καὶ ἀπ᾽ αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἥ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθεῖαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

θ΄. Όταν δὲ αἱ περιέχουσαι τὴν γωνίαν εὐθεῖαι ἀπολαμβάνωσί τινα περιφέρειαν, ἐπ' ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

ι'. Τομεὺς δὲ χύχλου ἐστίν, ὅταν πρὸς τῷ χέντρῷ τοῦ χύχλου συσταθῆ γωνία, τὸ περιεχόμενον σχῆμα ὑπό τε τῶν τὴν γωνίαν περιεχουσῶν εὐθειῶν χαὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῶν περιφερείας.

ια'. Όμοία τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἤ ἐν οῖς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

α΄.

Τοῦ δοθέντος κύκλου τὸ κέντρον εὑρεῖν.

Έστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ κύκλου τὸ κέντρον εὑρεῖν.

 Δ ιήχθω τις εἰς αὐτόν, ὡς ἔτυχεν, εὐθεῖα ἡ AB, καὶ τετμήσθω δίχα κατὰ τὸ Δ σημεῖον, καὶ ἀπὸ τοῦ Δ τῆ AB πρὸς ὀρθὰς ἦχθω ἡ Δ Γ καὶ διήχθω ἐπὶ τὸ E, καὶ τετμήσθω ἡ ΓΕ δίχα κατὰ τὸ Z· λέγω, ὅτι τὸ Z κέντρον ἐστὶ τοῦ ABΓ [κύκλου].

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστω τὸ Η, καὶ ἐπεζεύχθωσαν αἱ HA, HΔ, HB. καὶ ἐπεὶ ἴση ἐστὶν ἡ AΔ τῆ ΔB, κοινὴ δὲ ἡ ΔH, δύο δὴ αἱ AΔ, ΔH δύο ταῖς HΔ, ΔB ἴσαι εἰσὶν ἑκατέρα ἑκατέρα· καὶ βάσις ἡ HA βάσει τῆ HB ἐστιν ἴση· ἐκ κέντρου γάρ· γωνία ἄρα ἡ ὑπὸ AΔH γωνία τῆ ὑπὸ HΔB ἴση ἐστίν.

Definitions

1. Equal circles are (circles) whose diameters are equal, or whose (distances) from the centers (to the circumferences) are equal (i.e., whose radii are equal).

2. A straight-line said to touch a circle is any (straightline) which, meeting the circle and being produced, does not cut the circle.

3. Circles said to touch one another are any (circles) which, meeting one another, do not cut one another.

4. In a circle, straight-lines are said to be equally far from the center when the perpendiculars drawn to them from the center are equal.

5. And (that straight-line) is said to be further (from the center) on which the greater perpendicular falls (from the center).

6. A segment of a circle is the figure contained by a straight-line and a circumference of a circle.

7. And the angle of a segment is that contained by a straight-line and a circumference of a circle.

8. And the angle in a segment is the angle contained by the joined straight-lines, when any point is taken on the circumference of a segment, and straight-lines are joined from it to the ends of the straight-line which is the base of the segment.

9. And when the straight-lines containing an angle cut off some circumference, the angle is said to stand upon that (circumference).

10. And a sector of a circle is the figure contained by the straight-lines surrounding an angle, and the circumference cut off by them, when the angle is constructed at the center of a circle.

11. Similar segments of circles are those accepting equal angles, or in which the angles are equal to one another.

Proposition 1

To find the center of a given circle.

Let ABC be the given circle. So it is required to find the center of circle ABC.

Let some straight-line AB have been drawn through (ABC), at random, and let (AB) have been cut in half at point D [Prop. 1.9]. And let DC have been drawn from D, at right-angles to AB [Prop. 1.11]. And let (CD) have been drawn through to E. And let CE have been cut in half at F [Prop. 1.9]. I say that (point) F is the center of the [circle] ABC.

For (if) not then, if possible, let G (be the center of the circle), and let GA, GD, and GB have been joined. And since AD is equal to DB, and DG (is) common, the two

όταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῆ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΗΔΒ. ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΔΒ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΔΒ τῆ ὑπὸ ΗΔΒ, ἡ μείζων τῆ ἐλάττονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Η κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλο τι πλὴν τοῦ Ζ.



Τὸ Ζ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ [κύκλου].

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι ἐἀν ἐν χύχλῳ εὐθεῖά τις εὐθεῖάν τινα δίχα καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ χύχλου. — ὅπερ ἔδει ποιῆσαι.

[†] The Greek text has "*GD*, *DB*", which is obviously a mistake.

β΄.

Έαν κύκλου ἐπὶ τῆς περιφερείας ληφθῆ δύο τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Έστω κύκλος ὁ ABΓ, καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰλήφθω δύο τυχόντα σημεῖα τὰ A, B· λέγω, ὅτι ἡ ἀπὸ τοῦ A ἐπὶ τὸ B ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Μή γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ἐκτὸς ὡς ἡ ΑΕΒ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Δ, καὶ ἐπεζεύχθωσαν αἱ ΔΑ, ΔΒ, καὶ διήχθω ἡ ΔΖΕ.

Έπει οὖν ἴση ἐστιν ή ΔA τῆ ΔB , ἴση ἄρα και γωνία ή ὑπὸ ΔAE τῆ ὑπὸ ΔBE · και ἐπει τριγώνου τοῦ ΔAE μία

(straight-lines) AD, DG are equal to the two (straightlines) BD, DG,[†] respectively. And the base GA is equal to the base GB. For (they are both) radii. Thus, angle ADG is equal to angle GDB [Prop. 1.8]. And when a straight-line stood upon (another) straight-line make adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, GDBis a right-angle. And FDB is also a right-angle. Thus, FDB (is) equal to GDB, the greater to the lesser. The very thing is impossible. Thus, (point) G is not the center of the circle ABC. So, similarly, we can show that neither is any other (point) except F.



Thus, point F is the center of the [circle] ABC.

Corollary

So, from this, (it is) manifest that if any straight-line in a circle cuts any (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line). — (Which is) the very thing it was required to do.

Proposition 2

If two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle.

Let ABC be a circle, and let two points A and B have been taken at random on its circumference. I say that the straight-line joining A to B will fall inside the circle.

For (if) not then, if possible, let it fall outside (the circle), like AEB (in the figure). And let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) D. And let DA and DB have been joined, and let DFE have been drawn through.

Therefore, since DA is equal to DB, the angle DAE

πλευρὰ προσεκβέβληται ἡ ΑΕΒ, μείζων ἄρα ἡ ὑπὸ ΔΕΒ γωνία τῆς ὑπὸ ΔΑΕ. ἴση δὲ ἡ ὑπὸ ΔΑΕ τῆ ὑπὸ ΔΒΕ· μείζων ἄρα ἡ ὑπὸ ΔΕΒ τῆς ὑπὸ ΔΒΕ. ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΔΒ τῆς ΔΕ. ἴση δὲ ἡ ΔΒ τῆ ΔΖ. μείζων ἄρα ἡ ΔΖ τῆς ΔΕ ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Α ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται τοῦ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἐπ' αὐτῆς τῆς περιφερείας· ἐντὸς ἄρα.



Έὰν ἄρα χύχλου ἐπὶ τῆς περιφερείας ληφθῆ δύο τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ χύχλου· ὅπερ ἔδει δεῖξαι.

γ'.

Έὰν ἐν χύχλῷ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει.

Έστω κύκλος ὁ ABΓ, καὶ ἐν αὐτῷ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΓΔ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν AB δίχα τεμνέτω κατὰ τὸ Z σημεῖον λέγω, ὅτι καὶ πρὸς ὀρθὰς αὐτὴν τέμνει.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ AZ τῆ ZB, xοινὴ δὲ ἡ ZE, δύο δυσὶν ἴσαι [εἰσίν]· καὶ βάσις ἡ EA βάσει τῆ EB ἴση· γωνία ἄρα ἡ ὑπὸ AZE γωνία τῆ ὑπὸ BZE ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ[°] εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῆ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ἑκατέρα ἄρα τῶν ὑπὸ AZE, BZE ὀρθή ἐστιν. ἡ ΓΔ ἄρα διὰ τοῦ κέντρου οῦσα τὴν AB μὴ διὰ τοῦ κέντρου οὕσαν δίχα τέμνουσα καὶ πρὸς ὀρθὰς τέμνει. (is) thus also equal to DBE [Prop. 1.5]. And since in triangle DAE the one side, AEB, has been produced, angle DEB (is) thus greater than DAE [Prop. 1.16]. And DAE (is) equal to DBE [Prop. 1.5]. Thus, DEB (is) greater than DBE. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, DB (is) greater than DE. And DB (is) equal to DF. Thus, DF (is) greater than DE, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining A to B will not fall outside the circle. So, similarly, we can show that neither (will it fall) on the circumference itself. Thus, (it will fall) inside (the circle).

ELEMENTS BOOK 3



Thus, if two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle. (Which is) the very thing it was required to show.

Proposition 3

In a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half.

Let ABC be a circle, and, within it, let some straightline through the center, CD, cut in half some straight-line not through the center, AB, at the point F. I say that (CD) also cuts (AB) at right-angles.

For let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) E, and let EA and EB have been joined.

And since AF is equal to FB, and FE (is) common, two (sides of triangle AFE) [are] equal to two (sides of triangle BFE). And the base EA (is) equal to the base EB. Thus, angle AFE is equal to angle BFE [Prop. 1.8]. And when a straight-line stood upon (another) straightline makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, AFE and BFE are each right-angles. Thus, the


Άλλὰ δὴ ἡ ΓΔ τὴν ΑΒ πρὸς ὀρθὰς τεμνέτω· λέγω, ὅτι καὶ δίχα αὐτὴν τέμνει, τουτέστιν, ὅτι ἴση ἐστὶν ἡ ΑΖ τῆ ΖΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴση ἐστὶν ἡ EA τῆ EB, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ EAZ τῆ ὑπὸ EBZ. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ AZE ὀρθῆ τῆ ὑπὸ BZE ἴση· δύο ἄρα τρίγωνά ἐστι EAZ, EZB τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιῷ πλευρῷ ἴσην κοινὴν αὐτῶν τὴν EZ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ AZ τῆ ZB.

Έὰν ἄρα ἐν χύχλῷ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει· ὅπερ ἔδει δεῖξαι.

δ'.

Έὰν ἐν κύκλῷ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ δὶα τοῦ κέντρου οὕσαι, οὐ τέμνουσιν ἀλλήλας δίχα.

Έστω χύχλος ὁ ΑΒΓΔ, χαὶ ἐν αὐτῷ δύο εὐθεῖαι αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας χατὰ τὸ Ε μὴ διὰ τοῦ χέντρου οὕσαι· λέγω, ὅτι οὐ τέμνουσιν ἀλλήλας δίχα.

Eἰ γὰρ δυνατόν, τεμνέτωσαν ἀλλήλας δίχα ὥστε ἴσην εἶναι τὴν μὲν ΑΕ τῆ ΕΓ, τὴν δὲ ΒΕ τῆ ΕΔ· καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΕ.

Έπεὶ οὖν εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΖΕ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΕΑ· πάλιν, ἐπεὶ εὐθεῖά τις ἡ ΖΕ εὐθεῖάν τινα τὴν ΒΔ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἡ ὑπὸ ΖΕΒ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΕΑ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΕΑ τῆ ὑπὸ ΖΕΒ ἡ ἐλάττων τῆ (straight-line) *CD*, which is through the center and cuts

ELEMENTS BOOK 3

in half the (straight-line) AB, which is not through the center, also cuts (AB) at right-angles.



And so let CD cut AB at right-angles. I say that it also cuts (AB) in half. That is to say, that AF is equal to FB.

For, with the same construction, since EA is equal to EB, angle EAF is also equal to EBF [Prop. 1.5]. And the right-angle AFE is also equal to the right-angle BFE. Thus, EAF and EFB are two triangles having two angles equal to two angles, and one side equal to one side—(namely), their common (side) EF, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, AF (is) equal to FB.

Thus, in a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half. (Which is) the very thing it was required to show.

Proposition 4

In a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half.

Let ABCD be a circle, and within it, let two straightlines, AC and BD, which are not through the center, cut one another at (point) E. I say that they do not cut one another in half.

For, if possible, let them cut one another in half, such that AE is equal to EC, and BE to ED. And let the center of the circle ABCD have been found [Prop. 3.1], and let it be (at point) F, and let FE have been joined.

Therefore, since some straight-line through the center, FE, cuts in half some straight-line not through the center, AC, it also cuts it at right-angles [Prop. 3.3]. Thus, FEA is a right-angle. Again, since some straight-line FE

μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα αἱ ΑΓ, ΒΔ τέμνουσιν ἀλλήλας δίγα.



Έὰν ἄρα ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ δὶα τοῦ κέντρου οὖσαι, οὐ τέμνουσιν ἀλλήλας δίχα· ὅπερ ἔδει δεῖζαι.

ε΄.

Έὰν δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.



Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΗ τεμνέτωσαν ἀλλήλους κατὰ τὰ Β, Γ σημεῖα. λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΓ, καὶ διήχθω ἡ ΕΖΗ, ὡς ἔτυχεν. καὶ ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῆ ΕΖ. πάλιν, ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΓΔΗ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῆ ΕΗ· ἐδείχθη δὲ ἡ ΕΓ καὶ τῆ ΕΖ ἴση· καὶ ἡ ΕΖ ἄρα τῆ ΕΗ ἐστιν ἴση ἡ ἐλάσσων τῆ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ε σημεῖον κέντρον ἐστὶ τῶν ΑΒΓ, ΓΔΗ κύκλων.

Έαν άρα δύο χύχλοι τέμνωσιν άλλήλους, ούχ ἔστιν

cuts in half some straight-line BD, it also cuts it at rightangles [Prop. 3.3]. Thus, FEB (is) a right-angle. But FEA was also shown (to be) a right-angle. Thus, FEA(is) equal to FEB, the lesser to the greater. The very thing is impossible. Thus, AC and BD do not cut one another in half.



Thus, in a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half. (Which is) the very thing it was required to show.

Proposition 5

If two circles cut one another then they will not have the same center.



For let the two circles ABC and CDG cut one another at points B and C. I say that they will not have the same center.

For, if possible, let E be (the common center), and let EC have been joined, and let EFG have been drawn through (the two circles), at random. And since point E is the center of the circle ABC, EC is equal to EF. Again, since point E is the center of the circle CDG, ECis equal to EG. But EC was also shown (to be) equal to EF. Thus, EF is also equal to EG, the lesser to the greater. The very thing is impossible. Thus, point E is not αὐτῶν τὸ αὐτὸ κέντρον. ὅπερ ἔδει δεῖξαι.

ج'.

Έὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.



Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΕ ἐφαπτέσθωσαν ἀλλήλων κατὰ τὸ Γ σημεῖον· λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

El γὰρ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω, ὡς ἔτυχεν, ἡ ΖΕΒ.

Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ABΓ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῆ ΖΒ. πάλιν, ἐπεὶ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΓΔΕ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῆ ΖΕ. ἐδείχθη δὲ ἡ ΖΓ τῆ ΖΒ ἴση· καὶ ἡ ΖΕ ἄρα τῆ ΖΒ ἐστιν ἴση, ἡ ἐλάττων τῆ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ σημεῖον κέντρον ἐστὶ τῶν ABΓ, ΓΔΕ κύκλων.

Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ζ.

Έὰν χύχλου ἐπὶ τῆς διαμέτρου ληφθῆ τι σημεῖον, ὃ μή ἐστι κέντρον τοῦ κύχλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύχλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφʾ ῆς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς δὶα τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύχλον ἐφ' ἑκάτερα τῆς ἐλαχίστης. the (common) center of the circles ABC and CDG.

Thus, if two circles cut one another then they will not have the same center. (Which is) the very thing it was required to show.

Proposition 6

If two circles touch one another then they will not have the same center.



For let the two circles ABC and CDE touch one another at point C. I say that they will not have the same center.

For, if possible, let F be (the common center), and let FC have been joined, and let FEB have been drawn through (the two circles), at random.

Therefore, since point F is the center of the circle ABC, FC is equal to FB. Again, since point F is the center of the circle CDE, FC is equal to FE. But FC was shown (to be) equal to FB. Thus, FE is also equal to FB, the lesser to the greater. The very thing is impossible. Thus, point F is not the (common) center of the circles ABC and CDE.

Thus, if two circles touch one another then they will not have the same center. (Which is) the very thing it was required to show.

Proposition 7

If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straightlines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straightline) nearer[†] to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each



Έστω κύκλος ὁ ABΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ AΔ, καὶ ἐπὶ τῆς AΔ εἰλήφθω τι σημεῖον τὸ Z, ὃ μή ἐστι κέντρον τοῦ κύκλου, κέντρον δὲ τοῦ κύκλου ἔστω τὸ E, καὶ ἀπὸ τοῦ Z πρὸς τὸν ABΓΔ κύκλον προσπιπτέτωσαν εὐθεῖαί τινες αἱ ZB, ZΓ, ZH· λέγω, ὅτι μεγίστη μέν ἐστιν ἡ ZA, ἐλαχίστη δὲ ἡ ZΔ, τῶν δὲ ἄλλων ἡ μὲν ZB τῆς ZΓ μείζων, ἡ δὲ ZΓ τῆς ZH.

Έπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΕ, ΗΕ. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσιν, αἱ ἄρα ΕΒ, ΕΖ τῆς ΒΖ μείζονές εἰσιν. ἴση δὲ ἡ ΑΕ τῆ ΒΕ [αἱ ἄρα ΒΕ, ΕΖ ἴσαι εἰσὶ τῆ ΑΖ]· μείζων ἄρα ἡ ΑΖ τῆς ΒΖ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῆ ΓΕ, κοινὴ δὲ ἡ ΖΕ, δύο δὴ αἱ ΒΕ, ΕΖ δυσὶ ταῖς ΓΕ, ΕΖ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΒΕΖ γωνίας τῆς ὑπὸ ΓΕΖ μείζων· βάσις ἄρα ἡ ΒΖ βάσεως τῆς ΓΖ μείζων ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῆς ΖΗ μείζων ἐστίν.

Πάλιν, ἐπεὶ αί ΗΖ, ΖΕ τῆς ΕΗ μείζονές εἰσιν, ἴση δὲ ἡ ΕΗ τῆ ΕΔ, αί ἄρα ΗΖ, ΖΕ τῆς ΕΔ μείζονές εἰσιν. χοινὴ ἀφηρήσθω ἡ ΕΖ· λοιπὴ ἄρα ἡ ΗΖ λοιπῆς τῆς ΖΔ μείζων ἐστίν. μεγίστη μὲν ἄρα ἡ ΖΑ, ἐλαχίστη δὲ ἡ ΖΔ, μείζων δὲ ἡ μὲν ΖΒ τῆς ΖΓ, ἡ δὲ ΖΓ τῆς ΖΗ.

Λέγω, ὅτι καὶ ἀπὸ τοῦ Ζ σημείου δύο μόνον ἴσαι προσπεσοῦνται πρὸς τὸν ABΓΔ κύκλον ἐφ' ἑκάτερα τῆς ΖΔ ἑλαχίστης. συνεστάτω γὰρ πρὸς τῆ ΕΖ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Ε τῆ ὑπὸ HEZ γωνία ἴση ἡ ὑπὸ ΖΕΘ, καὶ ἑπεζεύχθω ἡ ΖΘ. ἐπεὶ οῦν ἴση ἐστὶν ἡ HE τῆ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἰ HE, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ HEZ γωνία τῆ ὑπὸ ΘΕΖ ἴση· βάσις ἄρα ἡ ΖΗ βάσει τῆ ΖΘ ἴση ἐστίν. λέγω δή, ὅτι τῆ ΖΗ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Ζ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω ἡ ΖΚ. καὶ ἐπεὶ ἡ ΖΚ τῆ ΖΗ ἴση ἐστίν, ἀλλὰ ἡ ΖΘ τῆ ΖΗ [ἴση ἐστίν], καὶ ἡ ΖΚ ἄρα τῆ ΖΘ ἐστιν ἴση, ἡ ἔγγιον τῆς διὰ τοῦ Χ σημείου ἑτέρα τις



Let ABCD be a circle, and let AD be its diameter, and let some point F, which is not the center of the circle, have been taken on AD. Let E be the center of the circle. And let some straight-lines, FB, FC, and FG, radiate from F towards (the circumference of) circle ABCD. I say that FA is the greatest (straight-line), FD the least, and of the others, FB (is) greater than FC, and FC than FG.

For let BE, CE, and GE have been joined. And since for every triangle (any) two sides are greater than the remaining (side) [Prop. 1.20], EB and EF is thus greater than BF. And AE (is) equal to BE [thus, BE and EFis equal to AF]. Thus, AF (is) greater than BF. Again, since BE is equal to CE, and FE (is) common, the two (straight-lines) BE, EF are equal to the two (straightlines) CE, EF (respectively). But, angle BEF (is) also greater than angle CEF.[‡] Thus, the base BF is greater than the base CF. Thus, the base BF is greater than the base CF [Prop. 1.24]. So, for the same (reasons), CF is also greater than FG.

Again, since GF and FE are greater than EG[Prop. 1.20], and EG (is) equal to ED, GF and FEare thus greater than ED. Let EF have been taken from both. Thus, the remainder GF is greater than the remainder FD. Thus, FA (is) the greatest (straight-line), FD the least, and FB (is) greater than FC, and FC than FG.

I also say that from point F only two equal (straightlines) will radiate towards (the circumference of) circle ABCD, (one) on each (side) of the least (straight-line) FD. For let the (angle) FEH, equal to angle GEF, have been constructed on the straight-line EF, at the point Eon it [Prop. 1.23], and let FH have been joined. Therefore, since GE is equal to EH, and EF (is) common, προσπεσεῖται πρὸς τὸν κύκλον ἴση τῆ ΗΖ· μία ἄρα μόνη.

Έλν ἄρα χύχλου ἐπὶ τῆς διαμέτρου ληφθῆ τι σημεῖον, δ μή ἐστι κέντρον τοῦ χύχλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν χύχλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφ᾽ ῆς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς δὶα τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ αὐτοῦ σημείου προσπεσοῦνται πρὸς τὸν χύχλον ἐφ᾽ ἑχάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

[†] Presumably, in an angular sense.

[‡] This is not proved, except by reference to the figure.

η'.

Έὰν χύχλου ληφθῆ τι σημεῖον ἐχτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν χύχλον διαχθῶσιν εὐθεῖαί τινες, ῶν μία μὲν διὰ τοῦ κέντρου, αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν χοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου, τῶν δὲ ἀλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν χυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἀλλων ἀεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν χύχλον ἐφ' ἑχάτερα τῆς ἐλαχίστης.

Έστω κύκλος ὁ ABΓ, καὶ τοῦ ABΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπ' αὐτοῦ διήχθωσαν εὐθεῖαί τινες αἱ ΔΑ, ΔΕ, ΔΖ, ΔΓ, ἔστω δὲ ἡ ΔΑ διὰ τοῦ κέντρου. λέγω, ὅτι τῶν μὲν πρὸς τὴν ΑΕΖΓ κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου ἡ ΔΑ, μείζων δὲ ἡ μὲν ΔΕ τῆς ΔΖ ἡ δὲ ΔΖ τῆς ΔΓ, τῶν δὲ πρὸς τὴν ΘΛΚΗ κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ ΔΗ ἡ μεταξὺ τοῦ σημείου καὶ τῆς διαμέτρου τῆς ΑΗ, ἀεὶ δὲ ἡ ἔγγιον τῆς ΔΗ ἐλαχίστης ἐλάττων ἐστὶ τῆς ἀπώτερον, ἡ μὲν ΔΚ τῆς ΔΛ, ἡ δὲ ΔΛ the two (straight-lines) GE, EF are equal to the two (straight-lines) HE, EF (respectively). And angle GEF (is) equal to angle HEF. Thus, the base FG is equal to the base FH [Prop. 1.4]. So I say that another (straight-line) equal to FG will not radiate towards (the circumference of) the circle from point F. For, if possible, let FK (so) radiate. And since FK is equal to FG, but FH [is equal] to FG, FK is thus also equal to FH, the nearer to the (straight-line) through the center equal to the further away. The very thing (is) impossible. Thus, another (straight-line) equal to GF will not radiate from the point F towards (the circumference of) the circle. Thus, (there is) only one (such straight-line).

Thus, if some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the same point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

Proposition 8

If some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer[†] to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straightlines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Let ABC be a circle, and let some point D have been taken outside ABC, and from it let some straight-lines, DA, DE, DF, and DC, have been drawn through (the circle), and let DA be through the center. I say that for the straight-lines radiating towards the concave (part of τῆς $\Delta \Theta$.



Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου καὶ ἔστω τὸ Μ· καὶ ἐπεζεύχθωσαν αἱ ΜΕ, ΜΖ, ΜΓ, ΜΚ, ΜΛ, ΜΘ.

Kai
ἐπεὶ ἴση ἐστὶν ἡ AM τῆ EM, xοινὴ προσχείσθω ἡ

MΔ· ἡ ἄρα AΔ ἴση ἐστὶ ταῖς EM, MΔ. ἀλλ' ai EM, MΔ

τῆς ΕΔ μείζον
ές εἰσιν· καὶ ἡ AΔ ἄρα τῆς ΕΔ μείζων ἐστίν.

πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ME τῆ MZ, xοινὴ δὲ ἡ MΔ, ai EM,

MΔ ἄρα ταῖς ZM, MΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ EMΔ

γωνίας τῆς ὑπὸ ZMΔ μείζων ἐστίν. βάσις ἄρα ἡ ΕΔ βάσεως

τῆς ZΔ μείζων ἐστίν· ὑμοίως δὴ δείξομεν, ὅτι καὶ ἡ ZΔ τῆς

ΓΔ μείζων ἐστίν· μεγίστη μὲν ἄρα ἡ ΔΑ, μείζων δὲ ἡ μὲν

ΔΕ τῆς ΔΖ, ἡ δὲ ΔΖ τῆς ΔΓ.

Καὶ ἐπεὶ αἱ ΜΚ, ΚΔ τῆς ΜΔ μείζονές εἰσιν, ἴση δὲ ἡ MH τῆ ΜΚ, λοιπὴ ἄρα ἡ ΚΔ λοιπῆς τῆς ΗΔ μείζων ἐστίν· ὥστε ἡ ΗΔ τῆς ΚΔ ἐλάττων ἐστίν· καὶ ἐπεὶ τριγώνου τοῦ MΛΔ ἐπὶ μιᾶς τῶν πλευρῶν τῆς ΜΔ δύο εὐθεĩαι ἐντὸς συνεστάθησαν αἱ ΜΚ, ΚΔ, αἱ ἄρα ΜΚ, ΚΔ τῶν ΜΛ, ΛΔ ἐλάττονές εἰσιν· ἴση δὲ ἡ ΜΚ τῆ ΜΛ· λοιπὴ ἄρα ἡ ΔΚ λοιπῆς τῆς ΔΛ ἐλάττων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΔΛ τῆς ΔΘ ἐλάττων ἐστίν· ἐλαχίστη μὲν ἄρα ἡ ΔΗ, ἐλάττων δὲ ἡ μὲν ΔΚ τῆς ΔΛ ἡ δὲ ΔΛ τῆς ΔΘ.

Λέγω, ὅτι καὶ δύο μόνον ἴσαι ἀπὸ τοῦ Δ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ' ἐκάτερα τῆς ΔΗ ἐλαχίστης· συνεστάτω πρὸς τῆ ΜΔ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Μ τῆ ὑπὸ ΚΜΔ γωνία ἴση γωνία ἡ ὑπὸ ΔΜΒ, καὶ ἐπεζεύχθω ἡ ΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΜΚ τῆ MB, κοινὴ δὲ ἡ MΔ, δύο δὴ αἱ KM, MΔ δύο ταῖς BM, MΔ the) circumference, AEFC, the greatest is the one (passing) through the center, (namely) AD, and (that) DE (is) greater than DF, and DF than DC. For the straight-lines radiating towards the convex (part of the) circumference, HLKG, the least is the one between the point and the diameter AG, (namely) DG, and a (straight-line) nearer to the least (straight-line) DG is always less than one farther away, (so that) DK (is less) than DL, and DL than than DH.



For let the center of the circle have been found [Prop. 3.1], and let it be (at point) *M* [Prop. 3.1]. And let *ME*, *MF*, *MC*, *MK*, *ML*, and *MH* have been joined.

And since AM is equal to EM, let MD have been added to both. Thus, AD is equal to EM and MD. But, EM and MD is greater than ED [Prop. 1.20]. Thus, AD is also greater than ED. Again, since ME is equal to MF, and MD (is) common, the (straight-lines) EM, MD are thus equal to FM, MD. And angle EMD is greater than angle FMD.[‡] Thus, the base ED is greater than the base FD [Prop. 1.24]. So, similarly, we can show that FD is also greater than CD. Thus, AD (is) the greatest (straight-line), and DE (is) greater than DF, and DF than DC.

And since MK and KD is greater than MD [Prop. 1.20], and MG (is) equal to MK, the remainder KD is thus greater than the remainder GD. So GD is less than KD. And since in triangle MLD, the two internal straight-lines MK and KD were constructed on one of the sides, MD, then MK and KD are thus less than ML and LD [Prop. 1.21]. And MK (is) equal to ML. Thus, the remainder DK is less than the remainder DL. So, similarly, we can show that DL is also less than DH. Thus, DG (is) the least (straight-line), and DK (is) less than DL, and DL than DH.

I also say that only two equal (straight-lines) will radi-

ίσαι εἰσὶν ἑχατέρα ἑχατέρα[.] χαὶ γωνία ἡ ὑπὸ ΚΜΔ γωνία τῆ ὑπὸ BMΔ ἴση· βάσις ἄρα ἡ ΔΚ βάσει τῆ ΔΒ ἴση ἐστίν. λέγω [δή], ὅτι τῆ ΔΚ εὐθεία ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν χύχλον ἀπὸ τοῦ Δ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω χαὶ ἔστω ἡ ΔΝ. ἐπεὶ οῦν ἡ ΔΚ τῆ ΔΝ ἐστιν ἴση, ἀλλ' ἡ ΔΚ τῆ ΔΒ ἐστιν ἴση, χαὶ ἡ ΔΒ ἄρα τῆ ΔΝ ἐστιν ἴση, ἡ ἔγγιον τῆς ΔΗ ἐλαχίστης τῆ ἀπώτερον [ἐστιν] ἴση· ὅπερ ἀδύνατον ἐδείχθη. οὐχ ἄρα πλείους ἢ δύο ἴσαι πρὸς τὸν ΑΒΓ χύχλον ἀπὸ τοῦ Δ σημείου ἐφ' ἑχάτερα τῆς ΔΗ ἐλαχίστης προσπεσοῦνται.

Έλν ἄρα χύχλου ληφθῆ τι σημεῖον ἐχτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν χύχλον διαχθῶσιν εὐθεῖαί τινες, ῶν μία μὲν διὰ τοῦ χέντρου αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν χοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ χέντου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ χέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν χυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ μεταξὺ τοῦ τε σημείου χαὶ τῆς διαμέτρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν χύχλον ἐφ' ἑχάτερα τῆς ἐλαχίστης· ὅπερ ἕδει δεῖξαι.

[†] Presumably, in an angular sense.

‡ This is not proved, except by reference to the figure.

θ'.

Έὰν χύχλου ληφθῆ τι σημεῖον ἐντός, ἀπο δὲ τοῦ σημείου πρὸς τὸν χύχλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον χέντρον ἐστὶ τοῦ χύχλου.

Έστω κύκλος ὁ ABΓ, ἐντὸς δὲ αὐτοῦ σημεῖον τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ABΓ κύκλον προσπιπτέτωσαν πλείους ἢ δύο ἴσαι εὐϑεῖαι αἱ ΔΑ, ΔΒ, ΔΓ· λέγω, ὅτι τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ABΓ κύκλου.

ate from point D towards (the circumference of) the circle, (one) on each (side) on the least (straight-line), DG. Let the angle DMB, equal to angle KMD, have been constructed on the straight-line MD, at the point M on it [Prop. 1.23], and let DB have been joined. And since MK is equal to MB, and MD (is) common, the two (straight-lines) KM, MD are equal to the two (straightlines) BM, MD, respectively. And angle KMD (is) equal to angle BMD. Thus, the base DK is equal to the base DB [Prop. 1.4]. [So] I say that another (straightline) equal to DK will not radiate towards the (circumference of the) circle from point D. For, if possible, let (such a straight-line) radiate, and let it be DN. Therefore, since DK is equal to DN, but DK is equal to DB, then DB is thus also equal to DN, (so that) a (straightline) nearer to the least (straight-line) DG [is] equal to one further away. The very thing was shown (to be) impossible. Thus, not more than two equal (straight-lines) will radiate towards (the circumference of) circle ABC from point D, (one) on each side of the least (straightline) DG.

Thus, if some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straightlines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer to the (straightline) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straightlines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

Proposition 9

If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle.

Let ABC be a circle, and D a point inside it, and let more than two equal straight-lines, DA, DB, and DC, radiate from D towards (the circumference of) circle ABC.



Έπεζεύχθωσαν γὰρ a
ἱ AB, BΓ καὶ τετμήσθωσαν δίχα κατὰ τὰ E, Z σημεĩα, καὶ ἐπιζευχθεῖσαι a
ἱ EΔ, ZΔ διήχθωσαν ἐπὶ τὰ H, K, Θ, Λ σημεῖα.

Έπεὶ οὖν ἴση ἐστὶν ἡ ΑΕ τῆ ΕΒ, Χοινὴ δὲ ἡ ΕΔ, δύο δὴ ai ΑΕ, ΕΔ δύο ταῖς ΒΕ, ΕΔ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΑ βάσει τῆ ΔΒ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΕΔ γωνία τῆ ὑπὸ ΒΕΔ ἴση ἐστίν· ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΑΕΔ, ΒΕΔ γωνιῶν· ἡ ΗΚ ἄρα τὴν ΑΒ τέμνει δίχα καὶ πρὸς ὀρθάς. καὶ ἐπεί, ἐὰν ἐν κύκλῳ εὐθεῖά τις εὐθεῖάν τινα δίχα τε καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ κύκλου, ἐπὶ τῆς ΗΚ ἄρα ἐστὶ τὸ κέντρον τοῦ κύκλου. διὰ τὰ αὐτὰ δὴ καὶ ἐπὶ τῆς ΘΛ ἐστι τὸ κέντρον τοῦ ΑΒΓ κύκλου. καὶ οὐδὲν ἔτερον κοινὸν ἔχουσιν αἱ ΗΚ, ΘΛ εὐθεῖαι ἢ τὸ Δ σημεῖον· τὸ Δ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου.

Έὰν ἄρα κύκλου ληφθῆ τι σημεῖον ἐντός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

ι´.

Κύχλος χύχλον οὐ τέμνει χατὰ πλείονα σημεῖα ἢ δύο.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΒΓ κύκλον τὸν ΔΕΖ τεμνέτω κατὰ πλείονα σημεῖα ἢ δύο τὰ Β, Η, Ζ, Θ, καὶ ἐπιζευχθεῖσαι αἱ ΒΘ, ΒΗ δίχα τεμνέσθωσαν κατὰ τὰ Κ, Λ σημεῖα καὶ ἀπὸ τῶν Κ, Λ ταῖς ΒΘ, ΒΗ πρὸς ὀρθὰς ἀχθεῖσαι αἱ ΚΓ, ΛΜ διήχθωσαν ἐπὶ τὰ Α, Ε σημεῖα. I say that point D is the center of circle ABC.



For let AB and BC have been joined, and (then) have been cut in half at points E and F (respectively) [Prop. 1.10]. And ED and FD being joined, let them have been drawn through to points G, K, H, and L.

Therefore, since AE is equal to EB, and ED (is) common, the two (straight-lines) AE, ED are equal to the two (straight-lines) BE, ED (respectively). And the base DA (is) equal to the base DB. Thus, angle AED is equal to angle BED [Prop. 1.8]. Thus, angles AED and BED (are) each right-angles [Def. 1.10]. Thus, GK cuts AB in half, and at right-angles. And since, if some straight-line in a circle cuts some (other) straight-line in half, and at right-angles the center of the circle is on the former (straight-line) [Prop. 3.1 corr.], the center of the circle is thus on GK. So, for the same (reasons), the center of circle ABC is also on HL. And the straight-lines GK and HL have no common (point) other than point D. Thus, point D is the center of circle ABC.

Thus, if some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle. (Which is) the very thing it was required to show.

Proposition 10

A circle does not cut a(nother) circle at more than two points.

For, if possible, let the circle ABC cut the circle DEF at more than two points, B, G, F, and H. And BH and BG being joined, let them (then) have been cut in half at points K and L (respectively). And KC and LM being drawn at right-angles to BH and BG from K and L (respectively) [Prop. 1.11], let them (then) have been drawn through to points A and E (respectively).



Έπεὶ οὖν ἐν κύκλῳ τῷ ABΓ εὐθεῖά τις ἡ AΓ εὐθεῖάν τινα τὴν BΘ δίχα καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς AΓ ἄρα ἐστὶ τὸ κέντρον τοῦ ABΓ κύκλου. πάλιν, ἐπεὶ ἐν κύκλῳ τῷ αὐτῷ τῷ ABΓ εὐθεῖά τις ἡ NΞ εὐθεῖάν τινα τὴν BH δίχα καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς NΞ ἄρα ἐστὶ τὸ κέντρον τοῦ ABΓ κύκλου. ἐδείχθη δὲ καὶ ἐπὶ τῆς AΓ, καὶ κατ' οὐδὲν συμβάλλουσιν αἱ AΓ, NΞ εὐθεῖαι ἢ κατὰ τὸ O· τὸ Ο ἄρα σημεῖον κέντρον ἐστὶ τοῦ ABΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι καὶ τοῦ ΔΕΖ κύκλου κέντρον ἐστὶ τὸ Ο· δύο ἄρα κύκλων τεμνόντων ἀλλήλους τῶν ABΓ, ΔΕΖ τὸ αὐτό ἐστι κέντρον τὸ O· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα κύκλος κύκλον τέμνει κατὰ πλείονα σημεῖα ἢ δύο· ὅπερ ἔδει δεῖξαι.

ια'.

Έαν δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐντός, καὶ ληφθῆ αὐτῶν τὰ κέντρα, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα καὶ ἐκβαλλομένη ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων.

Δύο γὰρ κύκλοι οἱ ABΓ, AΔΕ ἐφαπτέσθωσαν ἀλλήλων ἐντὸς κατὰ τὸ A σημεῖον, καὶ εἰλήφθω τοῦ μὲν ABΓ κύκλου κέντρον τὸ Z, τοῦ δὲ AΔΕ τὸ H· λέγω, ὅτι ἡ ἀπὸ τοῦ H ἐπὶ τὸ Z ἐπιζευγνυμένη εὐθεῖα ἐκβαλλομένη ἐπὶ τὸ A πεσεῖται.

Μή γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ὡς ἡ ΖΗΘ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΑΗ.

Έπει οὕν αί AH, HZ τῆς ZA, τουτέστι τῆς ZΘ, μείζονές εἰσιν, χοινὴ ἀφηρήσθω ἡ ZH· λοιπὴ ἄρα ἡ AH λοιπῆς τῆς HΘ μείζων ἐστίν. ἴση δὲ ἡ AH τῆ HΔ· χαὶ ἡ HΔ ἄρα τῆς HΘ μείζων ἐστίν ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον· οὐχ ἄρα ἡ ἀπὸ τοῦ Z ἐπὶ τὸ H ἐπιζευγνυμένη εὐθεὶα ἐχτὸς πεσεῖται· χατὰ τὸ A ἄρα ἐπὶ τῆς συναφῆς πεσεῖται.



ELEMENTS BOOK 3

Therefore, since in circle ABC some straight-line AC cuts some (other) straight-line BH in half, and at right-angles, the center of circle ABC is thus on AC [Prop. 3.1 corr.]. Again, since in the same circle ABC some straight-line NO cuts some (other straight-line) BG in half, and at right-angles, the center of circle ABC is thus on NO [Prop. 3.1 corr.]. And it was also shown (to be) on AC. And the straight-lines AC and NO meet at no other (point) than P. Thus, point P is the center of circle ABC. So, similarly, we can show that P is also the center of circle DEF. Thus, two circles cutting one another, ABC and DEF, have the same center P. The very thing is impossible [Prop. 3.5].

Thus, a circle does not cut a(nother) circle at more than two points. (Which is) the very thing it was required to show.

Proposition 11

If two circles touch one another internally, and their centers are found, then the straight-line joining their centers, being produced, will fall upon the point of union of the circles.

For let two circles, ABC and ADE, touch one another internally at point A, and let the center F of circle ABChave been found [Prop. 3.1], and (the center) G of (circle) ADE [Prop. 3.1]. I say that the straight-line joining G to F, being produced, will fall on A.

For (if) not then, if possible, let it fall like FGH (in the figure), and let AF and AG have been joined.

Therefore, since AG and GF is greater than FA, that is to say FH [Prop. 1.20], let FG have been taken from both. Thus, the remainder AG is greater than the remainder GH. And AG (is) equal to GD. Thus, GD is also greater than GH, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining Fto G will not fall outside (one circle but inside the other). Thus, it will fall upon the point of union (of the circles)



Έὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐντός, [καὶ ληφθῆ αὐτῶν τὰ κέντρα], ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα [καὶ ἐκβαλλομένη] ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων. ὅπερ ἔδει δεῖξαι.

ιβ'.

Έαν δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη διὰ τῆς ἐπαφῆς ἐλεύσεται.



Δύο γὰρ κύκλοι οἱ ABΓ, AΔE ἐφαπτέσθωσαν ἀλλήλων ἐκτὸς κατὰ τὸ A σημεῖον, καὶ εἰλήφθω τοῦ μὲν ABΓ κέντρον τὸ Z, τοῦ δὲ AΔE τὸ H· λέγω, ὅτι ἡ ἀπὸ τοῦ Z ἐπὶ τὸ H ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ A ἐπαφῆς ἐλεύσεται.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἐρχέσθω ὡς ἡ ΖΓΔΗ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΑΗ.

Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΖΑ τῇ ΖΓ. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΔΕ κύκλου, ἴση ἐστὶν ἡ ΗΑ τῇ ΗΔ. ἐδείχθη



Thus, if two circles touch one another internally, [and their centers are found], then the straight-line joining their centers, [being produced], will fall upon the point of union of the circles. (Which is) the very thing it was required to show.

Proposition 12

If two circles touch one another externally then the (straight-line) joining their centers will go through the point of union.



For let two circles, ABC and ADE, touch one another externally at point A, and let the center F of ABC have been found [Prop. 3.1], and (the center) G of ADE [Prop. 3.1]. I say that the straight-line joining F to G will go through the point of union at A.

For (if) not then, if possible, let it go like FCDG (in the figure), and let AF and AG have been joined.

Therefore, since point F is the center of circle ABC, FA is equal to FC. Again, since point G is the center of circle ADE, GA is equal to GD. And FA was also shown

δὲ καὶ ἡ ΖΑ τῆ ΖΓ ἴση· αἱ ἄρα ΖΑ, ΑΗ ταῖς ΖΓ, ΗΔ ἴσαι εἰσίν· ὥστε ὅλη ἡ ΖΗ τῶν ΖΑ, ΑΗ μείζων ἐστίν· ἀλλὰ καὶ ἐλάττων· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ Α ἐπαφῆς οὐκ ἐλεύσεται· δι' αὐτῆς ἄρα.

Ἐἀν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη [εὐθεῖα] διὰ τῆς ἐπαφῆς ἐλεύσεται· ὅπερ ἔδει δεῖξαι.

ιγ΄.

Κύκλος κύκλου οὐκ ἐφάπτεται κατὰ πλείονα σημεῖα ἢ καϑ ἕν, ἐάν τε ἐντὸς ἐάν τε ἐκτὸς ἐφάπτηται.



Εἰ γὰρ δυνατόν, κύκλος ὁ $AB\Gamma\Delta$ κύκλου τοῦ $EBZ\Delta$ ἐφαπτέσθω πρότερον ἐντὸς κατὰ πλείονα σημεĩα ἢ ἕν τὰ $\Delta,$ B.

Καὶ εἰλήφθω τοῦ μὲν ΑΒΓΔ κύκλου κέντρον τὸ Η, τοῦ δὲ ΕΒΖΔ τὸ Θ.

Ή ἄρα ἀπὸ τοῦ Η ἐπὶ τὸ Θ ἐπιζευγνυμένη ἐπὶ τὰ Β, Δ πεσεῖται. πιπτέτω ὡς ἡ ΒΗΘΔ. καὶ ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓΔ κύκλου, ἴση ἐστὶν ἡ ΒΗ τῆ ΗΔ· μείζων ἄρα ἡ ΒΗ τῆς ΘΔ· πολλῷ ἄρα μείζων ἡ ΒΘ τῆς ΘΔ. πάλιν, ἐπεὶ τὸ Θ σημεῖον κέντρον ἐστὶ τοῦ ΕΒΖΔ κύκλου, ἴση ἐστὶν ἡ ΒΘ τῆ ΘΔ· ἐδείχθη δὲ αὐτῆς καὶ πολλῷ μείζων· ὅπερ ἀδύνατον· οὐκ ἄρα κύκλος κύκλου ἐφάπτεται ἐντὸς κατὰ πλείονα σημεῖα ἢ ἕν.

Λέγω δή, ὅτι οὐδὲ ἐκτός.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΓΚ κύκλου τοῦ ΑΒΓΔ ἐφαπτέσθω ἐκτὸς κατὰ πλείονα σημεῖα ἢ ἐν τὰ Α, Γ, καὶ ἐπεζεύχθω ἡ ΑΓ.

Έπεὶ οὖν κύκλων τῶν ΑΒΓΔ, ΑΓΚ εἴληπται ἐπὶ τῆς περιφερείας ἑκατέρου δύο τυχόντα σημεῖα τὰ Α, Γ, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς ἑκατέρου πεσεῖται· ἀλλὰ τοῦ μὲν ΑΒΓΔ ἐντὸς ἔπεσεν, τοῦ δὲ ΑΓΚ ἐκτός· ὅπερ ἄτοπον· οὐκ ἄρα κύκλος κύκλου ἐφάπτεται ἐκτὸς κατὰ πλείονα σημεῖα ἢ ἕν. ἐδείχϑη δέ, ὅτι οὐδὲ ἐντός. (to be) equal to FC. Thus, the (straight-lines) FA and AG are equal to the (straight-lines) FC and GD. So the whole of FG is greater than FA and AG. But, (it is) also less [Prop. 1.20]. The very thing is impossible. Thus, the straight-line joining F to G cannot not go through the point of union at A. Thus, (it will go) through it.

Thus, if two circles touch one another externally then the [straight-line] joining their centers will go through the point of union. (Which is) the very thing it was required to show.

Proposition 13

A circle does not touch a(nother) circle at more than one point, whether they touch internally or externally.



For, if possible, let circle $ABDC^{\dagger}$ touch circle EBFD—first of all, internally—at more than one point, D and B.

And let the center *G* of circle ABDC have been found [Prop. 3.1], and (the center) *H* of EBFD [Prop. 3.1].

Thus, the (straight-line) joining G and H will fall on B and D [Prop. 3.11]. Let it fall like BGHD (in the figure). And since point G is the center of circle ABDC, BG is equal to GD. Thus, BG (is) greater than HD. Thus, BH (is) much greater than HD. Again, since point H is the center of circle EBFD, BH is equal to HD. But it was also shown (to be) much greater than it. The very thing (is) impossible. Thus, a circle does not touch a(nother) circle internally at more than one point.

So, I say that neither (does it touch) externally (at more than one point).

For, if possible, let circle ACK touch circle ABDC externally at more than one point, A and C. And let AC have been joined.

Therefore, since two points, A and C, have been taken at random on the circumference of each of the circles ABDC and ACK, the straight-line joining the points will fall inside each (circle) [Prop. 3.2]. But, it fell inside ABDC, and outside ACK [Def. 3.3]. The very thing Κύχλος ἄρα χύχλου οὐχ ἐφάπτεται χατὰ πλείονα σημεῖα η̈ [χαθ'] ἕν, ἐάν τε ἐντὸς ἐάν τε ἐχτὸς ἐφάπτηται· ὅπερ ἔδει δεῖξαι.

[†] The Greek text has "*ABCD*", which is obviously a mistake.

ιδ'.

Έν κύκλω αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ ἀίσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν.



Έστω χύχλος ὁ $AB\Gamma\Delta$, χαὶ ἐν αὐτῷ ἴσαι εὐθεῖαι ἔστωσαν αἱ AB, $\Gamma\Delta$ · λέγω, ὅτι αἱ AB, $\Gamma\Delta$ ἴσον ἀπέχουσιν ἀπὸ τοῦ χέντρου.

Εἰλήφθω γὰρ τὸ κέντον τοῦ ΑΒΓΔ κύκλου καὶ ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὰς ΑΒ, ΓΔ κάθετοι ἤχθωσαν αἱ ΕΖ, ΕΗ, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΓ.

Έπεὶ οῦν εὐθεῖά τις δἰα τοῦ xέντρου ἡ ΕΖ εὐθεῖάν τινα μὴ διὰ τοῦ xέντρου τὴν AB πρὸς ὀpθὰς τέμνει, xaì δίχα αὐτὴν τέμνει. ẳση ἄρα ἡ AZ τῆ ZB· διπλῆ ἄρα ἡ AB τῆς AZ. διὰ τὰ αὐτὰ δὴ xaì ἡ ΓΔ τῆς ΓΗ ἐστι διπλῆ· xaí ἐστιν ἴση ἡ AB τῆ ΓΔ· ἴση ἄρα xaì ἡ AZ τῆ ΓΗ. xaì ἐπεὶ ἴση ἐστιν ἡ AE τῆ EΓ, ἴσον xaì τὸ ἀπὸ τῆς AE τῷ ἀπὸ τῆς EΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς AE ἴσα τὰ ἀπὸ τῶν AZ, EZ· ὀpθὴ γὰρ ἡ πρὸς τῷ Z γωνία· τῷ δὲ ἀπὸ τῆς EΓ ἴσα τὰ ἀπὸ τῶν EH, HΓ· ὀpθὴ γὰρ ἡ πρὸς τῷ Η γωνία· τὰ ἄρα ἀπὸ τῶν AZ, ZE ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, HE, ῶν τὸ ἀπὸ τῆς AZ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση γάρ ἐστιν ἡ AZ τῆ ΓΗ· λοιπὸν ӑρα τὸ ἀπὸ τῆς ZE τῷ ἀπὸ τῆς EH ἴσον ἐστίν· ἴση ἄρα ἡ EZ τῆ EH. ἐν δὲ xύxλῷ ἴσον ἀπέχειν ἀπὸ τοῦ xέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ xέντρου ἑπ² αὐτὰς xάθετοι ἀγόμεναι ἴσαι ῶσιν· αἱ ἄρα AB, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ xέντρου.

Άλλὰ δὴ αἱ AB, ΓΔ εὐθεῖαι ἴσον ἀπεχέτωσαν ἀπὸ τοῦ κέντρου, τουτέστιν ἴση ἔστω ἡ ΕΖ τῆ ΕΗ. λέγω, ὅτι ἴση ἐστὶ καὶ ἡ AB τῆ ΓΔ. (is) absurd. Thus, a circle does not touch a(nother) circle externally at more than one point. And it was shown that neither (does it) internally.

Thus, a circle does not touch a(nother) circle at more than one point, whether they touch internally or externally. (Which is) the very thing it was required to show.

Proposition 14

In a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another.



Let $ABDC^{\dagger}$ be a circle, and let AB and CD be equal straight-lines within it. I say that AB and CD are equally far from the center.

For let the center of circle ABDC have been found [Prop. 3.1], and let it be (at) E. And let EF and EG have been drawn from (point) E, perpendicular to AB and CD (respectively) [Prop. 1.12]. And let AE and EC have been joined.

Therefore, since some straight-line, EF, through the center (of the circle), cuts some (other) straight-line, AB, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, AF (is) equal to FB. Thus, AB(is) double AF. So, for the same (reasons), CD is also double CG. And AB is equal to CD. Thus, AF (is) also equal to CG. And since AE is equal to EC, the (square) on AE (is) also equal to the (square) on EC. But, the (sum of the squares) on AF and EF (is) equal to the (square) on AE. For the angle at F (is) a rightangle [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on EC. For the angle at G (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AF and FE is equal to the (sum of the squares) on CG and GE, of which the (square) on AFis equal to the (square) on CG. For AF is equal to CG.

Tῶν γὰρ αὐτῶν κατασκευασθέντων ὑμοίως δείξομεν, ὅτι διπλῆ ἐστιν ἡ μὲν AB τῆς AZ, ἡ δὲ ΓΔ τῆς ΓΗ· καὶ ἐπεὶ ἴση ἐστὶν ἡ AE τῆ ΓΕ, ἴσον ἐστὶ τὸ ἀπὸ τῆς AE τῷ ἀπὸ τῆς ΓΕ· ἀλλὰ τῷ μὲν ἀπὸ τῆς AE ἴσα ἐστὶ τὰ ἀπὸ τῶν EZ, ZA, τῷ δὲ ἀπὸ τῆς ΓΕ ἴσα τὰ ἀπὸ τῶν EH, HΓ. τὰ ἄρα ἀπὸ τῶν EZ, ZA ἴσα ἐστὶ τοῖς ἀπὸ τῶν EH, HΓ· τὰ ἄρα ἀπὸ τῶν EZ, ZA ἴσα ἐστὶ τοῖς ἀπὸ τῶν EH, HΓ· τὸ ἀπὸ τῆς EZ τῷ ἀπὸ τῆς EH ἐστιν ἴσον· ἴση γὰρ ἡ EZ τῆ EH· λοιπὸν ἄρα τὸ ἀπὸ τῆς AZ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση ἄρα ἡ AZ τῆ ΓΗ· καί ἐστι τῆς μὲν AZ διπλῆ ἡ AB, τῆς δὲ ΓΗ διπλῆ ἡ ΓΔ· ἴση ἄρα ἡ AB τῆ ΓΔ.

Έν κύκλω ἄρα αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν. ὅπερ ἔδει δεῖξαι.

^{\dagger} The Greek text has "*ABCD*", which is obviously a mistake.

ιε΄.

Έν κύκλω μεγίστη μὲν ἡ διάμετρος, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν.

Έστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΑΔ, κέντρον δὲ τὸ Ε, καὶ ἔγγιον μὲν τῆς ΑΔ διαμέτρου ἔστω ἡ ΒΓ, ἀπώτερον δὲ ἡ ΖΗ· λέγω, ὅτι μεγίστη μέν ἐστιν ἡ ΑΔ, μείζων δὲ ἡ ΒΓ τῆς ΖΗ.

Ήχθωσαν γὰρ ἀπὸ τοῦ Ε κέντρου ἐπὶ τὰς ΒΓ, ΖΗ κάθετοι αἱ ΕΘ, ΕΚ. καὶ ἐπεὶ ἔγγιον μὲν τοῦ κέντρου ἐστὶν ἡ ΒΓ, ἀπώτερον δὲ ἡ ΖΗ, μείζων ἄρα ἡ ΕΚ τῆς ΕΘ. κείσθω τῆ ΕΘ ἴση ἡ ΕΛ, καὶ διὰ τοῦ Λ τῆ ΕΚ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΛΜ διήχθω ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αί ΜΕ, ΕΝ, ΖΕ, ΕΗ.

Kaì
ἐπεὶ ἴση ἐστὶν ἡ ΕΘ τῆ ΕΛ, ἴση ἐστὶ κaì ἡ ΒΓ τῆ
MN. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῆ ΕΜ, ἡ δὲ ΕΔ τῆ
EN, ἡ ἄρα ΑΔ ταῖς ΜΕ, ΕΝ ἴση ἐστίν. ἀλλ' ai μὲν ΜΕ, ΕΝ
τῆς MN μείζονές εἰσιν [καὶ ἡ ΑΔ τῆς MN μείζων ἐστίν],
ἴση δὲ ἡ MN τῆ ΒΓ· ἡ ΑΔ ἄρα τῆς ΒΓ μείζων ἐστίν. καὶ
ἐπεὶ δύο ai ME, ΕΝ δύο ταῖς ΖΕ, ΕΗ ἴσαι εἰσίν, καὶ γωνία
ἡ ὑπὸ MEN γωνίας τῆς ὑπὸ ΖΕΗ μείζων [ἐστίν], βάσις ἄρα

Thus, the remaining (square) on FE is equal to the (remaining square) on EG. Thus, EF (is) equal to EG. And straight-lines in a circle are said to be equally far from the center when perpendicular (straight-lines) which are drawn to them from the center are equal [Def. 3.4]. Thus, AB and CD are equally far from the center.

So, let the straight-lines AB and CD be equally far from the center. That is to say, let EF be equal to EG. I say that AB is also equal to CD.

For, with the same construction, we can, similarly, show that AB is double AF, and CD (double) CG. And since AE is equal to CE, the (square) on AE is equal to the (square) on CE. But, the (sum of the squares) on EF and FA is equal to the (square) on AE [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on CE [Prop. 1.47]. Thus, the (sum of the squares) on EF and FA is equal to the (square) on CE [Prop. 1.47]. Thus, the (sum of the squares) on EF and FA is equal to the (square) on EF and FA is equal to the (square) on EF and FA is equal to the (square) on EF is equal to the (square) on EG. For EF (is) equal to EG. Thus, the remaining (square) on AF is equal to the (remaining square) on CG. Thus, AF (is) equal to CG. And AB is double AF, and CD double CG. Thus, AB (is) equal to CD.

Thus, in a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another. (Which is) the very thing it was required to show.

Proposition 15

In a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away.

Let ABCD be a circle, and let AD be its diameter, and E (its) center. And let BC be nearer to the diameter AD,[†] and FG further away. I say that AD is the greatest (straight-line), and BC (is) greater than FG.

For let EH and EK have been drawn from the center E, at right-angles to BC and FG (respectively) [Prop. 1.12]. And since BC is nearer to the center, and FG further away, EK (is) thus greater than EH [Def. 3.5]. Let EL be made equal to EH [Prop. 1.3]. And LM being drawn through L, at right-angles to EK [Prop. 1.11], let it have been drawn through to N. And let ME, EN, FE, and EG have been joined.

And since EH is equal to EL, BC is also equal to MN [Prop. 3.14]. Again, since AE is equal to EM, and ED to EN, AD is thus equal to ME and EN. But, ME and EN is greater than MN [Prop. 1.20] [also AD is

ή MN βάσεως τῆς ΖΗ μείζων ἐστίν. ἀλλὰ ἡ MN τῆ ΒΓ ἐδείχθη ἴση [xaì ἡ ΒΓ τῆς ΖΗ μείζων ἐστίν]. μεγίστη μὲν ἄρα ἡ ΑΔ διάμετρος, μείζων δὲ ἡ ΒΓ τῆς ΖΗ.



Έν κύκλω ἄρα μεγίστη μὲν έστιν ἡ διάμετρος, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

greater than MN], and MN (is) equal to BC. Thus, AD is greater than BC. And since the two (straight-lines) ME, EN are equal to the two (straight-lines) FE, EG (respectively), and angle MEN [is] greater than angle FEG,[‡] the base MN is thus greater than the base FG [Prop. 1.24]. But, MN was shown (to be) equal to BC [(so) BC is also greater than FG]. Thus, the diameter AD (is) the greatest (straight-line), and BC (is) greater than FG.



Thus, in a circle, a diameter (is) the greatest (straightline), and for the others, a (straight-line) nearer to the center is always greater than one further away. (Which is) the very thing it was required to show.

[†] Euclid should have said "to the center", rather than "to the diameter AD", since BC, AD and FG are not necessarily parallel. [‡] This is not proved, except by reference to the figure.

ເຈ′.

Ή τῆ διαμέτρω τοῦ χύχλου πρὸς ὀρθὰς ἀπ' ἄχρας ἀγομένη ἐχτὸς πεσεῖται τοῦ χύχλου, χαὶ εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας χαὶ τῆς περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται, χαὶ ἡ μὲν τοῦ ἡμιχυχλίου γωνία ἀπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἐλάττων.

Έστω κύκλος ὁ ΑΒΓ περὶ κέντρον τὸ Δ καὶ διάμετρον τὴν ΑΒ· λέγω, ὅτι ἡ ἀπὸ τοῦ Α τῆ ΑΒ πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου.

Mỳ γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ἐντὸς ὡς ἡ ΓΑ, καὶ ἐπεζεύχθω ἡ ΔΓ.

Έπεὶ ἴση ἐστὶν ἡ ΔA τῆ $\Delta \Gamma$, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ $\Delta A\Gamma$ γωνία τῆ ὑπὸ $A\Gamma\Delta$. ὀρθὴ δὲ ἡ ὑπὸ $\Delta A\Gamma$ · ὀρθὴ ἄρα καὶ ἡ ὑπὸ $A\Gamma\Delta$ · τριγώνου δὴ τοῦ $A\Gamma\Delta$ αἱ δύο γωνίαι αἱ ὑπὸ $\Delta A\Gamma$, $A\Gamma\Delta$ δύο ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἐστὶν ἀδύνατον. οὐχ ἄρα ἡ ἀπὸ τοῦ A σημείου τῆ BA πρὸς ὀρθὰς ἀγομένη ἐντὸς πεσεῖται τοῦ χύχλου. ὁμοίως δὴ δεῖξομεν, ὅτι οὐδ' ἐπὶ τῆς περιφερείας· ἐχτὸς ἄρα.

Proposition 16

A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).

Let ABC be a circle around the center D and the diameter AB. I say that the (straight-line) drawn from A, at right-angles to AB [Prop 1.11], from its end, will fall outside the circle.

For (if) not then, if possible, let it fall inside, like CA (in the figure), and let DC have been joined.

Since DA is equal to DC, angle DAC is also equal to angle ACD [Prop. 1.5]. And DAC (is) a right-angle. Thus, ACD (is) also a right-angle. So, in triangle ACD, the two angles DAC and ACD are equal to two rightangles. The very thing is impossible [Prop. 1.17]. Thus, the (straight-line) drawn from point A, at right-angles



Πιπτέτω ὡς ἡ ΑΕ· λέγω δή, ὅτι εἰς τὸν μεταξὺ τόπον τῆς τε ΑΕ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται.

El γὰρ δυνατόν, παρεμπιπτέτω ὡς ἡ ZA, καὶ ἦχθω ἀπὸ τοῦ Δ σημείου ἐπὶ τῆν ZA κάθετος ἡ ΔΗ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ AHΔ, ἐλάττων δὲ ὀρθῆς ἡ ὑπὸ ΔAH, μείζων ἄρα ἡ AΔ τῆς ΔΗ. ἴση δὲ ἡ ΔΑ τῆ ΔΘ· μείζων ἄρα ἡ ΔΘ τῆς ΔΗ, ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα παρεμπεσεῖται.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἡ περιεχομένη ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἀπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἡ περιεχομένη ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας ἁπάσης γωνίας ὀξείας εὐθυγράμμου ἐλάττων ἐστίν.

Εἰ γὰρ ἐστί τις γωνία εὐθύγραμμος μείζων μὲν τῆς περιεχομένης ὑπό τε τῆς BA εὐθείας καὶ τῆς ΓΘΑ περιφερείας, ἐλάττων δὲ τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τὴς AE εὐθείας, εἰς τὸν μεταξὺ τόπον τῆς τε ΓΘΑ περιφερείας καὶ τῆς AE εὐθείας εὐθεῖα παρεμπεσεῖται, ἤτις ποιήσει μείζονα μὲν τῆς περιεχομένης ὑπὸ τε τῆς BA εὐθείας καὶ τῆς ΓΘΑ περιφερείας ὑπὸ εὐθειῶν περιεχομένην, ἐλάττονα δὲ τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς AE εὐθείας. οὐ παρεμπίπτει δέ· οὐx ἄρα τῆς περιεχομένης γωνίας ὑπό τε τῆς BA εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἔσται μείζων ὀξεῖα ὑπὸ εὐθειῶν περιεχομένη, οὐδὲ μὴν ἐλάττων τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς AE εὐθείας. to BA, will not fall inside the circle. So, similarly, we can show that neither (will it fall) on the circumference. Thus, (it will fall) outside (the circle).



Let it fall like AE (in the figure). So, I say that another straight-line cannot be inserted into the space between the straight-line AE and the circumference CHA.

For, if possible, let it be inserted like FA (in the figure), and let DG have been drawn from point D, perpendicular to FA [Prop. 1.12]. And since AGD is a right-angle, and DAG (is) less than a right-angle, AD (is) thus greater than DG [Prop. 1.19]. And DA (is) equal to DH. Thus, DH (is) greater than DG, the lesser than the greater. The very thing is impossible. Thus, another straight-line cannot be inserted into the space between the straight-line (AE) and the circumference.

And I also say that the semi-circular angle contained by the straight-line BA and the circumference CHA is greater than any acute rectilinear angle whatsoever, and the remaining (angle) contained by the circumference CHA and the straight-line AE is less than any acute rectilinear angle whatsoever.

For if any rectilinear angle is greater than the (angle) contained by the straight-line BA and the circumference CHA, or less than the (angle) contained by the circumference CHA and the straight-line AE, then a straight-line can be inserted into the space between the circumference CHA and the straight-line AE—anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line BAand the circumference CHA, or less than the (angle) contained by the circumference CHA and the straightline AE. But (such a straight-line) cannot be inserted. Thus, an acute (angle) contained by straight-lines cannot be greater than the angle contained by the straight-line BA and the circumference CHA, neither (can it be) less than the (angle) contained by the circumference CHAand the straight-line AE.

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι ἡ τῆ διαμέτρω τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου [καὶ ὅτι εὐθεῖα κύκλου καθ' ἕν μόνον ἐφάπτεται σημεῖον, ἐπειδήπερ καὶ ἡ κατὰ δύο αὐτῷ συμβάλλουσα ἐντὸς αὐτοῦ πίπτουσα ἐδείχθη]. ὅπερ ἔδει δεῖξαι.

ιζ΄.

Άπὸ τοῦ δοθέντος σημείου τοῦ δοθέντος κύκλου ἐφαπτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.



Έστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ὁ δὲ δοθεὶς κύκλος ὁ ΒΓΔ· δεῖ δὴ ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ κέντρω μὲν τῷ Ε διαστήματι δὲ τῷ ΕΑ κύκλος γεγράφθω ὁ ΑΖΗ, καὶ ἀπὸ τοῦ Δ τῆ ΕΑ πρὸς ὀρθὰς ἦχθω ἡ ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΑΒ· λέγω, ὅτι ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένη ἦκται ἡ ΑΒ.

Έπεὶ γὰρ τὸ Ε κέντρον ἐστὶ τῶν ΒΓΔ, ΑΖΗ κύκλων, ἴση ἄρα ἐστὶν ἡ μὲν ΕΑ τῆ ΕΖ, ἡ δὲ ΕΔ τῆ ΕΒ· δύο δὴ αἱ ΑΕ, ΕΒ δύο ταῖς ΖΕ, ΕΔ ἴσαι εἰσίν· καὶ γωνίαν κοινὴν περιέχουσι τὴν πρὸς τῷ Ε· βάσις ἄρα ἡ ΔΖ βάσει τῆ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΕΒΑ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΕΔΖ τῆ ὑπὸ ΕΒΑ. ὀρθὴ δὲ ἡ ὑπὸ ΕΔΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΒΑ. καί ἐστιν ἡ ΕΒ ἐκ τοῦ κέντρου· ἡ δὲ τῆ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ² ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΑΒ ἄρα ἐφάπτεται τοῦ ΒΓΔ κύκλου.

Άπὸ τοῦ ἄρα δοθέντος σημείου τοῦ Α τοῦ δοθέντος κύκλου τοῦ ΒΓΔ ἐφαπτομένη εὐθεῖα γραμμὴ ῆκται ἡ ΑΒ· ὅπερ ἔδει ποιῆσαι.

Corollary

So, from this, (it is) manifest that a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [and that the straight-line touches the circle at a single point, inasmuch as it was also shown that a (straight-line) meeting (the circle) at two (points) falls inside it [Prop. 3.2]]. (Which is) the very thing it was required to show.

Proposition 17

To draw a straight-line touching a given circle from a given point.



Let A be the given point, and BCD the given circle. So it is required to draw a straight-line touching circle BCD from point A.

For let the center E of the circle have been found [Prop. 3.1], and let AE have been joined. And let (the circle) AFG have been drawn with center E and radius EA. And let DF have been drawn from from (point) D, at right-angles to EA [Prop. 1.11]. And let EF and AB have been joined. I say that the (straight-line) AB has been drawn from point A touching circle BCD.

For since E is the center of circles BCD and AFG, EA is thus equal to EF, and ED to EB. So the two (straight-lines) AE, EB are equal to the two (straight-lines) FE, ED (respectively). And they contain a common angle at E. Thus, the base DF is equal to the base AB, and triangle DEF is equal to triangle EBA, and the remaining angles (are equal) to the (corresponding) remaining angles [Prop. 1.4]. Thus, (angle) EDF (is) equal to EBA. And EDF (is) a right-angle. Thus, EBA (is) also a right-angle. And EB is a radius. And a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [Prop. 3.16 corr.]. Thus, AB touches circle BCD.

Thus, the straight-line AB has been drawn touching

ιη´.

Έαν χύχλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπιζευχθῆ τις εὐθεῖα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην.



Κύχλου γὰρ τοῦ ABΓ ἐφαπτέσθω τις εὐθεῖα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ εἰλήφθω τὸ κέντρον τοῦ ABΓ κύκλου τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὸ Γ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι ἡ ΖΓ κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Εἰ γὰρ μή, ἤχθω ἀπὸ τοῦ Ζ ἐπὶ τὴν ΔΕ κάθετος ἡ ΖΗ.

Ἐπεὶ οῦν ἡ ὑπὸ ZHΓ γωνία ὀρθή ἐστιν, ὀξεῖα ἄρα ἐστὶν ἡ ὑπὸ ZΓΗ· ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ZΓ τῆς ZH· ἴση δὲ ἡ ZΓ τῆ ZB· μείζων ἄρα xaὶ ἡ ZB τῆς ZH ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐx ἄρα ἡ ZH xάθετός ἐστιν ἐπὶ τὴν ΔΕ. ὁμοίως δὴ δεῖξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς ZΓ· ἡ ZΓ ἄρα xάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπιζευχθῆ τις εὐθεῖα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην. ὅπερ ἔδει δεῖξαι.

ıθ'.

Έὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς τῆ ἐφαπτομένῃ πρὸς ὀρθὰς [γωνίας] εὐθεῖα γραμμὴ ἀχθῆ, ἐπὶ τῆς ἀγθείσης ἔσται τὸ κέντρον τοῦ κύκλου.

Κύχλου γὰρ τοῦ ABΓ ἐφαπτέσθω τις εὐθεῖα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ ἀπὸ τοῦ Γ τῆ ΔΕ πρὸς ὀρθὰς ἤχθω ἡ ΓΑ· λέγω, ὅτι ἐπὶ τῆς ΑΓ ἐστι τὸ κέντρον τοῦ κύκλου. the given circle BCD from the given point A. (Which is) the very thing it was required to do.

Proposition 18

If some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent.



For let some straight-line DE touch the circle ABC at point C, and let the center F of circle ABC have been found [Prop. 3.1], and let FC have been joined from F to C. I say that FC is perpendicular to DE.

For if not, let FG have been drawn from F, perpendicular to DE [Prop. 1.12].

Therefore, since angle FGC is a right-angle, (angle) FCG is thus acute [Prop. 1.17]. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, FC (is) greater than FG. And FC (is) equal to FB. Thus, FB (is) also greater than FG, the lesser than the greater. The very thing is impossible. Thus, FG is not perpendicular to DE. So, similarly, we can show that neither (is) any other (straight-line) except FC. Thus, FC is perpendicular to DE.

Thus, if some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent. (Which is) the very thing it was required to show.

Proposition 19

If some straight-line touches a circle, and a straightline is drawn from the point of contact, at right-[angles] to the tangent, then the center (of the circle) will be on the (straight-line) so drawn.

For let some straight-line DE touch the circle ABC at point C. And let CA have been drawn from C, at right-



Mỳ γάρ, ἀλλ' εἰ δυνατόν, ἔστω τὸ Z, καὶ ἐπεζεύχθω ἡ ΓΖ.

Έπεὶ [οὖν] xúxλου τοῦ ABΓ ἐφάπτεταί τις εὐθεĩα ἡ ΔE , ἀπὸ δὲ τοῦ xέντρου ἐπὶ τὴν ἁφὴν ἐπέζευxται ἡ ZΓ, ἡ ZΓ ἄρα xάθετός ἐστιν ἐπὶ τὴν ΔE · ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ZΓE. ἐστὶ δὲ xαὶ ἡ ὑπὸ AΓE ὀρθή· ἴση ἄρα ἐστὶν ἡ ὑπὸ ZΓE τῆ ὑπὸ AΓE ἡ ἐλάττων τῆ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐx ἄρα τὸ Z κέντρον ἐστὶ τοῦ ABΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλο τι πλὴν ἐπὶ τῆς AΓ.

Ἐἀν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς τῆ ἐφαπτομένῃ πρὸς ὀρθὰς εὐθεῖα γραμμὴ ἀχθῆ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου. ὅπερ ἔδει δεῖζαι.

χ.

Έν χύχλω ή πρὸς τῷ χέντρω γωνία διπλασίων ἐστὶ τῆς πρὸς τῆ περιφερεία, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν αἱ γωνίαι.

Έστω χύχλος ὁ ΑΒΓ, χαὶ πρὸς μὲν τῷ χέντρῳ αὐτοῦ γωνία ἔστω ἡ ὑπὸ ΒΕΓ, πρὸς δὲ τῆ περιφερεία ἡ ὑπὸ ΒΑΓ, ἐχέτωσαν δὲ τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓ· λέγω, ὅτι διπλασίων ἐστὶν ἡ ὑπὸ ΒΕΓ γωνία τῆς ὑπὸ ΒΑΓ.

Ἐπιζευχθεῖσα γὰρ ἡ ΑΕ διήχθω ἐπὶ τὸ Ζ.

Έπεὶ οῦν ἴση ἐστὶν ἡ ΕΑ τῆ ΕΒ, ἴση καὶ γωνία ἡ ὑπὸ EAB τῆ ὑπὸ EBA· αἱ ἄρα ὑπὸ EAB, EBA γωνίαι τῆς ὑπὸ EAB διπλασίους εἰσίν. ἴση δὲ ἡ ὑπὸ BEZ ταῖς ὑπὸ EAB, EBA· καὶ ἡ ὑπὸ BEZ ἄρα τῆς ὑπὸ EAB ἐστι διπλῆ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ZEΓ τῆς ὑπὸ EAΓ ἐστι διπλῆ. ὅλη ἄρα ἡ ὑπὸ BEΓ ὅλης τῆς ὑπὸ BAΓ ἐστι διπλῆ. angles to DE [Prop. 1.11]. I say that the center of the circle is on AC.



For (if) not, if possible, let F be (the center of the circle), and let CF have been joined.

[Therefore], since some straight-line DE touches the circle ABC, and FC has been joined from the center to the point of contact, FC is thus perpendicular to DE [Prop. 3.18]. Thus, FCE is a right-angle. And ACE is also a right-angle. Thus, FCE is equal to ACE, the lesser to the greater. The very thing is impossible. Thus, F is not the center of circle ABC. So, similarly, we can show that neither is any (point) other (than one) on AC.

Thus, if some straight-line touches a circle, and a straightline is drawn from the point of contact, at right-angles to the tangent, then the center (of the circle) will be on the (straight-line) so drawn. (Which is) the very thing it was required to show.

Proposition 20

In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.

Let ABC be a circle, and let BEC be an angle at its center, and BAC (one) at (its) circumference. And let them have the same circumference base BC. I say that angle BEC is double (angle) BAC.

For being joined, let AE have been drawn through to F.

Therefore, since EA is equal to EB, angle EAB (is) also equal to EBA [Prop. 1.5]. Thus, angle EAB and EBA is double (angle) EAB. And BEF (is) equal to EAB and EBA [Prop. 1.32]. Thus, BEF is also double EAB. So, for the same (reasons), FEC is also double EAC. Thus, the whole (angle) BEC is double the whole (angle) BAC.

ΣΤΟΙΧΕΙΩΝ γ΄.



Κεκλάσθω δὴ πάλιν, καὶ ἔστω ἑτέρα γωνία ἡ ὑπὸ ΒΔΓ, καὶ ἐπιζευχθεῖσα ἡ ΔΕ ἐκβεβλήσθω ἐπὶ τὸ Η. ὁμοίως δὴ δείζομεν, ὅτι διπλῆ ἐστιν ἡ ὑπὸ ΗΕΓ γωνία τῆς ὑπὸ ΕΔΓ, ῶν ἡ ὑπὸ ΗΕΒ διπλῆ ἐστι τῆς ὑπὸ ΕΔΒ· λοιπὴ ἄρα ἡ ὑπὸ ΒΕΓ διπλῆ ἐστι τῆς ὑπὸ ΒΔΓ.

Έν κύκλω ἄρα ή πρός τῷ κέντρω γωνία διπλασίων ἐστὶ τῆς πρὸς τῆ περιφερεία, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν [αἰ γωνίαι]· ὅπερ ἔδει δεῖξαι.

χα΄.

Έν κύκλω αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν.



Έστω χύχλος ὁ ΑΒΓΔ, χαὶ ἐν τῷ αὐτῷ τμήματι τῷ ΒΑΕΔ γωνίαι ἔστωσαν αἱ ὑπὸ ΒΑΔ, ΒΕΔ· λέγω, ὅτι αἱ ὑπὸ ΒΑΔ, ΒΕΔ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Εἰλήφθω γὰρ τοῦ ΑΒΓΔ κύκλου τὸ κέντρον, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΒΖ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ὑπὸ ΒΖΔ γωνία πρὸς τῷ κέντρῳ ἐστίν, ἡ δὲ ὑπὸ ΒΑΔ πρὸς τῆ περιφερεία, καὶ ἔχουσι τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓΔ, ἡ ἄρα ὑπὸ ΒΖΔ γωνία διπλασίων ἐστὶ τῆς ὑπὸ ΒΑΔ. διὰ τὰ αὐτὰ δὴ ἡ ὑπὸ ΒΖΔ καὶ τῆς ὑπὸ



So let another (straight-line) have been inflected, and let there be another angle, BDC. And DE being joined, let it have been produced to G. So, similarly, we can show that angle GEC is double EDC, of which GEB is double EDB. Thus, the remaining (angle) BEC is double the (remaining angle) BDC.

Thus, in a circle, the angle at the center is double that at the circumference, when [the angles] have the same circumference base. (Which is) the very thing it was required to show.

Proposition 21

In a circle, angles in the same segment are equal to one another.



Let ABCD be a circle, and let BAD and BED be angles in the same segment BAED. I say that angles BAD and BED are equal to one another.

For let the center of circle ABCD have been found [Prop. 3.1], and let it be (at point) *F*. And let *BF* and *FD* have been joined.

And since angle BFD is at the center, and BAD at the circumference, and they have the same circumference base BCD, angle BFD is thus double BAD [Prop. 3.20].

ΒΕΔ ἐστι διπλσίων ἴση ἄρα ἡ ὑπὸ ΒΑΔ τῆ ὑπὸ ΒΕΔ.

Έν κύκλω ἄρα αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν ὅπερ ἔδει δεῖξαι.

хβ'.

Τῶν ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.



Έστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ τετράπλευρον ἔστω τὸ ΑΒΓΔ· λέγω, ὅτι αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἐπεζεύχθωσαν αἱ ΑΓ, ΒΔ.

Έπεὶ οὖν παντὸς τριγώνου aἰ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, τοῦ ABΓ ἄρα τριγώνου aἱ τρεῖς γωνίαι aἱ ὑπὸ ΓAB, ABΓ, BΓA δυσὶν ὀρθαῖς ἴσαι εἰσίν. ἴση δὲ ἡ μὲν ὑπὸ ΓAB τῆ ὑπὸ BΔΓ· ἐν γὰρ τῷ αὐτῷ τμήματί εἰσι τῷ BAΔΓ· ἡ δὲ ὑπὸ AΓB τῆ ὑπὸ AΔB· ἐν γὰρ τῷ αὐτῷ τμήματί εἰσι τῷ AΔΓB· ὅλη ἄρα ἡ ὑπὸ AΔΓ ταῖς ὑπὸ BAΓ, AΓB ἴση ἐστίν. χοινὴ προσχείσθω ἡ ὑπὸ ABΓ· aἱ ἄρα ὑπὸ ABΓ, BAΓ, AΓB ταῖς ὑπὸ ABΓ, AΔΓ ἴσαι εἰσίν. ἀλλỉ aἱ ὑπὸ ABΓ, BAΓ, AΓB δυσὶν ὀρθαῖς ἴσαι εἰσίν. χαὶ aἱ ὑπὸ ABΓ, AΔΓ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὑμοίως δὴ δείζομεν, ὅτι χαὶ aἱ ὑπὸ BAΔ, ΔΓΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Τῶν ἄρα ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

хγ′.

Έπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὄμοια καὶ ἄνισα οὐ συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη.

Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς AB δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συνεστάτω ἐπὶ τὰ αὐτὰ μέρη τὰ ΑΓΒ, ΑΔΒ, καὶ διήχθω ἡ ΑΓΔ, καὶ ἐπεζεύχθωσαν So, for the same (reasons), BFD is also double BED. Thus, BAD (is) equal to BED.

Thus, in a circle, angles in the same segment are equal to one another. (Which is) the very thing it was required to show.

Proposition 22

For quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles.



Let ABCD be a circle, and let ABCD be a quadrilateral within it. I say that the (sum of the) opposite angles is equal to two right-angles.

Let AC and BD have been joined.

Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles CAB, ABC, and BCA of triangle ABC are thus equal to two right-angles. And CAB (is) equal to BDC. For they are in the same segment BADC [Prop. 3.21]. And ACB (is equal) to ADB. For they are in the same segment ADCB [Prop. 3.21]. Thus, the whole of ADC is equal to BAC and ACB. Let ABC have been added to both. Thus, ABC, BAC, and ACB are equal to ABC and ADC. But, ABC, BAC, and ACB are equal to two right-angles. Thus, ABC and ADC are also equal to two right-angles. Similarly, we can show that angles BAD and DCB are also equal to two right-angles.

Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 23

Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

For, if possible, let the two similar and unequal segments of circles, ACB and ADB, have been constructed on the same side of the same straight-line AB. And let

αί ΓΒ, ΔB .



Έπεὶ οῦν ὅμοιόν ἐστι τὸ ΑΓΒ τμῆμα τῷ ΑΔΒ τμήματι, ὅμοια δὲ τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ γωνία τῃ ὑπὸ ΑΔΒ ἡ ἐκτὸς τῃ ἐντός· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

χδ'.

Τὰ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύλων ἴσα ἀλλήλοις ἐστίν.



Έστωσαν γὰρ ἐπὶ ἴσων εὐθειῶν τῶν ΑΒ, ΓΔ ὅμοια τμήματα κύκλων τὰ ΑΕΒ, ΓΖΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕΒ τμῆμα τῷ ΓΖΔ τμήματι.

Έφαρμοζομένου γὰρ τοῦ ΑΕΒ τμήματος ἐπὶ τὸ ΓΖΔ καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Γ τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΓΔ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Δ σημεῖον διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῆ ΓΔ· τῆς δὲ ΑΒ ἐπὶ τὴν ΓΔ ἐφαρμοσάσης ἐφαρμόσει καὶ τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ. εἰ γὰρ ἡ ΑΒ εὐθεῖα ἐπὶ τὴν ΓΔ ἐφαρμόσει, τὸ δὲ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ μὴ ἐφαρμόσει, ἤτοι ἐντὸς αὐτοῦ πεσεῖται ἢ ἐκτὸς ἢ παραλλάξει, ὡς τὸ ΓΗΔ, καὶ κύκλος κύκλον τέμνει κατὰ πλείονα σημεῖα ἢ δύο· ὅπερ ἐστίν ἀδύνατον. οὐκ ἄφαρμόσει καὶ ACD have been drawn through (the segments), and let CB and DB have been joined.



Therefore, since segment ACB is similar to segment ADB, and similar segments of circles are those accepting equal angles [Def. 3.11], angle ACB is thus equal to ADB, the external to the internal. The very thing is impossible [Prop. 1.16].

Thus, two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

Proposition 24

Similar segments of circles on equal straight-lines are equal to one another.



For let AEB and CFD be similar segments of circles on the equal straight-lines AB and CD (respectively). I say that segment AEB is equal to segment CFD.

For if the segment AEB is applied to the segment CFD, and point A is placed on (point) C, and the straight-line AB on CD, then point B will also coincide with point D, on account of AB being equal to CD. And if AB coincides with CD then the segment AEB will also coincide with CFD. For if the straight-line AB coincides with CD, and the segment AEB does not coincide with CFD, then it will surely either fall inside it, outside (it),[†] or it will miss like CGD (in the figure), and a circle (will) cut (another) circle at more than two points. The very

τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ· ἐφαρμόσει ἄρα, καὶ ἴσον αὐτῷ ἔσται.

Τὰ ἄρα ἐπὶ ἴσων εὐθειῶν ὄμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι. thing is impossible [Prop. 3.10]. Thus, if the straight-line AB is applied to CD, the segment AEB cannot not also coincide with CFD. Thus, it will coincide, and will be equal to it [C.N. 4].

Thus, similar segments of circles on equal straightlines are equal to one another. (Which is) the very thing it was required to show.

[†] Both this possibility, and the previous one, are precluded by Prop. 3.23.

χε΄.

Κύκλου τμήματος δοθέντος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.



Έστω τὸ δοθὲν τμῆμα κύκλου τὸ ABΓ· δεῖ δὴ τοῦ ABΓ τμήματος προσαναγράψαι τὸν κύκλον, οὖπέρ ἐστι τμῆμα.

Τετμήσθω γὰρ ἡ ΑΓ δίχα κατὰ τὸ Δ, καὶ ἦχθω ἀπὸ τοῦ Δ σημείου τῆ ΑΓ πρὸς ὀρθὰς ἡ ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· ἡ ὑπὸ ΑΒΔ γωνία ἄρα τῆς ὑπὸ ΒΑΔ ἤτοι μείζων ἐστὶν ἢ ἴση ἢ ἐλάττων.

Έστω πρότερον μείζων, καὶ συνεστάτω πρὸς τῆ ΒΑ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Α τῆ ὑπὸ ΑΒΔ γωνία ἴση ἡ ὑπὸ BAE, καὶ διήχθω ἡ ΔB ἐπὶ τὸ E, καὶ ἐπεζεύχθω ή ΕΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῃ ὑπὸ ΒΑΕ, ἴση ἄρα ἐστὶ καὶ ἡ EB εὐθεῖα τῆ EA. καὶ ἐπεὶ ἴση ἐστὶν ἡ $A\Delta$ τῆ $\Delta\Gamma$, κοινὴ δὲ ἡ ΔE , δύο δὴ αἱ $A\Delta$, ΔE δύο ταῖς $\Gamma\Delta, \Delta E$ ίσαι εἰσὶν ἑκατέρα ἑκατέρα· καὶ γωνία ἡ ὑπὸ $A\Delta E$ γωνία τῆ ὑπὸ ΓΔΕ ἐστιν ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ή ΑΕ βάσει τῆ ΓΕ ἐστιν ἴση. ἀλλὰ ἡ ΑΕ τῆ ΒΕ ἐδείχθη ἴση· καὶ ἡ ΒΕ ἄρα τῆ ΓΕ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΑΕ, ΕΒ, ΕΓ ίσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῷ τῷ Ε διαστήματι δὲ ένὶ τῶν ΑΕ, ΕΒ, ΕΓ κύκλος γραφόμενος ἤξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται προσαναγεγραμμένος. κύκλου άρα τμήματος δοθέντος προσαναγέγραπται ὁ κύκλος. καὶ δῆλον, ὡς τὸ ΑΒΓ τμῆμα ἔλαττόν ἐστιν ἡμικυκλίου διὰ τὸ τὸ Ε κέντρον ἐκτὸς αὐτοῦ τυγχάνειν.

Όμοίως [δἑ] Χἂν ἢ ἡ ὑπὸ ABΔ γωνία ἴση τῆ ὑπὸ BAΔ, τῆς AΔ ἴσης γενομένης ἑχατέρα τῶν BΔ, ΔΓ αἰ τρεῖς αἰ ΔA, ΔB, ΔΓ ἴσαι ἀλλήλαις ἔσονται, χαὶ ἔσται τὸ Δ κέντρον τοῦ προσαναπεπληρωμένου κύχλου, καὶ δηλαδὴ ἔσται τὸ ABΓ ἡμικύχλιον.

Ἐἀν δὲ ἡ ὑπὸ ABΔ ἐλάττων ἢ τῆς ὑπὸ BAΔ, καὶ συστησώμεθα πρὸς τῆ BA εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω

Proposition 25

For a given segment of a circle, to complete the circle, the very one of which it is a segment.



Let ABC be the given segment of a circle. So it is required to complete the circle for segment ABC, the very one of which it is a segment.

For let AC have been cut in half at (point) D [Prop. 1.10], and let DB have been drawn from point D, at right-angles to AC [Prop. 1.11]. And let AB have been joined. Thus, angle ABD is surely either greater than, equal to, or less than (angle) BAD.

First of all, let it be greater. And let (angle) BAE, equal to angle ABD, have been constructed on the straight-line *BA*, at the point *A* on it [Prop. 1.23]. And let DB have been drawn through to E, and let EC have been joined. Therefore, since angle ABE is equal to BAE, the straight-line EB is thus also equal to EA[Prop. 1.6]. And since AD is equal to DC, and DE (is) common, the two (straight-lines) AD, DE are equal to the two (straight-lines) CD, DE, respectively. And angle ADE is equal to angle CDE. For each (is) a right-angle. Thus, the base AE is equal to the base CE [Prop. 1.4]. But, AE was shown (to be) equal to BE. Thus, BE is also equal to CE. Thus, the three (straight-lines) AE, EB, and EC are equal to one another. Thus, if a circle is drawn with center E, and radius one of AE, EB, or EC, it will also go through the remaining points (of the segment), and the (associated circle) will have been completed [Prop. 3.9]. Thus, a circle has been completed from the given segment of a circle. And (it is) clear that the segment ABC is less than a semi-circle, because the center E happens to lie outside it.

τῷ Α τῆ ὑπὸ ΑΒΔ γωνία ἴσην, ἐντὸς τοῦ ΑΒΓ τμήματος πεσεῖται τὸ κέντρον ἐπὶ τῆς ΔΒ, καὶ ἔσται δηλαδὴ τὸ ΑΒΓ τμῆμα μεῖζον ἡμικυκλίου.

Κύκλου ἄρα τμήματος δοθέντος προσαναγέγραπται ό κύκλος· ὅπερ ἔδει ποιῆσαι.

χς΄.

Έν τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.



Έστωσαν ίσοι χύχλοι οἱ ABΓ, ΔΕΖ χαὶ ἐν αὐτοῖς ίσαι γωνίαι ἔστωσαν πρὸς μὲν τοῖς κέντροις αἱ ὑπὸ BHΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ BAΓ, ΕΔΖ· λέγω, ὅτι ἴση ἐστὶν ἡ BKΓ περιφέρεια τῆ ΕΛΖ περιφερεία.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΓ, ΕΖ.

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΑΒΓ, ΔΕΖ κύκλοι, ἴσαι εἰσὶν αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΒΗ, ΗΓ δύο ταῖς ΕΘ, ΘΖ ἴσαι· καὶ γωνία ἡ πρὸς τῷ Η γωνία τῆ πρὸς τῷ Θ ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῆ ΕΖ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ πρὸς τῷ Α γωνία τῆ πρὸς τῷ Δ, ὄμοιον ἄρα ἐστὶ τὸ ΒΑΓ τμῆμα τῷ ΕΔΖ τμήματι· καί εἰσιν ἐπὶ ἴσων εὐθειῶν [τῶν ΒΓ, ΕΖ]· τὰ δὲ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα τὸ ΒΑΓ τμῆμα τῷ ΕΔΖ. ἔστι δὲ καὶ ὅλος ὁ ΑΒΓ κύκλος ὅλῳ τῷ ΔΕΖ κύκλῳ ἴσος· λοιπὴ ἄρα ἡ ΒΚΓ περιφέρεια τῆ ΕΛΖ περιφερεία ἐστὶν ἴση.

Έν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείας ῶσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι. [And], similarly, even if angle ABD is equal to BAD, (since) AD becomes equal to each of BD [Prop. 1.6] and DC, the three (straight-lines) DA, DB, and DC will be equal to one another. And point D will be the center of the completed circle. And ABC will manifestly be a semi-circle.

And if ABD is less than BAD, and we construct (angle BAE), equal to angle ABD, on the straight-line BA, at the point A on it [Prop. 1.23], then the center will fall on DB, inside the segment ABC. And segment ABC will manifestly be greater than a semi-circle.

Thus, a circle has been completed from the given segment of a circle. (Which is) the very thing it was required to do.

Proposition 26

In equal circles, equal angles stand upon equal circumferences whether they are standing at the center or at the circumference.



Let ABC and DEF be equal circles, and within them let BGC and EHF be equal angles at the center, and BAC and EDF (equal angles) at the circumference. I say that circumference BKC is equal to circumference ELF.

For let BC and EF have been joined.

And since circles ABC and DEF are equal, their radii are equal. So the two (straight-lines) BG, GC (are) equal to the two (straight-lines) EH, HF (respectively). And the angle at G (is) equal to the angle at H. Thus, the base BC is equal to the base EF [Prop. 1.4]. And since the angle at A is equal to the (angle) at D, the segment BACis thus similar to the segment EDF [Def. 3.11]. And they are on equal straight-lines [BC and EF]. And similar segments of circles on equal straight-lines are equal to one another [Prop. 3.24]. Thus, segment BAC is equal to (segment) EDF. And the whole circle ABC is also equal to the whole circle DEF. Thus, the remaining circumference BKC is equal to the (remaining) circumference ELF.

Thus, in equal circles, equal angles stand upon equal circumferences, whether they are standing at the center

κζ΄.

Έν τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ῶσι βεβηκυῖαι.



Έν γὰρ ἴσοις κύκλοις τοῖς ΑΒΓ, ΔΕΖ ἐπὶ ἴσων περιφερειῶν τῶν ΒΓ, ΕΖ πρὸς μὲν τοῖς Η, Θ κέντροις γωνίαι βεβηκέτωσαν αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἡ μὲν ὑπὸ ΒΗΓ γωνία τῆ ὑπὸ ΕΘΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΒΑΓ τῆ ὑπὸ ΕΔΖ ἐστιν ἴση.

Εἰ γὰρ ἄνισός ἑστιν ἡ ὑπὸ ΒΗΓ τῆ ὑπὸ ΕΘΖ, μία αὐτῶν μείζων ἑστίν. ἕστω μείζων ἡ ὑπὸ ΒΗΓ, καὶ συνεστάτω πρὸς τῆ ΒΗ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Η τῆ ὑπὸ ΕΘΖ γωνία ἴση ἡ ὑπὸ ΒΗΚ· αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ῶσιν· ἴση ἄρα ἡ ΒΚ περιφέρεια τῆ ΕΖ περιφερεία. ἀλλὰ ἡ ΕΖ τῆ ΒΓ ἐστιν ἴση· καὶ ἡ ΒΚ ἄρα τῆ ΒΓ ἐστιν ἴση ἡ ἐλάττων τῆ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΒΗΓ γωνία τῆ ὑπὸ ΕΘΖ· ἴση ἄρα. καί ἐστι τῆς μὲν ὑπὸ ΒΗΓ ἡμίσεια ἡ πρὸς τῷ Α, τῆς δὲ ὑπὸ ΕΘΖ ἡμίσεια ἡ πρὸς τῷ Δ· ἴση ἄρα καὶ ἡ πρὸς τῷ Α γωνία τῆ πρὸς τῷ Δ.

Έν ἄρα τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὥσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

×η'.

Έν τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῆ μείζονι τὴν δὲ ἐλάττονα τῆ ἐλάττονι.

Έστωσαν ίσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ ἐν τοῖς κύκλοις ίσαι εὐθεῖαι ἔστωσαν αἱ ΑΒ, ΔΕ τὰς μὲν ΑΓΒ, ΑΖΕ περιφερείας μείζονας ἀφαιροῦσαι τὰς δὲ ΑΗΒ, ΔΘΕ ἐλάττονας· λέγω, ὅτι ἡ μὲν ΑΓΒ μείζων περιφέρεια ἴση ἐστὶ τῆ ΔΖΕ μείζονι περιφερεία ἡ δὲ ΑΗΒ ἐλάττων περιφέρεια τῆ ΔΘΕ. or at the circumference. (Which is) the very thing which it was required to show.

Proposition 27

In equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference.



For let the angles BGC and EHF at the centers G and H, and the (angles) BAC and EDF at the circumferences, stand upon the equal circumferences BC and EF, in the equal circles ABC and DEF (respectively). I say that angle BGC is equal to (angle) EHF, and BAC is equal to EDF.

For if BGC is unequal to EHF, one of them is greater. Let BGC be greater, and let the (angle) BGK, equal to angle EHF, have been constructed on the straight-line BG, at the point G on it [Prop. 1.23]. But equal angles (in equal circles) stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference BK (is) equal to circumference EF. But, EF is equal to BC. Thus, BK is also equal to BC, the lesser to the greater. The very thing is impossible. Thus, angle BGCis not unequal to EHF. Thus, (it is) equal. And the (angle) at A is half BGC, and the (angle) at D half EHF[Prop. 3.20]. Thus, the angle at A (is) also equal to the (angle) at D.

Thus, in equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference. (Which is) the very thing it was required to show.

Proposition 28

In equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser.

Let ABC and DEF be equal circles, and let ABand DE be equal straight-lines in these circles, cutting off the greater circumferences ACB and DFE, and the lesser (circumferences) AGB and DHE (respectively). I say that the greater circumference ACB is equal to the greater circumference DFE, and the lesser circumfer-



Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων τὰ Κ, Λ, καὶ ἐπεζεύχθωσαν αἰ ΑΚ, ΚΒ, ΔΛ, ΛΕ.

Καὶ ἐπεὶ ἴσοι κύκλοι εἰσίν, ἴσαι εἰσὶ καὶ αἰ ἐκ τῶν κέντρων· δύο δὴ αἱ AK, KB δυσὶ ταῖς ΔΛ, ΛΕ ἴσαι εἰσίν· καὶ βάσις ἡ AB βάσει τῆ ΔΕ ἴση· γωνία ἄρα ἡ ὑπὸ AKB γωνία τῆ ὑπὸ ΔΛΕ ἴση ἐστίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὥσιν· ἴση ἄρα ἡ AHB περιφέρεια τῆ ΔΘΕ. ἐστὶ δὲ καὶ ὅλος ὁ ABΓ κύκλος ὅλῳ τῷ ΔΕΖ κύκλῳ ἴσος· καὶ λοιπὴ ἄρα ἡ AΓB περιφέρεια λοιπῆ τῆ ΔΖΕ περιφερεία ἴση ἐστίν.

Έν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῆ μείζονι τὴν δὲ ἐλάττονα τῆ ἐλάττονι· ὅπερ ἔδει δεῖξαι.



For let the centers of the circles, K and L, have been found [Prop. 3.1], and let AK, KB, DL, and LE have been joined.

And since (*ABC* and *DEF*) are equal circles, their radii are also equal [Def. 3.1]. So the two (straightlines) AK, KB are equal to the two (straight-lines) DL, LE (respectively). And the base AB (is) equal to the base DE. Thus, angle AKB is equal to angle DLE[Prop. 1.8]. And equal angles stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference AGB (is) equal to DHE. And the whole circle ABC is also equal to the whole circle DEF. Thus, the remaining circumference ACB is also equal to the remaining circumference DFE.

Thus, in equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser. (Which is) the very thing it was required to show.

Proposition 29

In equal circles, equal straight-lines subtend equal circumferences.



Let ABC and DEF be equal circles, and within them let the equal circumferences BGC and EHF have been cut off. And let the straight-lines BC and EF have been joined. I say that BC is equal to EF.

For let the centers of the circles have been found [Prop. 3.1], and let them be (at) K and L. And let BK,

κθ΄.

Έν τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν.



Έστωσαν ίσοι χύχλοι οἱ ΑΒΓ, ΔΕΖ, καὶ ἐν αὐτοῖς ίσαι περιφέρειαι ἀπειλήφθωσαν αἱ ΒΗΓ, ΕΘΖ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΕΖ εὐθεῖαι· λέγω, ὅτι ἴση ἐστὶν ἡ ΒΓ τῆ ΕΖ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων, καὶ ἔστω τὰ Κ, Λ, καὶ ἐπεζεύχθωσαν αἱ ΒΚ, ΚΓ, ΕΛ, ΛΖ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΗΓ περιφέρεια τῆ ΕΘΖ περιφερεία,

Έν ἄρα τοῖς ἴσοις χύχλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

λ΄.

Τὴν δοθεῖσαν περιφέρειαν δίχα τεμεῖν.



Έστω ή δοθεῖσα περιφέρεια ή ΑΔΒ· δεῖ δὴ τὴν ΑΔΒ περιφέρειαν δίχα τεμεῖν.

Έπεζεύχθω ή AB, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ ἀπὸ τοῦ Γ σημείου τῆ AB εὐθεία πρὸς ὀρθὰς ἤχθω ή ΓΔ, καὶ ἐπεζεύχθωσαν αἱ AΔ, ΔB.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΒ, Χοινὴ δὲ ἡ ΓΔ, δύο δὴ aἱ ΑΓ, ΓΔ δυσὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσίν· Χαὶ γωνία ἡ ὑπὸ ΑΓΔ γωνία τῆ ὑπὸ ΒΓΔ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΔ βάσει τῆ ΔΒ ἴση ἐστίν. ai δὲ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῆ μείζονι τὴν δὲ ἐλάττονα τῆ ἐλάττονι· κάι ἐστιν ἑκατέρα τῶν ΑΔ, ΔΒ περιφερειῶν ἐλάττων ἡμιχυχλίου· ἴση ἄρα ἡ ΑΔ περιφέρεια τῆ ΔΒ περιφερεία.

Ή ἄρα δοθεῖσα περιφέρεια δίχα τέτμηται κατὰ τὸ Δ σημεῖον· ὅπερ ἔδει ποιῆσαι.

λα΄.

Έν κύκλω ή μὲν ἐν τῷ ήμικυκλίω γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι τμήματι μείζων ὀρθῆς· καὶ ἔπι ἡ μὲν τοῦ μείζονος τμήματος γωνία μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἐλάττων ὀρθῆς. **ELEMENTS BOOK 3**

KC, EL, and LF have been joined.

And since the circumference BGC is equal to the circumference EHF, the angle BKC is also equal to (angle) ELF [Prop. 3.27]. And since the circles ABC and DEF are equal, their radii are also equal [Def. 3.1]. So the two (straight-lines) BK, KC are equal to the two (straight-lines) EL, LF (respectively). And they contain equal angles. Thus, the base BC is equal to the base EF [Prop. 1.4].

Thus, in equal circles, equal straight-lines subtend equal circumferences. (Which is) the very thing it was required to show.

Proposition 30

To cut a given circumference in half.



Let ADB be the given circumference. So it is required to cut circumference ADB in half.

Let AB have been joined, and let it have been cut in half at (point) C [Prop. 1.10]. And let CD have been drawn from point C, at right-angles to AB [Prop. 1.11]. And let AD, and DB have been joined.

And since AC is equal to CB, and CD (is) common, the two (straight-lines) AC, CD are equal to the two (straight-lines) BC, CD (respectively). And angle ACD (is) equal to angle BCD. For (they are) each right-angles. Thus, the base AD is equal to the base DB [Prop. 1.4]. And equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser [Prop. 1.28]. And the circumferences AD and DB are each less than a semicircle. Thus, circumference AD (is) equal to circumference DB.

Thus, the given circumference has been cut in half at point D. (Which is) the very thing it was required to do.

Proposition 31

In a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser segment (is) greater than a rightangle. And, further, the angle of a segment greater (than a semi-circle) is greater than a right-angle, and the an-



Έστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΒΓ, κέντρον δὲ τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΒΑ, ΑΓ, ΑΔ, ΔΓ· λέγω, ὅτι ἡ μὲν ἐν τῷ ΒΑΓ ἡμικυκλίω γωνία ἡ ὑπὸ ΒΑΓ ὀρθή ἐστιν, ἡ δὲ ἐν τῷ ΑΒΓ μείζονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ ΑΒΓ ἐλάττων ἐστὶν ὀρθῆς, ἡ δὲ ἐν τῷ ΑΔΓ ἐλάττονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ ΑΔΓ μείζων ἐστὶν ὀρθῆς.

Έπεζεύχθω ή ΑΕ, καὶ διήχθω ή ΒΑ ἐπὶ τὸ Ζ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῆ ΕΑ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΒΕ τῆ ὑπὸ ΒΑΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΓΕ τῆ ΕΑ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΑΓΕ τῆ ὑπὸ ΓΑΕ· ὅλη ἄρα ἡ ὑπὸ ΒΑΓ δυσὶ ταῖς ὑπὸ ΑΒΓ, ΑΓΒ ἴση ἐστίν. ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΑΓ ἐκτὸς τοῦ ΑΒΓ τριγώνου δυσὶ ταῖς ὑπὸ ΑΒΓ, ΑΓΒ γωνίαις ἴση· ἴση ἄρα καὶ ἡ ὑπὸ ΒΑΓ γωνία τῆ ὑπὸ ΖΑΓ· ὀθὴ ἄρα ἑκατέρα· ἡ ἄρα ἐν τῷ ΒΑΓ ἡμικυκλίῳ γωνία ἡ ὑπὸ ΒΑΓ ὀβή ἐστιν.

Καὶ ἐπεὶ τοῦ ΑΒΓ τρίγωνου δύο γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΑΓ δύο ὀρθῶν ἐλάττονές εἰσιν, ὀρθὴ δὲ ἡ ὑπὸ ΒΑΓ, ἐλάττων ἄρα ὀρθῆς ἐστιν ἡ ὑπὸ ΑΒΓ γωνία καί ἐστιν ἐν τῷ ΑΒΓ μείζονι τοῦ ἡμιχυχλίου τμήματι.

Καὶ ἐπεὶ ἐν κύκλῷ τετράπλευρόν ἐστι τὸ ΑΒΓΔ, τῶν δὲ ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν [αἱ ἄρα ὑπὸ ΑΒΓ, ΑΔΓ γωνίαι δυσὶν ὀρθαῖς ἴσας εἰσίν], καί ἐστιν ἡ ὑπὸ ΑΒΓ ἐλάττων ὀρθῆς· λοιπὴ ἄρα ἡ ὑπὸ ΑΔΓ γωνία μείζων ὀρθῆς ἐστιν· καί ἐστιν ἐν τῷ ΑΔΓ ἐλάττονι τοῦ ἡμικυκλίου τμήματι.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ μείζονος τμήματος γωνία ἡ περιεχομένη ὑπό [τε] τῆς ΑΒΓ περιφερείας καὶ τῆς ΑΓ εὐθείας μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἡ περιεχομένη ὑπό [τε] τῆς ΑΔ[Γ] περιφερείας καὶ τῆς ΑΓ εὐθείας ἐλάττων ἐστὶν ὀρθῆς. καί ἐστιν αὐτόθεν φανερόν. ἐπεὶ γὰρ ἡ ὑπὸ τῶν ΒΑ, ΑΓ εὐθειῶν ὀρθή ἐστιν, ἡ ἄρα ὑπὸ τῆς ΑΒΓ περιφερείας καὶ τῆς ΑΓ εὐθείας περιεχομένη μείζων ἐστὶν ὀρθῆς. πάλιν, ἐπεὶ ἡ ὑπὸ τῶν ΑΓ, ΑΖ εὐθειῶν ὀρθή ἐστιν, ἡ ἄρα ὑπὸ τῆς ΓΑ εὐθείας καὶ τῆς ΑΔ[Γ] περιgle of a segment less (than a semi-circle) is less than a right-angle.

F



Let ABCD be a circle, and let BC be its diameter, and E its center. And let BA, AC, AD, and DC have been joined. I say that the angle BAC in the semi-circle BAC is a right-angle, and the angle ABC in the segment ABC, (which is) greater than a semi-circle, is less than a right-angle, and the angle ADC in the segment ADC, (which is) less than a semi-circle, is greater than a right-angle.

Let AE have been joined, and let BA have been drawn through to F.

And since BE is equal to EA, angle ABE is also equal to BAE [Prop. 1.5]. Again, since CE is equal to EA, ACE is also equal to CAE [Prop. 1.5]. Thus, the whole (angle) BAC is equal to the two (angles) ABCand ACB. And FAC, (which is) external to triangle ABC, is also equal to the two angles ABC and ACB[Prop. 1.32]. Thus, angle BAC (is) also equal to FAC. Thus, (they are) each right-angles. [Def. 1.10]. Thus, the angle BAC in the semi-circle BAC is a right-angle.

And since the two angles ABC and BAC of triangle ABC are less than two right-angles [Prop. 1.17], and BAC is a right-angle, angle ABC is thus less than a rightangle. And it is in segment ABC, (which is) greater than a semi-circle.

And since ABCD is a quadrilateral within a circle, and for quadrilaterals within circles the (sum of the) opposite angles is equal to two right-angles [Prop. 3.22] [angles ABC and ADC are thus equal to two rightangles], and (angle) ABC is less than a right-angle. The remaining angle ADC is thus greater than a right-angle. And it is in segment ADC, (which is) less than a semicircle.

I also say that the angle of the greater segment, (namely) that contained by the circumference ABC and the straight-line AC, is greater than a right-angle. And the angle of the lesser segment, (namely) that contained

φερείας περιεχομένη ἐλάττων ἐστὶν ὀρθῆς.

Έν κύκλω ἄρα ή μὲν ἐν τῷ ἡμικυκλίω γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι [τμήματι] μείζων ὀρθῆς· καὶ ἔπι ἡ μὲν τοῦ μείζονος τμήματος [γωνία] μείζων [ἐστὶν] ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος [γωνία] ἐλάττων ὀρθῆς· ὅπερ ἔδει δεῖξαι.

λβ'.

Έαν χύχλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν χύχλον διαχθῆ τις εὐθεῖα τέμνουσα τὸν χύχλον, ἂς ποιεῖ γωνίας πρὸς τῆ ἐφαπτομένῃ, ἶσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ χύχλου τμήμασι γωνίαις.



Κύχλου γὰρ τοῦ ΑΒΓΔ ἐφαπτέσθω τις εὐθεῖα ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ ἀπὸ τοῦ Β σημείου διήχθω τις εὐθεῖα εἰς τὸν ΑΒΓΔ χύχλον τέμνουσα αὐτὸν ἡ ΒΔ. λέγω, ὅτι ἀς ποιεῖ γωνίας ἡ ΒΔ μετὰ τῆς ΕΖ ἐφαπτομένης, ἴσας ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τμήμασι τοῦ χύχλου γωνίαις, τουτέστιν, ὅτι ἡ μὲν ὑπὸ ΖΒΔ γωνία ἴση ἐστὶ τῆ ἐν τῷ ΒΑΔ τμήματι συνισταμένῃ γωνία, ἡ δὲ ὑπὸ ΕΒΔ γωνία ἴση ἐστὶ τῆ ἐν τῷ ΔΓΒ τμήματι συνισταμένῃ γωνία.

Ήχθω γὰρ ἀπὸ τοῦ Β τῆ ΕΖ πρὸς ὀρθὰς ἡ ΒΑ, καὶ εἰλήφθω ἐπὶ τῆς ΒΔ περιφερείας τυχὸν σημεῖον τὸ Γ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, ΓΒ.

Καὶ ἐπεὶ κύκλου τοῦ ΑΒΓΔ ἐφάπτεταί τις εὐθεῖα ἡ ΕΖ

by the circumference AD[C] and the straight-line AC, is less than a right-angle. And this is immediately apparent. For since the (angle contained by) the two straight-lines BA and AC is a right-angle, the (angle) contained by the circumference ABC and the straight-line AC is thus greater than a right-angle. Again, since the (angle contained by) the straight-lines AC and AF is a right-angle, the (angle) contained by the circumference AD[C] and the straight-line CA is thus less than a right-angle.

Thus, in a circle, the angle in a semi-circle is a rightangle, and that in a greater segment (is) less than a right-angle, and that in a lesser [segment] (is) greater than a right-angle. And, further, the [angle] of a segment greater (than a semi-circle) [is] greater than a rightangle, and the [angle] of a segment less (than a semicircle) is less than a right-angle. (Which is) the very thing it was required to show.

Proposition 32

If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.



For let some straight-line EF touch the circle ABCDat the point B, and let some (other) straight-line BDhave been drawn from point B into the circle ABCD, cutting it (in two). I say that the angles BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle. That is to say, that angle FBD is equal to the angle constructed in segment BAD, and angle EBD is equal to the angle constructed in segment DCB.

For let BA have been drawn from B, at right-angles to EF [Prop. 1.11]. And let the point C have been taken at random on the circumference BD. And let AD, DC,

κατὰ τὸ B, καὶ ἀπὸ τῆς ἁφῆς ῆκται τῆ ἐφαπτομένῃ πρὸς ὀρθὰς ἡ BA, ἐπὶ τῆς BA ἄρα τὸ κέντρον ἐστὶ τοῦ ABΓΔ κύκλου. ἡ BA ἄρα διάμετός ἐστι τοῦ ABΓΔ κύκλου· ἡ ἄρα ὑπὸ AΔB γωνία ἐν ἡμικυκλίφ οὕσα ὀρθή ἐστιν. λοιπαὶ ἄρα αἱ ὑπὸ BAΔ, ABΔ μιῷ ὀρθῆ ἴσαι εἰσίν. ἐστὶ δὲ καὶ ἡ ὑπὸ ABZ ὀρθή· ἡ ἄρα ὑπὸ ABZ ἴση ἐστὶ ταῖς ὑπὸ BAΔ, ABΔ. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ABΔ· λοιπὴ ἄρα ἡ ὑπὸ ΔBZ γωνία ἴση ἐστὶ τῆ ἐν τῷ ἐναλλὰξ τμήματι τοῦ κύκλου γωνία τῆ ὑπὸ BAΔ. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ABΓΔ, αἱ ἀπεναντίον αὐτοῦ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ ἀ ὑπὸ ΔBZ, ΔBE δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ ἀ ὑπὸ ΔBZ, ΔBE δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΔBZ, ΔBE ταῖς ὑπὸ BAΔ, BΓΔ ἴσαι εἰσίν, ῶν ἡ ὑπὸ BAΔ τῆ ὑπὸ ΔBZ ἐδείχθη ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΔΓΕ τῆ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι τῷ ΔΓΒ τῆ ὑπὸ ΔΓΒ γωνία ἐστιν ἴση.

Έὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν κύκλον διαχθῆ τις εὐθεῖα τέμνουσα τὸν κύκλον, ἂς ποιεῖ γωνίας πρὸς τῆ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ κύκλου τμήμασι γωνίαις. ὅπερ ἔδει δεῖξαι.

λγ΄.

Έπὶ τῆς δοθείσης εὐθείας γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἶσην τῆ δοθείση γωνία εὐθυγράμμω.



Έστω ή δοθεϊσα εὐθεῖα ή AB, ή δὲ δοθεϊσα γωνία εὐθύγραμμος ή πρὸς τῷ Γ· δεῖ δὴ ἐπὶ τῆς δοθείσης εὐθείας τῆς AB γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῆ πρὸς τῷ Γ.

Ή δὴ πρὸς τῷ Γ [γωνία] ἤτοι ὀξεῖά ἐστιν ἢ ὀρθὴ ἢ ἀμβλεῖα· ἔστω πρότερον ὀξεῖα, καὶ ὡς ἐπὶ τῆς πρώτης καταγραφῆς συνεστάτω πρὸς τῆ ΑΒ εὐθεία καὶ τῷ Α σημείω τῆ πρὸς τῷ Γ γωνία ἴση ἡ ὑπὸ ΒΑΔ· ὀξεῖα ἄρα ἐστὶ καὶ ἡ ὑπὸ ΒΑΔ. ἤχθω τῆ ΔΑ πρὸς ὀρθὰς ἡ ΑΕ, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ ἤχθω ἀπὸ τοῦ Ζ σημείου τῆ ΑΒ and *CB* have been joined.

And since some straight-line EF touches the circle ABCD at point B, and BA has been drawn from the point of contact, at right-angles to the tangent, the center of circle ABCD is thus on BA [Prop. 3.19]. Thus, BAis a diameter of circle ABCD. Thus, angle ADB, being in a semi-circle, is a right-angle [Prop. 3.31]. Thus, the remaining angles (of triangle ADB) BAD and ABD are equal to one right-angle [Prop. 1.32]. And ABF is also a right-angle. Thus, ABF is equal to BAD and ABD. Let ABD have been subtracted from both. Thus, the remaining angle DBF is equal to the angle BAD in the alternate segment of the circle. And since ABCD is a quadrilateral in a circle, (the sum of) its opposite angles is equal to two right-angles [Prop. 3.22]. And DBF and DBE is also equal to two right-angles [Prop. 1.13]. Thus, DBF and DBE is equal to BAD and BCD, of which BADwas shown (to be) equal to DBF. Thus, the remaining (angle) DBE is equal to the angle DCB in the alternate segment DCB of the circle.

Thus, if some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle. (Which is) the very thing it was required to show.

Proposition 33

To draw a segment of a circle, accepting an angle equal to a given rectilinear angle, on a given straight-line.



Let AB be the given straight-line, and C the given rectilinear angle. So it is required to draw a segment of a circle, accepting an angle equal to C, on the given straight-line AB.

So the [angle] C is surely either acute, a right-angle, or obtuse. First of all, let it be acute. And, as in the first diagram (from the left), let (angle) BAD, equal to angle C, have been constructed on the straight-line AB, at the point A (on it) [Prop. 1.23]. Thus, BAD is also acute. Let AE have been drawn, at right-angles to DA [Prop. 1.11]. πρὸς ὀρθὰς ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῆ ΖΒ, ×οινὴ δὲ ἡ ΖΗ, δύο δὴ aἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσίν· ×αὶ γωνία ἡ ὑπὸ AΖΗ [γωνία] τῆ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῆ BH ἴση ἐστίν. ὁ ἄρα ×έντρῳ μὲν τῷ Η διαστήματι δὲ τῷ HA ×ύ×λος γραφόμενος ἥξει ×αὶ διὰ τοῦ Β. γεγράφθω ×αὶ ἔστω ὁ ABE, ×αὶ ἐπεζεύχθω ἡ ΕΒ. ἐπεὶ οῦν ἀπ᾽ ἄ×ρας τῆς AE διαμέτρου ἀπὸ τοῦ Α τῆ AE πρὸς ὀρθάς ἐστιν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ABE ×ύ×λου· ἐπεὶ οῦν ×ύ×λου τοῦ ABE ἐφάπτεται τις εὐθεῖα ἡ ΑΔ, ×αὶ ἀπὸ τῆς ×ατὰ τὸ A ἁφῆς εἰς τὸν ABE ×ύ×λον διῆ×ταί τις εὐθεῖα ἡ AB, ἡ ἄρα ὑπὸ ΔAB γωνία ἴση ἐστὶ τῆ ἐν τῷ ἐναλλὰξ τοῦ ×ύ×λου τμήματι γωνία τῆ ὑπὸ AEB. ἀλλ᾽ ἡ ὑπὸ ΔAB τῆ πρὸς τῷ Γ ἐστιν ἴση· ×αὶ ἡ πρὸς τῷ Γ ἅρα γωνία ἴση ἐστὶ τῆ ὑπὸ AEB.

Έπὶ τῆς δοθείσης ἄρα εὐθείας τῆς AB τμῆμα κύκλου γέγραπται τὸ AEB δεχόμενον γωνίαν τὴν ὑπὸ AEB ἴσην τῆ δοθείσῃ τῇ πρὸς τῷ Γ.

Άλλὰ δὴ ὀρθὴ ἔστω ἡ πρὸς τῷ Γ· καὶ δέον πάλιν ἔστω ἐπὶ τῆς AB γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῆ πρὸς τῷ Γ ὀρθῆ [γωνία]. συνεστάτω [πάλιν] τῆ πρὸς τῷ Γ ὀρθῆ γωνία ἴση ἡ ὑπὸ BAΔ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ τετμήσθω ἡ AB δίχα κατὰ τὸ Ζ, καὶ κέντρω τῷ Ζ, διαστήματι δὲ ὁποτέρῳ τῶν ZA, ZB, κύκλος γεγράφθω ὁ AEB.

Έφάπτεται ἄρα ή ΑΔ εὐθεῖα τοῦ ΑΒΕ χύχλου διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Α γωνίαν. καὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΔ γωνία τῆ ἐν τῷ ΑΕΒ τμήματι· ὀρθὴ γὰρ καὶ αὐτὴ ἐν ἡμιχυχλίω οῦσα. ἀλλὰ καὶ ἡ ὑπὸ ΒΑΔ τῆ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΕΒ ἄρα ἴση ἐστὶ τῆ πρὸς τῷ Γ.

Γέγραπται ἄρα πάλιν ἐπὶ τῆς AB τμῆμα κύκλου τὸ AEB δεχόμενον γωνίαν ἴσην τῆ πρὸς τῷ Γ.

Άλλὰ δὴ ἡ πρὸς τῷ Γ ἀμβλεῖα ἔστω· καὶ συνεστάτω αὐτῆ ἴση πρὸς τῆ ΑΒ εὐθεία καὶ τῷ Α σημείω ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ τῆ ΑΔ πρὸς ὀρθὰς ἤχθω ἡ ΑΕ, καὶ τετμήσθω πάλιν ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ τῆ ΑΒ πρὸς ὀρθὰς ἦχθω ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ HB.

Kaì
ἑπεὶ πάλιν ἴση ἐστὶν ἡ AZ τῆ ZB, xaì xoιvὴ ἡ ZH,
δύο δὴ aỉ AZ, ZH δύο ταῖς BZ, ZH ἴσαι εἰσίν· xaì γωνία ἡ
ὑπὸ AZH γωνία τῆ ὑπὸ BZH ἴση· βάσις ἄρα ἡ AH βάσει
τῆ BH ἴση ἐστίν· ὁ ἄρα xέντρῳ μὲν τῷ H διαστήματι δὲ τῷ
HA xύxλος γραφόμενος ἥξει xaì διὰ τοῦ B. ἐρχέσθω ὡς ὁ
AEB. xaì ἐπεὶ τῆ AE διαμέτρῳ ἀπ' ἄxρας πρὸς ὀρθάς ἐστιν
ἡ AΔ, ἡ AΔ ἄρα ἐφάπτεται τοῦ AEB xύxλου. xaì ἀπὸ τῆς
xaτὰ τὸ A ἐπαφῆς διῆxται ἡ AB· ἡ ἄρα ὑπὸ BAΔ γωνία
ίση ἐστὶ τῆ ἐν τῷ ἐναλλὰξ τοῦ xύxλου τμήματι τῷ AΘB
συνισταμένη γωνία. ἀλλ' ἡ ὑπὸ BAΔ γωνία τῆ πρὸς τῷ Γ
ίση ἐστίν. xaì ἡ ἐν τῷ AΘB ἄρα τμήματι γωνία ἴση ἑστὶ τῆ
πρὸς τῷ Γ.

Ἐπὶ τῆς ἄρα δοθείσης εὐθείας τῆς AB γέγραπται τμῆμα κύκλου τὸ AΘB δεχόμενον γωνίαν ἴσην τῆ πρὸς τῷ Γ· ὅπερ ἔδει ποιῆσαι.

And let AB have been cut in half at F [Prop. 1.10]. And let FG have been drawn from point F, at right-angles to AB [Prop. 1.11]. And let GB have been joined.

And since AF is equal to FB, and FG (is) common, the two (straight-lines) AF, FG are equal to the two (straight-lines) BF, FG (respectively). And angle AFG(is) equal to [angle] BFG. Thus, the base AG is equal to the base BG [Prop. 1.4]. Thus, the circle drawn with center G, and radius GA, will also go through B (as well as A). Let it have been drawn, and let it be (denoted) ABE. And let EB have been joined. Therefore, since AD is at the extremity of diameter AE, (namely, point) A, at right-angles to AE, the (straight-line) ADthus touches the circle ABE [Prop. 3.16 corr.]. Therefore, since some straight-line AD touches the circle ABE, and some (other) straight-line AB has been drawn across from the point of contact A into circle ABE, angle DABis thus equal to the angle AEB in the alternate segment of the circle [Prop. 3.32]. But, *DAB* is equal to *C*. Thus, angle C is also equal to AEB.

Thus, a segment AEB of a circle, accepting the angle AEB (which is) equal to the given (angle) C, has been drawn on the given straight-line AB.

And so let C be a right-angle. And let it again be necessary to draw a segment of a circle on AB, accepting an angle equal to the right-[angle] C. Let the (angle) BAD [again] have been constructed, equal to the rightangle C [Prop. 1.23], as in the second diagram (from the left). And let AB have been cut in half at F [Prop. 1.10]. And let the circle AEB have been drawn with center F, and radius either FA or FB.

Thus, the straight-line AD touches the circle ABE, on account of the angle at A being a right-angle [Prop. 3.16 corr.]. And angle BAD is equal to the angle in segment AEB. For (the latter angle), being in a semi-circle, is also a right-angle [Prop. 3.31]. But, BAD is also equal to C. Thus, the (angle) in (segment) AEB is also equal to C.

Thus, a segment AEB of a circle, accepting an angle equal to C, has again been drawn on AB.

And so let (angle) C be obtuse. And let (angle) BAD, equal to (C), have been constructed on the straight-line AB, at the point A (on it) [Prop. 1.23], as in the third diagram (from the left). And let AE have been drawn, at right-angles to AD [Prop. 1.11]. And let AB have again been cut in half at F [Prop. 1.10]. And let FG have been drawn, at right-angles to AB [Prop. 1.10]. And let GBhave been joined.

And again, since AF is equal to FB, and FG (is) common, the two (straight-lines) AF, FG are equal to the two (straight-lines) BF, FG (respectively). And angle AFG (is) equal to angle BFG. Thus, the base AG is

equal to the base BG [Prop. 1.4]. Thus, a circle of center G, and radius GA, being drawn, will also go through B (as well as A). Let it go like AEB (in the third diagram from the left). And since AD is at right-angles to the diameter AE, at its extremity, AD thus touches circle AEB [Prop. 3.16 corr.]. And AB has been drawn across (the circle) from the point of contact A. Thus, angle BAD is equal to the angle constructed in the alternate segment AHB of the circle [Prop. 3.32]. But, angle BAD is equal to C. Thus, the angle in segment AHB is also equal to C.

Thus, a segment AHB of a circle, accepting an angle equal to C, has been drawn on the given straight-line AB. (Which is) the very thing it was required to do.

Proposition 34

To cut off a segment, accepting an angle equal to a given rectilinear angle, from a given circle.



Let ABC be the given circle, and D the given rectilinear angle. So it is required to cut off a segment, accepting an angle equal to the given rectilinear angle D, from the given circle ABC.

Let EF have been drawn touching ABC at point B.[†] And let (angle) FBC, equal to angle D, have been constructed on the straight-line FB, at the point B on it [Prop. 1.23].

Therefore, since some straight-line EF touches the circle ABC, and BC has been drawn across (the circle) from the point of contact B, angle FBC is thus equal to the angle constructed in the alternate segment BAC [Prop. 1.32]. But, FBC is equal to D. Thus, the (angle) in the segment BAC is also equal to [angle] D.

Thus, the segment BAC, accepting an angle equal to the given rectilinear angle D, has been cut off from the given circle ABC. (Which is) the very thing it was required to do.



λδ'.

Ἀπὸ τοῦ δοθέντος κύκλου τμῆμα ἀφελεῖν δεχόμενον γωνίαν ἴσην τῆ δοθείση γωνία εὐθυγράμμω.



Έστω ὁ δοθεἰς κύκλος ὁ ABΓ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ πρὸς τῷ Δ· δεῖ δὴ ἀπὸ τοῦ ABΓ κύκλου τμῆμα ἀφελεῖν δεχόμενον γωνίαν ἴσην τῆ δοθείση γωνία εὐθυγράμμω τῆ πρὸς τῷ Δ.

Ήχθω τοῦ ΑΒΓ ἐφαπτομένη ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ συνεστάτω πρὸς τῆ ΖΒ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Β τῆ πρὸς τῷ Δ γωνία ἴση ἡ ὑπὸ ΖΒΓ.

Ἐπεὶ οὖν κύκλου τοῦ ABΓ ἐφάπτεταί τις εὐθεῖα ἡ EZ, καὶ ἀπὸ τῆς κατὰ τὸ B ἐπαφῆς διῆκται ἡ BΓ, ἡ ὑπὸ ZBΓ ἄρα γωνία ἴση ἐστὶ τῆ ἐν τῷ BAΓ ἐναλλὰξ τμήματι συνισταμένῃ γωνία. ἀλλ' ἡ ὑπὸ ZBΓ τῆ πρὸς τῷ Δ ἐστιν ἴση· καὶ ἡ ἐν τῷ BAΓ ἄρα τμήματι ἴση ἐστὶ τῆ πρὸς τῷ Δ [γωνία].

Άπὸ τοῦ δοθέντος ἄρα κύκλου τοῦ ΑΒΓ τμῆμα ἀφήρηται τὸ ΒΑΓ δεχόμενον γωνίαν ἴσην τῆ δοθείση γωνία εὐθυγράμμω τῆ πρὸς τῷ Δ· ὅπερ ἔδει ποιῆσαι.

point B, at right-angles to the aforementioned straight-line [Prop. 1.11].

λε΄.

Έὰν ἐν κύκλῷ δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἶσον ἐστὶ τῷ ὑπὸ τῶν τῆς ἑτέρας τμημάτων περιεχομένῷ ὀρθογωνίῷ.



Έν γὰρ κύκλω τῷ ΑΒΓΔ δύο εὐθεῖαι αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένω ὀρθογωνίω.

Εἰ μὲν οῦν αἱ ΑΓ, ΒΔ διὰ τοῦ κέντρου εἰσὶν ὥστε τὸ Ε κέντρον εἶναι τοῦ ΑΒΓΔ κύκλου, φανερόν, ὅτι ἴσων οὐσῶν τῶν ΑΕ, ΕΓ, ΔΕ, ΕΒ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

Μη ἕστωσαν δη αί ΑΓ, ΔΒ διὰ τοῦ κέντρου, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ, καὶ ἔστω τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὰς ΑΓ, ΔΒ εὐθείας κάθετοι ἤχθωσαν αἱ ΖΗ, ΖΘ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΓ, ΖΕ.

Καὶ ἐπεὶ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΗΖ εὐθεῖάν τινα μή διὰ τοῦ κέντρου τὴν ΑΓ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει· ἴση ἄρα ἡ ΑΗ τῇ ΗΓ. ἐπεὶ οὖν εὐθεῖα ἡ ΑΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ άρα ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΓ· [κοινὸν] προσχείσθω τὸ ἀπὸ τῆς ΗΖ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τῶν ἀπὸ τῶν ΗΕ, ΗΖ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, ΗΖ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΕΗ, ΗΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΕ, τοὶς δὲ ἀπὸ τῶν ΓΗ, ΗΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΓ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΓ. ἴση δὲ ἡ ΖΓ τῆ ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ άπὸ τῆς ΕΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΒ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν ΔE , EB μετὰ τοῦ ἀπὸ τῆς ZE ἰσον ἐστὶ τῷ ἀπὸ τῆς ΖΒ. ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον τῷ ἀπὸ τῆς ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ μετὰ τοῦ άπὸ τῆς ΖΕ. χοινὸν ἀφῆρήσθω τὸ ἀπὸ τῆς ΖΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ύπὸ τῶν ΔΕ, ΕΒ περιεχομένω ὀρθογωνίω.

Ἐἀν ἄρα ἐν κύκλῷ εὐθεῖαι δύο τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον

Proposition 35

If two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.



For let the two straight-lines AC and BD, in the circle ABCD, cut one another at point E. I say that the rectangle contained by AE and EC is equal to the rectangle contained by DE and EB.

In fact, if AC and BD are through the center (as in the first diagram from the left), so that E is the center of circle ABCD, then (it is) clear that, AE, EC, DE, and EB being equal, the rectangle contained by AE and ECis also equal to the rectangle contained by DE and EB.

So let AC and DB not be though the center (as in the second diagram from the left), and let the center of ABCD have been found [Prop. 3.1], and let it be (at) F. And let FG and FH have been drawn from F, perpendicular to the straight-lines AC and DB (respectively) [Prop. 1.12]. And let FB, FC, and FE have been joined.

And since some straight-line, GF, through the center, cuts at right-angles some (other) straight-line, AC, not through the center, then it also cuts it in half [Prop. 3.3]. Thus, AG (is) equal to GC. Therefore, since the straightline AC is cut equally at G, and unequally at E, the rectangle contained by AE and EC plus the square on EG is thus equal to the (square) on GC [Prop. 2.5]. Let the (square) on GF have been added [to both]. Thus, the (rectangle contained) by AE and EC plus the (sum of the squares) on GE and GF is equal to the (sum of the squares) on CG and GF. But, the (square) on FEis equal to the (sum of the squares) on EG and GF[Prop. 1.47], and the (square) on FC is equal to the (sum of the squares) on CG and GF [Prop. 1.47]. Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (square) on FC. And FC (is) equal to FB. Thus, the (rectangle contained) by AEand EC plus the (square) on FE is equal to the (square) on FB. So, for the same (reasons), the (rectangle contained) by DE and EB plus the (square) on FE is equal

έστι τῷ ὑπὸ τῶν τῆς ἑτέρας τμημάτων περιεχομένῳ ὀρθογωνίῳ. ὅπερ ἔδει δεῖξαι.

λς'.

Έαν χύχλου ληφθή τι σημεϊον ἐχτός, καὶ ἀπ' αὐτοῦ πρὸς τὸν χύχλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνη τὸν χύχλον, ἡ δὲ ἐφάπτηται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐχτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς χυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ.



Κύχλου γὰρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐχτὸς τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύχλον προσπιπτέτωσαν δύο εὐθεῖαι αἱ ΔΓ[Α], ΔΒ· καὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν ΑΒΓ κύχλον, ἡ δὲ ΒΔ ἐφαπτέσθω· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ τετραγώνῳ.

Ή ἄρα [Δ]ΓΑ ἤτοι διὰ τοῦ κέντρου ἐστὶν ἢ οὕ. ἔστω πρότερον διὰ τοῦ κέντρου, καὶ ἔστω τὸ Z κέντρον τοῦ ABΓ κύκλου, καὶ ἐπεζεύχθω ἡ ZB· ὀρθἡ ἄρα ἐστὶν ἡ ὑπὸ ZBΔ. καὶ ἐπεὶ εὐθεῖα ἡ AΓ δίχα τέτμηται κατὰ τὸ Z, πρόσκειται δὲ αὐτῆ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν AΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ZΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ZΔ. ἴση δὲ ἡ ZΓ τῆ ZB· τὸ ἄρα ὑπὸ τῶν AΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ZB ἴσον ἐστὶ τῷ ἀπὸ τὴς ZΔ. τῷ δὲ ἀπὸ τῆς ZΔ ἴσα ἐστὶ τὰ ἀπὸ τῶν ZB, BΔ· τὸ ἄρα ὑπὸ τῶν AΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ZB ἴσον ἐστὶ τοῖς ἀπὸ τῶν ZB, BΔ. κοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς ZB· λοιπὸν ἄρα τὸ ὑπὸ τῶν AΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔB to the (square) on FB. And the (rectangle contained) by AE and EC plus the (square) on FE was also shown (to be) equal to the (square) on FB. Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (rectangle contained) by DE and EB plus the (square) on FE. Let the (square) on FE have been taken from both. Thus, the remaining rectangle contained by AE and EC is equal to the rectangle contained by DE and EB.

Thus, if two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other. (Which is) the very thing it was required to show.

Proposition 36

If some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and the (other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line).



For let some point D have been taken outside circle ABC, and let two straight-lines, DC[A] and DB, radiate from D towards circle ABC. And let DCA cut circle ABC, and let BD touch (it). I say that the rectangle contained by AD and DC is equal to the square on DB.

[D]CA is surely either through the center, or not. Let it first of all be through the center, and let F be the center of circle ABC, and let FB have been joined. Thus, (angle) FBD is a right-angle [Prop. 3.18]. And since straight-line AC is cut in half at F, let CD have been added to it. Thus, the (rectangle contained) by AD and DC plus the (square) on FC is equal to the (square) on FD [Prop. 2.6]. And FC (is) equal to FB. Thus, the (rectangle contained) by AD and DC plus the (square) on FB is equal to the (square) on FD. And the (square) on FD [Prop. 1.47]. Thus, the (rectangle contained) by AD

Άλλὰ δὴ ἡ ΔΓΑ μὴ ἔστω διὰ τοῦ κέντρου τοῦ ΑΒΓ κύκλου, καὶ εἰλήφθω τὸ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΓ κάθετος ἤχθω ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΒ, ΕΓ, $\mathrm{E}\Delta^{.}$ ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ $\mathrm{EB}\Delta$. καὶ ἐπεὶ εὐθεῖά τις διὰ τοῦ κέντρου ή ΕΖ εὐθεῖάν τινα μή διὰ τοῦ κέντρου τὴν ΑΓ πρὸς όρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει· ἡ ΑΖ ἄρα τῇ ΖΓ ἐστιν ἴση. καὶ ἐπεὶ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ζ σημεῖον, πρόσκειται δὲ αὐτῆ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ άπὸ τῆς ΖΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΖΕ· τὸ ẳρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τῶν ἀπὸ τῶν ΓZ , ZE ἴσον ἐστὶ τοῖς ἀπὸ τῶν ZΔ, ZE. τοῖς δὲ ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΓ· ὀρθὴ γὰρ [ἐστιν] ἡ ὑπὸ EZΓ [γωνία]· τοῖς δὲ ἀπὸ τῶν ΔZ , ZE ἴσον ἐστὶ τὸ ἀπὸ τῆς $\rm E\Delta^{.}$ τὸ ἄρα ὑπὸ τῶν $\rm A\Delta, \, \Delta\Gamma$ μετὰ τοῦ ἀπὸ τῆς $\rm E\Gamma$ ἴσον έστι τῷ ἀπὸ τῆς ΕΔ. ἴση δὲ ἡ ΕΓ τὴ ΕΒ· τὸ ἄρα ὑπὸ τῶν $A\Delta$, $\Delta\Gamma$ μετὰ τοῦ ἀπὸ τῆς EB ἴσον ἐστὶ τῷ ẳπὸ τῆς $E\Delta$. τῷ δὲ ἀπὸ τῆς $E\Delta$ ἴσα ἐστὶ τὰ ἀπὸ τῶν EB, $B\Delta$ · ὀρθὴ γὰρ ή ὑπὸ $ext{EB}\Delta$ γωνία· τὸ ἄρα ὑπὸ τῶν $ext{A}\Delta,\,\Delta\Gamma$ μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΕΒ, ΒΔ. χοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς ΕΒ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ.

Έαν άρα κύκλου ληφθῆ τι σημεῖον ἐκτός, καὶ ἀπ' αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτηται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

ELEMENTS BOOK 3

and DC plus the (square) on FB is equal to the (sum of the squares) on FB and BD. Let the (square) on FB have been subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on the tangent DB.

And so let DCA not be through the center of circle ABC, and let the center E have been found, and let EF have been drawn from E, perpendicular to AC[Prop. 1.12]. And let *EB*, *EC*, and *ED* have been joined. (Angle) EBD (is) thus a right-angle [Prop. 3.18]. And since some straight-line, EF, through the center, cuts some (other) straight-line, AC, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, AF is equal to FC. And since the straight-line AC is cut in half at point F, let CD have been added to it. Thus, the (rectangle contained) by AD and DC plus the (square) on FC is equal to the (square) on FD [Prop. 2.6]. Let the (square) on FE have been added to both. Thus, the (rectangle contained) by AD and DC plus the (sum of the squares) on CF and FE is equal to the (sum of the squares) on FD and FE. But the (square) on EC is equal to the (sum of the squares) on CF and FE. For [angle] *EFC* [is] a right-angle [Prop. 1.47]. And the (square) on ED is equal to the (sum of the squares) on DF and FE [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on EC is equal to the (square) on ED. And EC (is) equal to EB. Thus, the (rectangle contained) by AD and DC plus the (square) on EBis equal to the (square) on ED. And the (sum of the squares) on EB and BD is equal to the (square) on ED. For *EBD* (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on EB is equal to the (sum of the squares) on EB and BD. Let the (square) on EB have been subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on BD.

Thus, if some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and (the other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line). (Which is) the very thing it was required to show.

Proposition 37

If some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-

Έὰν χύχλου ληφθῆ τι σημεῖον ἐχτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν χύχλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν χύχλον, ἡ δὲ προσπίπτῃ, ῇ δὲ τὸ ὑπὸ [τῆς] ὅλης τῆς τεμνούσης καὶ τῆς ἐχτὸς ἀπολαμβανομένης μεταξύ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἡ προσπίπτουσα ἐφάψεται τοῦ κύκλου.



Κύχλου γὰρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐχτὸς τὸ Δ, χαὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύχλον προσπιπτέτωσαν δύο εὐθεῖαι αἱ ΔΓΑ, ΔΒ, χαὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν κύχλον, ἡ δὲ ΔΒ προσπιπτέτω, ἔστω δὲ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ. λέγω, ὅτι ἡ ΔΒ ἐφάπτεται τοῦ ΑΒΓ κύχλου.

Ήχθω γὰρ τοῦ ΑΒΓ ἐφαπτομένη ή ΔΕ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν aἱ ZE, ZB, ZΔ. ἡ ἄρα ὑπὸ ZEΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ΔΕ ἐφάπτεται τοῦ ΑΒΓ κύκλου, τέμνει δὲ ἡ ΔΓΑ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΕ. ἤν δὲ καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσαι ἐστι δὲ mà ὑπὸ ΔΕΖ τῷ ΔΠ τῶν ἀλὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσαι ἐστὶ τῷ ἀπὸ τῆς ΔΒ· ἰση ἄρα ἡ ΔΕ τῆ ΔΒ. ἐστὶ δὲ καὶ ἡ ZE τῆ ZB ἴση· δύο δὴ αἱ ΔΕ, ΕΖ δύο ταῖς ΔB, BZ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ZΔ· γωνία ἄρα ἡ ὑπὸ ΔΕΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΔBZ. καί ἐστιν ἡ ZB ἐκβαλλομένη διάμετρος· ἡ δὲ τῆ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΔΒ ἄρα ἐφάπτεται τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δειχθήσεται, κἂν τὸ κέντρον ἐπὶ τῆς ΑΓ τυγχάνη.

Έὰν ἄρα χύχλου ληφθῆ τι σημεῖον ἐχτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν χύχλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν χύχλον, ἡ δὲ προσπίπτῃ, ἢ δὲ τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐχτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἡ προσπίπτουσα ἐφάψεται τοῦ χύχλου. ὅπερ ἔδει δεῖξαι. line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle.



For let some point D have been taken outside circle ABC, and let two straight-lines, DCA and DB, radiate from D towards circle ABC, and let DCA cut the circle, and let DB meet (the circle). And let the (rectangle contained) by AD and DC be equal to the (square) on DB. I say that DB touches circle ABC.

For let DE have been drawn touching ABC [Prop. 3.17], and let the center of the circle ABC have been found, and let it be (at) F. And let FE, FB, and FDhave been joined. (Angle) *FED* is thus a right-angle [Prop. 3.18]. And since DE touches circle ABC, and DCA cuts (it), the (rectangle contained) by AD and DCis thus equal to the (square) on DE [Prop. 3.36]. And the (rectangle contained) by AD and DC was also equal to the (square) on DB. Thus, the (square) on DE is equal to the (square) on DB. Thus, DE (is) equal to DB. And FE is also equal to FB. So the two (straight-lines) DE, EF are equal to the two (straight-lines) DB, BF (respectively). And their base, FD, is common. Thus, angle DEF is equal to angle DBF [Prop. 1.8]. And DEF (is) a right-angle. Thus, DBF (is) also a right-angle. And FB produced is a diameter, And a (straight-line) drawn at right-angles to a diameter of a circle, at its extremity, touches the circle [Prop. 3.16 corr.]. Thus, DB touches circle ABC. Similarly, (the same thing) can be shown, even if the center happens to be on AC.

Thus, if some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straightline) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle. (Which is) the very thing it

was required to show.
ELEMENTS BOOK 4

Construction of Rectilinear Figures In and Around Circles

Ὅροι.

α'. Σχῆμα εὐθύγραμμον εἰς σχῆμα εὐθύγραμμον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη τῶν τοῦ ἐγγραφομένου σχήματος γωνιῶν ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἄπτηται.

β'. Σχῆμα δὲ ὑμοίως περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἄπτηται.

γ'. Σχῆμα εὐθύγραμμον εἰς κύκλον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη γωνία τοῦ ἐγγραφομένου ἄπτηται τῆς τοῦ κύκλου περιφερείας.

δ'. Σχῆμα δὲ εὐθύγραμμον περὶ κύκλον περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἐφάπτηται τῆς τοῦ κύκλου περιφερείας.

ε'. Κύχλος δὲ εἰς σχῆμα ὑμοίως ἐγγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύχλου περιφέρεια ἑχάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἄπτηται.

γ΄. Κύχλος δὲ περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύχλου περιφέρεια ἐχάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἄπτηται.

ζ΄. Εὐθεῖα εἰς κύκλον ἐναρμόζεσθαι λέγεται, ὅταν τὰ πέρατα αὐτῆς ἐπὶ τῆς περιφερείας ἢ τοῦ κύκλου.

Definitions

1. A rectilinear figure is said to be inscribed in a(nother) rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the (figure) in which it is inscribed.

2. And, similarly, a (rectilinear) figure is said to be circumscribed about a(nother rectilinear) figure when the respective sides of the circumscribed (figure) touch the respective angles of the (figure) about which it is circumscribed.

3. A rectilinear figure is said to be inscribed in a circle when each angle of the inscribed (figure) touches the circumference of the circle.

4. And a rectilinear figure is said to be circumscribed about a circle when each side of the circumscribed (figure) touches the circumference of the circle.

5. And, similarly, a circle is said to be inscribed in a (rectilinear) figure when the circumference of the circle touches each side of the (figure) in which it is inscribed.

6. And a circle is said to be circumscribed about a rectilinear (figure) when the circumference of the circle touches each angle of the (figure) about which it is circumscribed.

7. A straight-line is said to be inserted into a circle when its extemities are on the circumference of the circle.

Proposition 1

To insert a straight-line equal to a given straight-line into a circle, (the latter straight-line) not being greater than the diameter of the circle.

D



Let ABC be the given circle, and D the given straightline (which is) not greater than the diameter of the circle. So it is required to insert a straight-line, equal to the straight-line D, into the circle ABC.

Let a diameter BC of circle ABC have been drawn.[†]

α΄.

Εἰς τὸν δοθέντα κύκλον τῆ δοθείσῃ εὐθεία μὴ μείζονι οὖσῃ τῆς τοῦ κύκλου διαμέτρου ἴσην εὐθεῖαν ἐναρμόσαι.



Έστω ὁ δοθεὶς κύκλος ὁ ABΓ, ἡ δὲ δοθεῖσα εὐθεῖα μὴ μείζων τῆς τοῦ κύκλου διαμέτρου ἡ Δ. δεῖ δὴ εἰς τὸν ABΓ κύκλον τῆ Δ εὐθεία ἴσην εὐθεῖαν ἐναρμόσαι.

Ηχθω τοῦ ABΓ κύκλου διάμετρος ή BΓ. εἰ μèν οὖν ἴση ἐστὶν ή BΓ τῆ Δ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· ἐνήρμοσται γὰρ εἰς τὸν ΑΒΓ κύκλον τῆ Δ εὐθεία ἴση ἡ ΒΓ. εἰ δὲ μείζων ἐστὶν ἡ ΒΓ τῆς Δ, κείσθω τῆ Δ ἴση ἡ ΓΕ, καὶ κέντρῳ τῷ Γ διαστήματι δὲ τῷ ΓΕ κύκλος γεγράφθω ὁ ΕΑΖ, καὶ ἐπεζεύχθω ἡ ΓΑ.

Ἐπεὶ οὖν το Γ σημεῖον κέντρον ἐστὶ τοῦ ΕΑΖ κύκλου, ἴση ἐστὶν ἡ ΓΑ τῆ ΓΕ. ἀλλὰ τῆ Δ ἡ ΓΕ ἐστιν ἴση· καὶ ἡ Δ ἀρα τῆ ΓΑ ἐστιν ἴση.

Εἰς ἄρα τὸν δοθέντα κύκλον τὸν ΑΒΓ τῆ δοθείση εὐθεία τῆ Δ ἴση ἐνήρμοσται ἡ ΓΑ· ὅπερ ἔδει ποιῆσαι. Therefore, if BC is equal to D then that (which) was prescribed has taken place. For the (straight-line) BC, equal to the straight-line D, has been inserted into the circle ABC. And if BC is greater than D then let CE be made equal to D [Prop. 1.3], and let the circle EAF have been drawn with center C and radius CE. And let CAhave been joined.

Therefore, since the point C is the center of circle EAF, CA is equal to CE. But, CE is equal to D. Thus, D is also equal to CA.

Thus, CA, equal to the given straight-line D, has been inserted into the given circle ABC. (Which is) the very thing it was required to do.

[†] Presumably, by finding the center of the circle [Prop. 3.1], and then drawing a line through it.

β'.

Εἰς τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.



Έστω ὁ δοθεὶς κύκλος ὁ ABΓ, τὸ δὲ δοθὲν τριγωνον τὸ ΔΕΖ· δεῖ δὴ εἰς τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῷ ἰσογώνιον τρίγωνον ἐγγράψαι.

Ήχθω τοῦ ABΓ κύκλου ἐφαπτομένη ἡ HΘ κατὰ τὸ A, καὶ συνεστάτω πρὸς τῆ AΘ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ A τῆ ὑπὸ ΔΕΖ γωνία ἴση ἡ ὑπὸ ΘΑΓ, πρὸς δὲ τῆ AH εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ A τῆ ὑπὸ ΔΖΕ [γωνία] ἴση ἡ ὑπὸ HAB, καὶ ἐπεζεύχθω ἡ BΓ.

Έπεὶ οὖν κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΑΘ, καὶ ἀπὸ τῆς κατὰ τὸ Α ἐπαφῆς εἰς τὸν κύκλον διῆκται εὐθεῖα ἡ ΑΓ, ἡ ἄρα ὑπὸ ΘΑΓ ἴση ἐστὶ τῆ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνία τῆ ὑπὸ ΑΒΓ. ἀλλ' ἡ ὑπὸ ΘΑΓ τῆ ὑπὸ ΔΕΖ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΒΓ ἄρα γωνία τῆ ὑπὸ ΔΕΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῆ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΑΓ λοιπῆ τῆ ὑπὸ ΕΔΖ ἐστιν ἰση καὶ λοιπὴ ἄρα ἡ ὑπὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἐγγέγραπται εἰς τὸν ΑΒΓ κύκλον].

Εἰς τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνω ἰσογώνιον τρίγωνον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 2

To inscribe a triangle, equiangular with a given triangle, in a given circle.



Let ABC be the given circle, and DEF the given triangle. So it is required to inscribe a triangle, equiangular with triangle DEF, in circle ABC.

Let GH have been drawn touching circle ABC at A.[†] And let (angle) HAC, equal to angle DEF, have been constructed on the straight-line AH at the point A on it, and (angle) GAB, equal to [angle] DFE, on the straightline AG at the point A on it [Prop. 1.23]. And let BChave been joined.

Therefore, since some straight-line AH touches the circle ABC, and the straight-line AC has been drawn across (the circle) from the point of contact A, (angle) HAC is thus equal to the angle ABC in the alternate segment of the circle [Prop. 3.32]. But, HAC is equal to DEF. Thus, angle ABC is also equal to DEF. So, for the same (reasons), ACB is also equal to DFE. Thus, the remaining (angle) BAC is equal to the remaining (angle) EDF [Prop. 1.32]. [Thus, triangle ABC is equiangular with triangle DEF, and has been inscribed in circle

ABC].

Thus, a triangle, equiangular with the given triangle, has been inscribed in the given circle. (Which is) the very thing it was required to do.

[†] See the footnote to Prop. 3.34.

γ΄.

Περὶ τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.



Έστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τρίγωνον τὸ ΔΕΖ· δεῖ δὴ περὶ τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῷ ἰσογώνιον τρίγωνον περιγράψαι.

Έκβεβλήσθω ή EZ ἐφ' ἑκάτερα τὰ μέρη κατὰ τὰ Η, Θ σημεῖα, καὶ εἰλήφθω τοῦ ABΓ κύκλου κέντρον τὸ Κ, καὶ διήχθω, ὡς ἔτυχεν, εὐθεῖα ἡ KB, καὶ συνεστάτω πρὸς τῆ KB εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Κ τῆ μὲν ὑπὸ ΔΕΗ γωνία ἴση ἡ ὑπὸ BKA, τῆ δὲ ὑπὸ ΔΖΘ ἴση ἡ ὑπὸ BKΓ, καὶ διὰ τῶν A, B, Γ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ABΓ κύκλου αἱ ΛAM, MBN, NΓΛ.

Καὶ ἐπεὶ ἐφάπτονται τοῦ ΑΒΓ χύχλου αἱ ΛΜ, ΜΝ, ΝΛ κατὰ τὰ Α, Β, Γ σημεῖα, ἀπὸ δὲ τοῦ Κ κέντρου ἐπὶ τὰ Α, Β, Γ σημεῖα ἐπεζευγμέναι εἰσὶν αἱ ΚΑ, ΚΒ, ΚΓ, ὀρθαὶ ἄρα εἰσὶν αἱ πρὸς τοῖς Α, Β, Γ σημείοις γωνίαι. καὶ ἐπεὶ τοῦ ΑΜΒΚ τετραπλεύρου αἱ τέσσαρες γωνίαι τέτρασιν ὀρθαῖς ἴσαι εἰσίν, ἐπειδήπερ καὶ εἰς δύο τρίγωνα διαιρεῖται τὸ ΑΜΒΚ, καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΚΑΜ, ΚΒΜ γωνίαι, λοιπαὶ ἄρα αἱ ὑπὸ ΑΚΒ, ΑΜΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΕΗ, ΔΕΖ δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΕΗ, ΔΕΖ ἴσαι εἰσίν, ῶν ἡ ὑπὸ ΑΚΒ τῆ ὑπὸ ΔΕΗ ἐστιν ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῆ τῆ ὑπὸ ΔΕΖ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἡ ὑπὸ ΛΝΒ τῆ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΜΛΝ [λοιπῆ] τῆ ὑπὸ ΕΔΖ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΑΝΝ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ περιγέγραπται περὶ τὸν ΑΒΓ κύχλον.

Περί τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῷ ἰσογώνιον τρίγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 3

To circumscribe a triangle, equiangular with a given triangle, about a given circle.



Let ABC be the given circle, and DEF the given triangle. So it is required to circumscribe a triangle, equiangular with triangle DEF, about circle ABC.

Let *EF* have been produced in each direction to points *G* and *H*. And let the center *K* of circle *ABC* have been found [Prop. 3.1]. And let the straight-line *KB* have been drawn, at random, across (*ABC*). And let (angle) *BKA*, equal to angle *DEG*, have been constructed on the straight-line *KB* at the point *K* on it, and (angle) *BKC*, equal to *DFH* [Prop. 1.23]. And let the (straight-lines) *LAM*, *MBN*, and *NCL* have been drawn through the points *A*, *B*, and *C* (respectively), touching the circle *ABC*.[†]

And since LM, MN, and NL touch circle ABC at points A, B, and C (respectively), and KA, KB, and KC are joined from the center K to points A, B, and C (respectively), the angles at points A, B, and C are thus right-angles [Prop. 3.18]. And since the (sum of the) four angles of quadrilateral AMBK is equal to four rightangles, inasmuch as AMBK (can) also (be) divided into two triangles [Prop. 1.32], and angles KAM and KBM are (both) right-angles, the (sum of the) remaining (angles), AKB and AMB, is thus equal to two right-angles. And DEG and DEF is also equal to two right-angles [Prop. 1.13]. Thus, AKB and AMB is equal to DEGand DEF, of which AKB is equal to DEG. Thus, the remainder AMB is equal to the remainder DEF. So, similarly, it can be shown that LNB is also equal to DFE. Thus, the remaining (angle) MLN is also equal to the LMN is equiangular with triangle DEF. And it has been drawn around circle ABC. Thus, a triangle, equiangular with the given triangle,

has been circumscribed about the given circle. (Which is) the very thing it was required to do.

[remaining] (angle) EDF [Prop. 1.32]. Thus, triangle

Proposition 4

To inscribe a circle in a given triangle.



Let ABC be the given triangle. So it is required to inscribe a circle in triangle ABC.

Let the angles ABC and ACB have been cut in half by the straight-lines BD and CD (respectively) [Prop. 1.9], and let them meet one another at point D, and let DE, DF, and DG have been drawn from point D, perpendicular to the straight-lines AB, BC, and CA (respectively) [Prop. 1.12].

And since angle ABD is equal to CBD, and the rightangle BED is also equal to the right-angle BFD, EBDand FBD are thus two triangles having two angles equal to two angles, and one side equal to one side—the (one) subtending one of the equal angles (which is) common to the (triangles)—(namely), BD. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, DE (is) equal to DF. So, for the same (reasons), DG is also equal to DF. Thus, the three straight-lines DE, DF, and DG are equal to one another. Thus, the circle drawn with center D, and radius one of E, F, or G^{\dagger} , will also go through the remaining points, and will touch the straight-lines AB, BC, and CA, on account of the angles at E, F, and G being right-angles. For if it cuts (one of) them then it will be a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, falling inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center D, and radius one of E, F,

[†] See the footnote to Prop. 3.34.

δ'.

Εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι.



Έστω τὸ δοθὲν τρίγωνον τὸ ΑΒΓ· δεῖ δὴ εἰς τὸ ΑΒΓ τρίγωνον χύχλον ἐγγράψαι.

Τετμήσθωσαν αί ὑπὸ ABΓ, AΓΒ γωνίαι δίχα ταῖς BΔ, ΓΔ εὐθείαις, καὶ συμβαλλέτωσαν ἀλλήλαις κατὰ τὸ Δ σημεῖον, καὶ ἤχθωσαν ἀπὸ τοῦ Δ ἐπὶ τὰς AB, BΓ, ΓΑ εὐθείας κάθετοι αί ΔΕ, ΔΖ, ΔΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΒΔ γωνία τῆ ὑπὸ ΓΒΔ, έστι δὲ και ὀρθή ή ὑπὸ ΒΕΔ ὀρθῆ τῆ ὑπὸ ΒΖΔ ἴση, δύο δή τρίγωνά ἐστι τὰ ΕΒΔ, ΖΒΔ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα χαὶ μίαν πλευρὰν μιῷ πλευρῷ ἴσην τὴν ύποτείνουσαν ύπὸ μίαν τῶν ἴσων γωνιῶν κοινὴν αὐτῶν τὴν $\mathrm{B}\Delta\cdot$ καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν· ἴση ἄρα ή ΔE τῆ ΔZ . διὰ τὰ αὐτὰ δὴ καὶ ή ΔH τῆ ΔZ ἐστιν ἴση.
 αἱ τρεῖς ἄρα εὐθεῖαι αἱ $\Delta E, \Delta Z, \Delta H$ ἴσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῷ τῷ Δ καὶ διαστήματι ἑνὶ τῶν Ε, Ζ, Η κύκλος γραφόμενος ἤξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΑ εὐθειῶν διὰ τὸ όρθὰς εἶναι τὰς πρὸς τοῖς Ε, Ζ, Η σημείοις γωνίας. εἰ γὰρ τεμεῖ αὐτάς, ἔσται ἡ τῆ διαμέτρω τοῦ κύκλου πρὸς ὀρθὰς άπ' ἄχρας ἀγομένη ἐντὸς πίπτουσα τοῦ χύχλου. ὅπερ ἄτοπον ἐδείχθη· οὐκ ἄρα ὁ κέντρ ω τ $\widetilde{\omega}$ Δ διαστήματι δὲ ἑνὶ τ $\widetilde{\omega}$ ν Ε, Ζ, Η γραφόμενος χύχλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΑ εὐθείας έφάψεται ἄρα αὐτῶν, καὶ ἔσται ὁ κύκλος ἐγγεγραμμένος εἰς τὸ ΑΒΓ τρίγωνον. ἐγγεγράφθω ὡς ὁ ΖΗΕ.

Εἰς ἄρα τὸ δοθὲν τρίγωνον τὸ ΑΒΓ κύκλος ἐγγέγραπται ὁ ΕΖΗ· ὅπερ ἔδει ποιῆσαι.

or G, does not cut the straight-lines AB, BC, and CA. Thus, it will touch them and will be the circle inscribed in triangle ABC. Let it have been (so) inscribed, like FGE (in the figure).

Thus, the circle EFG has been inscribed in the given triangle ABC. (Which is) the very thing it was required to do.

 † Here, and in the following propositions, it is understood that the radius is actually one of DE, DF, or DG.

ε΄.

Περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι.



Έστω τὸ δοθὲν τρίγωνον τὸ ABΓ· δεῖ δὲ περὶ τὸ δοθὲν τρίγωνον τὸ ABΓ κύκλον περιγράψαι.

Τετμήσθωσαν αί AB, AΓ εὐθεῖαι δίχα κατὰ τὰ Δ, Ε σημεῖα, καὶ ἀπὸ τῶν Δ, Ε σημείων ταῖς AB, AΓ πρὸς ὀρθὰς ἤχθωσαν αί ΔΖ, ΕΖ· συμπεσοῦνται δὴ ἤτοι ἐντὸς τοῦ ABΓ τριγώνου ἢ ἐπὶ τῆς BΓ εὐθείας ἢ ἐκτὸς τῆς BΓ.

Συμπιπτέτωσαν πρότερον ἐντὸς κατὰ τὸ Z, καὶ ἐπεζεύχθωσαν αἱ ZB, ZΓ, ZA. καὶ ἐπεὶ ἴση ἐστὶν ἡ AΔ τῆ ΔB, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔZ, βάσις ἄρα ἡ AZ βάσει τῆ ZB ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΓΖ τῆ AZ ἐστιν ἴση· ὥστε καὶ ἡ ZB τῆ ZΓ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ZA, ZB, ZΓ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρω τῷ Z διαστήματι δὲ ἑνὶ τῶν A, B, Γ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος ὁ κύκλος περὶ τὸ ABΓ τρίγωνον. περιγεγράφθω ὡς ὁ ABΓ.

Άλλὰ δὴ αἱ ΔΖ, ΕΖ συμπιπτέτωσαν ἐπὶ τῆς ΒΓ εὐθείας κατὰ τὸ Ζ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ ἐπεζεύχθω ἡ ΑΖ. ὑμοίως δὴ δείξομεν, ὅτι τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ περὶ τὸ ΑΒΓ τρίγωνον περιγραφομένου κύκλου.

Άλλὰ δὴ ai ΔΖ, ΕΖ συμπιπτέτωσαν ἐχτὸς τοῦ ΑΒΓ τριγώνου κατὰ τὸ Ζ πάλιν, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καί ἐπεζεύχθωσαν αi ΑΖ, ΒΖ, ΓΖ. καὶ ἐπεὶ πάλιν ἴση ἑστὶν ἡ ΑΔ τῆ ΔΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΖ, βάσις ἄρα ἡ ΑΖ βάσει τῆ ΒΖ ἐστιν ἴση. ὑμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΓΖ τῆ ΑΖ ἐστιν ἴση· ὥστε καὶ ἡ ΒΖ τῆ ΖΓ ἐστιν ἴση· ὁ ἄρα [πάλιν] κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν ΖΑ, ΖΒ, ΖΓ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓ τρίγωνον.

Περὶ τὸ δοθὲν ἄρα τρίγωνον κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 5

To circumscribe a circle about a given triangle.



Let ABC be the given triangle. So it is required to circumscribe a circle about the given triangle ABC.

Let the straight-lines AB and AC have been cut in half at points D and E (respectively) [Prop. 1.10]. And let DF and EF have been drawn from points D and E, at right-angles to AB and AC (respectively) [Prop. 1.11]. So (DF and EF) will surely either meet inside triangle ABC, on the straight-line BC, or beyond BC.

Let them, first of all, meet inside (triangle ABC) at (point) F, and let FB, FC, and FA have been joined. And since AD is equal to DB, and DF is common and at right-angles, the base AF is thus equal to the base FB[Prop. 1.4]. So, similarly, we can show that CF is also equal to AF. So that FB is also equal to FC. Thus, the three (straight-lines) FA, FB, and FC are equal to one another. Thus, the circle drawn with center F, and radius one of A, B, or C, will also go through the remaining points. And the circle will have been circumscribed about triangle ABC. Let it have been (so) circumscribed, like ABC (in the first diagram from the left).

And so, let DF and EF meet on the straight-line BC at (point) F, like in the second diagram (from the left). And let AF have been joined. So, similarly, we can show that point F is the center of the circle circumscribed about triangle ABC.

And so, let DF and EF meet outside triangle ABC, again at (point) F, like in the third diagram (from the left). And let AF, BF, and CF have been joined. And, again, since AD is equal to DB, and DF is common and at right-angles, the base AF is thus equal to the base BF [Prop. 1.4]. So, similarly, we can show that CF is also equal to AF. So that BF is also equal to FC. Thus,

В

[again] the circle drawn with center F, and radius one of FA, FB, and FC, will also go through the remaining points. And it will have been circumscribed about triangle ABC.

Thus, a circle has been circumscribed about the given triangle. (Which is) the very thing it was required to do.

Proposition 6

To inscribe a square in a given circle.



Έστω ή δοθείς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ Le κύκλον τετράγωνον ἐγγράψαι.

Ήχθωσαν τοῦ ΑΒΓΔ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΑΓ, ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ.

ج'.

Е

Δ

Εἰς τὸν δοθέντα κύκλον τετράγωνον ἐγγράψαι.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῆ ΕΔ· κέντρον γὰρ τὸ Ε· κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΑ, βάσις ἄρα ἡ ΑΒ βάσει τῆ ΑΔ ἴση ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ΒΓ, ΓΔ ἑκατέρα τῶν ΑΒ, ΑΔ ἴση ἐστίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔ τετράπλευρον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ ἡ ΒΔ εὐθεῖα διάμετρός ἐστι τοῦ ΑΒΓΔ κύκλου, ἡμικύκλιον ἄρα ἐστὶ τὸ ΒΑΔ· ὀρθὴ ἄρα ἡ ὑπὸ ΒΑΔ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ ὀρθή ἐστιν· ὀρθογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔ τετράπλευρον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν. καὶ ἐγγέγραπται εἰς τὸν ΑΒΓΔ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον τετράγωνον ἐγγέγραπται τὸ ΑΒΓΔ· ὅπερ ἔδει ποιῆσαι. Let ABCD be the given circle. So it is required to inscribe a square in circle ABCD.

Let two diameters of circle ABCD, AC and BD, have been drawn at right-angles to one another.[†] And let AB, BC, CD, and DA have been joined.

And since BE is equal to ED, for E (is) the center (of the circle), and EA is common and at right-angles, the base AB is thus equal to the base AD [Prop. 1.4]. So, for the same (reasons), each of BC and CD is equal to each of AB and AD. Thus, the quadrilateral ABCDis equilateral. So I say that (it is) also right-angled. For since the straight-line BD is a diameter of circle ABCD, BAD is thus a semi-circle. Thus, angle BAD (is) a rightangle [Prop. 3.31]. So, for the same (reasons), (angles) ABC, BCD, and CDA are also each right-angles. Thus, the quadrilateral ABCD is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been inscribed in circle ABCD.

Thus, the square ABCD has been inscribed in the given circle. (Which is) the very thing it was required to do.

[†] Presumably, by finding the center of the circle [Prop. 3.1], drawing a line through it, and then drawing a second line through it, at right-angles to the first [Prop. 1.11].

Περὶ τὸν δοθέντα κύκλον τετράγωνον περιγράψαι.

Proposition 7

To circumscribe a square about a given circle.

Έστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ περὶ τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψαι.



Ήχθωσαν τοῦ ΑΒΓΔ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΑΓ, ΒΔ, καὶ διὰ τῶν Α, Β, Γ, Δ σημείων ἦχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓΔ κύκλου αἱ ΖΗ, ΗΘ, ΘΚ, ΚΖ.

Ἐπεὶ οὖν ἐφάπτεται ἡ ΖΗ τοῦ ΑΒΓΔ κύκλου, ἀπὸ δὲ τοῦ Ε κέντρου ἐπὶ τὴν κατὰ τὸ Α ἐπαφὴν ἐπέζευκται ἡ ΕΑ, αί ἄρα πρὸς τῷ Α γωνίαι ὀρθαί εἰσιν. διὰ τὰ αὐτὰ δή και αί πρός τοῖς Β, Γ, Δ σημείοις γωνίαι ὀρθαί εἰσιν. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΑΕΒ γωνία, ἐστὶ δὲ ὀρθὴ καὶ ἡ ύπὸ ΕΒΗ, παράλληλος ἄρα ἐστιν ἡ ΗΘ τῆ ΑΓ. διὰ τὰ αὐτὰ δή καὶ ἡ ΑΓ τῆ ΖΚ ἐστι παράλληλος. ὥστε καὶ ἡ ΗΘ τῆ ΖΚ ἐστι παράλληλος. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΖ, ΘΚ τῆ ΒΕΔ ἐστι παράλληλος. παραλληλόγραμμα ἄρα ἐστὶ τὰ ΗΚ, ΗΓ, ΑΚ, ΖΒ, ΒΚ· ἴση ἄρα ἐστὶν ἡ μὲν ΗΖ τῆ ΘΚ, ἡ δὲ ΗΘ τῆ ΖΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ $B\Delta$, ἀλλὰ καὶ ἡ μὲν ΑΓ ἑκατέρ
α τῶν HΘ, ZK, ἡ δὲ $B\Delta$ έκατέρα τῶν ΗΖ, ΘΚ ἐστιν ἴση [καὶ ἑκατέρα ἄρα τῶν ΗΘ, ΖΚ ἑκατέρα τῶν ΗΖ, ΘΚ ἐστιν ἴση], ἰσόπλευρον ἄρα ἐστὶ τὸ ΖΗΘΚ τετράπλευρον. λέγω δή, ὄτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΗΒΕΑ, καί ἐστιν ὀρθή ἡ ύπὸ AEB, ὀρθὴ ἄρα καὶ ἡ ὑπὸ AHB. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ πρὸς τοῖς $\Theta, \mathrm{K}, \mathrm{Z}$ γωνίαι ὀρθαί εἰσιν. ὀρθογώνιον άρα ἐστὶ τὸ ΖΗΘΚ. ἐδείχϑη δὲ καὶ ἰσόπλευρον· τετράγωνον άρα ἐστίν. καὶ περιγέγραπται περὶ τὸν ΑΒΓΔ κύκλον.

Περὶ τὸν δοθέντα ἄρα κύκλον τετράγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

Let ABCD be the given circle. So it is required to circumscribe a square about circle ABCD.



Let two diameters of circle *ABCD*, *AC* and *BD*, have been drawn at right-angles to one another.[†] And let *FG*, *GH*, *HK*, and *KF* have been drawn through points *A*, *B*, *C*, and *D* (respectively), touching circle *ABCD*.[‡]

Therefore, since FG touches circle ABCD, and EAhas been joined from the center E to the point of contact A, the angles at A are thus right-angles [Prop. 3.18]. So, for the same (reasons), the angles at points B, C, and D are also right-angles. And since angle AEB is a rightangle, and EBG is also a right-angle, GH is thus parallel to AC [Prop. 1.29]. So, for the same (reasons), AC is also parallel to FK. So that GH is also parallel to FK[Prop. 1.30]. So, similarly, we can show that GF and HK are each parallel to BED. Thus, GK, GC, AK, FB, and BK are (all) parallelograms. Thus, GF is equal to HK, and GH to FK [Prop. 1.34]. And since AC is equal to BD, but AC (is) also (equal) to each of GH and FK, and BD is equal to each of GF and HK [Prop. 1.34] [and each of GH and FK is thus equal to each of GFand HK], the quadrilateral FGHK is thus equilateral. So I say that (it is) also right-angled. For since *GBEA* is a parallelogram, and AEB is a right-angle, AGB is thus also a right-angle [Prop. 1.34]. So, similarly, we can show that the angles at H, K, and F are also right-angles. Thus, FGHK is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been circumscribed about circle ABCD.

Thus, a square has been circumscribed about the given circle. (Which is) the very thing it was required to do.

[†] See the footnote to the previous proposition.

[‡] See the footnote to Prop. 3.34.

η΄.

Εἰς τὸ δοθὲν τετράγωνον κύκλον ἐγγράψαι.

Έστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ. δεῖ δὴ εἰς τὸ ΑΒΓΔ τετράγωνον χύχλον ἐγγράψαι.



Τετμήσθω ἑκατέρα τῶν ΑΔ, ΑΒ δίχα κατὰ τὰ Ε, Ζ σημεῖα, καὶ διὰ μὲν τοῦ Ε ὁποτέρα τῶν ΑΒ, ΓΔ παράλληλος ἦχθω ὁ ΕΘ, διὰ δὲ τοῦ Ζ ὁποτέρα τῶν Α $\Delta,$ ΒΓ παράλληλος ήχθω ή ZK· παραλληλόγραμμον άρα ἐστὶν ἕκαστον τῶν AK, KB, A Θ , $\Theta\Delta$, AH, H Γ , BH, H Δ , $\varkappa\alpha$ ì α ì $\dot{\alpha}\pi\epsilon\nu\alpha\nu\tau$ ίον αὐτῶν πλευραὶ δηλονότι ἴσαι [εἰσίν]. καὶ ἐπεὶ ἴση ἐστὶν ἡ $A\Delta$ τῆ AB, καί ἐστι τῆς μὲν $A\Delta$ ἡμίσεια ἡ AE, τῆς δὲ AB ἡμίσεια ἡ AZ, ἴση ἄρα καὶ ἡ ΑΕ τῆ AZ· ὥστε καὶ αἱ ἀπεναντίον ἴση ἄρα καὶ ἡ ΖΗ τῆ ΗΕ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΘ, ΗΚ ἑκατέρα τῶν ΖΗ, ΗΕ ἐστιν ἴση· αί τέσσαρες ἄρα αί ΗΕ, ΗΖ, ΗΘ, ΗΚ ἴσαι ἀλλήλαις [εἰσίν]. ό ἄρα χέντρω μέν τῷ Η διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος ήξει καὶ διὰ τῶν λοιπῶν σημείων· καὶ έφάψεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς $\mathbf{E}, \mathbf{Z}, \Theta, \mathbf{K}$ γωνίας· εἰ γὰρ τεμεῖ ὁ κύκλος τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ, ή τῆ διαμέτρω τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄχρας ἀγομένη ἐντὸς πεσεῖται τοῦ χύχλου· ὅπερ ἄτοπον έδείχθη. οὐκ ἄρα ὁ κέντρῳ τῷ Η διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ εύθείας. ἐφάψεται ἄρα αὐτῶν καὶ ἔσται ἐγγεγραμμένος εἰς τὸ ΑΒΓΔ τετράγωνον.

Εἰς ἄρα τὸ δοθὲν τετράγωνον κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

θ'.

Περὶ τὸ δοθὲν τετράγωνον χύχλον περιγράψαι.

Έστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ· δεῖ δὴ περὶ τὸ ΑΒΓΔ τετράγωνον κύκλον περιγράψαι.

Proposition 8

To inscribe a circle in a given square. Let the given square be ABCD. So it is required to inscribe a circle in square ABCD.



Let AD and AB each have been cut in half at points Eand F (respectively) [Prop. 1.10]. And let EH have been drawn through E, parallel to either of AB or CD, and let FK have been drawn through F, parallel to either of ADor BC [Prop. 1.31]. Thus, AK, KB, AH, HD, AG, GC, BG, and GD are each parallelograms, and their opposite sides [are] manifestly equal [Prop. 1.34]. And since AD is equal to AB, and AE is half of AD, and AF half of AB, AE (is) thus also equal to AF. So that the opposite (sides are) also (equal). Thus, FG (is) also equal to GE. So, similarly, we can also show that each of GH and GKis equal to each of FG and GE. Thus, the four (straightlines) GE, GF, GH, and GK [are] equal to one another. Thus, the circle drawn with center G, and radius one of E, F, H, or K, will also go through the remaining points. And it will touch the straight-lines AB, BC, CD, and DA, on account of the angles at E, F, H, and K being right-angles. For if the circle cuts AB, BC, CD, or DA, then a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, will fall inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center G, and radius one of E, F, H, or K, does not cut the straight-lines AB, BC, CD, or DA. Thus, it will touch them, and will have been inscribed in the square *ABCD*.

Thus, a circle has been inscribed in the given square. (Which is) the very thing it was required to do.

Proposition 9

To circumscribe a circle about a given square.

Let ABCD be the given square. So it is required to circumscribe a circle about square ABCD.

Έπιζευχθεΐσαι γὰρ αί ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε.



Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῆ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δή αί ΔΑ, ΑΓ δυσὶ ταῖς ΒΑ, ΑΓ ἴσαι εἰσίν καὶ βάσις ἡ ΔΓ βάσει τῆ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνία τῆ ὑπὸ ΒΑΓ ἴση ἐστίν· ἡ ἄρα ὑπὸ ΔΑΒ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΓ. ὑμοίως δὴ δείξομεν, ὅτι καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ δίχα τέτμηται ὑπὸ τῶν ΑΓ, ΔΒ εὐθειῶν. καὶ έπει ἴση ἐστιν ἡ ὑπὸ ΔΑΒ γωνία τῆ ὑπὸ ΑΒΓ, καί ἐστι τῆς μὲν ὑπὸ ΔΑΒ ἡμίσεια ἡ ὑπὸ ΕΑΒ, τῆς δὲ ὑπὸ ΑΒΓ ήμίσεια ή ὑπὸ EBA, καὶ ή ὑπὸ EAB ἄρα τῆ ὑπὸ EBA ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΕΑ τῆ ΕΒ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὄτι καὶ ἑκατέρα τῶν ΕΑ, ΕΒ [εὐθειῶν] ἑκατέρα τῶν ΕΓ, ΕΔ ἴση ἐστίν. αἱ τέσσαρες ἄρα αἱ ΕΑ, ΕΒ, ΕΓ, $\mathbf{E}\Delta$ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρω τῶ \mathbf{E} καὶ διαστήματι ένὶ τῶν Α, Β, Γ, Δ κύκλος γραφόμενος ἤξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓΔ τετράγωνον. περιγεγράφθω ὡς ὁ ΑΒΓΔ.

Περὶ τὸ δοθὲν ἄρα τετράγωνον κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

ι΄.

Ίσοσκελὲς τρίγωνον συστήσασθαι ἔχον ἑκατέραν τῶν πρὸς τῆ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς.

Έχχείσθω τις εύθεῖα ἡ AB, καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε τὸ ὑπὸ τῶν AB, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τῆς ΓΑ τετραγώνω· καὶ κέντρω τῷ Α καὶ διαστήματι τῷ AB κύκλος γεγράφθω ὁ BΔΕ, καὶ ἐνηρμόσθω εἰς τὸν BΔΕ κύκλον τῆ ΑΓ εὐθεία μὴ μείζονι οὖση τῆς τοῦ BΔΕ κύκλου διαμέτρου ἴση εὐθεῖα ἡ BΔ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, καὶ περιγεγράφθω περὶ τὸ ΑΓΔ τρίγωνον κύκλος ὁ ΑΓΔ. AC and BD being joined, let them cut one another at E.



And since DA is equal to AB, and AC (is) common, the two (straight-lines) DA, AC are thus equal to the two (straight-lines) BA, AC. And the base DC (is) equal to the base BC. Thus, angle DAC is equal to angle BAC[Prop. 1.8]. Thus, the angle DAB has been cut in half by AC. So, similarly, we can show that ABC, BCD, and CDA have each been cut in half by the straight-lines ACand DB. And since angle DAB is equal to ABC, and EAB is half of DAB, and EBA half of ABC, EAB is thus also equal to EBA. So that side EA is also equal to EB [Prop. 1.6]. So, similarly, we can show that each of the [straight-lines] EA and EB are also equal to each of EC and ED. Thus, the four (straight-lines) EA, EB, EC, and ED are equal to one another. Thus, the circle drawn with center E, and radius one of A, B, C, or D, will also go through the remaining points, and will have been circumscribed about the square ABCD. Let it have been (so) circumscribed, like *ABCD* (in the figure).

Thus, a circle has been circumscribed about the given square. (Which is) the very thing it was required to do.

Proposition 10

To construct an isosceles triangle having each of the angles at the base double the remaining (angle).

Let some straight-line AB be taken, and let it have been cut at point C so that the rectangle contained by AB and BC is equal to the square on CA [Prop. 2.11]. And let the circle BDE have been drawn with center A, and radius AB. And let the straight-line BD, equal to the straight-line AC, being not greater than the diameter of circle BDE, have been inserted into circle BDE[Prop. 4.1]. And let AD and DC have been joined. And let the circle ACD have been circumscribed about triangle ACD [Prop. 4.5].



Καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, ἴση δὲ ἡ ΑΓ τῆ ΒΔ, τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΓ ἴσον έστι τῷ ἀπὸ τῆς ΒΔ. και ἐπει κύκλου τοῦ ΑΓΔ εἴληπταί τι σημεῖον ἐκτὸς τὸ Β, καὶ ἀπὸ τοῦ Β πρὸς τὸν ΑΓΔ κύκλον προσπεπτώχασι δύο εὐθεῖαι αἱ ΒΑ, ΒΔ, καὶ ἡ μὲν αὐτῶν τέμνει, ή δὲ προσπίπτει, χαί ἐστι τὸ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τῷ ἀπὸ τῆς ΒΔ, ἡ ΒΔ ἄρα ἐφάπτεται τοῦ ΑΓΔ κύκλου. ἐπεὶ ούν ἐφάπτεται μὲν ἡ ΒΔ, ἀπὸ δὲ τῆς κατὰ τὸ Δ ἐπαφῆς διῆχται ή $\Delta \Gamma$, ή ἄρα ὑπὸ $B\Delta \Gamma$ γωνιά ἴση ἐστὶ τῆ ἐν τῷ έναλλὰξ τοῦ κύκλου τμήματι γωνία τῆ ὑπὸ ΔΑΓ. ἐπεὶ οὖν ίση ἐστὶν ἡ ὑπὸ ΒΔΓ τῇ ὑπὸ ΔΑΓ, κοινὴ προσκείσθω ἡ ύπὸ ΓΔΑ· ὅλη ἄρα ἡ ὑπὸ ΒΔΑ ἴση ἐστὶ δυσὶ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ. ἀλλὰ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ ἴση ἐστὶν ἡ ἐκτὸς ἡ ὑπὸ ΒΓΔ· καὶ ἡ ὑπὸ ΒΔΑ ἄρα ἴση ἐστὶ τῆ ὑπὸ ΒΓΔ. ἀλλὰ ἡ ύπὸ ΒΔΑ τῆ ὑπὸ ΓΒΔ ἐστιν ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΑΔ τῆ ΑΒ ἐστιν ἴση· ὥστε καὶ ἡ ὑπὸ ΔΒΑ τῆ ὑπὸ ΒΓΔ ἐστιν ίση. αί τρεῖς ἄρα αί ὑπὸ ΒΔΑ, ΔΒΑ, ΒΓΑ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῆ ὑπὸ ΒΓΔ, ἴση ἐστὶ καὶ πλευρὰ ἡ
 ${\rm B}\Delta$ πλευρῷ τῆ
 $\Delta\Gamma.$ ἀλλὰ ἡ
 ${\rm B}\Delta$ τῆ ΓΑ ὑπόκειται ἴση· καὶ ἡ ΓΑ ἄρα τῆ ΓΔ ἐστιν ἴση· ὥστε καὶ γωνία ἡ ὑπὸ ΓΔΑ γωνία τῆ ὑπὸ ΔΑΓ ἐστιν ἴση· αἱ ẳρα ύπὸ ΓΔΑ, ΔΑΓ τῆς ὑπὸ ΔΑΓ εἰσι διπλασίους. ἴση δὲ ἡ ύπὸ ΒΓΔ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ· καὶ ἡ ὑπὸ ΒΓΔ ἄρα τῆς ὑπὸ ΓΑΔ ἐστι διπλῆ. ἴση δὲ ἡ ὑπὸ ΒΓΔ ἑκατέρα τῶν ὑπὸ ΒΔΑ, ΔΒΑ· καὶ ἑκατέρα ἄρα τῶν ὑπὸ ΒΔΑ, ΔΒΑ τῆς ὑπὸ ΔΑΒ έστι διπλη.

Ίσοσχελὲς ἄρα τρίγωνον συνέσταται τὸ ABΔ ἔχον ἑχατέραν τῶν πρὸς τῆ ΔΒ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς ὅπερ ἔδει ποιῆσαι.



And since the (rectangle contained) by AB and BCis equal to the (square) on AC, and AC (is) equal to BD, the (rectangle contained) by AB and BC is thus equal to the (square) on BD. And since some point Bhas been taken outside of circle ACD, and two straightlines BA and BD have radiated from B towards the circle ACD, and (one) of them cuts (the circle), and (the other) meets (the circle), and the (rectangle contained) by AB and BC is equal to the (square) on BD, BD thus touches circle ACD [Prop. 3.37]. Therefore, since BD touches (the circle), and DC has been drawn across (the circle) from the point of contact D, the angle BDC is thus equal to the angle DAC in the alternate segment of the circle [Prop. 3.32]. Therefore, since BDC is equal to DAC, let CDA have been added to both. Thus, the whole of BDA is equal to the two (angles) CDA and DAC. But, the external (angle) BCD is equal to CDAand DAC [Prop. 1.32]. Thus, BDA is also equal to BCD. But, BDA is equal to CBD, since the side AD is also equal to AB [Prop. 1.5]. So that DBA is also equal to BCD. Thus, the three (angles) BDA, DBA, and BCD are equal to one another. And since angle DBC is equal to BCD, side BD is also equal to side DC [Prop. 1.6]. But, BD was assumed (to be) equal to CA. Thus, CAis also equal to CD. So that angle CDA is also equal to angle DAC [Prop. 1.5]. Thus, CDA and DAC is double DAC. But BCD (is) equal to CDA and DAC. Thus, BCD is also double CAD. And BCD (is) equal to to each of BDA and DBA. Thus, BDA and DBA are each double DAB.

Thus, the isosceles triangle ABD has been constructed having each of the angles at the base BD double the remaining (angle). (Which is) the very thing it was required to do.

ια'.

Εἰς τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ

Proposition 11

To inscribe an equilateral and equiangular pentagon





Έστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Έχχείσθω τρίγωνον ἰσοσχελὲς τὸ ΖΗΘ διπλασίονα ἔχον ἑχατέραν τῶν πρὸς τοῖς Η, Θ γωνιῶν τῆς πρὸς τῷ Ζ, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔΕ χύχλον τῷ ΖΗΘ τριγώνῳ ἰσογώνιον τρίγωνον τὸ ΑΓΔ, ὥστε τῆ μὲν πρὸς τῷ Ζ γωνία ἴσην εἶναι τὴν ὑπὸ ΓΑΔ, ἑχατέραν δὲ τῶν πρὸς τοῖς Η, Θ ἴσην ἑχατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ· καὶ ἑχατέρα ἄρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ τῆς ὑπὸ ΓΑΔ ἐστι διπλῆ. τετμήσθω δὴ ἑχατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ δίχα ὑπὸ ἑχατέρας τῶν ΓΕ, ΔΒ εὐθειῶν, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΔΕ, ΕΑ.

Έπεὶ οὖν ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ γωνιῶν διπλασίων ἐστὶ τῆς ὑπὸ ΓΑΔ, καὶ τετμημέναι εἰσὶ δίχα ὑπὸ τῶν ΓΕ, ΔB εὐθειῶν, αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ $\Delta A \Gamma$, ΑΓΕ, ΕΓΔ, ΓΔΒ, ΒΔΑ ἴσαι ἀλλήλαις εἰσίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήχασιν· αἱ πέντε ἄρα περιφέρειαι αί ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ίσαι ἀλλήλαις εἰσίν. ὑπὸ δὲ τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ πέντε άρα εὐθεῖαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσίν ίσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. λέγω δή, ŏτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἡ AB περιφέρεια τῆ ΔΕ περιφερεία ἐστίν ἴση, κοινὴ προσκείσθω ἡ ΒΓΔ· ὅλη ἄρα ἡ ΑΒΓΔ περιφέρια ὅλη τῆ ΕΔΓΒ περιφερεία ἐστὶν ἴση. καὶ βέβηχεν ἐπὶ μὲν τῆς ΑΒΓΔ περιφερείας γωνία ἡ ὑπὸ ΑΕΔ, έπι δὲ τῆς ΕΔΓΒ περιφερείας γωνία ἡ ὑπὸ ΒΑΕ· καὶ ἡ ὑπὸ ΒΑΕ ἄρα γωνία τῆ ὑπὸ ΑΕΔ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΕ γωνιῶν ἑκατέρα τῶν ύπὸ ΒΑΕ, ΑΕΔ ἐστιν ἴση ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. έδείχθη δὲ καὶ ἰσόπλευρον.

Εἰς ἄρα τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

ıβ'.

Περὶ τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι. in a given circle.



Let *ABCDE* be the given circle. So it is required to inscribed an equilateral and equiangular pentagon in circle *ABCDE*.

Let the the isosceles triangle FGH be set up having each of the angles at G and H double the (angle) at F [Prop. 4.10]. And let triangle ACD, equiangular to FGH, have been inscribed in circle ABCDE, such that CAD is equal to the angle at F, and the (angles) at G and H (are) equal to ACD and CDA, respectively [Prop. 4.2]. Thus, ACD and CDA are each double CAD. So let ACD and CDA have been cut in half by the straight-lines CE and DB, respectively [Prop. 1.9]. And let AB, BC, DE and EA have been joined.

Therefore, since angles ACD and CDA are each double CAD, and are cut in half by the straight-lines CE and DB, the five angles DAC, ACE, ECD, CDB, and BDA are thus equal to one another. And equal angles stand upon equal circumferences [Prop. 3.26]. Thus, the five circumferences AB, BC, CD, DE, and EA are equal to one another [Prop. 3.29]. Thus, the pentagon ABCDE is equilateral. So I say that (it is) also equiangular. For since the circumference AB is equal to the circumference DE, let BCD have been added to both. Thus, the whole circumference ABCD is equal to the whole circumference EDCB. And the angle AED stands upon circumference ABCD, and angle BAE upon circumference EDCB. Thus, angle BAE is also equal to AED[Prop. 3.27]. So, for the same (reasons), each of the angles ABC, BCD, and CDE is also equal to each of BAEand AED. Thus, pentagon ABCDE is equiangular. And it was also shown (to be) equilateral.

Thus, an equilateral and equiangular pentagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

Proposition 12

To circumscribe an equilateral and equiangular pentagon about a given circle.



Έστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὲ περὶ τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

Νενοήσθω τοῦ ἐγγεγραμμένου πενταγώνου τῶν γωνιῶν σημεῖα τὰ Α, Β, Γ, Δ, Ε, ὥστε ἴσας εἶναι τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ περιφερείας· καὶ διὰ τῶν Α, Β, Γ, Δ, Ε ἤχθωσαν τοῦ κύκλου ἐφαπτόμεναι αἱ ΗΘ, ΘΚ, ΚΛ, ΛΜ, ΜΗ, καὶ εἰλήφθω τοῦ ΑΒΓΔΕ κύκλου κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἰ ΖΒ, ΖΚ, ΖΓ, ΖΛ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ΚΛ εὐθεῖα ἐφάπτεται τοῦ ΑΒΓΔΕ κατὰ τὸ Γ, ἀπὸ δὲ τοῦ Ζ κέντρου ἐπὶ τὴν κατὰ τὸ Γ ἐπαφὴν ἐπέζευχται ή ZΓ, ή ZΓ ἄρα χάθετός ἐστιν ἐπὶ τὴν KΛ· ὀρθή άρα ἐστὶν ἑχατέρα τῶν πρὸς τῷ Γ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς B, Δ σημείοις γωνίαι ὀρθαί εἰσιν. καὶ ἐπεὶ όρθή ἐστιν ἡ ὑπὸ ΖΓΚ γωνία, τὸ ἄρα ἀπὸ τῆς ΖΚ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΓ, ΓΚ. διὰ τὰ αὐτὰ δὴ καὶ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΚ· ὥστε τὰ ἀπὸ τῶν ΖΓ, ΓΚ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἐστιν ἴσα, ῶν τὸ ἀπὸ τῆς ΖΓ τῷ ἀπὸ τῆς ΖΒ ἐστιν ἴσον· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΓΚ τῷ ἀπὸ τῆς ΒΚ ἐστιν ἴσον. ἴση ἄρα ἡ ΒΚ τῇ ΓΚ. καὶ ἐπεὶ ἴση ἐστὶν ή ZB τῆ ZΓ, καὶ κοινὴ ή ZK, δύο δὴ αἱ BZ, ZK δυσὶ ταῖς ΓΖ, ΖΚ ἴσαι εἰσίν· καὶ βάσις ἡ ΒΚ βάσει τῆ ΓΚ [ἐστιν] ἴση· γωνία ἄρα ή μέν ὑπὸ ΒΖΚ [γωνία] τῆ ὑπὸ ΚΖΓ ἐστιν ἴση· ή δὲ ὑπὸ BKZ τῆ ὑπὸ ZKΓ· διπλῆ ἄρα ἡ μὲν ὑπὸ BZΓ τῆς ύπὸ ΚΖΓ, ἡ δὲ ὑπὸ ΒΚΓ τῆς ὑπὸ ΖΚΓ. διὰ τὰ αὐτὰ δὴ καὶ ή μεν ύπο ΓΖΔ τῆς ὑπο ΓΖΛ ἐστι διπλῆ, ή δὲ ὑπο ΔΛΓ τῆς ὑπὸ ΖΑΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ περιφέρεια τῆ ΓΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΖΓ τῆ ὑπὸ ΓΖΔ. καί ἐστιν ἡ μέν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ διπλῆ, ἡ δὲ ὑπὸ ΔΖΓ τῆς ὑπὸ ΛΖΓ· ἴση ἄρα καὶ ἡ ὑπὸ ΚΖΓ τῇ ὑπὸ ΛΖΓ· ἐστὶ δὲ καὶ ἡ ύπὸ ΖΓΚ γωνία τῆ ὑπὸ ΖΓΛ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΖΚΓ, ΖΛΓ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιῷ πλευρῷ ἴσην κοινὴν αὐτῶν τὴν ΖΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει καὶ τὴν λοιπὴν γωνίαν τῆ λοιπῆ γωνία. ἴση ἄρα ἡ μὲν ΚΓ εύθεῖα τῆ ΓΛ, ἡ δὲ ὑπὸ ΖΚΓ γωνία τῆ ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση έστιν ή ΚΓ τῆ ΓΛ, διπλῆ ἄρα ή ΚΛ τῆς ΚΓ. διὰ τὰ αὐτα δὴ δειχθήσεται καὶ ἡ ΘΚ τῆς ΒΚ διπλῆ. καί ἐστιν ἡ ΒΚ τῆ ΚΓ ίση καὶ ἡ ΘΚ ἄρα τῆ ΚΛ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται



Let *ABCDE* be the given circle. So it is required to circumscribe an equilateral and equiangular pentagon about circle *ABCDE*.

Let A, B, C, D, and E have been conceived as the angular points of a pentagon having been inscribed (in circle ABCDE) [Prop. 3.11], such that the circumferences AB, BC, CD, DE, and EA are equal. And let GH, HK, KL, LM, and MG have been drawn through (points) A, B, C, D, and E (respectively), touching the circle.[†] And let the center F of the circle ABCDE have been found [Prop. 3.1]. And let FB, FK, FC, FL, and FD have been joined.

And since the straight-line KL touches (circle) ABCDE at C, and FC has been joined from the center F to the point of contact C, FC is thus perpendicular to KL[Prop. 3.18]. Thus, each of the angles at C is a rightangle. So, for the same (reasons), the angles at B and D are also right-angles. And since angle FCK is a rightangle, the (square) on FK is thus equal to the (sum of the squares) on FC and CK [Prop. 1.47]. So, for the same (reasons), the (square) on FK is also equal to the (sum of the squares) on FB and BK. So that the (sum of the squares) on FC and CK is equal to the (sum of the squares) on FB and BK, of which the (square) on FC is equal to the (square) on FB. Thus, the remaining (square) on CK is equal to the remaining (square) on BK. Thus, BK (is) equal to CK. And since FB is equal to FC, and FK (is) common, the two (straightlines) BF, FK are equal to the two (straight-lines) CF, FK. And the base BK [is] equal to the base CK. Thus, angle BFK is equal to [angle] KFC [Prop. 1.8]. And BKF (is equal) to FKC [Prop. 1.8]. Thus, BFC (is) double KFC, and BKC (is double) FKC. So, for the same (reasons), CFD is also double CFL, and DLC (is also double) FLC. And since circumference BC is equal to CD, angle BFC is also equal to CFD [Prop. 3.27]. And *BFC* is double *KFC*, and *DFC* (is double) *LFC*. Thus, KFC is also equal to LFC. And angle FCK is also equal to FCL. So, FKC and FLC are two triangles havκαὶ ἐκάστη τῶν ΘΗ, ΗΜ, ΜΛ ἑκατέρα τῶν ΘΚ, ΚΛ ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΗΘΚΛΜ πεντάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΖΚΓ γωνία τῆ ὑπὸ ΖΛΓ, καὶ ἐδείχθη τῆς μὲν ὑπὸ ΖΚΓ διπλῆ ἡ ὑπὸ ΘΚΛ, τῆς δὲ ὑπὸ ΖΛΓ διπλῆ ἡ ὑπὸ ΚΛΜ, καὶ ἡ ὑπὸ ΘΚΛ ἄρα τῆ ὑπὸ ΚΛΜ ἐστιν ἴση. ὑμοίως δὴ δειχθήσεται καὶ ἐκάστη τῶν ὑπὸ ΚΘΗ, ΘΗΜ, ΗΜΛ ἑκατέρα τῶν ὑπὸ ΘΚΛ, ΚΛΜ ἴση· αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ ΗΘΚ, ΘΚΛ, ΚΛΜ, ΛΜΗ, ΜΗΘ ἴσαι ἀλλήλαις εἰσίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΗΘΚΛΜ πεντάγωνον. ἑδείχθη δὲ καὶ ἰσόπλευρον, καὶ περιγέγραπται περὶ τὸν ΑΒΓΔΕ κύκλον.

[Περὶ τὸν δοθέντα ἄρα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγέγραπται]· ὅπερ ἔδει ποιῆσαι.

ing two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle to the remaining angle [Prop. 1.26]. Thus, the straight-line KC (is) equal to CL, and the angle FKC to FLC. And since KC is equal to CL, KL (is) thus double KC. So, for the same (reasons), it can be shown that HK (is) also double BK. And BK is equal to KC. Thus, HK is also equal to KL. So, similarly, each of HG, GM, and MLcan also be shown (to be) equal to each of HK and KL. Thus, pentagon GHKLM is equilateral. So I say that (it is) also equiangular. For since angle FKC is equal to FLC, and HKL was shown (to be) double FKC, and KLM double FLC, HKL is thus also equal to KLM. So, similarly, each of KHG, HGM, and GML can also be shown (to be) equal to each of HKL and KLM. Thus, the five angles GHK, HKL, KLM, LMG, and MGH are equal to one another. Thus, the pentagon GHKLM is equiangular. And it was also shown (to be) equilateral, and has been circumscribed about circle ABCDE.

[Thus, an equilateral and equiangular pentagon has been circumscribed about the given circle]. (Which is) the very thing it was required to do.

[†] See the footnote to Prop. 3.34.

ıγ'.

Εἰς τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον ἐγγράψαι.



Έστω τὸ δοθὲν πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸ ΑΒΓΔΕ πεντάγωνον κύκλον ἐγγράψαι.

Τετμήσθω γὰρ ἑκατέρα τῶν ὑπὸ ΒΓΔ, ΓΔΕ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν ΓΖ, ΔΖ εὐθειῶν· καὶ ἀπὸ τοῦ Ζ σημείου, καθ' ὃ συμβάλλουσιν ἀλλήλαις αἱ ΓΖ, ΔΖ εὐθεῖαι, ἐπεζεύχθωσαν αἱ ΖΒ, ΖΑ, ΖΕ εὐθεῖαι. καὶ ἐπεὶ ἴση ἐστὶν

Proposition 13

To inscribe a circle in a given pentagon, which is equilateral and equiangular.



Let *ABCDE* be the given equilateral and equiangular pentagon. So it is required to inscribe a circle in pentagon *ABCDE*.

For let angles BCD and CDE have each been cut in half by each of the straight-lines CF and DF (respectively) [Prop. 1.9]. And from the point F, at which the straight-lines CF and DF meet one another, let the

ή ΒΓ τῆ ΓΔ, κοινή δὲ ή ΓΖ, δύο δὴ αί ΒΓ, ΓΖ δυσὶ ταῖς ΔΓ, ΓΖ ίσαι εἰσίν καὶ γωνία ἡ ὑπὸ ΒΓΖ γωνία τῆ ὑπὸ ΔΓΖ [ἐστιν] ἴση· βάσις ἄρα ἡ ΒΖ βάσει τῆ ΔΖ ἐστιν ἴση, καὶ τὸ ΒΓΖ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἐστιν ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ὡς αἱ ίσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΓΒΖ γωνία τῆ ύπὸ ΓΔΖ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ὑπὸ ΓΔΕ τῆς ὑπὸ ΓΔΖ, ίση δὲ ἡ μὲν ὑπὸ ΓΔΕ τῆ ὑπὸ ΑΒΓ, ἡ δὲ ὑπὸ ΓΔΖ τῆ ὑπὸ ΓΒΖ, καὶ ἡ ὑπὸ ΓΒΑ ἄρα τῆς ὑπὸ ΓΒΖ ἐστι διπλῆ· ἴση άρα ἡ ὑπὸ ABZ γωνία τῆ ὑπὸ ZBΓ· ἡ ἄρα ὑπὸ ABΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΖ εὐθείας. ὑμοίως δὴ δειχθήσεται, ότι καὶ ἑκατέρα τῶν ὑπὸ ΒΑΕ, ΑΕΔ δίχα τέτμηται ὑπὸ έκατέρας τῶν ΖΑ, ΖΕ εὐθειῶν. ἤχθωσαν δὴ ἀπὸ τοῦ Ζ σημείου ἐπὶ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας κάθετοι αί ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΘΓΖ γωνία τῆ ὑπὸ ΚΓΖ, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΖΘΓ [ὀρθῆ] τῆ ὑπὸ ΖΚΓ ἴση, δύο δὴ τρίγωνά ἐστι τὰ ΖΘΓ, ΖΚΓ τὰς δύο γωνίας δυσί γωνίαις ίσας ἔχοντα καὶ μίαν πλευράν μιῷ πλευρα ίσην κοινήν αὐτῶν τήν ΖΓ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ ΖΘ κάθετος τὴ ΖΚ καθέτω. όμοίως δη δειχθήσεται, ὄτι καὶ ἑκάστη τῶν ΖΛ, ΖΜ, ΖΗ έκατέρα τῶν ΖΘ, ΖΚ ἴση ἐστίν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΖΗ, ZΘ, ZK, ZΛ, ZM ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρω τῶ Ζ διαστήματι δὲ ἑνὶ τῶν Η, Θ, Κ, Λ, Μ κύκλος γραφόμενος ΓΔ, ΔΕ, ΕΑ εύθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Η, Θ , K, Λ, M σημείοις γωνίας. εἰ γὰρ οὐκ ἐφάψεται αὐτῶν, άλλὰ τεμεῖ αὐτάς, συμβήσεται τὴν τῆ διαμέτρω τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄχρας ἀγομένην ἐντὸς πίπτειν τοῦ χύχλου. ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρ ω τ $\widetilde{\omega} ext{ Z}$ διαστήματι δὲ ένὶ τῶν Η, Θ, Κ, Λ, Μ σημείων γραφόμενος κύκλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας ἐφάψεται ἄρα αὐτῶν. γεγράφθω ὡς ὁ ΗΘΚΛΜ.

Εἰς ἄρα τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιδ'.

Περὶ τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον περιγράψαι.

Έστω τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ

straight-lines FB, FA, and FE have been joined. And since BC is equal to CD, and CF (is) common, the two (straight-lines) BC, CF are equal to the two (straightlines) DC, CF. And angle BCF [is] equal to angle DCF. Thus, the base BF is equal to the base DF, and triangle BCF is equal to triangle DCF, and the remaining angles will be equal to the (corresponding) remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle CBF (is) equal to CDF. And since CDEis double CDF, and CDE (is) equal to ABC, and CDFto CBF, CBA is thus also double CBF. Thus, angle ABF is equal to FBC. Thus, angle ABC has been cut in half by the straight-line BF. So, similarly, it can be shown that *BAE* and *AED* have been cut in half by the straight-lines FA and FE, respectively. So let FG, FH, FK, FL, and FM have been drawn from point F, perpendicular to the straight-lines AB, BC, CD, DE, and EA (respectively) [Prop. 1.12]. And since angle HCFis equal to KCF, and the right-angle FHC is also equal to the [right-angle] FKC, FHC and FKC are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, the perpendicular FH (is) equal to the perpendicular FK. So, similarly, it can be shown that FL, FM, and FG are each equal to each of FH and FK. Thus, the five straight-lines FG, FH, FK, FL, and FM are equal to one another. Thus, the circle drawn with center F, and radius one of G, H, K, L, or M, will also go through the remaining points, and will touch the straight-lines AB, BC, CD, DE, and EA, on account of the angles at points G, H, K, L, and *M* being right-angles. For if it does not touch them, but cuts them, it follows that a (straight-line) drawn at rightangles to the diameter of the circle, from its extremity, falls inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center F, and radius one of G, H, K, L, or M, does not cut the straight-lines AB, BC, CD, DE, or EA. Thus, it will touch them. Let it have been drawn, like GHKLM (in the figure).

Thus, a circle has been inscribed in the given pentagon which is equilateral and equiangular. (Which is) the very thing it was required to do.

Proposition 14

To circumscribe a circle about a given pentagon which is equilateral and equiangular.

Let ABCDE be the given pentagon which is equilat-

ίσογώνιον, τὸ ΑΒΓΔΕ· δεῖ δὴ περὶ τὸ ΑΒΓΔΕ πεντάγωνον κύκλον περιγράψαι.



Τετμήσθω δη έχατέρα τῶν ὑπὸ ΒΓΔ, ΓΔΕ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν ΓΖ, ΔΖ, καὶ ἀπὸ τοῦ Ζ σημείου, καθ' δ συμβάλλουσιν αί εύθεῖαι, ἐπὶ τὰ Β, Α, Ε σημεῖα έπεζεύχθωσαν εὐθεῖαι αί ZB, ZA, ZE. ὁμοίως δὴ τῷ πρὸ τούτου δειχθήσεται, ὄτι καὶ ἑκάστη τῶν ὑπὸ ΓΒΑ, ΒΑΕ, ΑΕΔ γωνιῶν δίχα τέτμηται ὑπὸ ἑκάστης τῶν ΖΒ, ΖΑ, ΖΕ εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΓΔ γωνία τῆ ὑπὸ ΓΔΕ, καί ἐστι τῆς μὲν ὑπὸ ΒΓΔ ἡμίσεια ἡ ὑπὸ ΖΓΔ, τῆς δὲ ὑπὸ ΓΔΕ ἡμίσεια ἡ ὑπὸ ΓΔΖ, καὶ ἡ ὑπὸ ΖΓΔ ἄρα τῃ ὑπὸ ΖΔΓ έστιν ἴση· ὥστε καὶ πλευρ
ὰ ή ${\rm Z}\Gamma$ πλευρᾶ τῆ ${\rm Z}\Delta$ ἐστιν ἴση. όμοίως δη δειχθήσεται, ὄτι και ἑκάστη τῶν ΖΒ, ΖΑ, ΖΕ έκατέρα τῶν ZΓ, ZΔ ἐστιν ἴση· αἰ πέντε ἄρα εύθεῖαι αἰ ZA, ZB, Z Γ , Z Δ , ZE ĭσαι ἀλλήλαις εἰσίν. ὀ ἄρα κέντρω τῷ Ζ καὶ διαστήματι ἑνὶ τῶν ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος. περιγεγράφθω καὶ ἔστω ὁ ΑΒΓΔΕ.

Περὶ ἄρα τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιε΄.

Εἰς τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Έστω ὁ δοθεἰς κύκλος ὁ ΑΒΓΔΕΖ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕΖ κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ήχθω τοῦ ΑΒΓΔΕΖ κύκλου διάμετρος ἡ ΑΔ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Η, καὶ κέντρῳ μὲν τῷ Δ διαστήματι δὲ τῷ ΔΗ κύκλος γεγράφθω ὁ ΕΗΓΘ, καὶ ἐπιζευχθεῖσαι αἱ ΕΗ, ΓΗ διήχθωσαν ἐπὶ τὰ Β, Ζ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΖ, ΖΑ· λέγω, ὅτι

eral and equiangular. So it is required to circumscribe a circle about the pentagon *ABCDE*.



So let angles BCD and CDE have been cut in half by the (straight-lines) CF and DF, respectively [Prop. 1.9]. And let the straight-lines FB, FA, and FE have been joined from point F, at which the straight-lines meet, to the points B, A, and E (respectively). So, similarly, to the (proposition) before this (one), it can be shown that angles CBA, BAE, and AED have also been cut in half by the straight-lines FB, FA, and FE, respectively. And since angle BCD is equal to CDE, and FCDis half of BCD, and CDF half of CDE, FCD is thus also equal to FDC. So that side FC is also equal to side FD [Prop. 1.6]. So, similarly, it can be shown that FB, FA, and FE are also each equal to each of FC and FD. Thus, the five straight-lines FA, FB, FC, FD, and FE are equal to one another. Thus, the circle drawn with center F, and radius one of FA, FB, FC, FD, or FE, will also go through the remaining points, and will have been circumscribed. Let it have been (so) circumscribed, and let it be ABCDE.

Thus, a circle has been circumscribed about the given pentagon, which is equilateral and equiangular. (Which is) the very thing it was required to do.

Proposition 15

To inscribe an equilateral and equiangular hexagon in a given circle.

Let *ABCDEF* be the given circle. So it is required to inscribe an equilateral and equiangular hexagon in circle *ABCDEF*.

Let the diameter AD of circle ABCDEF have been drawn,[†] and let the center G of the circle have been found [Prop. 3.1]. And let the circle EGCH have been drawn, with center D, and radius DG. And EG and CGbeing joined, let them have been drawn across (the cir-



τὸ ΑΒΓΔΕΖ ἑξάγωνον ἰσόπλευρόν τέ ἐστι καὶ ἰσογώνιον.

Έπεὶ γὰρ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓΔΕΖ κύκλου, ἴση ἐστὶν ἡ ΗΕ τῆ ΗΔ. πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΗΓΘ κύκλου, ἴση ἐστὶν
ἡ ΔE τῆ ΔH . άλλ
՝ ή ΗΕ τῆ ΗΔ ἐδείχθη ἴση· καὶ ή ΗΕ ἄρα τῆ ΕΔ ἴση έστίν· ἰσόπλευρον ἄρα έστὶ τὸ ${\rm EH}\Delta$ τρίγωνον· καὶ αἱ τρ
εῖς άρα αὐτοῦ γωνίαι αἱ ὑπὸ ΕΗΔ, ΗΔΕ, ΔΕΗ ἴσαι ἀλλήλαις εἰσίν, ἐπειδήπερ τῶν ἰσοσχελῶν τριγώνων αἱ πρὸς τῆ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν· καί εἰσιν αἱ τρεῖς τοῦ τριγώνου γωνίαι δυσιν όρθαῖς ἴσαι· ἡ ἄρα ὑπὸ ΕΗΔ γωνία τρίτον ἐστὶ δύο ὀρθῶν. ὑμοίως δὴ δειχθήσεται καὶ ἡ ὑπὸ ΔΗΓ τρίτον δύο ὀρθῶν. καὶ ἐπεὶ ἡ ΓΗ εὐθεῖα ἐπὶ τὴν ΕΒ σταθεῖσα τὰς έφεξῆς γωνίας τὰς ὑπὸ ΕΗΓ, ΓΗΒ δυσὶν ὀρθαῖς ἴσας ποιεῖ, καὶ λοιπὴ ẳρα ἡ ὑπὸ ΓΗΒ τρίτον ἐστὶ δύο ὀρθῶν· αἱ ẳρα ύπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὥστε καὶ αί κατὰ κορυφήν αὐταῖς αἱ ὑπὸ BHA, AHZ, ZHE ἴσαι εἰσὶν [ταῖς ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ]. αἱ ἕξ ἄρα γωνίαι αἱ ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ, ΒΗΑ, ΑΗΖ, ΖΗΕ ἴσαι ἀλλήλαις εἰσίν. αἱ δὲ ίσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήχασιν· αἱ ἕξ ἄρα περιφέρειαι αί AB, BΓ, ΓΔ, ΔΕ, ΕΖ, ΖΑ ίσαι ἀλλήλαις εἰσίν. ύπὸ δὲ τὰς ἴσας περιφερείας αἱ ἴσαι εὐθεῖαι ὑποτείνουσιν. αί ἕξ ἄρα εὐθεῖαι ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ το ΑΒΓΔΕΖ ἑξάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΖΑ περιφέρεια τῆ ΕΔ περιφερεία, κοινὴ προσκείσθω ή ΑΒΓΔ περιφέρεια δλη άρα ή ΖΑΒΓΔ όλη τῆ ΕΔΓΒΑ ἐστιν ἴση· καὶ βέβηκεν ἐπὶ μὲν τῆς ΖΑΒΓΔ περιφερείας ή ὑπὸ ΖΕΔ γωνία, ἐπὶ δὲ τῆς ΕΔΓΒΑ περιφερείας ή ὑπὸ ΑΖΕ γωνία· ἴση ἄρα ή ὑπὸ ΑΖΕ γωνία τῆ ύπὸ ΔΕΖ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ αἱ λοιπαὶ γωνίαι τοῦ ΑΒΓΔΕΖ ἑξαγώνου κατὰ μίαν ἴσαι εἰσὶν ἑκατέρα τῶν ύπὸ ΑΖΕ, ΖΕΔ γωνιῶν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ έξάγωνον. έδείχθη δὲ καὶ ἰσόπλευρον καὶ ἐγγέγραπται εἰς τὸν ΑΒΓΔΕΖ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε

cle) to points B and F (respectively). And let AB, BC, CD, DE, EF, and FA have been joined. I say that the hexagon ABCDEF is equilateral and equiangular.



For since point G is the center of circle ABCDEF, GE is equal to GD. Again, since point D is the center of circle GCH, DE is equal to DG. But, GE was shown (to be) equal to GD. Thus, GE is also equal to ED. Thus, triangle EGD is equilateral. Thus, its three angles EGD, GDE, and DEG are also equal to one another, inasmuch as the angles at the base of isosceles triangles are equal to one another [Prop. 1.5]. And the three angles of the triangle are equal to two right-angles [Prop. 1.32]. Thus, angle EGD is one third of two rightangles. So, similarly, DGC can also be shown (to be) one third of two right-angles. And since the straight-line CG, standing on EB, makes adjacent angles EGC and CGB equal to two right-angles [Prop. 1.13], the remaining angle *CGB* is thus also one third of two right-angles. Thus, angles EGD, DGC, and CGB are equal to one another. And hence the (angles) opposite to them BGA, AGF, and FGE are also equal [to EGD, DGC, and CGB (respectively)] [Prop. 1.15]. Thus, the six angles EGD, DGC, CGB, BGA, AGF, and FGE are equal to one another. And equal angles stand on equal circumferences [Prop. 3.26]. Thus, the six circumferences AB, BC, CD, DE, EF, and FA are equal to one another. And equal circumferences are subtended by equal straight-lines [Prop. 3.29]. Thus, the six straight-lines (AB, BC, CD, DE, EF, and FA) are equal to one another. Thus, hexagon ABCDEF is equilateral. So, I say that (it is) also equiangular. For since circumference FA is equal to circumference ED, let circumference ABCD have been added to both. Thus, the whole of FABCD is equal to the whole of EDCBA. And angle FED stands on circumference FABCD, and angle AFE on circumference EDCBA. Thus, angle AFE is equal καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Πόρισμα.

Έκ δή τούτου φανερόν, ὄτι ή τοῦ ἑξαγώνου πλευρὰ ἴση ἐστὶ τῆ ἐκ τοῦ κέντρου τοῦ κύκλου.

Όμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐἀν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἀκολούθως τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις. καὶ ἔτι διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις εἰς τὸ δοθὲν ἑξάγωνον κύκλον ἐγγράψομέν τε καὶ περιγράψομεν. ὅπερ ἔδει ποιῆσαι.

[†] See the footnote to Prop. 4.6.

เร'.

Εἰς τὸν δοθέντα κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.



Έστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Έγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τριγώνου μὲν ἰσοπλεύρου τοῦ εἰς αὐτὸν ἐγγραφομένου πλευρὰ ἡ ΑΓ, πενταγώνου δὲ ἰσοπλεύρου ἡ ΑΒ· οἴων ἄρα ἐστὶν ὁ ΑΒΓΔ κύκλος ἴσων τμήματων δεκαπέντε, τοιούτων ἡ μὲν ΑΒΓ περιφέρεια τρίτον οῦσα τοῦ κύκλου ἔσται πέντε, ἡ δὲ ΑΒ περιφέρεια πέμτον οῦσα τοῦ κύκλου ἔσται τριῶν· λοιπὴ ἄρα to DEF [Prop. 3.27]. Similarly, it can also be shown that the remaining angles of hexagon ABCDEF are individually equal to each of the angles AFE and FED. Thus, hexagon ABCDEF is equiangular. And it was also shown (to be) equilateral. And it has been inscribed in circle ABCDE.

Thus, an equilateral and equiangular hexagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

Corollary

So, from this, (it is) manifest that a side of the hexagon is equal to the radius of the circle.

And similarly to a pentagon, if we draw tangents to the circle through the (sixfold) divisions of the (circumference of the) circle, an equilateral and equiangular hexagon can be circumscribed about the circle, analogously to the aforementioned pentagon. And, further, by (means) similar to the aforementioned pentagon, we can inscribe and circumscribe a circle in (and about) a given hexagon. (Which is) the very thing it was required to do.

Proposition 16

To inscribe an equilateral and equiangular fifteensided figure in a given circle.



Let ABCD be the given circle. So it is required to inscribe an equilateral and equiangular fifteen-sided figure in circle ABCD.

Let the side AC of an equilateral triangle inscribed in (the circle) [Prop. 4.2], and (the side) AB of an (inscribed) equilateral pentagon [Prop. 4.11], have been inscribed in circle ABCD. Thus, just as the circle ABCDis (made up) of fifteen equal pieces, the circumference ABC, being a third of the circle, will be (made up) of five ή ΒΓ τῶν ἴσων δύο. τετμήσθω ή ΒΓ δίχα κατὰ τὸ Ε· ἑκατέρα ἄρα τῶν ΒΕ, ΕΓ περιφερειῶν πεντεκαιδέκατόν ἐστι τοῦ ΑΒΓΔ κύκλου.

Ἐἀν ἄρα ἐπιζεύξαντες τὰς BE, ΕΓ ἴσας αὐταῖς κατὰ τὸ συνεχὲς εὐθείας ἐναρμόσωμεν εἰς τὸν $AB\Gamma\Delta[E]$ κύκλον, ἔσται εἰς αὐτὸν ἐγγεγραμμένον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ὅπερ ἔδει ποιῆσαι.

Όμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐἀν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον. ἔτι δὲ διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου δείξεων καὶ εἰς τὸ δοθὲν πεντεκαιδεκάγωνον κύκλον ἐγγράψομέν τε καὶ περιγράψομεν. ὅπερ ἔδει ποιῆσαι.

such (pieces), and the circumference AB, being a fifth of the circle, will be (made up) of three. Thus, the remainder BC (will be made up) of two equal (pieces). Let (circumference) BC have been cut in half at E [Prop. 3.30]. Thus, each of the circumferences BE and EC is one fifteenth of the circle ABCDE.

Thus, if, joining BE and EC, we continuously insert straight-lines equal to them into circle ABCD[E][Prop. 4.1], then an equilateral and equiangular fifteensided figure will have been inserted into (the circle). (Which is) the very thing it was required to do.

And similarly to the pentagon, if we draw tangents to the circle through the (fifteenfold) divisions of the (circumference of the) circle, we can circumscribe an equilateral and equiangular fifteen-sided figure about the circle. And, further, through similar proofs to the pentagon, we can also inscribe and circumscribe a circle in (and about) a given fifteen-sided figure. (Which is) the very thing it was required to do.

ELEMENTS BOOK 5

 $Proportion^{\dagger}$

[†]The theory of proportion set out in this book is generally attributed to Eudoxus of Cnidus. The novel feature of this theory is its ability to deal with irrational magnitudes, which had hitherto been a major stumbling block for Greek mathematicians. Throughout the footnotes in this book, α , β , γ , *etc.*, denote general (possibly irrational) magnitudes, whereas *m*, *n*, *l*, *etc.*, denote positive integers.

Ὅροι.

α'. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῆ τὸ μεῖζον.

β'. Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

γ'. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.

δ'. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἂ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

ε'. Έν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καί τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ' ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ἤ ἢ ἅμα ἐλλείπῇ ληφθέντα κατάλληλα.

τ΄. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

ζ΄. Όταν δὲ τῶν ἰσάχις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον.

η'. Άναλογία δὲ ἐν τρισὶν ὄροις ἐλαχίστη ἐστίν.

θ'. Όταν δὲ τρία μεγέθη ἀνάλογον ἢ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον.

ι΄. Όταν δὲ τέσσαρα μεγέθη ἀνάλογον ἢ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον, καὶ ἀεὶ ἑξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχῃ.

ια΄. Όμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγουμένοις τὰ δὲ ἑπόμενα τοῖς ἑπομένοις.

ιβ΄. Ἐναλλὰξ λόγος ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὸ ἡγούμενον καὶ τοῦ ἑπομένου πρὸς τὸ ἑπόμενον.

ιγ΄. Ἀνάπαλιν λόγος ἐστὶ λῆψις τοῦ ἑπομένου ὡς ἡγουμένου πρὸς τὸ ἡγούμενον ὡς ἑπόμενον.

ιδ'. Σύνθεσις λόγου ἐστὶ λῆψις τοῦ ἡγουμένου μετὰ τοῦ ἑπομένου ὡς ἑνὸς πρὸς αὐτὸ τὸ ἑπόμενον.

ιε΄. Διαίρεσις λόγου ἐστὶ λῆψις τῆς ὑπεροχῆς, ῇ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου, πρὸς αὐτὸ τὸ ἑπόμενον.

ις'. Ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ἤ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου.

ιζ΄. Δι' ἴσου λόγος ἐστὶ πλειόνων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδυο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ἢ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον. ἢ ἄλλως. λῆψις τῶν ἄχρων

Definitions

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.^{\dagger}

2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.

3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.[‡]

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.[§]

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.¶

6. And let magnitudes having the same ratio be called proportional.*

7. And when for equal multiples (as in Def. 5), the multiple of the first (magnitude) exceeds the multiple of the second, and the multiple of the third (magnitude) does not exceed the multiple of the fourth, then the first (magnitude) is said to have a greater ratio to the second than the third (magnitude has) to the fourth.

8. And a proportion in three terms is the smallest (possible).^{\$}

9. And when three magnitudes are proportional, the first is said to have to the third the squared^{||} ratio of that (it has) to the second.^{††}

10. And when four magnitudes are (continuously) proportional, the first is said to have to the fourth the cubed^{‡‡} ratio of that (it has) to the second.^{§§} And so on, similarly, in successive order, whatever the (continuous) proportion might be.

11. These magnitudes are said to be corresponding (magnitudes): the leading to the leading (of two ratios), and the following to the following.

12. An alternate ratio is a taking of the (ratio of the) leading (magnitude) to the leading (of two equal ratios), and (setting it equal to) the (ratio of the) following (magnitude) to the following.

13. An inverse ratio is a taking of the (ratio of the) following (magnitude) as the leading and the leading (magnitude) as the following.**

14. A composition of a ratio is a taking of the (ratio of the) leading plus the following (magnitudes), as one, to the following (magnitude) by itself.^{§§}

χαθ' ὑπεξαίρεσιν τῶν μέσων.

ιη'. Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθών και άλλων αυτοῖς ἴσων τὸ πλῆθος γίνηται ὡς μὲν έν τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἐπόμενον, οὕτως έν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, ὡς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἑπόμενον πρὸς ἄλλο τι, οὕτως έν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

15. A separation of a ratio is a taking of the (ratio of the) excess by which the leading (magnitude) exceeds the following to the following (magnitude) by itself.

16. A conversion of a ratio is a taking of the (ratio of the) leading (magnitude) to the excess by which the leading (magnitude) exceeds the following.^{†††}

17. There being several magnitudes, and other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, a ratio via equality (or ex *aequali*) occurs when as the first is to the last in the first (set of) magnitudes, so the first (is) to the last in the second (set of) magnitudes. Or alternately, (it is) a taking of the (ratio of the) outer (magnitudes) by the removal of the inner (magnitudes).^{‡‡‡}

18. There being three magnitudes, and other (magnitudes) of equal number to them, a perturbed proportion occurs when as the leading is to the following in the first (set of) magnitudes, so the leading (is) to the following in the second (set of) magnitudes, and as the following (is) to some other (*i.e.*, the remaining magnitude) in the first (set of) magnitudes, so some other (is) to the leading in the second (set of) magnitudes.§§§

[†] In other words, α is said to be a part of β if $\beta = m \alpha$.

[‡] In modern notation, the ratio of two magnitudes, α and β , is denoted α : β .

§ In other words, α has a ratio with respect to β if $m \alpha > \beta$ and $n \beta > \alpha$, for some m and n.

¶ In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m \alpha > n \beta$ whenever $m \gamma > n \delta$, and $m \alpha = n \beta$ whenever $m \gamma = n \delta$, and $m \alpha < n \beta$ whenever $m\gamma < n\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α , β , etc., are irrational.

* Thus if α and β have the same ratio as γ and δ then they are proportional. In modern notation, $\alpha : \beta :: \gamma : \delta$.

[§] In modern notation, a proportion in three terms— α , β , and γ —is written: $\alpha : \beta :: \beta : \gamma$.

|| Literally, "double".

^{††} In other words, if $\alpha : \beta :: \beta : \gamma$ then $\alpha : \gamma :: \alpha^2 : \beta^2$.

^{‡‡} Literally, "triple".

§§ In other words, if $\alpha : \beta :: \beta : \gamma :: \gamma : \delta$ then $\alpha : \delta :: \alpha^3 : \beta^3$.

¶¶ In other words, if $\alpha : \beta :: \gamma : \delta$ then the alternate ratio corresponds to $\alpha : \gamma :: \beta : \delta$.

** In other words, if $\alpha : \beta$ then the inverse ratio corresponds to $\beta : \alpha$.

\$\$ In other words, if $\alpha : \beta$ then the composed ratio corresponds to $\alpha + \beta : \beta$.

If $\alpha : \beta$ then the separated ratio corresponds to $\alpha - \beta : \beta$.

^{†††} In other words, if $\alpha : \beta$ then the converted ratio corresponds to $\alpha : \alpha - \beta$.

 ‡‡‡ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha: \beta: \gamma: \delta: \epsilon: \zeta$, then the ratio via equality (or *ex* aequali) corresponds to $\alpha : \gamma :: \delta : \zeta$.

\$\$ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta :: \delta : \epsilon$ as well as $\beta : \gamma :: \zeta : \delta$, then the proportion is said to be perturbed.

α'.

Έαν ή όποσαοῦν μεγέθη όποσωνοῦν μεγεθῶν ἴσων τὸ

Proposition 1[†]

If there are any number of magnitudes whatsoever πλήθος ἕχαστον ἑχάστου ἰσάχις πολλαπλάσιον, ὁσαπλάσιόν (which are) equal multiples, respectively, of some (other) έστιν ἒν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ magnitudes, of equal number (to them), then as many πάντα τῶν πάντων.



Έστω ὁποσαοῦν μεγέθη τὰ AB, ΓΔ ὁποσωνοῦν μεγεθῶν τῶν Ε, Ζ ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον· λέγω, ὅτι ὁσαπλάσιόν ἐστι τὸ AB τοῦ Ε, τοσαυταπλάσια ἔσται καὶ τὰ AB, ΓΔ τῶν Ε, Ζ.

Έπεὶ γὰρ ἰσάχις ἐστὶ πολλαπλάσιον τὸ AB τοῦ E καὶ τὸ ΓΔ τοῦ Z, ὅσα ἄρα ἐστὶν ἐν τῷ AB μεγέθη ἴσα τῷ E, τοσαῦτα καὶ ἐν τῷ ΓΔ ἴσα τῷ Z. διῃρήσθω τὸ μὲν AB εἰς τὰ τῷ E μεγέθη ἴσα τὰ AH, HB, τὸ δὲ ΓΔ εἰς τὰ τῷ Z ἴσα τὰ ΓΘ, ΘΔ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH, HB τῷ πλήθει τῶν ΓΘ, ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν AH τῷ E, τὸ δὲ ΓΘ τῷ Z, ἴσον ἄρα τὸ AH τῷ E, καὶ τὰ AH, ΓΘ τοῖς E, Z. διὰ τὰ αὐτὰ δὴ ἴσον ἐστὶ τὸ HB τῷ E, καὶ τὰ HB, ΘΔ τοῖς E, Z· ὅσα ἄρα ἐστὶν ἐν τῷ AB ἴσα τῷ E, τοσαῦτα καὶ ἐν τοῖς AB, ΓΔ ἴσα τοῖς E, Z· ὁσαπλάσιον ἄρα ἐστὶ τὸ AB τοῦ E, τοσαυταπλάσια ἔσται καὶ τὰ AB, ΓΔ τῶν E, Z.

Έὰν ἄρα ἢ ὑποσαοῦν μεγέθη ὑποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕχαστον ἑχάστου ἰσάχις πολλαπλάσιον, ὑσαπλάσιόν ἐστιν ἕν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται χαὶ τὰ πάντα τῶν πάντων. ὅπερ ἔδει δεῖξαι. times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first

ELEMENTS BOOK 5



Let there be any number of magnitudes whatsoever, AB, CD, (which are) equal multiples, respectively, of some (other) magnitudes, E, F, of equal number (to them). I say that as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

For since AB, CD are equal multiples of E, F, thus as many magnitudes as (there) are in AB equal to E, so many (are there) also in CD equal to F. Let AB have been divided into magnitudes AG, GB, equal to E, and CD into (magnitudes) CH, HD, equal to F. So, the number of (divisions) AG, GB will be equal to the number of (divisions) CH, HD. And since AG is equal to E, and CH to F, AG (is) thus equal to E, and AG, CH to E, F. So, for the same (reasons), GB is equal to E, and GB, HD to E, F. Thus, as many (magnitudes) as (there) are in AB equal to E, so many (are there) also in AB, CDequal to E, F. Thus, as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

Thus, if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second). (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads $m \alpha + m \beta + \cdots = m (\alpha + \beta + \cdots)$.

β'.

Έὰν πρῶτον δευτέρου ἰσάχις ἢ πολλαπλάσιον καὶ τρίτον τετάρτου, ἢ δὲ καὶ πέμπτον δευτέρου ἰσάχις πολλαπλάσιον καὶ ἕχτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάχις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕχτον τετάρτου.

Πρῶτον γὰρ τὸ AB δευτέρου τοῦ Γ ἰσάχις ἔστω πολλαπλάσιον καὶ τρίτον τὸ ΔΕ τετάρτου τοῦ Ζ, ἔστω δὲ καὶ πέμπτον τὸ BH δευτέρου τοῦ Γ ἰσάχις πολλαπλάσιον καὶ ἔχτον τὸ EΘ τετάρτου τοῦ Ζ· λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ AH δευτέρου τοῦ Γ ἰσάχις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἔχτον τὸ ΔΘ τετάρτου τοῦ Ζ.

Proposition 2[†]

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and the sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively).

For let a first (magnitude) AB and a third DE be equal multiples of a second C and a fourth F (respectively). And let a fifth (magnitude) BG and a sixth EHalso be (other) equal multiples of the second C and the fourth F (respectively). I say that the first (magnitude) and the fifth, being added together, (to give) AG, and the third (magnitude) and the sixth, (being added together,



Έπει γὰρ ἰσάχις ἐστὶ πολλαπλάσιον τὸ AB τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ AB ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὅσα ἐστὶν ἐν τῷ BH ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΕΘ ἴσα τῷ Ζ· ὅσα ἄρα ἐστὶν ἐν ὅλῳ τῷ AH ἴσα τῷ Γ, τοσαῦτα καὶ ἐν ὅλῳ τῷ ΔΘ ἴσα τῷ Ζ· ὅσαπλάσιον ἄρα ἐστὶ τὸ AH τοῦ Γ, τοσαυταπλάσιον ἔσται καὶ τὸ ΔΘ τοῦ Ζ. καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ AH δευτέρου τοῦ Γ ἰσάχις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕχτον τὸ ΔΘ τετάρτου τοῦ Ζ.

Έὰν ἄρα πρῶτον δευτέρου ἰσάκις ἤ πολλαπλάσιον καὶ τρίτον τετάρτου, ἤ δὲ καὶ πέμπτον δευτέρου ἰσάκις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου. ὅπερ ἔδει δεῖξαι.

[†] In modern notation, this propostion reads $m \alpha + n \alpha = (m + n) \alpha$.

γ'.

Έὰν πρῶτον δευτέρου ἰσάχις ἢ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῆ δὲ ἰσάχις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου, καὶ δι' ἴσου τῶν ληφθέντων ἐχάτερον ἑχατέρου ἰσάχις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου.

Πρῶτον γὰρ τὸ Α δευτέρου τοῦ Β ἰσάχις ἔστω πολλαπλάσιον καὶ τρίτον τὸ Γ τετάρτου τοῦ Δ, καὶ εἰλήφθω τῶν Α, Γ ἰσάχις πολλαπλάσια τὰ ΕΖ, ΗΘ· λέγω, ὅτι ἰσάχις ἐστὶ πολλαπλάσιον τὸ ΕΖ τοῦ Β καὶ τὸ ΗΘ τοῦ Δ.

Έπει γὰρ ἰσάχις ἐστὶ πολλαπλάσιον τὸ ΕΖ τοῦ Α καὶ τὸ ΗΘ τοῦ Γ, ὅσα ἄρα ἐστὶν ἐν τῷ ΕΖ ἴσα τῷ Α, τοσαῦτα καὶ ἐν τῷ ΗΘ ἴσα τῷ Γ. διηρήσθω τὸ μὲν ΕΖ εἰς τὰ τῷ Α μεγέθη ἴσα τὰ ΕΚ, ΚΖ, τὸ δὲ ΗΘ εἰς τὰ τῷ Γ ἴσα τὰ ΗΛ, **ELEMENTS BOOK 5**

to give) DH, will also be equal multiples of the second (magnitude) C and the fourth F (respectively).



For since AB and DE are equal multiples of C and F (respectively), thus as many (magnitudes) as (there) are in AB equal to C, so many (are there) also in DE equal to F. And so, for the same (reasons), as many (magnitudes) as (there) are in BG equal to C, so many (are there) also in EH equal to F. Thus, as many (magnitudes) as (there) are in the whole of AG equal to C, so many (are there) also in the whole of DH equal to F. Thus, as many times as AG is (divisible) by C, so many times will DH also be divisible by F. Thus, the first (magnitude) and the fifth, being added together, (to give) AG, and the third (magnitude) and the sixth, (being added together, to give) DH, will also be equal multiples of the second (magnitude) C and the fourth F (respectively).

Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively). (Which is) the very thing it was required to show.

Proposition 3[†]

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively.

For let a first (magnitude) A and a third C be equal multiples of a second B and a fourth D (respectively), and let the equal multiples EF and GH have been taken of A and C (respectively). I say that EF and GH are equal multiples of B and D (respectively).

For since EF and GH are equal multiples of A and C (respectively), thus as many (magnitudes) as (there) are in EF equal to A, so many (are there) also in GH

ΛΘ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΕΚ, ΚΖ τῷ πλήθει τῶν ΗΛ, ΛΘ. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Α τοῦ Β καὶ τὸ Γ τοῦ Δ, ἴσον δὲ τὸ μὲν ΕΚ τῷ Α, τὸ δὲ ΗΛ τῷ Γ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΕΚ τοῦ Β καὶ τὸ ΗΛ τοῦ Δ. διὰ τὰ αὐτὰ δὴ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΚΖ τοῦ Β καὶ τὸ ΛΘ τοῦ Δ. ἐπεὶ οῦν πρῶτον τὸ ΕΚ δευτέρου τοῦ Β ἴσάκις ἐστὶ πολλαπλάσιον καὶ τρίτον τὸ ΗΛ τετάρτου τοῦ Δ, ἔστι δὲ καὶ πέμπτον τὸ ΚΖ δευτέρου τοῦ Β ἰσάκις πολλαπλάσιον καὶ ἕκτον τὸ ΑΘ τετάρτου τοῦ Δ, καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ ΕΖ δευτέρου τοῦ Β ἰσάκις ἐστὶ πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΗΘ τετάρτου τοῦ



Έὰν ἄρα πρῶτον δευτέρου ἰσάκις ἢ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῆ δὲ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια, καὶ δι᾽ ἴσου τῶν ληφθέντων ἐκάτερον ἐκατέρου ἰσάκις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου. ὅπερ ἔδει δεῖξαι.

[†] In modern notation, this proposition reads $m(n \alpha) = (m n) \alpha$.

δ'.

Έὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου καθ' ὑποιονοῦν πολλαπλασιασμὸν τὸν αὐτὸν ἔξει λόγον ληφθέντα κατάλληλα.

Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, καὶ εἰλήφθω τῶν μὲν Α, Γ ἰσάχις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Β, Δ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Η, Θ· λέγω, ὅτι ἐστὶν ὡς τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.

equal to C. Let EF have been divided into magnitudes EK, KF equal to A, and GH into (magnitudes) GL, LHequal to C. So, the number of (magnitudes) EK, KFwill be equal to the number of (magnitudes) GL, LH. And since A and C are equal multiples of B and D (respectively), and EK (is) equal to A, and GL to C, EKand GL are thus equal multiples of B and D (respectively). So, for the same (reasons), KF and LH are equal multiples of B and D (respectively). Therefore, since the first (magnitude) EK and the third GL are equal multiples of the second B and the fourth D (respectively), and the fifth (magnitude) KF and the sixth LH are also equal multiples of the second B and the fourth D (respectively), then the first (magnitude) and fifth, being added together, (to give) EF, and the third (magnitude) and sixth, (being added together, to give) GH, are thus also equal multiples of the second (magnitude) B and the fourth D (respectively) [Prop. 5.2].



Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively. (Which is) the very thing it was required to show.

Proposition 4[†]

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever.

For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D. And let equal multiples E and F have been taken of A and C(respectively), and other random equal multiples G and



Εἰλήφθω γὰρ τῶν μὲν Ε, Ζ ἰσάχις πολλαπλάσια τὰ Κ, Λ, τῶν δὲ Η, Θ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Μ, Ν.

[Kaì] ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσιον τὸ μὲν Ε τοῦ Α, τὸ δὲ Ζ τοῦ Γ, καὶ εἴληπται τῶν Ε, Ζ ἴσάχις πολλαπλάσια τὰ Κ, Λ, ἴσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ Κ τοῦ Α καὶ τὸ Λ τοῦ Γ. διὰ τὰ αὐτὰ δὴ ἰσάχις ἐστὶ πολλαπλάσιον τὸ Μ τοῦ Β καὶ τὸ Ν τοῦ Δ. καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάχις πολλαπλάσια τὰ Κ, Λ, τῶν δὲ Β, Δ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ μὲν Κ, Λ τῶν Ε, Ζ ἰσάχις πολλαπλάσια, τὰ δὲ Μ, Ν τῶν Η, Θ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.

Έὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάχις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάχις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου τὸν αὐτὸν ἕξει λόγον καθ' ὁποιονοῦν πολλαπλασιασμὸν ληφθέντα κατάλληλα· ὅπερ ἔδει δεῖξαι. *H* of *B* and *D* (respectively). I say that as *E* (is) to *G*, so *F* (is) to *H*.



For let equal multiples K and L have been taken of E and F (respectively), and other random equal multiples M and N of G and H (respectively).

[And] since E and F are equal multiples of A and C (respectively), and the equal multiples K and L have been taken of E and F (respectively), K and L are thus equal multiples of A and C (respectively) [Prop. 5.3]. So, for the same (reasons), M and N are equal multiples of B and D (respectively). And since as A is to B, so C (is) to D, and the equal multiples K and L have been taken of A and C (respectively), and the other random equal multiples M and N of B and D (respectively), then if K exceeds M then L also exceeds N, and if (K is) equal (to M then L is also) equal (to N), and if (K is) less (than M then L is also) less (than N) [Def. 5.5]. And K and L are equal multiples of E and F (respectively), and M and N other random equal multiples of G and H (respectively). Thus, as E (is) to G, so F (is) to H [Def. 5.5].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $m \alpha : n \beta :: m \gamma : n \delta$, for all m and n.

ε΄.

Έὰν μέγεθος μεγέθους ἰσάχις ἤ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, χαὶ τὸ λοιπὸν τοῦ λοιποῦ ἰσάχις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι τὸ ὅλον τοῦ ὅλου.



Μέγεθος γὰρ τὸ AB μεγέθους τοῦ ΓΔ ἰσάχις ἔστω πολλαπλάσιον, ὅπερ ἀφαιρεθὲν τὸ AE ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ EB λοιποῦ τοῦ ΖΔ ἰσάχις ἔσται πολλαπλάσιον, ὅσαπλάσιόν ἐστιν ὅλον τὸ AB ὅλου τοῦ ΓΔ.

Όσαπλάσιον γάρ ἐστι τὸ ΑΕ τοῦ ΓΖ, τοσαυταπλάσιον γεγονέτω καὶ τὸ ΕΒ τοῦ ΓΗ.

Kai èrei ỉσáxıç èστì πολλαπλάσιον τὸ AE τοῦ ΓΖ xaì τὸ EB τοῦ HΓ, ἰσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓΖ xaì τὸ AB τοῦ HΖ. κεῖται δὲ ἰσάχις πολλαπλάσιον τὸ AE τοῦ ΓΖ xaì τὸ AB τοῦ ΓΔ. ἰσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ AB ἑχατέρου τῶν HΖ, ΓΔ· ἴσον ἄρα τὸ HΖ τῷ ΓΔ. χοινὸν ἀφηρήσϑω τὸ ΓΖ· λοιπὸν ἄρα τὸ HΓ λοιπῷ τῷ ΖΔ ἴσον ἐστίν. xaì ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓΖ xaì τὸ EB τοῦ HΓ, ἴσον δὲ τὸ HΓ τῷ ΔΖ, ἰσάχις ắρα ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓΖ xaì τὸ EB τοῦ ΖΔ. ἰσάχις δὲ ὑπόχειται πολλαπλάσιον τὸ AE τοῦ ΓΖ xaì τὸ AB τοῦ ΓΔ· ἰσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ EB τοῦ ΖΔ κaì τὸ AB τοῦ ΓΔ. xaì λοιπὸν ἄρα τὸ EB λοιποῦ τοῦ ΖΔ ἰσάχις ἔσται πολλαπλάσιον, ὁσαπλάσιον ἐστιν ὅλον τὸ AB ὅλου τοῦ ΓΔ.

Έὰν ἄρα μέγεθος μεγέθους ἰσάχις ἢ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, χαὶ τὸ λοιπὸν τοῦ λοιποῦ ἰσάχις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι χαὶ τὸ ὅλον τοῦ ὅλου. ὅπερ ἔδει δεῖξαι.

[†] In modern notation, this proposition reads $m \alpha - m \beta = m (\alpha - \beta)$.

۶'.

Έὰν δύο μεγέθη δύο μεγεθῶν ἰσάχις ἢ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ἰσάχις ἢ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἤτοι ἶσα ἐστὶν ἢ ἰσάχις αὐτῶν πολλαπλάσια.

Δύο γὰρ μεγέθη τὰ ΑΒ, ΓΔ δύο μεγεθῶν τῶν Ε, Ζ

Proposition 5[†]

If a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively).



For let the magnitude AB be the same multiple of the magnitude CD that the (part) taken away AE (is) of the (part) taken away CF (respectively). I say that the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

For as many times as AE is (divisible) by CF, so many times let EB also have been made (divisible) by CG.

And since AE and EB are equal multiples of CF and GC (respectively), AE and AB are thus equal multiples of CF and GF (respectively) [Prop. 5.1]. And AE and AB are assumed (to be) equal multiples of CF and CD(respectively). Thus, AB is an equal multiple of each of GF and CD. Thus, GF (is) equal to CD. Let CFhave been subtracted from both. Thus, the remainder GC is equal to the remainder FD. And since AE and EB are equal multiples of CF and GC (respectively), and GC (is) equal to DF, AE and EB are thus equal multiples of CF and FD (respectively). And AE and AB are assumed (to be) equal multiples of CF and CD(respectively). Thus, *EB* and *AB* are equal multiples of FD and CD (respectively). Thus, the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

Thus, if a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively). (Which is) the very thing it was required to show.

Proposition 6[†]

If two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples ίσάχις ἕστω πολλαπλάσια, καὶ ἀφαιρεθέντα τὰ ΑΗ, ΓΘ τῶν αὐτῶν τῶν Ε, Ζ ἰσάχις ἔστω πολλαπλάσια· λέγω, ὅτι καὶ λοιπὰ τὰ ΗΒ, ΘΔ τοῖς Ε, Ζ ἤτοι ἴσα ἐστὶν ἢ ἰσάχις αὐτῶν πολλαπλάσια.



Z⊢

Έστω γὰρ πρότερον τὸ HB τῷ E ἴσον· λέγω, ὅτι καὶ τὸ ΘΔ τῷ Ζ ἴσον ἐστίν.

Κείσθω γὰρ τῷ Ζ ἴσον τὸ ΓΚ. ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσιον τὸ AH τοῦ Ε χαὶ τὸ ΓΘ τοῦ Ζ, ἴσον δὲ τὸ μὲν HB τῷ Ε, τὸ δὲ ΚΓ τῷ Ζ, ἰσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ AB τοῦ Ε χαὶ τὸ KΘ τοῦ Ζ. ἰσάχις δὲ ὑπόχειται πολλαπλάσιον τὸ AB τοῦ Ε χαὶ τὸ ΓΔ τοῦ Ζ· ἴσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ KΘ τοῦ Ζ χαὶ τὸ ΓΔ τοῦ Ζ. ἐπεὶ οῦν ἑχάτερον τῶν KΘ, ΓΔ τοῦ Ζ ἰσάχις ἐστὶ πολλαπλάσιον, ἴσον ἄρα ἐστὶ τὸ KΘ τῷ ΓΔ. χοινὸν ἀφηρήσθω τὸ ΓΘ· λοιπὸν ἄρα τὸ ΚΓ λοιπῷ τῷ ΘΔ ἴσον ἐστίν. ἀλλὰ τὸ Ζ τῷ ΚΓ ἐστιν ἴσον· χαὶ τὸ ΘΔ ἄρα τῷ Ζ ἴσον ἑστίν. ὥστε εἰ τὸ HB τῷ Ε ἴσον ἐστίν, χαὶ τὸ ΘΔ ἴσον ἑσται τῷ Ζ.

Όμοίως δη δείξομεν, ὅτι, κἂν πολλαπλάσιον ἤ τὸ ΗΒ τοῦ Ε, τοσαυταπλάσιον ἔσται καὶ τὸ ΘΔ τοῦ Ζ.

Έὰν ἄρα δύο μεγέθη δύο μεγεθῶν ἰσάχις ἢ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ἰσάχις ἢ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἤτοι ἴσα ἐστὶν ἢ ἰσάχις αὐτῶν πολλαπλάσια· ὅπερ ἔδει δεῖξαι.

[†] In modern notation, this proposition reads $m \alpha - n \alpha = (m - n) \alpha$.

ζ'.

Τὰ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

Έστω ίσα μεγέθη τὰ A, B, ἄλλο δέ τι, δ ἔτυχεν, μέγεθος τὸ Γ · λέγω, ὅτι ἑχάτερον τῶν A, B πρὸς τὸ Γ τὸν αὐτὸν ἔχει λόγον, καὶ τὸ Γ πρὸς ἑχάτερον τῶν A, B. of them (respectively).

For let two magnitudes AB and CD be equal multiples of two magnitudes E and F (respectively). And let the (parts) taken away (from the former) AG and CH be equal multiples of E and F (respectively). I say that the remainders GB and HD are also either equal to E and F (respectively), or (are) equal multiples of them.



For let GB be, first of all, equal to E. I say that HD is also equal to F.

For let CK be made equal to F. Since AG and CH are equal multiples of E and F (respectively), and GB (is) equal to E, and KC to F, AB and KH are thus equal multiples of E and F (respectively) [Prop. 5.2]. And AB and CD are assumed (to be) equal multiples of E and F (respectively). Thus, KH and CD are equal multiples of F and F (respectively). Therefore, KH and CD are each equal multiples of F. Thus, KH is equal to CD. Let CH have be taken away from both. Thus, the remainder KC is equal to the remainder HD. But, F is equal to KC. Thus, HD is also equal to F.

So, similarly, we can show that even if GB is a multiple of E then HD will also be the same multiple of F.

Thus, if two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively). (Which is) the very thing it was required to show.

Proposition 7

Equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Let A and B be equal magnitudes, and C some other random magnitude. I say that A and B each have the



Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάχις πολλαπλάσια τὰ Δ,
Ε, τοῦ δὲ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον τὸ Ζ.

Έπεὶ οὖν ἰσάχις ἑστὶ πολλαπλάσιον τὸ Δ τοῦ A xaì τὸ E τοῦ B, ἴσον δὲ τὸ A τῷ B, ἴσον ἄρα xaì τὸ Δ τῷ E. ἄλλο δἑ, ὅ ἔτυχεν, τὸ Z. Eἰ ἄρα ὑπερέχει τὸ Δ τοῦ Z, ὑπερέχει xaì τὸ E τοῦ Z, xaì εἰ ἴσον, ἴσον, xaì εἰ ἔλαττον, ἕλαττον. xaí ἐστι τὰ μὲν Δ , E τῶν A, B ἰσάχις πολλαπλάσια, τὸ δὲ Z τοῦ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον ἕστιν ἄρα ὡς τὸ A πρὸς τὸ Γ, οὕτως τὸ B πρὸς τὸ Γ.

Λέγω [δή],
 ὅτι καὶ τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν ἔχει λόγον.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἴσον ἐστὶ τὸ Δ τῷ Ε· ἄλλο δέ τι τὸ Ζ· εἰ ἄρα ὑπερέχει τὸ Ζ τοῦ Δ , ὑπερέχει καὶ τοῦ Ε, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὸ μὲν Ζ τοῦ Γ πολλαπλάσιον, τὰ δὲ Δ , Ε τῶν Α, Β ἄλλα, ἂ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως τὸ Γ πρὸς τὸ Β.

Τὰ ἴσα ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι ἐἀν μεγέθη τινὰ ἀνάλογον ἤ, καὶ ἀνάπαλιν ἀνάλογον ἔσται. ὅπερ ἔδει δεῖξαι.

 † The Greek text has "*E*", which is obviously a mistake.

[‡] In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\beta : \alpha :: \delta : \gamma$.

η΄.

Τῶν ἀνίσων μεγεθῶν τὸ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον. καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἤπερ πρὸς τὸ μεῖζον.

Έστω ἄνισα μεγέθη τὰ AB, Γ , καὶ ἔστω μεῖζον τὸ AB, ἄλλο δέ, ὃ ἔτυχεν, τὸ Δ · λέγω, ὅτι τὸ AB πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Δ , καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ πρὸς τὸ AB. same ratio to C, and (that) C (has the same ratio) to each of A and B.



For let the equal multiples D and E have been taken of A and B (respectively), and the other random multiple F of C.

Therefore, since D and E are equal multiples of A and B (respectively), and A (is) equal to B, D (is) thus also equal to E. And F (is) different, at random. Thus, if D exceeds F then E also exceeds F, and if (D is) equal (to F then E is also) equal (to F), and if (D is) less (than F then E is also) less (than F). And D and E are equal multiples of A and B (respectively), and F another random multiple of C. Thus, as A (is) to C, so B (is) to C [Def. 5.5].

[So] I say that C^{\dagger} also has the same ratio to each of A and B.

For, similarly, we can show, by the same construction, that D is equal to E. And F (has) some other (value). Thus, if F exceeds D then it also exceeds E, and if (F is) equal (to D then it is also) equal (to E), and if (F is) less (than D then it is also) less (than E). And F is a multiple of C, and D and E other random equal multiples of A and B. Thus, as C (is) to A, so C (is) to B [Def. 5.5].

Thus, equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Corollary[‡]

So (it is) clear, from this, that if some magnitudes are proportional then they will also be proportional inversely. (Which is) the very thing it was required to show.

Proposition 8

For unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater.

Let AB and C be unequal magnitudes, and let AB be the greater (of the two), and D another random magnitude. I say that AB has a greater ratio to D than C (has) to D, and (that) D has a greater ratio to C than (it has) to AB.



Έπεὶ γὰρ μεῖζόν ἐστι τὸ AB τοῦ Γ, κείσθω τῷ Γ ἴσον τὸ BE· τὸ δὴ ἔλασσον τῶν AE, EB πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μεῖζον. ἔστω πρότερον τὸ AE ἔλαττον τοῦ EB, καὶ πεπολλαπλασιάσθω τὸ AE, καὶ ἔστω αὐτοῦ πολλαπλάσιον τὸ ZH μεῖζον ὄν τοῦ Δ, καὶ ὑσαπλάσιόν ἐστι τὸ ZH τοῦ AE, τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν HΘ τοῦ EB τὸ δὲ K τοῦ Γ· καὶ εἰλήφθω τοῦ Δ διπλάσιον μὲν τὸ Λ, τριπλάσιον δὲ τὸ M, καὶ ἑξῆς ἑνὶ πλεῖον, ἔως ἂν τὸ λαμβανόμενον πολλαπλάσιον μὲν γένηται τοῦ Δ, πρώτως δὲ μεῖζον τοῦ K. εἰλήφθω, καὶ ἔστω τὸ N τετραπλάσιον μὲν τοῦ Δ, πρώτως δὲ μεῖζον τοῦ K.

Ἐπεὶ οὖν τὸ Κ τοῦ Ν πρώτως ἐστὶν ἔλαττον, τὸ Κ ἄρα τοῦ Μ οὔχ ἐστιν ἕλαττον. χαὶ ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ ΗΘ τοῦ ΕΒ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ ΖΘ τοῦ ΑΒ. ἰσάκις δέ έστι πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ Κ τοῦ Γ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΖΘ τοῦ AB καὶ τὸ K τοῦ Γ. τὰ ΖΘ, Κ ἄρα τῶν ΑΒ, Γ ἰσάχις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ίσάχις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΕΒ χαὶ τὸ Κ τοῦ Γ, ἴσον δὲ τὸ EB τῷ Γ, ἴσον ἄρα καὶ τὸ HΘ τῷ Κ. τὸ δὲ K τοῦ Μ οὖχ ἐστιν ἔλαττον· οὐδ' ἄρα τὸ ΗΘ τοῦ Μ ἔλαττόν έστιν. με
ίζον δὲ τὸ ZH τοῦ Δ· ὅλον ἄρα τὸ ZΘ συναμφοτέρων τῶν Δ, Μ μεῖζόν ἐστιν. ἀλλὰ συναμφότερα τὰ Δ, M τῷ N ἐστιν ἴσα, ἐπειδήπερ τὸ M τοῦ Δ τριπλάσιόν ἐστιν, συναμφότερα δὲ τὰ M, Δ τοῦ Δ ἐστι τετραπλάσια, ἔστι δὲ καὶ τὸ N τοῦ Δ τετραπλάσιον· συναμφότερα ẳρα τὰ M, Δ τῷ N ἴσα ἐστίν. ἀλλὰ τὸ ΖΘ τῶν M, Δ με
ῖζόν ἐστιν· τὸ ΖΘ ἄρα τοῦ Ν ὑπερέχει· τὸ δὲ Κ τοῦ Ν οὐχ ὑπερέχει. καί έστι τὰ μὲν ΖΘ, Κ τῶν ΑΒ, Γ ἰσάχις πολλαπλάσια, τὸ δὲ Ν τοῦ Δ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· τὸ AB ἄρα πρὸς τὸ Δ μείζονα λόγον
 ἔχει ἤπερ τὸ Γ πρὸς τὸ Δ .

Λέγω δή, ὅτι καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἦπερ τὸ Δ πρὸς τὸ ΑΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι τὸ μὲν Ν τοῦ Κ ὑπερέχει, τὸ δὲ Ν τοῦ ΖΘ οὐχ ὑπερέχει. καί ἐστι τὸ μὲν Ν τοῦ Δ πολλαπλάσιον, τὰ δὲ ΖΘ, Κ τῶν AB, Γ ἄλλα, ἂ ἔτυχεν, ἰσάκις πολλαπλάσια· τὸ Δ ἄρα πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ τὸ Δ πρὸς τὸ AB.

Άλλὰ δὴ τὸ ΑΕ τοῦ ΕΒ μεῖζον ἔστω. τὸ δὴ ἔλαττον τὸ ΕΒ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μεῖζον. πε-



For since AB is greater than C, let BE be made equal to C. So, the lesser of AE and EB, being multiplied, will sometimes be greater than D [Def. 5.4]. First of all, let AE be less than EB, and let AE have been multiplied, and let FG be a multiple of it which (is) greater than D. And as many times as FG is (divisible) by AE, so many times let GH also have become (divisible) by EB, and K by C. And let the double multiple L of D have been taken, and the triple multiple M, and several more, (each increasing) in order by one, until the (multiple) taken becomes the first multiple of D (which is) greater than K. Let it have been taken, and let it also be the quadruple multiple N of D—the first (multiple) greater than K.

Therefore, since K is less than N first, K is thus not less than M. And since FG and GH are equal multiples of AE and EB (respectively), FG and FH are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. And FG and K are equal multiples of AE and C (respectively). Thus, FH and K are equal multiples of ABand C (respectively). Thus, FH, K are equal multiples of AB, C. Again, since GH and K are equal multiples of EB and C, and EB (is) equal to C, GH (is) thus also equal to K. And K is not less than M. Thus, GH not less than M either. And FG (is) greater than D. Thus, the whole of FH is greater than D and M (added) together. But, D and M (added) together is equal to N, inasmuch as M is three times D, and M and D (added) together is four times D, and N is also four times D. Thus, M and D(added) together is equal to N. But, FH is greater than M and D. Thus, FH exceeds N. And K does not exceed N. And FH, K are equal multiples of AB, C, and N another random multiple of D. Thus, AB has a greater ratio to D than C (has) to D [Def. 5.7].

So, I say that D also has a greater ratio to C than D (has) to AB.

For, similarly, by the same construction, we can show that N exceeds K, and N does not exceed FH. And N is a multiple of D, and FH, K other random equal multiples of AB, C (respectively). Thus, D has a greater

πολλαπλασιάσθω, καὶ ἔστω τὸ ΗΘ πολλαπλάσιον μὲν τοῦ EB, μεῖζον δὲ τοῦ Δ · καὶ ὑσαπλάσιόν ἐστι τὸ HΘ τοῦ EB, τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν ΖΗ τοῦ ΑΕ, τὸ δὲ Κ τοῦ Γ. ὑμοίως δὴ δείξομεν, ὅτι τὰ ΖΘ, Κ τῶν ΑΒ, Γ ἰσάχις έστὶ πολλαπλάσια καὶ εἰλήφθω ὁμοίως τὸ Ν πολλαπλάσιον μέν τοῦ Δ, πρώτως δὲ μεῖζον τοῦ ΖΗ· ὥστε πάλιν τὸ ΖΗ τοῦ M ο
ửχ ἐστιν ἕλασσον. μεῖζον δὲ τὸ HΘ τοῦ
 $\Delta^{.}$ ὅλον άρα τὸ ΖΘ τῶν Δ, Μ, τουτέστι τοῦ Ν, ὑπερέχει. τὸ δὲ Κ τοῦ Ν οὐχ ὑπερέχει, ἐπειδήπερ καὶ τὸ ΖΗ μεῖζον ὂν τοῦ ΗΘ, τουτέστι τοῦ Κ, τοῦ Ν οὐχ ὑπερέχει. καὶ ὡσαύτως κατακολουθοῦντες τοῖς ἐπάνω περαίνομεν τὴν ἀπόδειξιν.

Τῶν ἄρα ἀνίσων μεγεθῶν τὸ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔγει ἤπερ τὸ ἔλαττον· καὶ τὸ αὐτὸ πρὸς τὸ έλαττον μείζονα λόγον έχει ήπερ πρός τὸ μεῖζον. ὅπερ ἔδει δεĩξαι.

θ'.

Τὰ πρός τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λὸγον ἴσα ἀλλήλοις έστίν και πρός α τὸ αὐτὸ τὸν αὐτὸν ἕχει λόγον, ἐκεῖνα ἴσα ἐστίν.



Έχέτω γὰρ ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν λόγον λέγω, ὅτι ἴσον ἐστὶ τὸ Α τῷ Β.

Εἰ γὰρ μή, οὐκ ἂν ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ Α τῷ Β.

Ἐχέτω δὴ πάλιν τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν λόγον λέγω, ὅτι ἴσον ἐστὶ τὸ Α τῷ Β.

Εἰ γὰρ μή, οὐκ ἂν τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν εἶγε λόγον ἔγει δέ ἴσον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα άλλήλοις ἐστίν· καὶ πρὸς ἅ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, έχεῖνα ἴσα ἐστίν· ὅπερ ἔδει δεῖξαι.

ι'.

Τῶν πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον

ratio to C than D (has) to AB [Def. 5.5].

And so let AE be greater than EB. So, the lesser, *EB*, being multiplied, will sometimes be greater than *D*. Let it have been multiplied, and let GH be a multiple of EB (which is) greater than D. And as many times as GH is (divisible) by EB, so many times let FG also have become (divisible) by AE, and K by C. So, similarly (to the above), we can show that FH and K are equal multiples of AB and C (respectively). And, similarly (to the above), let the multiple N of D, (which is) the first (multiple) greater than FG, have been taken. So, FGis again not less than M. And GH (is) greater than D. Thus, the whole of FH exceeds D and M, that is to say N. And K does not exceed N, inasmuch as FG, which (is) greater than GH—that is to say, K—also does not exceed N. And, following the above (arguments), we (can) complete the proof in the same manner.

Thus, for unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater. (Which is) the very thing it was required to show.

Proposition 9

(Magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal.



For let A and B each have the same ratio to C. I say that A is equal to B.

For if not, A and B would not each have the same ratio to C [Prop. 5.8]. But they do. Thus, A is equal to B.

So, again, let C have the same ratio to each of A and B. I say that A is equal to B.

For if not, C would not have the same ratio to each of A and B [Prop. 5.8]. But it does. Thus, A is equal to B.

Thus, (magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal. (Which is) the very thing it was required to show.

Proposition 10

For (magnitudes) having a ratio to the same (magέχον ἐχεῖνο μεῖζόν ἐστιν· πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον nitude), that (magnitude which) has the greater ratio is



Έχέτω γὰρ τὸ Α πρὸς τὸ Γ μείζονα λόγον ἤπερ τὸ Β πρὸς τὸ Γ· λέγω, ὅτι μεῖζόν ἐστι τὸ Α τοῦ Β.

El γὰρ μή, ἤτοι ἴσον ἐστὶ τὸ Α τῷ B ἢ ἔλασσον. ἴσον μὲν οῦν οὖx ἐστὶ τὸ Α τῷ B· ἑxάτερον γὰρ ἂν τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον. οὐx ἔχει δέ· οὐx ἄρα ἴσον ἐστὶ τὸ Α τῷ B. οὐδὲ μὴν ἔλασσόν ἐστι τὸ Α τοῦ B· τὸ Α γὰρ ἂν πρὸς τὸ Γ ἐλάσσονα λόγον εἶχεν ἤπερ τὸ Β πρὸς τὸ Γ. οὐx ἔχει δέ· οὐx ἄρα ἔλασσόν ἑστι τὸ Α τοῦ B. ἐδείχθη δὲ οὐδὲ ἴσον· μεῖζον ἄρα ἑστὶ τὸ Α τοῦ B.

Έχέτω δὴ πάλιν τὸ Γ πρὸς τὸ Β μείζονα λόγον ἤπερ τὸ Γ πρὸς τὸ Α· λέγω, ὅτι ἔλασσόν ἐστι τὸ Β τοῦ Α.

Eἰ γὰρ μή, ἤτοι ἴσον ἐστὶν ἢ μεῖζον. ἴσον μὲν οὖν οὕχ ἐστι τὸ B τῷ A· τὸ Γ γὰρ ἂν πρὸς ἑχάτερον τῶν A, B τὸν αὐτὸν εἶχε λόγον. οὐχ ἔχει δέ· οὐχ ἄρα ἴσον ἐστὶ τὸ A τῷ B. οὐδὲ μὴν μεῖζόν ἐστι τὸ B τοῦ A· τὸ Γ γὰρ ἂν πρὸς τὸ B ἐλάσσονα λόγον είχεν ἤπερ πρὸς τὸ A. οὐχ ἔχει δέ· οὐχ ἄρα μεῖζόν ἐστι τὸ B τοῦ A. ἐδείχθη δέ, ὅτι οὐδὲ ἴσον· ἕλαττον ἄρα ἐστὶ τὸ B τοῦ A.

Τῶν ἄρα πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μεῖζόν ἐστιν· καὶ πρὸς ὃ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλαττόν ἐστιν· ὅπερ ἔδει δεῖξαι.

Οἱ τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί.



Έστωσαν γὰρ ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ , ὡς δὲ τὸ Γ πρὸς τὸ Δ , οὕτως τὸ E πρὸς τὸ Z· λέγω, ὅτι ἐστὶν ὡς τὸ A πρὸς τὸ B, οὕτως τὸ E πρὸς τὸ Z.

Εἰλήφθω γὰρ τῶν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Δ ἄλλα, ἂ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, εἰ ἄρα ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Θ τοῦ Μ, καὶ εἰ ἴσον ἐστίν, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. πάλιν, ἐπεί ἐστιν (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser.



For let A have a greater ratio to C than B (has) to C. I say that A is greater than B.

For if not, A is surely either equal to or less than B. In fact, A is not equal to B. For (then) A and B would each have the same ratio to C [Prop. 5.7]. But they do not. Thus, A is not equal to B. Neither, indeed, is A less than B. For (then) A would have a lesser ratio to C than B (has) to C [Prop. 5.8]. But it does not. Thus, A is not less than B. And it was shown not (to be) equal either. Thus, A is greater than B.

So, again, let C have a greater ratio to B than C (has) to A. I say that B is less than A.

For if not, (it is) surely either equal or greater. In fact, B is not equal to A. For (then) C would have the same ratio to each of A and B [Prop. 5.7]. But it does not. Thus, A is not equal to B. Neither, indeed, is B greater than A. For (then) C would have a lesser ratio to B than (it has) to A [Prop. 5.8]. But it does not. Thus, B is not greater than A. And it was shown that (it is) not equal (to A) either. Thus, B is less than A.

Thus, for (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser. (Which is) the very thing it was required to show.

Proposition 11[†]

(Ratios which are) the same with the same ratio are also the same with one another.

A	С	E
B i i i i i i i i i i i i i i i i i i i	$D \longmapsto$	F
G	H⊢−−−−	K
$\Gamma \longmapsto$	$M \longmapsto \cdots \longmapsto$	N

For let it be that as A (is) to B, so C (is) to D, and as C (is) to D, so E (is) to F. I say that as A is to B, so E (is) to F.

For let the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).

And since as A is to B, so C (is) to D, and the equal multiples G and H have been taken of A and C (respectively), and the other random equal multiples L and M of B and D (respectively), thus if G exceeds L then H also exceeds M, and if (G is) equal (to L then H is also)

ώς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν Γ, Ε ἰσάχις πολλαπλάσια τὰ Θ, Κ, τῶν δὲ Δ, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Θ τοῦ Μ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλλατον, ἕλαττον. ἀλλὰ εἰ ὑπερεῖχε τὸ Θ τοῦ Μ, ὑπερεῖχε καὶ τὸ Η τοῦ Λ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἕλαττον ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἕλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Ε ἰσάχις πολλαπλάσια, τὰ δὲ Λ, Ν τῶν Β, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια⁻ ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ.

Οί ἄρα τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί ὅπερ ἔδει δεῖξαι.

equal (to M), and if (G is) less (than L then H is also) less (than M) [Def. 5.5]. Again, since as C is to D, so E (is) to F, and the equal multiples H and K have been taken of C and E (respectively), and the other random equal multiples M and N of D and F (respectively), thus if H exceeds M then K also exceeds N, and if (H is) equal (to M then K is also) equal (to N), and if (H is) less (than M then K is also) less (than N) [Def. 5.5]. But (we saw that) if H was exceeding M then G was also exceeding L, and if (H was) equal (to M then G was also) equal (to L), and if (H was) less (than M then G was also) less (than L). And, hence, if G exceeds L then Kalso exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N). And G and K are equal multiples of Aand E (respectively), and L and N other random equal multiples of B and F (respectively). Thus, as A is to B, so *E* (is) to *F* [Def. 5.5].

Thus, (ratios which are) the same with the same ratio are also the same with one another. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta :: \epsilon : \zeta$ then $\alpha : \beta :: \epsilon : \zeta$.

ıβ'.

Έὰν ἢ ὑποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἕν τῶν ἡγουμένων πρὸς ἕν τῶν ἑπομένων, οὕτως ἄπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα.



Έστωσαν ὑποσαοῦν μεγέθη ἀνάλογον τὰ A, B, Γ, Δ, E, Z, ὡς τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ E πρὸς το Z· λέγω, ὅτι ἐστὶν ὡς τὸ A πρὸς τὸ B, οὕτως τὰ A, Γ, E πρὸς τὰ B, Δ, Z.

Εἰλήφθω γὰρ τῶν μὲν Α, Γ, Ε ἰσάχις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Λ, Μ, Ν.

Kal ἐπεί ἐστιν ὡς τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ E πρὸς τὸ Z, καὶ εἴληπται τῶν μὲν A, Γ, E ἰσάκις πολλαπλάσια τὰ H, Θ, K τῶν δὲ B, Δ, Z ӑλλα, ἂ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, M, N, εἰ ἄρα ὑπερέχει τὸ H τοῦ Λ, ὑπερέχει καὶ τὸ Θ τοῦ M, καὶ τὸ K τοῦ N, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἑλαττον, ἕλαττον. ὥστε καὶ εἰ ὑπερέχει τὸ H τοῦ Λ,

Proposition 12[†]

If there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following.



Let there be any number of magnitudes whatsoever, A, B, C, D, E, F, (which are) proportional, (so that) as A (is) to B, so C (is) to D, and E to F. I say that as A is to B, so A, C, E (are) to B, D, F.

For let the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).

And since as A is to B, so C (is) to D, and E to F, and the equal multiples G, H, K have been taken of A, C, E(respectively), and the other random equal multiples L, M, N of B, D, F (respectively), thus if G exceeds L then H also exceeds M, and K (exceeds) N, and if (G is) equal (to L then H is also) equal (to M, and K to N), ύπερέχει καὶ τὰ Η, Θ, Κ τῶν Λ, Μ, Ν, καὶ εἰ ἴσον, ἴσα, καὶ εἰ ἔλαττον, ἕλαττονα. καί ἐστι τὸ μὲν Η καὶ τὰ Η, Θ, Κ τοῦ Α καὶ τῶν Α, Γ, Ε ἰσάχις πολλαπλάσια, ἐπειδήπερ ἐὰν ἢ ὅποσαοῦν μεγέθη ὅποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕχαστον ἑχάστου ἰσάχις πολλαπλάσιον, ὅσαπλάσιόν ἐστιν ἐν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων. διὰ τὰ αὐτὰ δὴ καὶ τὸ Λ καὶ τὰ Λ, Μ, Ν τοῦ Β καὶ τῶν Β, Δ, Ζ ἰσάχις ἐστὶ πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Έὰν ἄρα ἢ ὑποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἕν τῶν ἡγουμένων πρὸς ἕν τῶν ἑπομένων, οὕτως ἄπαντα τὰ ἡγούμενα πρὸς ἄπαντα τὰ ἑπόμενα· ὅπερ ἔδει δεῖζαι. and if (G is) less (than L then H is also) less (than M, and K than N) [Def. 5.5]. And, hence, if G exceeds Lthen G, H, K also exceed L, M, N, and if (G is) equal (to L then G, H, K are also) equal (to L, M, N) and if (G is) less (than L then G, H, K are also) less (than L, M, N). And G and G, H, K are equal multiples of A and A, C, E (respectively), inasmuch as if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second) [Prop. 5.1]. So, for the same (reasons), L and L, M, N are also equal multiples of B and B, D, F (respectively). Thus, as A is to B, so A, C, E (are) to B, D, F (respectively).

Thus, if there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \alpha' :: \beta : \beta' :: \gamma : \gamma'$ etc. then $\alpha : \alpha' :: (\alpha + \beta + \gamma + \cdots) : (\alpha' + \beta' + \gamma' + \cdots)$.

ıγ'.

Έὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχῃ ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἔξει ἢ πέμπτον πρὸς ἕκτον.

A	$\Gamma \longmapsto$	E ———
$B \longmapsto$	$\Delta \longmapsto$	Z
M	⊣ H ⊢−−−−+	н Ө нн
N⊢⊢⊢	K⊢−−−−	$\Lambda \longmapsto$

Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, τρίτον δὲ τὸ Γ πρὸς τέταρτον τὸ Δ μείζονα λόγον ἐχέτω ἢ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ. λέγω, ὅτι καὶ πρῶτον τὸ Α πρὸς δεύτερον τὸ Β μείζονα λόγον ἕξει ἤπερ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ.

Έπεὶ γὰρ ἔστι τινὰ τῶν μὲν Γ, Ε ἰσάχις πολλαπλάσια, τῶν δὲ Δ , Ζ ἄλλα, ἀ ἔτυχεν, ἰσάχις πολλαπλάσια, καὶ τὸ μὲν τοῦ Γ πολλαπλάσιον τοῦ τοῦ Δ πολλαπλασίου ὑπερέχει, τὸ δὲ τοῦ Ε πολλαπλάσιον τοῦ τοῦ Ζ πολλαπλασίου οὐχ ὑπερέχει, εἰλήφθω, καὶ ἔστω τῶν μὲν Γ, Ε ἰσάχις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Δ , Ζ ἄλλα, ὰ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Κ, Λ, ὥστε τὸ μὲν Η τοῦ Κ ὑπερέχειν, τὸ δὲ Θ τοῦ Λ μὴ ὑπερέχειν· καὶ ὑσαπλάσιον μέν ἐστι τὸ Η τοῦ Γ, τοσαυταπλάσιον ἔστω καὶ τὸ Μ τοῦ Α, ὑσαπλάσιον δὲ τὸ Κ τοῦ Δ , τοσαυταπλάσιον ἔστω καὶ τὸ Ν τοῦ Β.

Proposition 13[†]

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the third (magnitude) has a greater ratio to the fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth.

A	С	E
B ⊢−−−	D	F
M	G I I I I I I I I I I I I I I I I I I I	H · · · · · · · · · · · · · · · · · · ·
$N \longmapsto \cdots \longmapsto$	K	L

For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D, and let the third (magnitude) C have a greater ratio to the fourth D than a fifth E (has) to a sixth F. I say that the first (magnitude) A will also have a greater ratio to the second B than the fifth E (has) to the sixth F.

For since there are some equal multiples of C and E, and other random equal multiples of D and F, (for which) the multiple of C exceeds the (multiple) of D, and the multiple of E does not exceed the multiple of F [Def. 5.7], let them have been taken. And let G and H be equal multiples of C and E (respectively), and K and L other random equal multiples of D and F (respectively), such that G exceeds K, but H does not exceed L. And as many times as G is (divisible) by C, so many times let M be (divisible) by A. And as many times as K (is divisible)

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάχις πολλαπλάσια τὰ Μ, Η, τῶν δὲ Β, Δ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Ν, Κ, εἰ ἄρα ὑπερέχει τὸ Μ τοῦ Ν, ὑπερέχει καὶ τὸ Η τοῦ Κ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλλατον. ὑπερέχει δὲ τὸ Η τοῦ Κ· ὑπερέχει ἄρα καὶ τὸ Μ τοῦ Ν. τὸ δὲ Θ τοῦ Λ οὐχ ὑπερέχει· καί ἐστι τὰ μὲν Μ, Θ τῶν Α, Ε ἰσάχις πολλαπλάσια, τὰ δὲ Ν, Λ τῶν Β, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὸ ἄρα Α πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Ζ.

Έὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχῃ ἢ πέμπτον πρὸς ἔκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἔξει ἢ πέμπτον πρὸς ἔκτον. ὅπερ ἔδει δεῖξαι. by D, so many times let N be (divisible) by B.

And since as A is to B, so C (is) to D, and the equal multiples M and G have been taken of A and C (respectively), and the other random equal multiples N and K of B and D (respectively), thus if M exceeds N then G exceeds K, and if (M is) equal (to N then G is also) equal (to K), and if (M is) less (than N then G is also) less (than K) [Def. 5.5]. And G exceeds K. Thus, M also exceeds N. And H does not exceeds L. And M and H are equal multiples of A and E (respectively), and N and L other random equal multiples of B and F (respectively). Thus, A has a greater ratio to B than E (has) to F [Def. 5.7].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and a third (magnitude) has a greater ratio to a fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta > \epsilon : \zeta$ then $\alpha : \beta > \epsilon : \zeta$.

ιδ'.

Έὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ἦ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.



Πρῶτον γὰρ τὸ A πρὸς δεύτερον τὸ B αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ , μεῖζον δὲ ἔστω τὸ A τοῦ Γ · λέγω, ὅτι καὶ τὸ B τοῦ Δ μεῖζόν ἐστιν.

Έπεὶ γὰρ τὸ Α τοῦ Γ μεῖζόν ἐστιν, ἄλλο δέ, ὅ ἔτυχεν, [μέγεθος] τὸ B, τὸ A ἄρα πρὸς τὸ B μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ B. ὡς δὲ τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ B. πρὸς ὅ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἕλασσόν ἐστιν· ἕλασσον ἄρα τὸ Δ τοῦ B· ὥστε μεῖζόν ἐστι τὸ B τοῦ Δ.

Όμοίως δη δε
ίζομεν, ὅτι κάν ἴσον η τὸ Α τῷ Γ, ἴσον ἕσται καὶ τὸ Β τῷ
 Δ , κάν ἕλασσον η τὸ Α τοῦ Γ, ἕλασσον ἔσται καὶ τὸ Β τοῦ
 Δ .

Έὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ἢ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον. ὅπερ ἔδει δεῖξαι.

Proposition 14[†]

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth).



For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D. And let A be greater than C. I say that B is also greater than D.

For since A is greater than C, and B (is) another random [magnitude], A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. And as A (is) to B, so C (is) to D. Thus, C also has a greater ratio to D than C (has) to B. And that (magnitude) to which the same (magnitude) has a greater ratio is the lesser [Prop. 5.10]. Thus, D (is) less than B. Hence, B is greater than D.

So, similarly, we can show that even if A is equal to C then B will also be equal to D, and even if A is less than C then B will also be less than D.

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is)
equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth). (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha \stackrel{\geq}{=} \gamma$ as $\beta \stackrel{\geq}{=} \delta$.

ιε΄.

Τὰ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα.



Έστω γὰρ ἰσάχις πολλαπλάσιον τὸ AB τοῦ Γ χαὶ το ΔE τοῦ Z· λέγω, ὅτι ἐστὶν ὡς τὸ Γ πρὸς τὸ Z, οὕτως τὸ AB πρὸς τὸ ΔE .

Ἐπεὶ γὰρ ἰσάχις ἑστὶ πολλαπλάσιον τὸ AB τοῦ Γ xaì τὸ ΔE τοῦ Z, ὅσα ἄρα ἐστὶν ἐν τῷ AB μεγέθη ἴσα τῷ Γ, τοσαῦτα xaì ἐν τῷ ΔE ἴσα τῷ Z. διῃρήσθω τὸ μὲν AB εἰς τὰ τῷ Γ ἴσα τὰ AH, HΘ, ΘB, τὸ δὲ ΔE εἰς τὰ τῷ Z ἴσα τὰ ΔK , KΛ, ΛE · ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH, HΘ, ΘB τῷ πλήθει τῶν ΔK , KΛ, ΛE . xaì ἐπεὶ ἴσα ἐστὶ τὰ AH, HΘ, ΘB ἀλλήλοις, ἔστι δὲ xaì τὰ ΔK , KΛ, ΛE ἴσα ἀλλήλοις, ἔστιν ἄρα ὡς τὸ AH πρὸς τὸ ΔK , οὕτως τὸ HΘ πρὸς τὸ ΚΛ, xaì τὸ ΘB πρὸς τὸ ΛE . ἔσται ἄρα xaì ὡς ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἑπομένων, οὕτως ἄπαντα τὰ ἡγουμένω πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα ὡς τὸ AH πρὸς τὸ ΔK , οῦτως τὸ AB πρὸς τὸ ΔE . ἴσον δὲ τὸ μὲν AH τῷ Γ, τὸ δὲ ΔK τῷ Z· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Zο ὕτως τὸ AB πρὸς τὸ ΔE .

Τὰ ἄρα μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα ὅπερ ἔδει δεῖξαι.

[†] In modern notation, this proposition reads that $\alpha : \beta :: m \alpha : m \beta$.

ເຈ່.

Ἐἀν τέσσαρα μεγέθη ἀνάλογον ἦ, καὶ ἐναλλὰξ ἀνάλογον ἔσται.

Έστω τέσσαρα μεγέθη ἀνάλογον τὰ A, B, Γ , Δ , ὡς τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ · λέγω, ὅτι καὶ ἐναλλὰξ [ἀνάλογον] ἔσται, ὡς τὸ A πρὸς τὸ Γ , οὕτως τὸ B πρὸς τὸ Δ .

Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάχις πολλαπλάσια τὰ Ε, Ζ, τῶν δ
ὲ Γ, Δ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Η, Θ.

Proposition 15[†]

Parts have the same ratio as similar multiples, taken in corresponding order.



For let AB and DE be equal multiples of C and F (respectively). I say that as C is to F, so AB (is) to DE.

For since AB and DE are equal multiples of C and F (respectively), thus as many magnitudes as there are in AB equal to C, so many (are there) also in DE equal to F. Let AB have been divided into (magnitudes) AG, GH, HB, equal to C, and DE into (magnitudes) DK, *KL*, *LE*, equal to *F*. So, the number of (magnitudes) AG, GH, HB will equal the number of (magnitudes) DK, KL, LE. And since AG, GH, HB are equal to one another, and DK, KL, LE are also equal to one another, thus as AG is to DK, so GH (is) to KL, and HB to LE[Prop. 5.7]. And, thus (for proportional magnitudes), as one of the leading (magnitudes) will be to one of the following, so all of the leading (magnitudes will be) to all of the following [Prop. 5.12]. Thus, as AG is to DK, so AB(is) to DE. And AG is equal to C, and DK to F. Thus, as C is to F, so AB (is) to DE.

Thus, parts have the same ratio as similar multiples, taken in corresponding order. (Which is) the very thing it was required to show.

Proposition 16[†]

If four magnitudes are proportional then they will also be proportional alternately.

Let A, B, C and D be four proportional magnitudes, (such that) as A (is) to B, so C (is) to D. I say that they will also be [proportional] alternately, (so that) as A (is) to C, so B (is) to D.

For let the equal multiples E and F have been taken of A and B (respectively), and the other random equal multiples G and H of C and D (respectively).



Καὶ ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσιον τὸ Ε τοῦ Α χαὶ τὸ Ζ τοῦ Β, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν έχει λόγον, έστιν άρα ώς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ · καὶ ὡς ἄρα τὸ Γ πρὸς τὸ Δ , οὕτως τὸ $\mathbf E$ πρὸς τὸ $\mathbf Z$. πάλιν, ἐπεὶ τὰ Η, Θ τῶν Γ, Δ ἰσάχις ἐστὶ πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Δ , οὕτως τὸ H πρὸς τὸ Θ. ὡς δὲ τὸ Γ πρὸς τὸ Δ , [οὕτως] τὸ Ε πρὸς τὸ Ζ· καὶ ὡς ἄρα τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Η πρὸς τὸ Θ. ἐὰν δὲ τέσσαρα μεγέθη ἀνάλογον ἢ, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ἢ, καὶ τὸ δεύτερον τοῦ τετάρτου μεϊζον έσται, καν ίσον, ίσον, κάν έλαττον, έλαττον. εί άρα ύπερέχει τὸ Ε τοῦ Η, ὑπερέχει καὶ τὸ Ζ τοῦ Θ, καὶ εἰ ἴσον, ίσον, καὶ εἰ ἕλαττον, ἕλαττον. καί ἐστι τὰ μὲν Ε, Ζ τῶν Α, Β ἰσάχις πολλαπλάσια, τὰ δὲ Η, Θ τῶν Γ, Δ ἄλλα, ἂ έτυχεν, ἰσάχις πολλαπλάσια έστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ B πρὸς τὸ Δ .

Έὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ἢ, καὶ ἐναλλὰξ ἀνάλογον ἔσται ὅπερ ἔδει δεῖξαι.



And since E and F are equal multiples of A and B(respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A is to B, so E (is) to F. But as A (is) to B, so C (is) to D. And, thus, as C (is) to D, so E (is) to F [Prop. 5.11]. Again, since G and H are equal multiples of C and D (respectively), thus as Cis to D, so G (is) to H [Prop. 5.15]. But as C (is) to D, [so] E (is) to F. And, thus, as E (is) to F, so G (is) to *H* [Prop. 5.11]. And if four magnitudes are proportional, and the first is greater than the third then the second will also be greater than the fourth, and if (the first is) equal (to the third then the second will also be) equal (to the fourth), and if (the first is) less (than the third then the second will also be) less (than the fourth) [Prop. 5.14]. Thus, if E exceeds G then F also exceeds H, and if (E is) equal (to G then F is also) equal (to H), and if (E is) less (than G then F is also) less (than H). And E and F are equal multiples of A and B (respectively), and G and Hother random equal multiples of C and D (respectively). Thus, as A is to C, so B (is) to D [Def. 5.5].

Thus, if four magnitudes are proportional then they will also be proportional alternately. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \gamma :: \beta : \delta$.

Έὰν συγκείμενα μεγέθη ἀνάλογον ἢ, καὶ διαιρεθέντα ἀνάλογον ἔσται.



Έστω συγχείμενα μεγέθη ἀνάλογον τὰ AB, BE, ΓΔ, ΔΖ, ὡς τὸ AB πρὸς τὸ BE, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ· λέγω, ὅτι καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ AE πρὸς τὸ EB, οὕτως τὸ ΓΖ πρὸς τὸ ΔΖ.

Eἰλήφθω γὰρ τῶν μὲν ΑΕ, ΕΒ, ΓΖ, ΖΔ ἰσάχις πολλαπλάσια τὰ ΗΘ, ΘΚ, ΛΜ, ΜΝ, τῶν δὲ ΕΒ, ΖΔ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ ΚΞ, ΝΠ.

Καὶ ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΘΚ τοῦ ΕΒ, ἰσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ

Proposition 17[†]

If composed magnitudes are proportional then they will also be proportional (when) separarted.



Let AB, BE, CD, and DF be composed magnitudes (which are) proportional, (so that) as AB (is) to BE, so CD (is) to DF. I say that they will also be proportional (when) separated, (so that) as AE (is) to EB, so CF (is) to DF.

For let the equal multiples GH, HK, LM, and MN have been taken of AE, EB, CF, and FD (respectively), and the other random equal multiples KO and NP of EB and FD (respectively).

ΑΕ καὶ τὸ ΗΚ τοῦ ΑΒ. ἰσάκις δέ ἐστι πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΛΜ τοῦ ΓΖ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΛΜ τοῦ ΓΖ. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ MN τοῦ ΖΔ, ἰσάκις ắρα έστι πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΛΝ τοῦ ΓΔ. ἰσάκις δὲ ἦν πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΗΚ τοῦ ΑΒ· ίσάχις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ χαὶ τὸ ΛΝ τοῦ ΓΔ. τὰ ΗΚ, ΛΝ ἄρα τῶν ΑΒ, ΓΔ ἰσάχις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλασίον τὸ ΘΚ τοῦ ΕΒ καὶ τὸ MN τοῦ $Z\Delta$, ἔστι δὲ καὶ τὸ $K\Xi$ τοῦ EB ἰσάκις πολλαπλάσιον καὶ τὸ ΝΠ τοῦ ZΔ, καὶ συντεθὲν τὸ $\Theta\Xi$ τοῦ EB ἰσάχις ἐστὶ πολλαπλάσιον καὶ τὸ ΜΠ τοῦ ΖΔ. καὶ ἐπεί ἐστιν ὡς τὸ AB πρός τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, καὶ εἴληπται τῶν μέν ΑΒ, ΓΔ ἰσάχις πολλαπλάσια τὰ ΗΚ, ΛΝ, τῶν δὲ ΕΒ, ΖΔ ἰσάχις πολλαπλάσια τὰ ΘΞ, ΜΠ, εἰ ἄρα ὑπερέχει τὸ ΗΚ τοῦ ΘΞ, ὑπερέχει καὶ τὸ ΛΝ τοῦ ΜΠ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἕλαττον, ἕλαττον. ὑπερεγέτω δὴ τὸ ΗΚ τοῦ ΘΞ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΘΚ ὑπερέγει ἄρα καὶ τὸ ΗΘ τοῦ ΚΞ. ἀλλα εἰ ὑπερεῖχε τὸ ΗΚ τοῦ ΘΞ ὑπερεῖχε καὶ τὸ ΛΝ τοῦ ΜΠ· ὑπερέγει ἄρα καὶ τὸ ΛΝ τοῦ ΜΠ, καὶ κοινοῦ άφαιρεθέντος τοῦ MN ὑπερέχει καὶ τὸ ΛΜ τοῦ NΠ· ὥστε εἰ ὑπερέχει τὸ ΗΘ τοῦ ΚΞ, ὑπερέχει καὶ τὸ ΛΜ τοῦ ΝΠ. όμοίως δή δεῖξομεν, ὅτι κἂν ἴσον ἤ τὸ ΗΘ τῷ ΚΞ, ἴσον ἔσται καὶ τὸ ΛΜ τῷ ΝΠ, κἂν ἔλαττον, ἔλαττον. καί ἐστι τὰ μέν ΗΘ, ΛΜ τῶν ΑΕ, ΓΖ ἰσάχις πολλαπλάσια, τὰ δὲ ΚΞ, ΝΠ τῶν ΕΒ, ΖΔ ἄλλα, ἁ ἔτυχεν, ἰσάχις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ.

Έὰν ἄρα συγκείμενα μεγέθη ἀνάλογον ἤ, καὶ διαιρεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

And since GH and HK are equal multiples of AE and EB (respectively), GH and GK are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. But GH and LM are equal multiples of AE and CF (respectively). Thus, GK and LM are equal multiples of AB and CF(respectively). Again, since LM and MN are equal multiples of CF and FD (respectively), LM and LN are thus equal multiples of *CF* and *CD* (respectively) [Prop. 5.1]. And LM and GK were equal multiples of CF and AB(respectively). Thus, GK and LN are equal multiples of AB and CD (respectively). Thus, GK, LN are equal multiples of AB, CD. Again, since HK and MN are equal multiples of EB and FD (respectively), and KOand NP are also equal multiples of EB and FD (respectively), then, added together, HO and MP are also equal multiples of EB and FD (respectively) [Prop. 5.2]. And since as AB (is) to BE, so CD (is) to DF, and the equal multiples GK, LN have been taken of AB, CD, and the equal multiples HO, MP of EB, FD, thus if GK exceeds HO then LN also exceeds MP, and if (GK is) equal (to HO then LN is also) equal (to MP), and if (GK is) less (than HO then LN is also) less (than MP) [Def. 5.5]. So let GK exceed HO, and thus, HK being taken away from both, GH exceeds KO. But (we saw that) if GKwas exceeding HO then LN was also exceeding MP. Thus, LN also exceeds MP, and, MN being taken away from both, LM also exceeds NP. Hence, if GH exceeds KO then LM also exceeds NP. So, similarly, we can show that even if GH is equal to KO then LM will also be equal to NP, and even if (GH is) less (than KO then LM will also be) less (than NP). And GH, LM are equal multiples of AE, CF, and KO, NP other random equal multiples of EB, FD. Thus, as AE is to EB, so CF (is) to FD [Def. 5.5].

Thus, if composed magnitudes are proportional then they will also be proportional (when) separarted. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha + \beta : \beta :: \gamma + \delta : \delta$ then $\alpha : \beta :: \gamma : \delta$.



Έαν διηρημένα μεγέθη ανάλογον ή, και συντεθέντα ἀνάλογον ἔσται.





Proposition 18[†]

If separated magnitudes are proportional then they will also be proportional (when) composed.



Let AE, EB, CF, and FD be separated magnitudes (which are) proportional, (so that) as AE (is) to EB, so

οὕτως τὸ Γ Δ πρὸς τὸ Ζ Δ .

Εἰ γὰρ μή ἐστὶν ὡς τὸ AB πρὸς τὸ BE, οὕτως τὸ ΓΔ πρὸς τὸ ΔZ , ἔσται ὡς τὸ AB πρὸς τὸ BE, οὕτως τὸ ΓΔ ἤτοι πρὸς ἕλασσόν τι τοῦ ΔZ ἢ πρὸς μεῖζον.

Έστω πρότερον πρὸς ἕλασσον τὸ ΔΗ. καὶ ἐπεί ἐστιν ὡς τὸ AB πρὸς τὸ BE, οὕτως τὸ ΓΔ πρὸς τὸ ΔΗ, συγκείμενα μεγέθη ἀνάλογόν ἐστιν· ὥστε καὶ διαιρεθέντα ἀνάλογον ἔσται. ἔστιν ἄρα ὡς τὸ AE πρὸς τὸ EB, οὕτως τὸ ΓΗ πρὸς τὸ HΔ. ὑπόκειται δὲ καὶ ὡς τὸ AE πρὸς τὸ EB, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. καὶ ὡς ἄρα τὸ ΓΗ πρὸς τὸ HΔ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. μεῖζον δὲ τὸ πρῶτον τὸ ΓΗ τοῦ τρίτου τοῦ ΖΔ. ἀλλὰ καὶ ἕλαττον· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἐστὶν ὡς τὸ AB πρὸς τὸ BE, οὕτως τὸ ΓΔ πρὸς ἕλασσον τοῦ ΖΔ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ πρὸς μεῖζον· πρὸς αὐτὸ ἄρα.

Έὰν ἄρα διηρημένα μεγέθη ἀνάλογον ἤ, καὶ συντεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι. (when) composed, (so that) as AB (is) to BE, so CD (is) to FD.

For if (it is) not (the case that) as AB is to BE, so CD (is) to FD, then it will surely be (the case that) as AB (is) to BE, so CD is either to some (magnitude) less than DF, or (some magnitude) greater (than DF).[‡]

Let it, first of all, be to (some magnitude) less (than DF), (namely) DG. And since composed magnitudes are proportional, (so that) as AB is to BE, so CD (is) to DG, they will thus also be proportional (when) separated [Prop. 5.17]. Thus, as AE is to EB, so CG (is) to GD. But it was also assumed that as AE (is) to EB, so CF (is) to FD. Thus, (it is) also (the case that) as CG (is) to GD, so CF (is) to FD [Prop. 5.11]. And the first (magnitude) CG (is) greater than the third CF. Thus, the second (magnitude) GD (is) also greater than the fourth FD [Prop. 5.14]. But (it is) also less. The very thing is impossible. Thus, (it is) not (the case that) as AB is to BE, so CD (is) to less than FD. Similarly, we can show that neither (is it the case) to greater (than FD). Thus, (it is the case) to the same (as FD).

Thus, if separated magnitudes are proportional then they will also be proportional (when) composed. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha + \beta : \beta :: \gamma + \delta : \delta$.

[‡] Here, Euclid assumes, without proof, that a fourth magnitude proportional to three given magnitudes can always be found.

*ι*θ'.

Έὰν ἢ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον.



Έστω γὰρ ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ ΑΕ πρὸς ἀφειρεθὲν τὸ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Έπεὶ γάρ ἐστιν ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ AE πρὸς τὸ ΓΖ, καὶ ἐναλλὰξ ὡς τὸ BA πρὸς τὸ AE, οὕτως τὸ $\Delta\Gamma$ πρὸς τὸ ΓΖ. καὶ ἐπεὶ συγκείμενα μεγέϑη ἀνάλογόν ἐστιν, καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ BE πρὸς τὸ EA, οὕτως τὸ Δ Z πρὸς τὸ ΓΖ· καὶ ἐναλλάξ, ὡς τὸ BE πρὸς τὸ ΔΖ, οὕτως τὸ EA πρὸς τὸ ΖΓ. ὡς δὲ τὸ AE πρὸς τὸ ΓΖ, οὕτως ὑπόκειται ὅλον τὸ AB πρὸς ὅλον τὸ ΓΔ. καὶ λοιπὸν ἄρα τὸ EB πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ AB πρὸς ὅλον τὸ ΓΔ.

Έαν ἄρα ἤ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς

Proposition 19[†]

If as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole.



For let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF. I say that the remainder EB to the remainder FD will also be as the whole AB (is) to the whole CD.

For since as AB is to CD, so AE (is) to CF, (it is) also (the case), alternately, (that) as BA (is) to AE, so DC (is) to CF [Prop. 5.16]. And since composed magnitudes are proportional then they will also be proportional (when) separated, (so that) as BE (is) to EA, so DF (is) to CF [Prop. 5.17]. Also, alternately, as BE (is) to DF, so EA (is) to FC [Prop. 5.16]. And it was assumed that as AE (is) to CF, so the whole AB (is) to the whole CD. And, thus, as the remainder EB (is) to the remainder FD, so the whole AB will be to the whole CD. ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον [ὅπερ ἔδει δεῖξαι].

[Καὶ ἐπεὶ ἐδείχθη ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΕΒ πρὸς τὸ ΖΔ, καὶ ἐναλλὰξ ὡς τὸ ΑΒ πρὸς τὸ ΒΕ οὕτως τὸ ΓΔ πρὸς τὸ ΖΔ, συγκείμενα ἄρα μεγέθη ἀνάλογόν ἐστιν· ἐδείχθη δὲ ὡς τὸ ΒΑ πρὸς τὸ ΑΕ, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ· καί ἐστιν ἀναστρέψαντι].

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι ἐἀν συγκείμενα μεγέθη ἀνάλογον ἢ, καὶ ἀναστρέψαντι ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \beta :: \alpha - \gamma : \beta - \delta$.

[‡] In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \alpha - \beta :: \gamma : \gamma - \delta$.

κ΄.

Έὰν ἢ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγω, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ἢ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.



Έστω τρία μεγέθη τὰ A, B, Γ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, E, Z, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ Δ πρὸς τὸ E, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ E πρὸς τὸ Z, δι᾽ ἴσου δὲ μεῖζον ἔστω τὸ A τοῦ Γ· λέγω, ὅτι καὶ τὸ Δ τοῦ Z μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἕλαττον, ἕλαττον.

Έπεὶ γὰρ μεῖζόν ἐστι τὸ Α τοῦ Γ, ἄλλο δέ τι τὸ B, τὸ δὲ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἕλαττον, τὸ Α ἄρα πρὸς τὸ B μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ B. ἀλλ' ὡς μὲν τὸ Α πρὸς τὸ B [οὕτως] τὸ Δ πρὸς τὸ E, ὡς δὲ τὸ Γ πρὸς τὸ B, ἀνάπαλιν οὕτως τὸ Ζ πρὸς τὸ E, ὡς ῶὲ τὸ Γ πρὸς τὸ B μείζονα λόγον ἔχει ἤπερ τὸ Ζ πρὸς τὸ E. τῶν δὲ πρὸς τὸ E μείζονα λόγον ἔχει ἤπερ τὸ Ζ πρὸς τὸ E. τῶν δὲ πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μεῖζόν ἐστιν. μεῖζον ἄρα τὸ Δ τοῦ Z. ὑμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ἢ τὸ A τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Z, κἂν

Thus, if as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole. [(Which is) the very thing it was required to show.]

[And since it was shown (that) as AB (is) to CD, so EB (is) to FD, (it is) also (the case), alternately, (that) as AB (is) to BE, so CD (is) to FD. Thus, composed magnitudes are proportional. And it was shown (that) as BA (is) to AE, so DC (is) to CF. And (the latter) is converted (from the former).]

Corollary[‡]

So (it is) clear, from this, that if composed magnitudes are proportional then they will also be proportional (when) converted. (Which is) the very thing it was required to show.

Proposition 20[†]

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Let A, B, and C be three magnitudes, and D, E, F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, (so that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. And let A be greater than C, via equality. I say that D will also be greater than F. And if (A is) equal (to C then D will also be) equal (to F). And if (A is) less (than C then D will also be) less (than F).

For since A is greater than C, and B some other (magnitude), and the greater (magnitude) has a greater ratio than the lesser to the same (magnitude) [Prop. 5.8], A thus has a greater ratio to B than C (has) to B. But as A (is) to B, [so] D (is) to E. And, inversely, as C (is) to B, so F (is) to E [Prop. 5.7 corr.]. Thus, D also has a greater ratio to E than F (has) to E [Prop. 5.13]. And for (mag-

ἕλαττον, ἕλαττον.

Έαν ἄρα ἤ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγω, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ἤ, καὶ τὸ τέταρτον τοῦ ἔκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον[.] ὅπερ ἔδει δεῖξαι. nitudes) having a ratio to the same (magnitude), that having the greater ratio is greater [Prop. 5.10]. Thus, D (is) greater than F. Similarly, we can show that even if A is equal to C then D will also be equal to F, and even if (A is) less (than C then D will also be) less (than F).

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third, then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And (if the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \delta : \epsilon$ and $\beta : \gamma :: \epsilon : \zeta$ then $\alpha \gtrless \gamma$ as $\delta \gtrless \zeta$.

κα΄.

Έὰν ἢ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ἢ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ἢ, καὶ τὸ τέταρτον τοῦ ἔκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.



Έστω τρία μεγέθη τὰ A, B, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, E, Z, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ E πρὸς τὸ Z, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ E, δι᾽ ἴσου δὲ τὸ A τοῦ Γ μεῖζον ἔστω· λέγω, ὅτι καὶ τὸ Δ τοῦ Z μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἒλαττον, ἒλαττον.

Έπεὶ γὰρ μεῖζόν ἐστι τὸ Α τοῦ Γ, ἄλλο δέ τι τὸ Β, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ἀλλ' ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Γ πρὸς τὸ Β, ἀνάπαλιν οὕτως τὸ Ε πρὸς τὸ Δ. καὶ τὸ Ε ἄρα πρὸς τὸ Ζ μείζονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Δ. πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλασσόν ἐστιν· ἔλασσον ἄρα ἐστὶ τὸ Ζ τοῦ Δ· μεῖζον ἄρα ἐστὶ τὸ Δ τοῦ Ζ. ὑμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ἢ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Ζ, κἂν ἔλαττον, ἕλαττον.

Έὰν ἄρα ἢ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ἢ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ἦ, καὶ τὸ τέταρτον τοῦ ἔκτου μεῖζον ἔσται, κἂν ἴσον,

Proposition 21[†]

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Let A, B, and C be three magnitudes, and D, E, F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. And let A be greater than C, via equality. I say that D will also be greater than F. And if (A is) equal (to C then D will also be) equal (to F). And if (A is) less (than C then D will also be) less (than F).

For since A is greater than C, and B some other (magnitude), A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. But as A (is) to B, so E (is) to F. And, inversely, as C (is) to B, so E (is) to D [Prop. 5.7 corr.]. Thus, E also has a greater ratio to F than E (has) to D [Prop. 5.13]. And that (magnitude) to which the same (magnitude) has a greater ratio is (the) lesser (magnitude) [Prop. 5.10]. Thus, F is less than D. Thus, D is greater than F. Similarly, we can show that even if A is equal to C then D will also be equal to F, and even if (A is) less (than C then D will also be) less (than F).

ίσον, καν έλαττον, έλαττον όπερ έδει δεϊξαι.

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha \gtrless \gamma$ as $\delta \gtrless \zeta$.

хβ'.

Έὰν ἢ ὑποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.

A	B ———	$\Gamma \longmapsto$
$\Delta \vdash$	E	Z
H	$K \longmapsto \cdots \longmapsto$	M
$\Theta \longmapsto$	$\Lambda \longmapsto$	N

Έστω ὁποσαοῦν μεγέθη τὰ A, B, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, E, Z, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ Δ πρὸς τὸ E, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ E πρὸς τὸ Z· λέγω, ὅτι καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῷ ἔσται.

Εἰλήφθω γὰρ τῶν μὲν Α, Δ ἰσάχις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Ε ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Κ, Λ, καὶ ἔτι τῶν Γ, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, καὶ εἴληπται τῶν μὲν Α, Δ ἰσάχις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Ε ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Κ, Λ, ἔστιν ἄρα ὡς τὸ Η πρὸς τὸ Κ, οὕτως τὸ Θ πρὸς τὸ Λ. δὶα τὰ αὐτὰ δὴ καὶ ὡς τὸ Κ πρὸς τὸ Μ, οὕτως τὸ Λ πρὸς τὸ Ν. ἐπεὶ οὕν τρία μεγέθη ἐστὶ τὰ Η, Κ, Μ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Θ, Λ, Ν, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι' ἴσου ἄρα, εἰ ὑπερέχει τὸ Η τοῦ Μ, ὑπερέχει καὶ τὸ Θ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Θ τῶν Α, Δ ἰσάχις πολλαπλάσια, τὰ δὲ Μ, Ν τῶν Γ, Ζ ἄλλα, ἂ ἔτυχεν, ἰσάχις πολλαπλάσια. ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

Έὰν ἄρα ἦ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

Proposition 22[†]

If there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.

A	В	C
D	E	Fi
G	K	M
H	L	N

Let there be any number of magnitudes whatsoever, A, B, C, and (some) other (magnitudes), D, E, F, of equal number to them, (which are) in the same ratio taken two by two, (so that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that they will also be in the same ratio via equality. (That is, as A is to C, so D is to F.)

For let the equal multiples G and H have been taken of A and D (respectively), and the other random equal multiples K and L of B and E (respectively), and the yet other random equal multiples M and N of C and F(respectively).

And since as A is to B, so D (is) to E, and the equal multiples G and H have been taken of A and D (respectively), and the other random equal multiples K and Lof B and E (respectively), thus as G is to K, so H (is) to L [Prop. 5.4]. And, so, for the same (reasons), as K (is) to M, so L (is) to N. Therefore, since G, K, and M are three magnitudes, and H, L, and N other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, thus, via equality, if G exceeds Mthen H also exceeds N, and if (G is) equal (to M then His also) equal (to N), and if (G is) less (than M then H is also) less (than N) [Prop. 5.20]. And G and H are equal multiples of A and D (respectively), and M and N other random equal multiples of C and F (respectively). Thus, as A is to C, so D (is) to F [Def. 5.5].

Thus, if there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by

two, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \zeta : \eta$ and $\gamma : \delta :: \eta : \theta$ then $\alpha : \delta :: \epsilon : \theta$.

×γ′.

Ἐἀν ἢ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ἢ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.



Έστω τρία μεγέθη τὰ A, B, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ τὰ Δ, E, Z, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ E πρὸς τὸ Z, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ E· λέγω, ὅτι ἐστὶν ὡς τὸ A πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Z.

Εἰλήφθω τῶν μὲν Α, Β, Δ ἰσάχις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Γ, Ε, Ζ ἄλλα, ἀ ἔτυχεν, ἰσάχις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεὶ ἰσάχις ἐστὶ πολλαπλάσια τὰ Η, Θ τῶν Α, Β, τὰ δὲ μέρη τοὶς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, έστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Η πρὸς τὸ Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Μ πρὸς τὸ Ν \cdot καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ' καὶ ὡς ἄρα τὸ H πρὸς τὸ Θ, οὕτως τὸ M πρὸς τὸ Ν. καὶ ἐπεί ἐστιν ώς τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε, καὶ ἐναλλὰξ ώς τὸ Β πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ Ε. καὶ ἐπεὶ τὰ Θ, Κ τῶν Β, Δ ἰσάχις ἐστὶ πολλαπλάσια, τὰ δὲ μέρη τοῖς ίσάχις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ B πρὸς τὸ Δ , οὕτως τὸ Θ πρὸς τὸ K. ἀλλ
՝ ὡς τὸ B πρὸς τὸ Δ , οὕτως τὸ Γ πρὸς τὸ E· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ K, οὕτως τὸ Γ πρὸς τὸ Ε. πάλιν, ἐπεὶ τὰ Λ, Μ τῶν Γ, Ε ἰσάχις έστι πολλαπλάσια, ἕστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ε, οὕτως τὸ Λ πρὸς τὸ Μ. ἀλλ' ὡς τὸ Γ πρὸς τὸ Ε, οὕτως τὸ Θ πρὸς τὸ K· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ K, οὕτως τὸ Λ πρὸς τὸ M, καὶ ἐναλλὰξ ὡς τὸ Θ πρὸς τὸ Λ, τὸ Κ πρὸς τὸ Μ. ἐδείχθη δὲ καὶ ὡς τὸ Η πρὸς τὸ Θ, οὕτως τὸ Μ πρὸς τὸ Ν. ἐπεὶ ούν τρία μεγέθη έστι τὰ Η, Θ, Λ, και ἄλλα αὐτοις ἴσα τὸ πλῆθος τὰ Κ, Μ, Ν σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καί ἐστιν αὐτῶν τεταραγμένη ἡ ἀναλογία, δι' ἴσου ἄρα, εἰ ύπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ίσον, καὶ εἰ ἕλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Δ ἰσάχις πολλαπλάσια, τὰ δ
ὲ Λ, Ν τῶν Γ, Ζ. ἔστιν ἄρα ὡς τὸ A πρὸς τὸ Γ , οὕτως τὸ Δ πρὸς τὸ Z.

Έὰν ἄρα ἢ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ἢ δὲ τεταραγμένη

Proposition 23[†]

If there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality.



Let A, B, and C be three magnitudes, and D, E and F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. I say that as A is to C, so D (is) to F.

Let the equal multiples G, H, and K have been taken of A, B, and D (respectively), and the other random equal multiples L, M, and N of C, E, and F (respectively).

And since G and H are equal multiples of A and B(respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A (is) to B, so G (is) to H. And, so, for the same (reasons), as E (is) to F, so M(is) to N. And as A is to B, so E (is) to F. And, thus, as G (is) to H, so M (is) to N [Prop. 5.11]. And since as Bis to C, so D (is) to E, also, alternately, as B (is) to D, so C (is) to E [Prop. 5.16]. And since H and K are equal multiples of B and D (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as B is to D, so H (is) to K. But, as B (is) to D, so C (is) to E. And, thus, as H (is) to K, so C (is) to E [Prop. 5.11]. Again, since L and M are equal multiples of C and E (respectively), thus as C is to E, so L (is) to M [Prop. 5.15]. But, as C (is) to E, so H (is) to K. And, thus, as H (is) to K, so L (is) to M [Prop. 5.11]. Also, alternately, as H(is) to L, so K (is) to M [Prop. 5.16]. And it was also shown (that) as G (is) to H, so M (is) to N. Therefore, since G, H, and L are three magnitudes, and K, M, and N other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, and their proportion is perturbed, thus, via equality, if G exceeds L then Kalso exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N) [Prop. 5.21]. And G and K are equal multiples of A and D (respectively), and L and N of C and αὐτῶν ἡ ἀναλογία, xαὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι. F (respectively). Thus, as A (is) to C, so D (is) to F [Def. 5.5].

Thus, if there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha : \gamma :: \delta : \zeta$.

χδ'.

Έὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἕξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον.



Πρῶτον γὰρ τὸ AB πρὸς δεύρερον τὸ Γ τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ ΔΕ πρὸς τέταρτον τὸ Ζ, ἐχέτω δὲ καὶ πέμπτον τὸ BH πρὸς δεύτερον τὸ Γ τὸν αὐτὸν λόγον καὶ ἔκτον τὸ ΕΘ πρὸς τέταρτον τὸ Ζ· λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ AH πρὸς δεύτερον τὸ Γ τὸν αὐτὸν ἔξει λόγον, καὶ τρίτον καὶ ἕκτον τὸ ΔΘ πρὸς τέταρτον τὸ Ζ.

Ἐπεὶ γάρ ἐστιν ὡς τὸ BH πρὸς τὸ Γ, οὕτως τὸ ΕΘ πρὸς τὸ Ζ, ἀνάπαλιν ἄρα ὡς τὸ Γ πρὸς τὸ BH, οὕτως τὸ Ζ πρὸς τὸ ΕΘ. ἐπεὶ οῦν ἐστιν ὡς τὸ AB πρὸς τὸ Γ, οὕτως τὸ ΔΕ πρὸς τὸ Ζ, ὡς δὲ τὸ Γ πρὸς τὸ BH, οὕτως τὸ Ζ πρὸς τὸ ΕΘ, δι' ἴσου ἄρα ἐστὶν ὡς τὸ AB πρὸς τὸ BH, οὕτως τὸ ΔΕ πρὸς τὸ ΕΘ. καὶ ἐπεὶ διηρημένα μεγέθη ἀνάλογόν ἐστιν, καὶ συντεθέντα ἀνάλογον ἔσται· ἔστιν ἄρα ὡς τὸ AH πρὸς τὸ HB, οὕτως τὸ ΔΘ πρὸς τὸ ΘΕ. ἔστι δὲ καὶ ὡς τὸ BH πρὸς τὸ Γ, οὕτως τὸ ΔΘ πρὸς τὸ Ζ· δι' ἴσου ἄρα ἐστὶν ὡς τὸ AH πρὸς τὸ Γ, οὕτως τὸ ΔΘ πρὸς τὸ Ζ.

Έὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἔκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἕξει λόγον καὶ τρίτον καὶ ἔκτον πρὸς τέταρτον. ὅπερ ἔδει δεἰξαι.

Proposition 24[†]

If a first (magnitude) has to a second the same ratio that third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and sixth (added together, have) to the fourth.



For let a first (magnitude) AB have the same ratio to a second C that a third DE (has) to a fourth F. And let a fifth (magnitude) BG also have the same ratio to the second C that a sixth EH (has) to the fourth F. I say that the first (magnitude) and the fifth, added together, AG, will also have the same ratio to the second C that the third (magnitude) and the sixth, (added together), DH, (has) to the fourth F.

For since as BG is to C, so EH (is) to F, thus, inversely, as C (is) to BG, so F (is) to EH [Prop. 5.7 corr.]. Therefore, since as AB is to C, so DE (is) to F, and as C (is) to BG, so F (is) to EH, thus, via equality, as AB is to BG, so DE (is) to EH [Prop. 5.22]. And since separated magnitudes are proportional then they will also be proportional (when) composed [Prop. 5.18]. Thus, as AG is to GB, so DH (is) to HE. And, also, as BG is to C, so EH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F. Thus, via equality, as AG is to C.

Thus, if a first (magnitude) has to a second the same ratio that a third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and the sixth (added

together, have) to the fourth. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\epsilon : \beta :: \zeta : \delta$ then $\alpha + \epsilon : \beta :: \gamma + \zeta : \delta$.

κε΄.

Έὰν τέσσαρα μεγέθη ἀνάλογον ἦ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν.



Έστω τέσσαρα μεγέθη ἀνάλογον τὰ AB, $\Gamma\Delta$, E, Z, ὡς τὸ AB πρὸς τὸ $\Gamma\Delta$, οὕτως τὸ E πρὸς τὸ Z, ἔστω δὲ μέγιστον μὲν αὐτῶν τὸ AB, ἐλάχιστον δὲ τὸ Z[.] λέγω, ὅτι τὰ AB, Z τῶν $\Gamma\Delta$, E μείζονά ἐστιν.

Κείσθω γὰρ τῷ μὲν Ε ἴσον τὸ AH, τῷ δὲ Z ἴσον τὸ $\Gamma\Theta$.

Έπεὶ $[0\tilde{0}v]$ ἐστιν ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ E πρὸς τὸ Z, ἴσον δὲ τὸ μὲν E τῷ AH, τὸ δὲ Z τῷ ΓΘ, ἔστιν ἄρα ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ AH πρὸς τὸ ΓΘ. καὶ ἐπεί ἐστιν ὡς ὅλον τὸ AB πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ AH πρὸς ἀφαιρεθὲν τὸ ΓΘ, καὶ λοιπὸν ἄρα τὸ HB πρὸς λοιπὸν τὸ ΘΔ ἔσται ὡς ὅλον τὸ AB πρὸς ὅλον τὸ ΓΔ. μεῖζον δὲ τὸ AB τοῦ ΓΔ· μεῖζον ἄρα καὶ τὸ HB τοῦ ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν AH τῷ E, τὸ δὲ ΓΘ τῷ Z, τὰ ἄρα AH, Z ἴσα ἐστὶ τοῖς ΓΘ, E. καὶ [ἐπεὶ] ἐὰν [ἀνίσοις ἴσα προστεθῆ, τὰ ὅλα ἄνισά ἐστιν, ἐὰν ἄρα] τῶν HB, ΘΔ ἀνίσων ὄντων καὶ μείζονος τοῦ HB τῷ μὲν HB προστεθῆ τὰ AH, Z, τῷ δὲ ΘΔ προστεθῆ τὰ ΓΘ, E, συνάγεται τὰ AB, Z μείζονα τῶν ΓΔ, E.

Έὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ἢ, τὸ μέγιστον αὐτῶν καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν. ὅπερ ἔδει δεῖξαι.

Proposition 25[†]

If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes).



Let AB, CD, E, and F be four proportional magnitudes, (such that) as AB (is) to CD, so E (is) to F. And let AB be the greatest of them, and F the least. I say that AB and F is greater than CD and E.

For let AG be made equal to E, and CH equal to F. [In fact,] since as AB is to CD, so E (is) to F, and E (is) equal to AG, and F to CH, thus as AB is to CD, so AG (is) to CH. And since the whole AB is to the whole CD as the (part) taken away AG (is) to the (part) taken away CH, thus the remainder GB will also be to the remainder HD as the whole AB (is) to the whole CD[Prop. 5.19]. And AB (is) greater than CD. Thus, GB(is) also greater than HD. And since AG is equal to E, and CH to F, thus AG and F is equal to CH and E. And [since] if [equal (magnitudes) are added to unequal (magnitudes) then the wholes are unequal, thus if] AGand F are added to GB, and CH and E to HD—GBand HD being unequal, and GB greater—it is inferred that AB and F (is) greater than CD and E.

Thus, if four magnitudes are proportional then the (sum of the) largest and the smallest of them is greater than the (sum of the) remaining two (magnitudes). (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if α : β :: γ : δ , and α is the greatest and δ the least, then $\alpha + \delta > \beta + \gamma$.

ELEMENTS BOOK 6

Similar Figures

Οροι.

α'. Όμοια σχήματα εὐθύγραμμά ἐστιν, ὄσα τάς τε γωνίας ἴσας ἔχει κατὰ μίαν καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον.

β΄. Άκρον καὶ μέσον λόγον εὐθεῖα τετμῆσθαι λέγεται, ὅταν ἢ ὡς ἡ ὅλη πρὸς τὸ μεῖζον τμῆμα, οὕτως τὸ μεῖζον πρὸς τὸ ἔλαττὸν.

γ'. Ύψος ἐστὶ πάντος σχήματος ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἀγομένη.

α'.

Τὰ τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



Έστω τρίγωνα μὲν τὰ ABΓ, AΓΔ, παραλληλόγραμμα δὲ τὰ ΕΓ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος τὸ AΓ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσις, οὕτως τὸ ABΓ τρίγωνον πρὸς τὸ AΓΔ τρίγωνον, καὶ τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον.

Έκβεβλήσθω γὰρ ἡ ΒΔ ἐφ' ἐκάτερα τὰ μέρη ἐπὶ τὰ Θ, Λ σημεῖα, καὶ κείσθωσαν τῆ μὲν ΒΓ βάσει ἴσαι [ὁσαιδηποτοῦν] αἱ ΒΗ, ΗΘ, τῆ δὲ ΓΔ βάσει ἴσαι ὑσαιδηποτοῦν αἱ ΔΚ, ΚΛ, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΑΘ, ΑΚ, ΑΛ.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΓΒ, ΒΗ, ΗΘ ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ ΑΘΗ, ΑΗΒ, ΑΒΓ τρίγωνα ἀλλήλοις. ὁσαπλασίων ἄρα ἐστὶν ἡ ΘΓ βάσις τῆς ΒΓ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΒΓ τριγώνου. διὰ τὰ αὐτὰ δὴ ἱσαπλασίων ἐστὶν ἡ ΛΓ βάσις τῆς ΓΔ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΑΛΓ τρίγωνον τοῦ ΑΓΔ τριγώνου[.] καὶ εἰ ἴση ἐστὶν ἡ ΘΓ βάσις τῆ ΓΛ βάσει, ἴσον ἐστὶ καὶ τὸ ΑΘΓ τρίγωνον τῷ ΑΓΛ τριγώνῷ, καὶ εἰ ὑπερέχει ἡ ΘΓ βάσις τῆς ΓΛ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΓΛ τριγώνου, καὶ εἰ ἐλάσσων, ἕλασσον. τεσσάρων δὴ ὄντων μεγεθῶν δύο μὲν βάσεων τῶν ΒΓ, ΓΔ, δύο δὲ τριγώνων τῶν ΑΒΓ, ΑΓΔ εἴληπται ἰσάκις πολλαπλάσια τῆς μὲν ΒΓ βάσεως καὶ τοῦ ΑΒΓ τριγώνου ἤ τε ΘΓ βάσις καὶ τὸ ΑΘΓ τρίγωνον, τῆς δὲ ΓΔ βάσεως καὶ τοῦ ΑΔΓ τριγώνου ἄλλα,

Definitions

1. Similar rectilinear figures are those (which) have (their) angles separately equal and the (corresponding) sides about the equal angles proportional.

2. A straight-line is said to have been cut in extreme and mean ratio when as the whole is to the greater segment so the greater (segment is) to the lesser.

3. The height of any figure is the (straight-line) drawn from the vertex perpendicular to the base.

Proposition 1[†]

Triangles and parallelograms which are of the same height are to one another as their bases.



Let ABC and ACD be triangles, and EC and CF parallelograms, of the same height AC. I say that as base BC is to base CD, so triangle ABC (is) to triangle ACD, and parallelogram EC to parallelogram CF.

For let the (straight-line) BD have been produced in each direction to points H and L, and let [any number] (of straight-lines) BG and GH be made equal to base BC, and any number (of straight-lines) DK and KL equal to base CD. And let AG, AH, AK, and AL have been joined.

And since CB, BG, and GH are equal to one another, triangles AHG, AGB, and ABC are also equal to one another [Prop. 1.38]. Thus, as many times as base HCis (divisible by) base BC, so many times is triangle AHCalso (divisible) by triangle ABC. So, for the same (reasons), as many times as base LC is (divisible) by base CD, so many times is triangle ALC also (divisible) by triangle ACD. And if base HC is equal to base CL then triangle AHC is also equal to triangle ACL [Prop. 1.38]. And if base HC exceeds base CL then triangle AHCalso exceeds triangle ACL.[‡] And if (HC is) less (than CL then AHC is also) less (than ACL). So, their being four magnitudes, two bases, BC and CD, and two trianἁ ἔτυχεν, ἰσάχις πολλαπλάσια ή τε ΛΓ βάσις καὶ τὸ ΑΛΓ τρίγωνον· καὶ δέδεικται, ὅτι, εἰ ὑπερέχει ή ΘΓ βάσις τῆς ΓΛ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΛΓ τριγώνου, καί εἰ ἴση, ἴσον, καὶ εἰ ἔλασσων, ἕλασσον· ἔστιν ἄρα ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον.

Καὶ ἐπεὶ τοῦ μὲν ΑΒΓ τριγώνου διπλάσιόν ἐστι τὸ ΕΓ παραλληλόγραμμον, τοῦ δὲ ΑΓΔ τριγώνου διπλάσιόν ἐστι τὸ ΖΓ παραλληλόγραμμον, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΖΓ παραλληλόγραμμον. ἐπεὶ οὕν ἐδείχϑη, ὡς μὲν ἡ ΒΓ βάσις πρὸς τὴν ΓΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, ὡς δὲ τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, οῦτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον, καὶ ὡς ἄρα ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΖΓ παραλληλόγραμμον.

Τὰ ἄρα τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

gles, ABC and ACD, equal multiples have been taken of base BC and triangle ABC—(namely), base HC and triangle AHC—and other random equal multiples of base *CD* and triangle *ADC*—(namely), base *LC* and triangle ALC. And it has been shown that if base HC exceeds base CL then triangle AHC also exceeds triangle ALC, and if (HC is) equal (to CL then AHC is also) equal (to ALC), and if (HC is) less (than CL then AHC is also) less (than ALC). Thus, as base BC is to base CD, so triangle ABC (is) to triangle ACD [Def. 5.5]. And since parallelogram EC is double triangle ABC, and parallelogram FC is double triangle ACD [Prop. 1.34], and parts have the same ratio as similar multiples [Prop. 5.15], thus as triangle ABC is to triangle ACD, so parallelogram EC(is) to parallelogram FC. In fact, since it was shown that as base BC (is) to CD, so triangle ABC (is) to triangle ACD, and as triangle ABC (is) to triangle ACD, so parallelogram EC (is) to parallelogram CF, thus, also, as base BC (is) to base CD, so parallelogram EC (is) to parallelogram FC [Prop. 5.11].

Thus, triangles and parallelograms which are of the same height are to one another as their bases. (Which is) the very thing it was required to show.

[†] As is easily demonstrated, this proposition holds even when the triangles, or parallelograms, do not share a common side, and/or are not right-angled.

[‡] This is a straight-forward generalization of Prop. 1.38.

β΄.

Έὰν τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῆ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν.



Τριγώνου γὰρ τοῦ ΑΒΓ παράλληλος μιᾶ τῶν πλευρῶν τῆ ΒΓ ἦχθω ἡ ΔΕ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ.

Proposition 2

If some straight-line is drawn parallel to one of the sides of a triangle then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle.



For let DE have been drawn parallel to one of the sides BC of triangle ABC. I say that as BD is to DA, so CE (is) to EA.

Έπεζεύχθωσαν γὰρ αί ΒΕ, ΓΔ.

Ίσον ἄρα ἐστὶ τὸ ΒΔΕ τρίγωνον τῷ ΓΔΕ τριγώνῳ· ἐπὶ γὰρ τῆς αὐτῆς βάσεώς ἐστι τῆς ΔΕ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΔΕ, ΒΓ· ἄλλο δέ τι τὸ ΑΔΕ τρίγωνον. τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον· ἔστιν ἄρα ὡς τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ [τρίγωνον], οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον. αλλ' ὡς μὲν τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ· ὑπὸ γὰρ τὸ αὐτὸ ὕψος ὄντα τὴν ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΒ κάθετον ἀγομένην πρὸς ἄλληλά εἰσιν ὡς αἱ βάσεις. διὰ τὰ αὐτὰ δὴ ὡς τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ· καὶ ὡς ἄρα ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ.

Άλλὰ δὴ αἱ τοῦ ΑΒΓ τριγώνου πλευραὶ αἱ ΑΒ, ΑΓ ἀνάλογον τετμήσθωσαν, ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ, καὶ ἐπεζεύχθω ἡ ΔΕ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΔΕ τῆ ΒΓ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ, ἀλλ' ὡς μὲν ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, ὡς δὲ ἡ ΓΕ πρὸς τὴν ΕΑ, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, καὶ ὡς ἄρα τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον. ἐκάτερον ἄρα τῶν ΒΔΕ, ΓΔΕ τριγώνων πρὸς τὸ ΑΔΕ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΒΔΕ τρίγωνον τῷ ΓΔΕ τριγώνω· καί εἰσιν ἐπὶ τῆς αὐτῆς βάσεως τῆς ΔΕ. τὰ δὲ ἴσα τρίγωνα καὶ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν. παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΓ.

Έὰν ἄρα τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῆ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν. ὅπερ ἔδει δεῖξαι.

γ'.

Έὰν τριγώνου ἡ γωνία δίχα τμηθῆ, ἡ δὲ τέμνουσα τὴν γωνίαν εὐθεῖα τέμνῃ καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἕξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένῃ εὐθεῖα δίχα τεμεῖ τὴν τοῦ τριγώνου γωνίαν.

Έστω τρίγωνον τὸ ΑΒΓ, καὶ τετμήσθω ἡ ὑπὸ ΒΑΓ γωνία δίχα ὑπὸ τῆς ΑΔ εὐθείας· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ.

Ήχθω γὰρ διὰ τοῦ Γ τῆ ΔΑ παράλληλος ἡ ΓΕ, καὶ διαχθεῖσα ἡ ΒΑ συμπιπτέτω αὐτῆ κατὰ τὸ Ε. For let BE and CD have been joined.

Thus, triangle BDE is equal to triangle CDE. For they are on the same base DE and between the same parallels DE and BC [Prop. 1.38]. And ADE is some other triangle. And equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7]. Thus, as triangle BDE is to [triangle] ADE, so triangle CDE (is) to triangle ADE. But, as triangle BDE (is) to triangle ADE, so (is) BD to DA. For, having the same height—(namely), the (straight-line) drawn from E perpendicular to AB they are to one another as their bases [Prop. 6.1]. So, for the same (reasons), as triangle CDE (is) to ADE, so CE(is) to EA. And, thus, as BD (is) to DA, so CE (is) to EA [Prop. 5.11].

And so, let the sides AB and AC of triangle ABC have been cut proportionally (such that) as BD (is) to DA, so CE (is) to EA. And let DE have been joined. I say that DE is parallel to BC.

For, by the same construction, since as BD is to DA, so CE (is) to EA, but as BD (is) to DA, so triangle BDE(is) to triangle ADE, and as CE (is) to EA, so triangle CDE (is) to triangle ADE [Prop. 6.1], thus, also, as triangle BDE (is) to triangle ADE [Prop. 6.1], thus, also, as triangle BDE (is) to triangle ADE, so triangle CDE (is) to triangle ADE [Prop. 5.11]. Thus, triangles BDE and CDE each have the same ratio to ADE. Thus, triangle BDE is equal to triangle CDE [Prop. 5.9]. And they are on the same base DE. And equal triangles, which are also on the same base, are also between the same parallels [Prop. 1.39]. Thus, DE is parallel to BC.

Thus, if some straight-line is drawn parallel to one of the sides of a triangle, then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally, then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle. (Which is) the very thing it was required to show.

Proposition 3

If an angle of a triangle is cut in half, and the straightline cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half.

Let ABC be a triangle. And let the angle BAC have been cut in half by the straight-line AD. I say that as BDis to CD, so BA (is) to AC.

For let CE have been drawn through (point) C parallel to DA. And, BA being drawn through, let it meet



Kai

 έπεὶ εἰς παραλλήλους τὰς ΑΔ, ΕΓ εὐθεĩa ἐν
έπεσεν
 ἡ ΑΓ, ἡ ἄρα ὑπὸ ΑΓΕ γωνία ἴση ἐστὶ τῆ ὑπὸ ΓΑΔ. ἀλλ

ἡ ὑπὸ ΓΑΔ τῆ ὑπὸ ΒΑΔ ὑπό
κειται ἴση· καὶ ἡ ὑπὸ ΒΑΔ

ἄρα τῆ ὑπὸ ΑΓΕ ἐστιν ἴση. πάλιν, ἐπεὶ εἰς παραλλήλους

τὰς ΑΔ, ΕΓ εὐθεĩa ἐν
έπεσεν ἡ ΒΑΕ, ἡ ἐκτὸς γωνία ἡ ὑπὸ

ΒΑΔ ἴση ἐστὶ τῆ ἐντὸς τῆ ὑπὸ ΑΕΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ

ΑΓΕ τῆ ὑπὸ ΒΑΔ ἴση· καὶ ἡ ὑπὸ ΑΓΕ ἄρα γωνία τῆ ὑπὸ

ΑΕΓ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΓΕ τῆ ὑπὸ ΒΑΔ

άση ἐστι τῆ ἐντὸς τῆ ὑπὸ ΑΕΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ

ΑΕΓ ἐστιν ἴση· ἀστε καὶ πλευρὰ ἡ ΑΕ πλευρᾶ τῆ ΑΓ ἐστιν

ἴση. καὶ ἐπεὶ τριγώνου τοῦ ΒΓΕ παρὰ μίαν τῶν πλευρῶν

τὴν ΕΓ ῆχται ἡ ΑΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν

ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΕ. ἴση δὲ ἡ ΑΕ τῆ ΑΓ· ὡς ἄρα

ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ.

Άλλὰ δὴ ἔστω ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ, καὶ ἐπεζεύχθω ἡ ΑΔ· λέγω, ὅτι δίχα τέτμηται ἡ ὑπὸ ΒΑΓ γωνία ὑπὸ τῆς ΑΔ εὐθείας.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ BΔ πρὸς τὴν ΔΓ, οὕτως ἡ BA πρὸς τὴν ΑΓ, ἀλλὰ καὶ ὡς ἡ BΔ πρὸς τὴν ΔΓ, οὕτως ἐστὶν ἡ BA πρὸς τὴν ΑΕ· τριγώνου γὰρ τοῦ BΓΕ παρὰ μίαν τὴν ΕΓ ἦκται ἡ ΑΔ· καὶ ὡς ἄρα ἡ BA πρὸς τὴν ΑΓ, οὕτως ἡ BA πρὸς τὴν ΑΕ. ἴση ἄρα ἡ ΑΓ τῆ ΑΕ· ὥστε καὶ γωνία ἡ ὑπὸ ΑΕΓ τῆ ὑπὸ ΑΓΕ ἐστιν ἴση. ἀλλ' ἡ μὲν ὑπὸ ΑΕΓ τῆ ἐκτὸς τῆ ὑπὸ BAΔ [ἐστιν] ἴση, ἡ δὲ ὑπὸ ΑΓΕ τῆ ἐναλλὰξ τῆ ὑπὸ ΓΑΔ ἐστιν ἴση· καὶ ἡ ὑπὸ BAΔ ἄρα τῆ ὑπὸ ΓΑΔ ἐστιν ἴση. ἡ ἄρα ὑπὸ BAΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΔ εὐθείας.

Έὰν ἄρα τριγώνου ἡ γωνία δίχα τμηθῆ, ἡ δὲ τέμνουσα τὴν γωνίαν εὐθεῖα τέμνῃ καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἕξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένῃ εὐθεῖα δίχα τέμνει τὴν τοῦ τριγώνου γωνίαν· ὅπερ ἔδει δεῖξαι.





And since the straight-line AC falls across the parallel (straight-lines) AD and EC, angle ACE is thus equal to CAD [Prop. 1.29]. But, (angle) CAD is assumed (to be) equal to BAD. Thus, (angle) BAD is also equal to ACE. Again, since the straight-line BAE falls across the parallel (straight-lines) AD and EC, the external angle BAD is equal to the internal (angle) AEC [Prop. 1.29]. And (angle) ACE was also shown (to be) equal to BAD. Thus, angle ACE is also equal to AEC. And, hence, side AE is equal to side AC [Prop. 1.6]. And since AD has been drawn parallel to one of the sides EC of triangle BCE, thus, proportionally, as BD is to DC, so BA (is) to AE [Prop. 6.2]. And AE (is) equal to AC. Thus, as BD (is) to DC, so BA (is) to AC.

And so, let BD be to DC, as BA (is) to AC. And let AD have been joined. I say that angle BAC has been cut in half by the straight-line AD.

For, by the same construction, since as BD is to DC, so BA (is) to AC, then also as BD (is) to DC, so BA is to AE. For AD has been drawn parallel to one (of the sides) EC of triangle BCE [Prop. 6.2]. Thus, also, as BA (is) to AC, so BA (is) to AE [Prop. 5.11]. Thus, AC (is) equal to AE [Prop. 5.9]. And, hence, angle AEC is equal to ACE [Prop. 1.5]. But, AEC [is] equal to the external (angle) BAD, and ACE is equal to the alternate (angle) CAD [Prop. 1.29]. Thus, (angle) BAD is also equal to CAD. Thus, angle BAC has been cut in half by the straight-line AD.

Thus, if an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half. (Which is) the very thing it was required to show.

[†] The fact that the two straight-lines meet follows because the sum of ACE and CAE is less than two right-angles, as can easily be demonstrated.

See Post. 5.

δ'.

Τῶν ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.



Έστω ίσογώνια τρίγωνα τὰ ABΓ, ΔΓΕ ἴσην ἔχοντα τὴν μὲν ὑπὸ ABΓ γωνίαν τῆ ὑπὸ ΔΓΕ, τὴν δὲ ὑπὸ BAΓ τῆ ὑπὸ ΓΔΕ xαὶ ἔτι τὴν ὑπὸ AΓΒ τῆ ὑπὸ ΓΕΔ· λέγω, ὅτι τῶν ABΓ, ΔΓΕ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας xαὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.

Κείσθω γὰρ ἐπ' εὐθείας ἡ ΒΓ τῆ ΓΕ. καὶ ἐπεὶ αἱ ὑπὸ ABΓ, AΓΒ γωνίαι δύο ὀρθῶν ἐλάττονές εἰσιν, ἴση δὲ ἡ ὑπὸ AΓΒ τῆ ὑπὸ ΔΕΓ, αἱ ἄρα ὑπὸ ABΓ, ΔΕΓ δύο ὀρθῶν ἐλάττονές εἰσιν· αἱ BA, ΕΔ ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Ζ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΓΕ γωνία τῆ ὑπὸ ΑΒΓ, παράλληλός ἐστιν ἡ ΒΖ τῆ ΓΔ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΓΒ τῆ ὑπὸ ΔΕΓ, παράλληλός ἐστιν ἡ ΑΓ τῆ ΖΕ. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΑΓΔ· ἴση ἄρα ἡ μὲν ΖΑ τῃ ΔΓ, ή δὲ ΑΓ τῆ ΖΔ. καὶ ἐπεὶ τριγώνου τοῦ ΖΒΕ παρὰ μίαν τὴν ΖΕ ῆκται ἡ ΑΓ, ἐστιν ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΖ, οὕτως ἡ BΓ πρὸς τὴν ΓΕ. ἴση δὲ ἡ AZ τỹ Γ Δ · ὡς ἄρα ἡ BA πρὸς τὴν ΓΔ, οὕτως ἡ ΒΓ πρὸς τὴν ΓΕ, καὶ ἐναλλὰξ ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΕ. πάλιν, ἐπεὶ παράλληλός έστιν ή ΓΔ τῆ BZ, ἔστιν ἄρα ὡς ἡ BΓ πρὸς τὴν ΓΕ, οὕτως ή Z Δ πρὸς τὴν ΔE . ἴση δὲ ή Z Δ τῃ $A\Gamma$ · ὡς ἄρα ή $B\Gamma$ πρὸς τὴν ΓΕ, οὕτως ἡ ΑΓ πρὸς τὴν ΔΕ, καὶ ἐναλλὰξ ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΔ. ἐπεὶ οὖν ἐδείχθη ώς μὲν ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΕ, ὡς δὲ ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΔ, δι ἴσου ἄρα ὡς ἡ BA πρὸς τὴν AΓ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΕ.

Τῶν ἄρα ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἰ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι· ὅπερ ἔδει δεῖξαι.

Proposition 4

In equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.



Let ABC and DCE be equiangular triangles, having angle ABC equal to DCE, and (angle) BAC to CDE, and, further, (angle) ACB to CED. I say that in triangles ABC and DCE the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.

Let BC be placed straight-on to CE. And since angles ABC and ACB are less than two right-angles [Prop 1.17], and ACB (is) equal to DEC, thus ABCand DEC are less than two right-angles. Thus, BA and ED, being produced, will meet [C.N. 5]. Let them have been produced, and let them meet at (point) F.

And since angle DCE is equal to ABC, BF is parallel to CD [Prop. 1.28]. Again, since (angle) ACB is equal to DEC, AC is parallel to FE [Prop. 1.28]. Thus, FACDis a parallelogram. Thus, FA is equal to DC, and AC to FD [Prop. 1.34]. And since AC has been drawn parallel to one (of the sides) FE of triangle FBE, thus as BAis to AF, so BC (is) to CE [Prop. 6.2]. And AF (is) equal to CD. Thus, as BA (is) to CD, so BC (is) to CE, and, alternately, as AB (is) to BC, so DC (is) to CE[Prop. 5.16]. Again, since CD is parallel to BF, thus as BC (is) to CE, so FD (is) to DE [Prop. 6.2]. And FD(is) equal to AC. Thus, as BC is to CE, so AC (is) to DE, and, alternately, as BC (is) to CA, so CE (is) to ED [Prop. 6.2]. Therefore, since it was shown that as AB (is) to BC, so DC (is) to CE, and as BC (is) to CA, so CE (is) to ED, thus, via equality, as BA (is) to AC, so CD (is) to DE [Prop. 5.22].

Thus, in equiangular triangles the sides about the equal angles are proportional, and those (sides) subtend-

ε΄.

Έὰν δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὑφ' ἂς αἱ ὑμόλογοι πλευραὶ ὑποτείνουσιν.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον ἔχοντα, ὡς μὲν τὴν AB πρὸς τὴν BΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὡς δὲ τὴν BΓ πρὸς τὴν ΓΑ, οὕτως τὴν ΕΖ πρὸς τὴν ΖΔ, καὶ ἔτι ὡς τὴν BA πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ πρὸς τὴν ΔΖ. λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ABΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσας ἕξουσι τὰς γωνίας, ὑφ' ἂς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν, τὴν μὲν ὑπὸ ABΓ τῆ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ BΓΑ τῆ ὑπὸ ΕΖΔ καὶ ἔτι τὴν ὑπὸ BAΓ τῆ ὑπὸ ΕΔΖ.

Συνεστάτω γὰρ πρὸς τῆ ΕΖ εὐθεία καὶ τοῖς πρὸς αὐτῆ σημείοις τοῖς Ε, Ζ τῆ μὲν ὑπὸ ΑΒΓ γωνία ἴση ἡ ὑπὸ ΖΕΗ, τῆ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΕΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Α λοιπῆ τῆ πρὸς τῷ Η ἐστιν ἴση.

Ίσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΕΗΖ [τριγών-[ω]. τῶν ἄρα ΑΒΓ, ΕΗΖ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αί περί τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ύποτείνουσαι· ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, [οὕτως] ή ΗΕ πρός την ΕΖ. ἀλλ' ὡς ἡ ΑΒ πρὸς την ΒΓ, οὕτως ύπόχειται ή ΔΕ πρὸς τὴν ΕΖ· ὡς ἄρα ἡ ΔΕ πρὸς τὴν ΕΖ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΖ. ἑκατέρα ἄρα τῶν ΔΕ, ΗΕ πρὸς τὴν ΕΖ τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἐστὶν ἡ ΔΕ τῆ ΗΕ. διὰ τὰ αὐτὰ δὴ χαὶ ἡ ΔZ τῆ HZ ἐστιν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ή ΔE τῆ EH, χοινὴ δὲ ἡ EZ, δύο δὴ αἱ ΔE , EZ δυσὶ ταῖς HE, EZ ἴσαι εἰσίν· καὶ βάσις ἡ ΔZ βάσει τῆ ZH [ἐστιν] ἴση· γωνία άρα ή ὑπὸ ΔEZ γωνία τῆ ὑπὸ HEZ ἐστιν ἴση, καὶ τὸ ΔEZ τρίγωνον τῷ ΗΕΖ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι, ὑφ' ὡς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶ χαὶ ἡ μὲν ὑπὸ ΔΖΕ γωνία τῆ ὑπὸ ΗΖΕ, ἡ δὲ ύπὸ ΕΔΖ τῆ ὑπὸ ΕΗΖ. καὶ ἐπεὶ ἡ μὲν ὑπὸ ΖΕΔ τῆ ὑπὸ ΗΕΖ ἐστιν ἴση, ἀλλ' ἡ ὑπὸ ΗΕΖ τῆ ὑπὸ ΑΒΓ, καὶ ἡ ὑπὸ

ing equal angles correspond. (Which is) the very thing it was required to show.

Proposition 5

If two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.



Let ABC and DEF be two triangles having proportional sides, (so that) as AB (is) to BC, so DE (is) to EF, and as BC (is) to CA, so EF (is) to FD, and, further, as BA (is) to AC, so ED (is) to DF. I say that triangle ABC is equiangular to triangle DEF, and (that the triangles) will have the angles which corresponding sides subtend equal. (That is), (angle) ABC (equal) to DEF, BCA to EFD, and, further, BAC to EDF.

For let (angle) FEG, equal to angle ABC, and (angle) EFG, equal to ACB, have been constructed on the straight-line EF at the points E and F on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at A is equal to the remaining (angle) at G [Prop. 1.32].

Thus, triangle ABC is equiangular to [triangle] EGF. Thus, for triangles ABC and EGF, the sides about the equal angles are proportional, and (those) sides subtending equal angles correspond [Prop. 6.4]. Thus, as ABis to BC, [so] GE (is) to EF. But, as AB (is) to BC, so, it was assumed, (is) DE to EF. Thus, as DE (is) to EF, so GE (is) to EF [Prop. 5.11]. Thus, DE and GEeach have the same ratio to EF. Thus, DE is equal to GE [Prop. 5.9]. So, for the same (reasons), DF is also equal to GF. Therefore, since DE is equal to EG, and EF (is) common, the two (sides) DE, EF are equal to the two (sides) GE, EF (respectively). And base DF[is] equal to base FG. Thus, angle DEF is equal to angle GEF [Prop. 1.8], and triangle DEF (is) equal to triangle GEF, and the remaining angles (are) equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle DFE is also equal to GFE, and ΑΒΓ ἄρα γωνία τῆ ὑπὸ ΔΕΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῆ ὑπὸ ΔΖΕ ἐστιν ἴση, καὶ ἔτι ἡ πρὸς τῷ Α τῆ πρὸς τῷ Δ· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Έὰν ἄρα δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὑφ' ἁς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν. ὅπερ ἔδει δεῖξαι.

ኖ′.

Έὰν δύο τρίγωνα μίαν γωνίαν μιἂ γωνία ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ' ἂς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ μίαν γωνίαν τὴν ὑπὸ BAΓ μιῷ γωνία τῆ ὑπὸ ΕΔΖ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὡς τὴν BA πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ πρὸς τὴν ΔΖ· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ABΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσην ἕξει τὴν ὑπὸ ABΓ γωνίαν τῆ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ AΓΒ τῆ ὑπὸ ΔΖΕ.

Συνεστάτω γὰρ πρὸς τῆ ΔΖ εὐθεία καὶ τοῖς πρὸς αὐτῆ σημείοις τοῖς Δ, Ζ ὁποτέρα μὲν τῶν ὑπὸ ΒΑΓ, ΕΔΖ ἴση ἡ ὑπὸ ΖΔΗ, τῆ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΔΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Β γωνία λοιπῆ τῆ πρὸς τῷ Η ἴση ἐστίν.

Τσογώνιον ἄρα ἐστὶ τὸ ABΓ τρίγωνον τῷ Δ HZ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ BA πρὸς τὴν AΓ, οὕτως ἡ HΔ πρὸς τὴν ΔΖ. ὑπόκειται δὲ καὶ ὡς ἡ BA πρὸς τὴν AΓ, οὕτως ἡ EΔ πρὸς τὴν ΔΖ· καὶ ὡς ἄρα ἡ EΔ πρὸς τὴν ΔΖ, οὕτως ἡ HΔ πρὸς τὴν ΔΖ. ἴση ἄρα ἡ EΔ τῆ ΔΗ· καὶ κοινὴ ἡ ΔΖ· δύο δὴ αἱ EΔ, ΔΖ δυσὶ ταῖς HΔ, ΔΖ ἴσας εἰσίν· καὶ γωνία ἡ ὑπὸ EΔΖ γωνία τῆ ὑπὸ HΔΖ [ἐστιν] ἴση· βάσις ἄρα ἡ EZ βάσει τῆ HΖ ἐστιν ἴση, καὶ τὸ ΔΕΖ τρίγωνον τῷ HΔΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσας ἔσονται, ὑφ' ἂς ἴσας πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΔΖΗ τῆ ὑπο ΔΖΕ, ἡ δὲ ὑπὸ ΔΗΖ (angle) EDF to EGF. And since (angle) FED is equal to GEF, and (angle) GEF to ABC, angle ABC is thus also equal to DEF. So, for the same (reasons), (angle) ACB is also equal to DFE, and, further, the (angle) at Ato the (angle) at D. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

Proposition 6

If two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.



Let ABC and DEF be two triangles having one angle, BAC, equal to one angle, EDF (respectively), and the sides about the equal angles proportional, (so that) as BA(is) to AC, so ED (is) to DF. I say that triangle ABC is equiangular to triangle DEF, and will have angle ABCequal to DEF, and (angle) ACB to DFE.

For let (angle) FDG, equal to each of BAC and EDF, and (angle) DFG, equal to ACB, have been constructed on the straight-line AF at the points D and F on it (respectively) [Prop. 1.23]. Thus, the remaining angle at B is equal to the remaining angle at G [Prop. 1.32].

Thus, triangle ABC is equiangular to triangle DGF. Thus, proportionally, as BA (is) to AC, so GD (is) to DF [Prop. 6.4]. And it was also assumed that as BAis) to AC, so ED (is) to DF. And, thus, as ED (is) to DF, so GD (is) to DF [Prop. 5.11]. Thus, ED (is) equal to DG [Prop. 5.9]. And DF (is) common. So, the two (sides) ED, DF are equal to the two (sides) GD, DF (respectively). And angle EDF [is] equal to angle GDF. Thus, base EF is equal to base GF, and triangle DEF is equal to triangle GDF, and the remaining angles τῆ ὑπὸ ΔΕΖ. ἀλλ' ἡ ὑπὸ ΔΖΗ τῆ ὑπὸ ΑΓΒ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΓΒ ἄρα τῆ ὑπὸ ΔΖΕ ἐστιν ἴση. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΒΑΓ τῆ ὑπὸ ΕΔΖ ἴση· καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῆ τῆ πρὸς τῷ Ε ἴση ἐστίν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Έλν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾶ γωνία ἴσην ἔχη, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὑφ' ἂς αἱ ὑμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

ζ.

Έὰν δύο τρίγωνα μίαν γωνίαν μιἂ γωνία ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἄμα ἤτοι ἐλάσσονα ἢ μὴ ἐλάσσονα ὀρϑῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, περὶ ἁς ἀνάλογόν εἰσιν αἱ πλευραί.



Έστω δύο τρίγωνα τὰ ABΓ, ΔΕΖ μίαν γωνίαν μιῷ γωνία ἴσην ἔχοντα τὴν ὑπὸ BAΓ τῆ ὑπὸ ΕΔΖ, περὶ δὲ ἄλλας γωνίας τὰς ὑπὸ ABΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον, ὡς τὴν AB πρὸς τὴν BΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, τῶν δὲ λοιπῶν τῶν πρὸς τοῖς Γ, Ζ πρότερον ἑκατέραν ἅμα ἐλάσσονα ὀρθῆς· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ABΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἴση ἔσται ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ, καὶ λοιπὴ δηλονότι ἡ πρὸς τῷ Γ λοιπῆ τῆ πρὸς τῷ Ζ ἴση.

Eἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ABΓ. καὶ συνεστάτω πρὸς τῆ AB εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ B τῆ ὑπὸ ΔΕΖ γωνία ἴση ἡ ὑπὸ ABH.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν Α γωνία τῃ Δ, ἡ δὲ ὑπὸ ABH τῃ̃ ὑπὸ ΔΕΖ, λοιπὴ ἄρα ἡ ὑπὸ AHB λοιπῃ̃ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ABH τρίγωνον τῷ ΔΕΖ will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, (angle) DFG is equal to DFE, and (angle) DGF to DEF. But, (angle) DFG is equal to ACB. Thus, (angle) ACB is also equal to DFE. And (angle) BAC was also assumed (to be) equal to EDF. Thus, the remaining (angle) at B is equal to the remaining (angle) at E [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

Proposition 7

If two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides are proportional equal.



Let ABC and DEF be two triangles having one angle, BAC, equal to one angle, EDF (respectively), and the sides about (some) other angles, ABC and DEF (respectively), proportional, (so that) as AB (is) to BC, so DE (is) to EF, and the remaining (angles) at C and F, first of all, both less than right-angles. I say that triangle ABC is equiangular to triangle DEF, and (that) angle ABC will be equal to DEF, and (that) the remaining (angle) at C (will be) manifestly equal to the remaining (angle) at F.

For if angle ABC is not equal to (angle) DEF then one of them is greater. Let ABC be greater. And let (angle) ABG, equal to (angle) DEF, have been constructed on the straight-line AB at the point B on it [Prop. 1.23].

And since angle A is equal to (angle) D, and (angle) ABG to DEF, the remaining (angle) AGB is thus equal

τριγώνω. ἔστιν ἄρα ὡς ἡ AB πρὸς τὴν BH, οὕτως ἡ ΔΕ πρὸς τὴν EZ. ὡς δὲ ἡ ΔΕ πρὸς τὴν EZ, [οὕτως] ὑπόκειται ἡ AB πρὸς τὴν BΓ· ἡ AB ἄρα πρὸς ἑκατέραν τῶν BΓ, BH τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἡ BΓ τῆ BH. ὥστε καὶ γωνία ἡ πρὸς τῷ Γ γωνία τῆ ὑπὸ BHΓ ἐστιν ἴση. ἐλάττων δὲ ὀρθῆς ὑπόκειται ἡ πρὸς τῷ Γ· ἐλάττων ἄρα ἐστὶν ὀρθῆς καὶ ὑπὸ BHΓ· ὥστε ἡ ἐφεξῆς αὐτῆ γωνία ἡ ὑπὸ AHB μείζων ἐστὶν ὀρθῆς. καὶ ἐδείχθη ἴση οῦσα τῆ πρὸς τῷ Ζ· καὶ ἡ πρὸς τῷ Ζ ἄρα μείζων ἐστὶν ὀρθῆς. ὑπόκειται δὲ ἐλάσσων ὀρθῆς· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ· ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ Ζ ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ ABΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Άλλὰ δὴ πάλιν ὑποκείσθω ἑκατέρα τῶν πρὸς τοῖς Γ, Ζ μὴ ἐλάσσων ὀρθῆς· λέγω πάλιν, ὅτι καὶ οὕτως ἐστὶν ἰσογώνιον τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἴση ἐστὶν ἡ BΓ τῆ BH· ὥστε καὶ γωνία ἡ πρὸς τῷ Γ τῆ ὑπὸ BHΓ ἴση ἐστίν. οὐκ ἐλάττων δὲ ὀρθῆς ἡ πρὸς τῷ Γ· οὐκ ἐλάττων ἄρα ὀρθῆς οὐδὲ ἡ ὑπὸ BHΓ. τριγώνου δὴ τοῦ BHΓ αἱ δύο γωνίαι δύο ὀρθῶν οὕκ εἰσιν ἐλάττονες· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα πάλιν ἄνισός ἐστιν ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ· ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ Α τῆ πρὸς τῷ Δ ἴση· λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῆ τῆ πρὸς τῷ ΔΕΖ τριγώνον τῷ ΔΕΖ τριγώνο.

Έὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾶ γωνία ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἄμα ἐλάττονα ἢ μὴ ἐλάττονα ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕζει τὰς γωνίας, περὶ ἂς ἀνάλογόν εἰσιν αἱ πλευραί· ὅπερ ἔδει δεῖξαι.

η΄.

Έὰν ἐν ὀρθογωνίω τριγώνω ἀπό τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῆ, τὰ πρὸς τῆ καθέτω τρίγωνα ὅμοιά ἐστι τῷ τε ὅλω καὶ ἀλλήλοις.

Έστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν, καὶ ἦχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ· λέγω, ὅτι ὅμοιόν ἐστιν ἑκάτερον τῶν ΑΒΔ, ΑΔΓ to the remaining (angle) DFE [Prop. 1.32]. Thus, triangle ABG is equiangular to triangle DEF. Thus, as AB is to BG, so DE (is) to EF [Prop. 6.4]. And as DE (is) to EF, [so] it was assumed (is) AB to BC. Thus, AB has the same ratio to each of BC and BG [Prop. 5.11]. Thus, BC (is) equal to BG [Prop. 5.9]. And, hence, the angle at C is equal to angle BGC [Prop. 1.5]. And the angle at C was assumed (to be) less than a right-angle. Thus, (angle) BGC is also less than a right-angle. Hence, the adjacent angle to it, AGB, is greater than a right-angle [Prop. 1.13]. And (AGB) was shown to be equal to the (angle) at F. Thus, the (angle) at F is also greater than a right-angle. But it was assumed (to be) less than a rightangle. The very thing is absurd. Thus, angle ABC is not unequal to (angle) DEF. Thus, (it is) equal. And the (angle) at A is also equal to the (angle) at D. And thus the remaining (angle) at C is equal to the remaining (angle) at F [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

But, again, let each of the (angles) at C and F be assumed (to be) not less than a right-angle. I say, again, that triangle ABC is equiangular to triangle DEF in this case also.

For, with the same construction, we can similarly show that BC is equal to BG. Hence, also, the angle at C is equal to (angle) BGC. And the (angle) at C (is) not less than a right-angle. Thus, BGC (is) not less than a right-angle either. So, in triangle BGC the (sum of) two angles is not less than two right-angles. The very thing is impossible [Prop. 1.17]. Thus, again, angle ABCis not unequal to DEF. Thus, (it is) equal. And the (angle) at A is also equal to the (angle) at D. Thus, the remaining (angle) at C is equal to the remaining (angle) at F [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides (are) proportional equal. (Which is) the very thing it was required to show.

Proposition 8

If, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the triangles around the perpendicular are similar to the whole (triangle), and to one another.

Let ABC be a right-angled triangle having the angle BAC a right-angle, and let AD have been drawn from

τριγώνων ὅλῳ τῷ ΑΒΓ καὶ ἔτι ἀλλήλοις.



Έπει γαρ ίση έστιν ή ύπο ΒΑΓ τῆ ύπο ΑΔΒ· όρθή γὰρ ἑκατέρα· καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΒΓ καὶ τοῦ ΑΒΔ ἡ πρὸς τῷ Β, λοιπὴ ἄρα ἡ ὑπὸ ΑΓΒ λοιπῃ τῃ ύπο ΒΑΔ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνω. ἔστιν ἄρα ὡς ἡ ΒΓ ὑποτείνουσα τὴν όρθήν τοῦ ΑΒΓ τριγώνου πρὸς τὴν ΒΑ ὑποτείνουσαν τὴν όρθήν τοῦ ΑΒΔ τριγώνου, οὕτως αὐτή ἡ ΑΒ ὑποτείνουσα τὴν πρὸς τῷ Γ γωνίαν τοῦ AB
Γ τριγώνου πρὸς τὴν B $\!\Delta$ ύποτείνουσαν την ίσην την ύπό ΒΑΔ τοῦ ΑΒΔ τριγώνου, καὶ ἔτι ἡ ΑΓ πρὸς τὴν ΑΔ ὑποτείνουσαν τὴν πρὸς τῷ B γωνίαν κοινήν τῶν δύο τριγώνων. τὸ ΑΒΓ ἄρα τρίγωνον τῷ ΑΒΔ τριγώνω ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευράς ἀνάλογον ἔχει. ὅμοιον ἄμα [ἐστὶ] τὸ ΑΒΓ τρίγωνον τῷ ABΔ τριγώνω. ὑμοίως δη δείξομεν, ὅτι καὶ τῷ ΑΔΓ τριγώνω ὄμοιόν ἐστι τὸ ΑΒΓ τρίγωνον ἑκάτερον άρα τῶν ΑΒΔ, ΑΔΓ [τριγώνων] ὄμοιόν ἐστιν ὅλω τῷ ΑΒΓ. Λέγω δή, ὄτι καὶ ἀλλήλοις ἐστὶν ὅμοια τὰ ΑΒΔ, ΑΔΓ

Λεγω οη, ότι και αλληλοίς έστιν ομοία τα ΑΒΔ, ΑΔΙ τρίγωνα.

Έπεὶ γὰρ ὀρθὴ ἡ ὑπὸ ΒΔΑ ὀρθῆ τῆ ὑπὸ ΑΔΓ ἐστιν ἴση, ἀλλὰ μὴν xaì ἡ ὑπὸ ΒΑΔ τῆ πρὸς τῷ Γ ἐδείχθη ἴση, xaì λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῆ τῆ ὑπὸ ΔΑΓ ἐστιν ἴση ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΑΔΓ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΒΔ τοῦ ΑΒΔ τριγώνου ὑποτείνουσα τὴν ὑπὸ ΒΑΔ πρὸς τὴν ΔΑ τοῦ ΑΔΓ τριγώνου ὑποτείνουσαν τὴν πρὸς τῷ Γ ἴσην τῆ ὑπὸ ΒΑΔ, οὕτως αὐτὴ ἡ ΑΔ τοῦ ΑΒΔ τριγώνου ὑποτείνουσα τὴν πρὸς τῷ Β γωνίαν πρὸς τὴν ΔΓ ὑποτείνουσαν τὴν ὑπὸ ΔΑΓ τοῦ ΑΔΓ τριγώνου ἴσην τῆ πρὸς τῷ Β, xaì ἔτι ἡ ΒΑ πρὸς τὴν ΑΓ ὑποτείνουσαι τὰς ὀρθάς· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΑΔΓ τριγώνῳ.

Έὰν ἄρα ἐν ὀρθογωνίω τριγώνω ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῆ, τὰ πρὸς τῆ καθέτω τρίγωνα ὅμοιά ἐστι τῷ τε ὅλω καὶ ἀλλήλοις [ὅπερ ἔδει δεῖξαι]. A, perpendicular to BC [Prop. 1.12]. I say that triangles ABD and ADC are each similar to the whole (triangle) ABC and, further, to one another.



For since (angle) BAC is equal to ADB—for each (are) right-angles—and the (angle) at B (is) common to the two triangles ABC and ABD, the remaining (angle) ACB is thus equal to the remaining (angle) BAD[Prop. 1.32]. Thus, triangle ABC is equiangular to triangle ABD. Thus, as BC, subtending the right-angle in triangle ABC, is to BA, subtending the right-angle in triangle ABD, so the same AB, subtending the angle at Cin triangle ABC, (is) to BD, subtending the equal (angle) BAD in triangle ABD, and, further, (so is) AC to AD, (both) subtending the angle at B common to the two triangles [Prop. 6.4]. Thus, triangle ABC is equiangular to triangle ABD, and has the sides about the equal angles proportional. Thus, triangle ABC [is] similar to triangle ABD [Def. 6.1]. So, similarly, we can show that triangle ABC is also similar to triangle ADC. Thus, [triangles] ABD and ADC are each similar to the whole (triangle) ABC.

So I say that triangles *ABD* and *ADC* are also similar to one another.

For since the right-angle BDA is equal to the rightangle ADC, and, indeed, (angle) BAD was also shown (to be) equal to the (angle) at C, thus the remaining (angle) at B is also equal to the remaining (angle) DAC[Prop. 1.32]. Thus, triangle ABD is equiangular to triangle ADC. Thus, as BD, subtending (angle) BAD in triangle ABD, is to DA, subtending the (angle) at C in triangle ADC, (which is) equal to (angle) BAD, so (is) the same AD, subtending the angle at B in triangle ABD, to DC, subtending (angle) DAC in triangle ADC, (which is) equal to the (angle) at B, and, further, (so is) BA to AC, (each) subtending right-angles [Prop. 6.4]. Thus, triangle ABD is similar to triangle ADC [Def. 6.1].

Thus, if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base

then the triangles around the perpendicular are similar to the whole (triangle), and to one another. [(Which is) the very thing it was required to show.]

Corollary

So (it is) clear, from this, that if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the (straight-line so) drawn is in mean proportion to the pieces of the base.[†] (Which is) the very thing it was required to show.

Proposition 9

To cut off a prescribed part from a given straight-line.



Let AB be the given straight-line. So it is required to cut off a prescribed part from AB.

So let a third (part) have been prescribed. [And] let some straight-line AC have been drawn from (point) A, encompassing a random angle with AB. And let a random point D have been taken on AC. And let DE and EC be made equal to AD [Prop. 1.3]. And let BC have been joined. And let DF have been drawn through Dparallel to it [Prop. 1.31].

Therefore, since FD has been drawn parallel to one of the sides, BC, of triangle ABC, then, proportionally, as CD is to DA, so BF (is) to FA [Prop. 6.2]. And CD(is) double DA. Thus, BF (is) also double FA. Thus, BA (is) triple AF.

Thus, the prescribed third part, AF, has been cut off from the given straight-line, AB. (Which is) the very thing it was required to do.

Proposition 10

To cut a given uncut straight-line similarly to a given cut (straight-line).

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι ἐἀν ἐν ὀρθογωνίω τριγώνω ἀπὸ τῆς ὀρθῆς γωνάις ἐπὶ τὴν βάσις χάθετος ἀχθῆ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν. ὅπερ ἔδει δεῖξαι.

[†] In other words, the perpendicular is the geometric mean of the pieces.

θ'.

Τῆς δοθείσης εὐθείας τὸ προσταχθὲν μέρος ἀφελεῖν.



Έστω ή δοθεῖσα εὐθεῖα ή AB· δεῖ δὴ τῆς AB τὸ προσταχθὲν μέρος ἀφελεῖν.

Έπιτετάχθω δὴ τὸ τρίτον. [καὶ] διήθχω τις ἀπὸ τοῦ Α εὐθεῖα ἡ ΑΓ γωνίαν περιέχουσα μετὰ τῆς ΑΒ τυχοῦσαν· καὶ εἰλήφθω τυχὸν σημεῖον ἐπὶ τῆς ΑΓ τὸ Δ, καὶ κείσθωσαν τῆ ΑΔ ἴσαι αἰ ΔΕ, ΕΓ. καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῆ ἦχθω ἡ ΔΖ.

Έπεὶ οὖν τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἦκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΒΖ πρὸς τὴν ΖΑ. διπλῆ δὲ ἡ ΓΔ τῆς ΔΑ· διπλῆ ἄρα καὶ ἡ ΒΖ τῆς ΖΑ· τριπλῆ ἄρα ἡ ΒΑ τῆς ΑΖ.

Τῆς ἄρα δοθείσης εὐθείας τῆς AB τὸ ἐπιταχθὲν τρίτον μέρος ἀφήρηται τὸ AZ· ὅπερ ἔδει ποιῆσαι.

ι΄.

Τὴν δοθεϊσαν εὐθεῖαν ἄτμητον τῆ δοθείσῃ τετμημένῃ ὁμοίως τεμεῖν.



Έστω ή μὲν δοθεῖσα εὐθεῖα ἄτμητος ή AB, ή δὲ τετμημένη ή AΓ κατὰ τὰ Δ, Ε σημεῖα, καὶ κείσθωσαν ὥστε γωνίαν τυχοῦσαν περιέχειν, καὶ ἐπεζεύχθω ή ΓΒ, καὶ διὰ τῶν Δ, Ε τῆ BΓ παράλληλοι ἤχθωσαν αἱ ΔΖ, ΕΗ, διὰ δὲ τοῦ Δ τῆ AB παράλληλος ἤχθω ή ΔΘΚ.

Παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΖΘ, ΘΒ· ἴση ἄρα ἡ μὲν ΔΘ τῆ ΖΗ, ἡ δὲ ΘΚ τῆ ΗΒ. καὶ ἐπεὶ τριγώνου τοῦ ΔΚΓ παρὰ μίαν τῶν πλευρῶν τὴν ΚΓ εὐθεῖα ἤκται ἡ ΘΕ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΔ. ἴση δὲ ἡ μὲν ΚΘ τῆ ΒΗ, ἡ δὲ ΘΔ τῆ ΗΖ. ἔστιν ἄρα ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν HZ. πάλιν, ἐπεὶ τριγώνου τοῦ ΑΗΕ παρὰ μίαν τῶν πλευρῶν τὴν ΗΕ ἦκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΕΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ. ἐδείχθη δὲ καὶ ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ· ἔστιν ἄρα ὡς μὲν ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ, ὡς δὲ ἡ ΕΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ.

Ή ἄρα δοθεῖσα εὐθεῖα ἄτμητος ἡ AB τῆ δοθείση εὐθεία τετμημένη τῆ ΑΓ ὁμοίως τέτμηται· ὅπερ ἔδει ποιῆσαι·

ια'.

Δύο δοθεισῶν εὐθειῶν τρίτην ἀνάλογον προσευρεῖν.

Έστωσαν αἱ δοθεῖσαι [δύο εὐθεῖαι] αἱ ΒΑ, ΑΓ καὶ κείσθωσαν γωνίαν περιέχουσαι τυχοῦσαν. δεῖ δὴ τῶν ΒΑ, ΑΓ τρίτην ἀνάλογον προσευρεῖν. ἐκβεβλήσθωσαν γὰρ ἐπὶ τὰ Δ, Ε σημεῖα, καὶ κείσθω τῆ ΑΓ ἴση ἡ ΒΔ, καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῆ ἤχθω ἡ ΔΕ.

Έπεὶ οὖν τριγώνου τοῦ ΑΔΕ παρὰ μίαν τῶν πλευρῶν τὴν ΔΕ ἦμται ἡ ΒΓ, ἀνάλογόν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως ἡ ΑΓ πρὸς τὴν ΓΕ. ἴση δὲ ἡ ΒΔ τῇ ΑΓ. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΑΓ, οὕτως ἡ ΑΓ πρὸς τὴν ΓΕ.



Let AB be the given uncut straight-line, and AC a (straight-line) cut at points D and E, and let (AC) be laid down so as to encompass a random angle (with AB). And let CB have been joined. And let DF and EG have been drawn through (points) D and E (respectively), parallel to BC, and let DHK have been drawn through (point) D, parallel to AB [Prop. 1.31].

Thus, FH and HB are each parallelograms. Thus, DH (is) equal to FG, and HK to GB [Prop. 1.34]. And since the straight-line HE has been drawn parallel to one of the sides, KC, of triangle DKC, thus, proportionally, as CE is to ED, so KH (is) to HD [Prop. 6.2]. And KH (is) equal to BG, and HD to GF. Thus, as CE is to ED, so BG (is) to GF. Again, since FD has been drawn parallel to one of the sides, GE, of triangle AGE, thus, proportionally, as ED is to DA, so GF (is) to FA [Prop. 6.2]. And it was also shown that as CE (is) to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF. Thus, as CF is to ED, so BG (is) to GF. Thus, as CF is to ED, so BG (is) to FA.

Thus, the given uncut straight-line, AB, has been cut similarly to the given cut straight-line, AC. (Which is) the very thing it was required to do.

Proposition 11

To find a third (straight-line) proportional to two given straight-lines.

Let BA and AC be the [two] given [straight-lines], and let them be laid down encompassing a random angle. So it is required to find a third (straight-line) proportional to BA and AC. For let (BA and AC) have been produced to points D and E (respectively), and let BD be made equal to AC [Prop. 1.3]. And let BC have been joined. And let DE have been drawn through (point) D parallel to it [Prop. 1.31].

Therefore, since BC has been drawn parallel to one of the sides DE of triangle ADE, proportionally, as AB is to BD, so AC (is) to CE [Prop. 6.2]. And BD (is) equal



Δύο ἄρα δοθεισῶν εὐθειῶν τῶν AB, AΓ τρίτη ἀνάλογον αὐταῖς προσεύρηται ἡ ΓΕ· ὅπερ ἔδει ποιῆσαι.

ıβ'.

Τριῶν δοθεισῶν εὐθειῶν τετάρτην ἀνάλογον προσευρεῖν.



Έστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ A, B, Γ· δεῖ δὴ τῶν A, B, Γ τετράτην ἀνάλογον προσευρεῖν.

Έκκείσθωσαν δύο εὐθεῖαι αἱ ΔE , ΔZ γωνίαν περιέχουςαι [τυχοῦσαν] τὴν ὑπὸ $E\Delta Z$ · καὶ κείσθω τῆ μὲν A ἴση ἡ ΔH , τῆ δὲ B ἴση ἡ HE, καὶ ἔτι τῆ Γ ἴση ἡ $\Delta \Theta$ · καὶ ἐπιζευχθείσης τῆς HΘ παράλληλος αὐτῆ ἦχθω διὰ τοῦ E ἡ EZ.

Έπεὶ οὖν τριγώνου τοῦ ΔEZ παρὰ μίαν τὴν EZ ῆχται ἡ HΘ, ἔστιν ἄρα ὡς ἡ ΔH πρὸς τὴν HE, οὕτως ἡ $\Delta Θ$ πρὸς τὴν ΘΖ. ἴση δὲ ἡ μὲν ΔH τῆ A, ἡ δὲ HE τῆ B, ἡ δὲ $\Delta Θ$ τῆ Γ· ἔστιν ἄρα ὡς ἡ A πρὸς τὴν B, οὕτως ἡ Γ πρὸς τὴν ΘΖ.

Τριῶν ἄρα δοθεισῶν εὐθειῶν τῶν Α, Β, Γ τετάρτη ἀνάλογον προσεύρηται ἡ ΘΖ· ὅπερ ἔδει ποιῆσαι. to AC. Thus, as AB is to AC, so AC (is) to CE.



Thus, a third (straight-line), CE, has been found (which is) proportional to the two given straight-lines, AB and AC. (Which is) the very thing it was required to do.

Proposition 12

To find a fourth (straight-line) proportional to three given straight-lines.



Let A, B, and C be the three given straight-lines. So it is required to find a fourth (straight-line) proportional to A, B, and C.

Let the two straight-lines DE and DF be set out encompassing the [random] angle EDF. And let DG be made equal to A, and GE to B, and, further, DH to C [Prop. 1.3]. And GH being joined, let EF have been drawn through (point) E parallel to it [Prop. 1.31].

Therefore, since GH has been drawn parallel to one of the sides EF of triangle DEF, thus as DG is to GE, so DH (is) to HF [Prop. 6.2]. And DG (is) equal to A, and GE to B, and DH to C. Thus, as A is to B, so C (is)

to HF.

Thus, a fourth (straight-line), HF, has been found (which is) proportional to the three given straight-lines, A, B, and C. (Which is) the very thing it was required to do.

Proposition 13

To find the (straight-line) in mean proportion to two given straight-lines.^{\dagger}



Let AB and BC be the two given straight-lines. So it is required to find the (straight-line) in mean proportion to AB and BC.

Let (AB and BC) be laid down straight-on (with respect to one another), and let the semi-circle ADC have been drawn on AC [Prop. 1.10]. And let BD have been drawn from (point) B, at right-angles to AC [Prop. 1.11]. And let AD and DC have been joined.

And since ADC is an angle in a semi-circle, it is a right-angle [Prop. 3.31]. And since, in the right-angled triangle ADC, the (straight-line) DB has been drawn from the right-angle perpendicular to the base, DB is thus the mean proportional to the pieces of the base, AB and BC [Prop. 6.8 corr.].

Thus, DB has been found (which is) in mean proportion to the two given straight-lines, AB and BC. (Which is) the very thing it was required to do.

ıγ'.

Δύο δοθεισῶν εὐθειῶν μέσην ἀνάλογον προσευρεῖν.



Έστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΒ, ΒΓ· δεῖ δὴ τῶν ΑΒ, ΒΓ μέσην ἀνάλογον προσευρεῖν.

Κείσθωσαν ἐπ' εὐθείας, καὶ γεγράφθω ἐπὶ τῆς ΑΓ ἡμικύκλιον τὸ ΑΔΓ, καὶ ἦχθω ἀπὸ τοῦ Β σημείου τῆ ΑΓ εὐθεία πρὸς ὀρθὰς ἡ ΒΑ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ.

Έπεὶ ἐν ἡμιχυχλίω γωνία ἐστὶν ἡ ὑπὸ ΑΔΓ, ὀρθή ἐστιν. καὶ ἐπεὶ ἐν ὀρθογωνίω τριγώνω τῷ ΑΔΓ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἤχται ἡ ΔΒ, ἡ ΔΒ ἄρα τῶν τῆς βάσεως τμημάτων τῶν ΑΒ, ΒΓ μέση ἀνάλογόν ἐστιν.

Δύο ἄρα δοθεισῶν εὐθειῶν τῶν ΑΒ, ΒΓ μέση ἀνάλογον προσεύρηται ἡ ΔΒ· ὅπερ ἔδει ποιῆσαι.

[†] In other words, to find the geometric mean of two given straight-lines.

ιδ'.

Τῶν ἴσων τε καὶ ἴσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ῶν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Έστω ίσα τε καὶ ἰσογώνια παραλληλόγραμμα τὰ AB, BΓ ἴσας ἔχοντα τὰς πρὸς τῷ B γωνίας, καὶ κείσθωσαν ἐπ' εὐθείας αἰ ΔB, BE· ἐπ' εὐθείας ἄρα εἰσὶ καὶ αἱ ZB, BH. λέγω, ὅτι τῶν AB, BΓ ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΔB πρὸς τὴν BE, οὕτως ἡ HB πρὸς τὴν BZ.

Proposition 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let AB and BC be equal and equiangular parallelograms having the angles at B equal. And let DB and BEbe laid down straight-on (with respect to one another). Thus, FB and BG are also straight-on (with respect to one another) [Prop. 1.14]. I say that the sides of AB and



Συμπεπληρώσθω γὰρ τὸ ΖΕ παραλληλόγραμμον. ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ, ἄλλο δέ τι τὸ ΖΕ, ἔστιν ἄρα ὡς τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως τὸ ΒΓ πρὸς τὸ ΖΕ. ἀλλ᾽ ὡς μὲν τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΕ, ὡς δὲ τὸ ΒΓ πρὸς τὸ ΖΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ· καὶ ὡς ἄρα ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ. τῶν ἄρα ΑΒ, ΒΓ παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Άλλὰ δὴ ἔστω ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμω.

Έπεὶ γάρ ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ, ἀλλ' ὡς μὲν ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως τὸ ΑΒ παραλληλόγραμμον πρὸς τὸ ΖΕ παραλληλόγραμμον, ὡς δὲ ἡ ΗΒ πρὸς τὴν ΒΖ, οὕτως τὸ ΒΓ παραλληλόγραμμον πρὸς τὸ ΖΕ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως τὸ ΒΓ πρὸς τὸ ΖΕ· ἴσον ἄρα ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ.

Τῶν ἄρα ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ῶν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα· ὅπερ ἔδει δεῖξαι.

ιε΄.

Τῶν ἴσων καὶ μίαν μιῷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ῶν μίαν μιῷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Έστω ίσα τρίγωνα τὰ ABΓ, ΑΔΕ μίαν μιἂ ίσην ἔχοντα γωνίαν τὴν ὑπὸ BAΓ τῆ ὑπὸ ΔΑΕ· λέγω, ὅτι τῶν ABΓ, ΑΔΕ τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἶσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως

ELEMENTS BOOK 6

BC about the equal angles are reciprocally proportional, that is to say, that as DB is to BE, so GB (is) to BF.



For let the parallelogram FE have been completed. Therefore, since parallelogram AB is equal to parallelogram BC, and FE (is) some other (parallelogram), thus as (parallelogram) AB is to FE, so (parallelogram) BC(is) to FE [Prop. 5.7]. But, as (parallelogram) AB (is) to FE, so DB (is) to BE, and as (parallelogram) BC (is) to FE, so GB (is) to BF [Prop. 6.1]. Thus, also, as DB (is) to BE, so GB (is) to BF. Thus, in parallelograms ABand BC the sides about the equal angles are reciprocally proportional.

And so, let DB be to BE, as GB (is) to BF. I say that parallelogram AB is equal to parallelogram BC.

For since as DB is to BE, so GB (is) to BF, but as DB (is) to BE, so parallelogram AB (is) to parallelogram FE, and as GB (is) to BF, so parallelogram BC (is) to parallelogram FE [Prop. 6.1], thus, also, as (parallelogram) AB (is) to FE, so (parallelogram) BC (is) to FE [Prop. 5.11]. Thus, parallelogram AB is equal to parallelogram BC [Prop. 5.9].

Thus, in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 15

In equal triangles also having one angle equal to one (angle) the sides about the equal angles are reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal.

Let ABC and ADE be equal triangles having one angle equal to one (angle), (namely) BAC (equal) to DAE. I say that, in triangles ABC and ADE, the sides about the ή ΕΑ πρός τὴν ΑΒ.



Κείσθω γὰρ ὥστε ἐπ' εὐθείας εἶναι τὴν ΓΑ τῆ ΑΔ· ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ ΕΑ τῆ ΑΒ. καὶ ἐπεζεύχθω ἡ ΒΔ.

Έπεὶ οὕν ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΔΕ τριγώνῳ, ἄλλο δέ τι τὸ ΒΑΔ, ἔστιν ἄρα ὡς τὸ ΓΑΒ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον. ἀλλ' ὡς μὲν τὸ ΓΑΒ πρὸς τὸ ΒΑΔ, οὕτως ἡ ΓΑ πρὸς τὴν ΑΔ, ὡς δὲ τὸ ΕΑΔ πρὸς τὸ ΒΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ. καὶ ὡς ἄρα ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ. τῶν ΑΒΓ, ΑΔΕ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Άλλὰ δὴ ἀντιπεπονθέτωσαν αἱ πλευραὶ τῶν ΑΒΓ, ΑΔΕ τριγώνων, καὶ ἔστω ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΔΕ τριγώνῳ.

Έπιζευχθείσης γὰρ πάλιν τῆς ΒΔ, ἐπεί ἐστιν ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ, ἀλλ' ὡς μὲν ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, ὡς δὲ ἡ ΕΑ πρὸς τὴν ΑΒ, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον. ἐκάτερον ἄρα τῶν ΑΒΓ, ΕΑΔ πρὸς τὸ ΒΑΔ τὸν αὐτὸν ἔχει λόγον. ἴσων ἄρα ἐστὶ τὸ ΑΒΓ [τρίγωνον] τῷ ΕΑΔ τριγώνῳ.

Τῶν ἄρα ἴσων καὶ μίαν μιῷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἰ πλευραὶ αἰ περὶ τὰς ἴσας γωνίας· καὶ ῶς μίαν μιῷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἐκεῖνα ἴσα ἐστὶν· ὅπερ ἔδει δεῖξαι.

เร'.

Έὰν τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῷ ὀρθογωνίῷ· κἂν τὸ ὑπὸ τῶν ἄχρων equal angles are reciprocally proportional, that is to say, that as CA is to AD, so EA (is) to AB.



For let CA be laid down so as to be straight-on (with respect) to AD. Thus, EA is also straight-on (with respect) to AB [Prop. 1.14]. And let BD have been joined.

Therefore, since triangle ABC is equal to triangle ADE, and BAD (is) some other (triangle), thus as triangle CAB is to triangle BAD, so triangle EAD (is) to triangle BAD [Prop. 5.7]. But, as (triangle) CAB (is) to BAD, so CA (is) to AD, and as (triangle) EAD (is) to BAD, so EA (is) to AB [Prop. 6.1]. And thus, as CA (is) to AD, so EA (is) to AB. Thus, in triangles ABC and ADE the sides about the equal angles (are) reciprocally proportional.

And so, let the sides of triangles ABC and ADE be reciprocally proportional, and (thus) let CA be to AD, as EA (is) to AB. I say that triangle ABC is equal to triangle ADE.

For, BD again being joined, since as CA is to AD, so EA (is) to AB, but as CA (is) to AD, so triangle ABC (is) to triangle BAD, and as EA (is) to AB, so triangle EAD (is) to triangle BAD [Prop. 6.1], thus as triangle ABC (is) to triangle BAD, so triangle EAD (is) to triangle BAD, so triangle EAD (is) to triangle BAD. Thus, (triangles) ABC and EAD each have the same ratio to BAD. Thus, [triangle] ABC is equal to triangle EAD [Prop. 5.9].

Thus, in equal triangles also having one angle equal to one (angle) the sides about the equal angles (are) reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 16

If four straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rectπεριεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ὑπὸ τῶν μέσων περιεχομένω ὀρθογωνίω, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται.



Έστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ AB, ΓΔ, Ε, Ζ, ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ· λέγω, ὅτι τὸ ὑπὸ τῶν AB, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένω ὀρθογωνίω.

Ήχθωσαν [γὰρ] ἀπὸ τῶν Α, Γ σημείων ταῖς ΑΒ, ΓΔ εὐθείαις πρὸς ὀρθὰς αἱ ΑΗ, ΓΘ, καὶ κείσθω τῆ μὲν Ζ ἴση ἡ ΑΗ, τῆ δὲ Ε ἴση ἡ ΓΘ. καὶ συμπεπληρώσθω τὰ ΒΗ, ΔΘ παραλληλόγραμμα.

Καὶ ἐπεί ἐστιν ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ, ἴση δὲ ἡ μὲν Ε τῆ ΓΘ, ἡ δὲ Ζ τῆ AH, ἔστιν ἄρα ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ πρὸς τὴν AH. τῶν BH, ΔΘ ἄρα παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ῶν δὲ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραί αἱ περὶ τὰς ἴσας γωνάις, ἴσα ἐστὶν ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ BH παραλληλόγραμμον τῷ ΔΘ παραλληλογράμμῳ. καί ἐστι τὸ μὲν BH τὸ ὑπὸ τῶν AB, Ζ· ἴση γὰρ ἡ AH τῆ Ζ· τὸ δὲ ΔΘ τὸ ὑπὸ τῶν ΓΔ, Ε· ἴση γὰρ ἡ Ε τῆ ΓΘ· τὸ ἄρα ὑπὸ τῶν AB, Ζ περιεχόμενον ὀρθογώνιων ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένω ὀρθογώνιω.

Άλλὰ δὴ τὸ ὑπὸ τῶν AB, Z περιεχόμενον ὀρθογώνιον ἴσον ἔστω τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογωνίῳ. λέγω, ὅτι αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται, ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ.

Tῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν AB, Z ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, E, καί ἐστι τὸ μὲν ὑπὸ τῶν AB, Z τὸ BH· ἴση γάρ ἐστιν ἡ AH τῆ Z· τὸ δὲ ὑπὸ τῶν ΓΔ, E τὸ $\Delta \Theta$ · ἴση γὰρ ἡ $\Gamma \Theta$ τῆ E· τὸ ἄρα BH ἴσον ἐστὶ τῷ $\Delta \Theta$. καί ἐστιν ἰσογώνια. τῶν δὲ ἴσων καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ἔστιν ἄρα ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ $\Gamma Θ$ πρὸς τὴν AH. ἴση δὲ ἡ μὲν $\Gamma Θ$ τῆ E, ἡ δὲ AH τῆ Z· ἔστιν ἄρα ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ E πρὸς τὴν Z.

Έὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ· κἂν τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἕδει δεῖξαι. angle contained by the (two) outermost is equal to the rectangle contained by the middle (two) then the four straight-lines will be proportional.



Let AB, CD, E, and F be four proportional straightlines, (such that) as AB (is) to CD, so E (is) to F. I say that the rectangle contained by AB and F is equal to the rectangle contained by CD and E.

[For] let AG and CH have been drawn from points A and C at right-angles to the straight-lines AB and CD (respectively) [Prop. 1.11]. And let AG be made equal to F, and CH to E [Prop. 1.3]. And let the parallelograms BG and DH have been completed.

And since as AB is to CD, so E (is) to F, and E(is) equal CH, and F to AG, thus as AB is to CD, so CH (is) to AG. Thus, in the parallelograms BG and DHthe sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.14]. Thus, parallelogram BGis equal to parallelogram DH. And BG is the (rectangle contained) by AB and F. For AG (is) equal to F. And DH (is) the (rectangle contained) by CD and E. For E(is) equal to CH. Thus, the rectangle contained by ABand F is equal to the rectangle contained by CD and E.

And so, let the rectangle contained by AB and F be equal to the rectangle contained by CD and E. I say that the four straight-lines will be proportional, (so that) as AB (is) to CD, so E (is) to F.

For, with the same construction, since the (rectangle contained) by AB and F is equal to the (rectangle contained) by CD and E. And BG is the (rectangle contained) by AB and F. For AG is equal to F. And DH (is) the (rectangle contained) by CD and E. For CH (is) equal to E. BG is thus equal to DH. And they are equiangular. And in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as AB is to CD, so CH (is) to AG. And CH (is) equal to E, and AG to F. Thus, as AB is to CD, so E (is) to F.

Thus, if four straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rectangle contained by the (two) outermost is equal to ιζ΄.

Έὰν τρεῖς εὐθεῖαι ἀνάλογον ὥσιν, τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνω· κἂν τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ἀπὸ τῆς μέσης τετραγώνω, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσονται.



Έστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ A, B, Γ, ὡς ἡ A πρὸς τὴν B, οὕτως ἡ B πρὸς τὴν Γ· λέγω, ὅτι τὸ ὑπὸ τῶν A, Γ περιεχόμενον ὀρθογώνιον ἶσον ἐστὶ τῷ ἀπὸ τῆς B τετραγώνῳ.

Κείσθω τῆ B ἴση ἡ Δ .

Καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν B, οὕτως ἡ B πρὸς τὴν Γ, ἴση δὲ ἡ B τῆ Δ, ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν B, ἡ Δ πρὸς τὴν Γ. ἐὰν δὲ τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, τὸ ὑπὸ τῶν ἄχρων περιεχόμενον [ὀθθογώνιον] ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ. τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν B, Δ. ἀλλὰ τὸ ὑπὸ τῶν B, Δ τὸ ἀπὸ τῆς B ἐστιν· ἴση γὰρ ἡ B τῆ Δ· τὸ ἄρα ὑπὸ τῶν Α, Γ περιεχόμενον ὀθθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς B τετραγώνῳ.

Tῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν A, Γ ἴσον ἐστὶ τῷ ἀπὸ τῆς B, ἀλλὰ τὸ ἀπὸ τῆς B τὸ ὑπὸ τῶν B, Δ ἐστιν· ἴση γὰρ ἡ B τῆ Δ· τὸ ἄρα ὑπὸ τῶν A, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν B, Δ. ἐὰν δὲ τὸ ὑπὸ τῶν ἄκρων ἴσον ἢ τῷ ὑπὸ τῶν μέσων, αἱ τέσσαρες εὐθεῖαι ἀνάλογόν εἰσιν. ἔστιν ἄρα ὡς ἡ A πρὸς τὴν B, οὕτως ἡ Δ πρὸς τὴν Γ. ἴση δὲ ἡ B τῆ Δ· ὡς ἄρα ἡ A πρὸς τὴν B, οὕτως ἡ B πρὸς τὴν Γ.

Έὰν ἄρα τρεῖς εὐθεῖαι ἀνάλογον ῶσιν, τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνω· κἂν τὸ ὑπὸ τῶν ἄχρων περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ἀπὸ τῆς μέσης τετραγώνω, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι. the rectangle contained by the middle (two) then the four straight-lines will be proportional. (Which is) the very thing it was required to show.

Proposition 17

If three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional.



Let A, B and C be three proportional straight-lines, (such that) as A (is) to B, so B (is) to C. I say that the rectangle contained by A and C is equal to the square on B.

Let D be made equal to B [Prop. 1.3].

And since as A is to B, so B (is) to C, and B (is) equal to D, thus as A is to B, (so) D (is) to C. And if four straight-lines are proportional then the [rectangle] contained by the (two) outermost is equal to the rectangle contained by the middle (two) [Prop. 6.16]. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by B and D. But, the (rectangle contained) by B and D is the (square) on B. For B (is) equal to D. Thus, the rectangle contained by A and C is equal to the square on B.

And so, let the (rectangle contained) by A and C be equal to the (square) on B. I say that as A is to B, so B (is) to C.

For, with the same construction, since the (rectangle contained) by A and C is equal to the (square) on B. But, the (square) on B is the (rectangle contained) by B and D. For B (is) equal to D. The (rectangle contained) by A and C is thus equal to the (rectangle contained) by B and D. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four straight-lines are proportional [Prop. 6.16]. Thus, as A is to B, so D (is) to C. And B (is) equal to D. Thus, as A (is) to B, so B (is) to C.

Thus, if three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional. (Which is) the very thing it was required to

show.

*ι*η′.

Απὸ τῆς δοθείσης εὐθείας τῷ δοθέντι εὐθυγράμμω ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγράψαι.



Έστω ή μὲν δοθεϊσα εὐθεῖα ή AB, τὸ δὲ δοθὲν εὐθύγραμμον τὸ ΓΕ· δεῖ δὴ ἀπὸ τῆς AB εὐθείας τῷ ΓΕ εὐθυγράμμῷ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγράψαι.

Ἐπεζεύχθω ἡ ΔΖ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθεία καὶ τοῖς πρὸς αὐτῆ σημείοις τοῖς Α, Β τῆ μὲν πρὸς τῷ Γ γωνία ἴση ἡ ὑπὸ ΗΑΒ, τῇ δὲ ὑπὸ ΓΔΖ ἴση ἡ ὑπὸ ΑΒΗ. λοιπὴ ἄρα ἡ ὑπὸ ΓΖ Δ τῇ ὑπὸ m AHB ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΓΔ τρίγωνον τῷ ΗΑΒ τριγώνω. ἀνάλογον ẳρα ἐστὶν ώς ή ΖΔ πρός την ΗΒ, οὕτως ή ΖΓ πρός την ΗΑ, καὶ ή ΓΔ πρός τὴν ΑΒ. πάλιν συνεστάτω πρός τῆ ΒΗ εὐθεία καὶ τοῖς πρὸς αὐτῆ σημείοις τοῖς Β, Η τῆ μὲν ὑπὸ ΔΖΕ γωνία ἴση ἡ ὑπὸ ΒΗΘ, τῇ δὲ ὑπὸ ΖΔΕ ἴση ἡ ὑπὸ ΗΒΘ. λοιπὴ ἄρα ή πρὸς τῷ Ε λοιπῆ τῆ πρὸς τῷ Θ ἐστιν ἴση· ἰσογώνιον ἄρα έστι τὸ ΖΔΕ τρίγωνον τῷ ΗΘΒ τριγώνω· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν HB, οὕτως ἡ ΖΕ πρὸς τὴν HΘ καὶ ή ΕΔ πρός τὴν ΘΒ. ἐδείχθη δὲ καὶ ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΓ πρὸς τὴν ΗΑ καὶ ἡ ΓΔ πρὸς τὴν ΑΒ· καὶ ὡς ἄρα ἡ ΖΓ πρὸς τὴν AH, οὕτως ἤ τε ΓΔ πρὸς τὴν AB xαὶ ἡ ΖΕ πρός τὴν ΗΘ καὶ ἔτι ἡ ΕΑ πρός τὴν ΘΒ. καὶ ἐπεὶ ἴση έστιν ή μεν ύπο ΓΖΔ γωνία τῆ ύπο AHB, ή δε ύπο ΔΖΕ τῆ ύπὸ BHΘ, ὅλη ἄρα ἡ ὑπὸ ΓΖΕ ὅλῃ τῇ ὑπὸ AHΘ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ ϰαὶ ἡ ὑπὸ ΓΔΕ τῆ ὑπὸ ΑΒΘ ἐστιν ἴση. ἔστι δ
ἑ καὶ ἡ μὲν πρὸς τῷ Γ τῇ πρὸς τῷ Α ἴση, ἡ δὲ πρὸς τῷ Ε τῆ πρὸς τῷ Θ. ἰσογώνιον ẳρα ἐστὶ τὸ ΑΘ τῷ ΓΕ· καὶ τὰς περί τὰς ἴσας γωνίας αὐτῶν πλευρὰς ἀνάλογον ἔχει· ὅμοιον ἄρα ἐστὶ τὸ ΑΘ εὐθύγραμμον τῷ ΓΕ εὐθυγράμμω.

Άπὸ τῆς δοθείσης ἄρα εὐθείας τῆς AB τῷ δοθέντι εὐθυγράμμω τῷ ΓΕ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγέγραπται τὸ ΑΘ· ὅπερ ἔδει ποιῆσαι.

Proposition 18

To describe a rectilinear figure similar, and similarly laid down, to a given rectilinear figure on a given straight-line.



Let AB be the given straight-line, and CE the given rectilinear figure. So it is required to describe a rectilinear figure similar, and similarly laid down, to the rectilinear figure CE on the straight-line AB.

Let DF have been joined, and let GAB, equal to the angle at C, and ABG, equal to (angle) CDF, have been constructed on the straight-line AB at the points A and B on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) CFD is equal to AGB [Prop. 1.32]. Thus, triangle FCD is equiangular to triangle GAB. Thus, proportionally, as FD is to GB, so FC (is) to GA, and CD to AB [Prop. 6.4]. Again, let BGH, equal to angle DFE, and GBH equal to (angle) FDE, have been constructed on the straight-line BG at the points G and B on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at E is equal to the remaining (angle) at H[Prop. 1.32]. Thus, triangle FDE is equiangular to triangle GHB. Thus, proportionally, as FD is to GB, so FE (is) to GH, and ED to HB [Prop. 6.4]. And it was also shown (that) as FD (is) to GB, so FC (is) to GA, and CD to AB. Thus, also, as FC (is) to AG, so CD (is) to AB, and FE to GH, and, further, ED to HB. And since angle CFD is equal to AGB, and DFE to BGH, thus the whole (angle) CFE is equal to the whole (angle) AGH. So, for the same (reasons), (angle) CDE is also equal to ABH. And the (angle) at C is also equal to the (angle) at A, and the (angle) at E to the (angle) at H. Thus, (figure) AH is equiangular to CE. And (the two figures) have the sides about their equal angles proportional. Thus, the rectilinear figure AH is similar to the rectilinear figure CE [Def. 6.1].

Thus, the rectilinear figure AH, similar, and similarly laid down, to the given rectilinear figure CE has been constructed on the given straight-line AB. (Which is) the *ι*θ'.

Τὰ ὅμοια τρίγωνα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν.



Έστω ὄμοια τρίγωνα τὰ ΑΒΓ, ΔΕΖ ἴσην ἕχοντα τὴν πρὸς τῷ Β γωνίαν τῆ πρὸς τῷ Ε, ὡς δὲ τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὥστε ὁμόλογον εἶναι τὴν ΒΓ τῆ ΕΖ· λέγω, ὅτι τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον έχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰλήφθω γὰρ τῶν ΒΓ, ΕΖ τρίτη ἀνάλογον ἡ ΒΗ, ὥστε είναι ώς τὴν ΒΓ πρὸς τὴν ΕΖ, οὕτως τὴν ΕΖ πρὸς τὴν ΒΗ· καὶ ἐπεζεύχθω ἡ ΑΗ.

Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρός τὴν ΕΖ. ἀλλ' ὡς ἡ ΒΓ πρός ΕΖ, οὕτως ἐστιν ἡ ΕΖ πρὸς BH. καὶ ὡς ἄρα ἡ AB πρὸς ΔE , οὕτως ἡ EZ πρὸς BH· τῶν ΑΒΗ, ΔΕΖ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περί τὰς ἴσας γωνάις. ῶν δὲ μίαν μιῷ ἴσην ἐχόντων γωνίαν τριγώνων αντιπεπόνθασιν αί πλευραί αί περί τὰς ἴσας γωνάις, ἴσα ἐστὶν ἐκεῖνα. ἴσον ἄρα ἐστὶ τὸ m ABH τρίγωνον τῷ $m \Delta EZ$ τριγώνω. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΕΖ, οὕτως ἡ ΕΖ πρός την ΒΗ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ή πρώτη πρός την τρίτην διπλασίονα λόγον έχει ήπερ πρός τὴν δευτέραν, ἡ ΒΓ ἄρα πρὸς τὴν ΒΗ διπλασίονα λόγον έχει ήπερ ή ΓΒ πρός την ΕΖ. ώς δὲ ή ΓΒ πρός την ΒΗ, ούτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΒΗ τρίγωνον καὶ τὸ ΑΒΓ ἄρα τρίγωνον πρὸς τὸ ΑΒΗ διπλασίονα λόγον ἔχει ήπερ ή BΓ πρός την ΕΖ. ἴσον δὲ τὸ ABH τρίγωνον τῶ ΔΕΖ τριγώνω. καὶ τὸ ΑΒΓ ἄρα τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Τὰ ἄρα ὄμοια τρίγωνα πρὸς ἄλληλα ἐν διπλασίονι λόγω έστι τῶν ὁμολόγων πλευρῶν. [ὅπερ ἔδει δεῖξαι.]

very thing it was required to do.

Proposition 19

Similar triangles are to one another in the squared[†] ratio of (their) corresponding sides.



Let ABC and DEF be similar triangles having the angle at B equal to the (angle) at E, and AB to BC, as DE (is) to EF, such that BC corresponds to EF. I say that triangle ABC has a squared ratio to triangle DEFwith respect to (that side) BC (has) to EF.

For let a third (straight-line), BG, have been taken (which is) proportional to BC and EF, so that as BC(is) to EF, so EF (is) to BG [Prop. 6.11]. And let AGhave been joined.

Therefore, since as AB is to BC, so DE (is) to EF, thus, alternately, as AB is to DE, so BC (is) to EF[Prop. 5.16]. But, as BC (is) to EF, so EF is to BG. And, thus, as AB (is) to DE, so EF (is) to BG. Thus, for triangles ABG and DEF, the sides about the equal angles are reciprocally proportional. And those triangles having one (angle) equal to one (angle) for which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.15]. Thus, triangle ABG is equal to triangle DEF. And since as BC (is) to EF, so EF (is) to BG, and if three straight-lines are proportional then the first has a squared ratio to the third with respect to the second [Def. 5.9], BC thus has a squared ratio to BG with respect to (that) CB (has) to EF. And as CB (is) to BG, so triangle ABC (is) to triangle ABG [Prop. 6.1]. Thus, triangle ABC also has a squared ratio to (triangle) ABG with respect to (that side) BC (has) to EF. And triangle ABG (is) equal to triangle DEF. Thus, triangle ABC also has a squared ratio to triangle DEF with respect to (that side) BC (has) to EF.

Thus, similar triangles are to one another in the squared ratio of (their) corresponding sides. [(Which is) the very thing it was required to show].

Corollary

Πόρισμα.

Έκ δη τούτου φανερόν, ὄτι, ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ώσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ proportional, then as the first is to the third, so the figure

So it is clear, from this, that if three straight-lines are

τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὑμοίως ἀναγραφόμενον. ὅπερ ἔδει δεῖξαι.

[†] Literally, "double".

κ΄.

Τὰ ὅμοια πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.



Έστω ὄμοια πολύγωνα τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, ὁμόλογος δὲ ἔστω ἡ ΑΒ τῆ ΖΗ· λέγω, ὅτι τὰ ΑΒΓΔΕ, ΖΗΘΚΛ πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΖΗ.

Έπεζεύχθωσαν αί ΒΕ, ΕΓ, ΗΛ, ΛΘ.

Καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνω, ἴση ἐστὶν ἡ ὑπὸ ΒΑΕ γωνία τῆ ὑπὸ ΗΖΛ. καί ἐστιν ὡς ἡ ΒΑ πρὸς ΑΕ, οὕτως ἡ ΗΖ πρὸς ΖΛ. έπει ούν δύο τρίγωνά έστι τὰ ΑΒΕ, ΖΗΛ μίαν γωνίαν μιᾶ γωνία ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς άνάλογον, ἰσογώνιον ἄρα ἐστὶ τὸ ABE τρίγωνον τῷ ZHΛ τριγώνω. ὥστε καὶ ὅμοιον. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῆ ὑπὸ ΖΗΛ. ἔστι δὲ καὶ ὅλη ἡ ὑπὸ ΑΒΓ ὅλη τῆ ὑπὸ ΖΗΘ ἴση διὰ τὴν ὁμοιότητα τῶν πολυγώνων λοιπὴ ἄρα ἡ ύπὸ ΕΒΓ γωνία τῆ ὑπὸ ΛΗΘ ἐστιν ἴση. καὶ ἐπεὶ διὰ τὴν όμοιότητα τῶν ABE, ΖΗΛ τριγώνων ἐστὶν ὡς ἡ ΕΒ πρὸς BA, οὕτως ἡ ΛΗ πρὸς ΗΖ, ἀλλὰ μὴν καὶ διὰ τὴν ὑμοιότητα τῶν πολυγώνων ἐστὶν ὡς ἡ ΑΒ πρὸς ΒΓ, οὕτως ἡ ΖΗ πρὸς ΗΘ, δι' ἴσου ἄρα ἐστὶν ὡς ἡ ΕΒ πρὸς ΒΓ, οὕτως ἡ ΛΗ πρὸς ΗΘ, καὶ περὶ τὰς ἴσας γωνάις τὰς ὑπὸ ΕΒΓ, ΛΗΘ αἱ πλευραὶ άνάλογόν είσιν ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΓ τρίγωνον τῷ ΛΗΘ τριγώνω. ὥστε καὶ ὅμοιόν ἐστι τὸ ΕΒΓ τρίγωνον τῷ ΛΗΘ τριγώνω. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΕΓΔ τρίγωνον ὄμοιόν ἐστι τῷ ΛΘΚ τριγώνῳ. τὰ ἄρα ὅμοια πολύγωνα τὰ $AB\Gamma\Delta E$, $ZH\Theta K\Lambda$ εἴς τε ὅμοια τρίγωνα διήρηται καὶ εἰς ἴσα

(described) on the first (is) to the similar, and similarly described, (figure) on the second. (Which is) the very thing it was required to show.

Proposition 20

Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let ABCDE and FGHKL be similar polygons, and let AB correspond to FG. I say that polygons ABCDEand FGHKL can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and (that) polygon ABCDE has a squared ratio to polygon FGHKL with respect to that AB (has) to FG.

Let BE, EC, GL, and LH have been joined.

And since polygon ABCDE is similar to polygon FGHKL, angle BAE is equal to angle GFL, and as BAis to AE, so GF (is) to FL [Def. 6.1]. Therefore, since ABE and FGL are two triangles having one angle equal to one angle and the sides about the equal angles proportional, triangle ABE is thus equiangular to triangle FGL[Prop. 6.6]. Hence, (they are) also similar [Prop. 6.4, Def. 6.1]. Thus, angle ABE is equal to (angle) FGL. And the whole (angle) ABC is equal to the whole (angle) FGH, on account of the similarity of the polygons. Thus, the remaining angle EBC is equal to LGH. And since, on account of the similarity of triangles ABE and FGL, as EB is to BA, so LG (is) to GF, but also, on account of the similarity of the polygons, as AB is to BC, so FG (is) to GH, thus, via equality, as EB is to BC, so LG (is) to GH [Prop. 5.22], and the sides about the equal angles, EBC and LGH, are proportional. Thus, triangle EBC is equiangular to triangle LGH [Prop. 6.6]. Hence, triangle *EBC* is also similar to triangle *LGH* [Prop. 6.4, Def. 6.1]. So, for the same (reasons), triangle ECD is also similar

τὸ πλῆθος.

Λέγω, ὅτι καὶ ὑμόλογα τοῖς ὅλοις, τουτέστιν ὥστε ἀνάλογον εἶναι τὰ τρίγωνα, καὶ ἡγούμενα μὲν εἶναι τὰ ABE, EBΓ, ΕΓΔ, ἑπόμενα δὲ αὐτῶν τὰ ZHΛ, ΛΗΘ, ΛΘΚ, καὶ ὅτι τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ZHΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἡ AB πρὸς τὴν ZH.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΖΘ. καὶ ἐπεὶ διὰ τὴν όμοιότητα τῶν πολυγώνων ἴση ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῆ ύπὸ ΖΗΘ, καί ἐστιν ὡς ἡ ΑΒ πρὸς ΒΓ, οὕτως ἡ ΖΗ πρὸς ΗΘ, ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῶ ΖΗΘ τριγώνω. ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΗΖΘ, ἡ δὲ ὑπὸ ΒΓΑ τῆ ὑπὸ ΗΘΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΜ γωνία τῆ ὑπὸ HZN, ἔστι δὲ καὶ ἡ ὑπὸ ABM τῆ ὑπὸ ZHN ἴση, καὶ λοιπὴ ἄρα ἡ ὑπὸ AMB λοιπῆ τῆ ὑπὸ ZNH ἴση ἐστίν ἰσογώνιον ἄρα ἐστὶ τὸ ABM τρίγωνον τῷ ZHN τριγώνῳ. όμοίως δή δεῖξομεν, ὅτι καὶ τὸ ΒΜΓ τρίγωνον ἰσογώνιόν έστι τῷ ΗΝΘ τριγώνω. ἀνάλογον ἄρα ἐστίν, ὡς μὲν ἡ ΑΜ πρός MB, οὕτως ἡ ΖΝ πρός NH, ὡς δὲ ἡ BM πρός MΓ, οὕτως ἡ ΗΝ πρὸς ΝΘ· ὥστε καὶ δι' ἴσου, ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως ἡ ΖΝ πρὸς ΝΘ. ἀλλ' ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως τὸ ΑΒΜ [τρίγωνον] πρὸς τὸ ΜΒΓ, καὶ τὸ ΑΜΕ πρὸς τὸ ΕΜΓ· πρὸς ἄλληλα γάρ εἰσιν ὡς αἱ βάσεις. καὶ ὡς ἄρα εν τῶν ἡγουμένων πρὸς εν τῶν ἑπόμενων, οὕτως ἄπαντα τὰ ἡγούμενα πρὸς ἄπαντα τὰ ἑπόμενα ὡς ἄρα τὸ ΑΜΒ τρίγωνον πρός τὸ ΒΜΓ, οὕτως τὸ ΑΒΕ πρὸς τὸ ΓΒΕ. αλλ ώς τὸ ΑΜΒ πρὸς τὸ ΒΜΓ, οὕτως ἡ ΑΜ πρὸς ΜΓ· καὶ ώς ἄρα ή ΑΜ πρός ΜΓ, οὕτως τὸ ΑΒΕ τρίγωνον πρός τὸ ΕΒΓ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ ΖΝ πρὸς ΝΘ, οὕτως τὸ ΖΗΛ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον. καί ἐστιν ώς ή ΑΜ πρός ΜΓ, οὕτως ή ΖΝ πρός ΝΘ· καὶ ὡς ἄρα τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΒΕΓ τρίγωνον, οὕτως τὸ ΖΗΛ τρίγωνον πρός τὸ ΗΛΘ τρίγωνον, καὶ ἐναλλὰξ ὡς τὸ ΑΒΕ τρίγωνον πρός τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΒΕΓ τρίγωνον πρός τὸ ΗΛΘ τρίγωνον. ὁμοίως δὴ δείξομεν ἐπιζευχθεισῶν τῶν ΒΔ, ΗΚ, ὅτι καὶ ὡς τὸ ΒΕΓ τρίγωνον πρὸς τὸ ΛΗΘ τρίγωνον, οὕτως τὸ ΕΓΔ τρίγωνον πρὸς τὸ ΛΘΚ τρίγωνον. καὶ ἐπεί ἐστιν ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΕΒΓ πρὸς τὸ $ΛH\Theta$, καὶ ἔτι τὸ ΕΓ Δ πρὸς τὸ $Λ\Theta$ K, καὶ ὡς ἄρα ἕν τῶν ἡγουμένων πρὸς ἕν τῶν ἑπομένων, οὕτως άπαντα τὰ ἡγούμενα πρὸς ἄπαντα τὰ ἑπόμενα· ἔστιν ἄρα ώς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρός τὸ ΖΗΘΚΛ πολύγωνον. ἀλλὰ τὸ ΑΒΕ τρίγωνον πρός τὸ ΖΗΛ τρίγωνον διπλασίονα λόγον έχει ήπερ ή AB όμόλογος πλευρά πρός την ZH όμόλογον πλευράν τὰ γὰρ ὄμοια τρίγωνα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὑμολόγων πλευρῶν. καὶ τὸ ΑΒΓΔΕ ἄρα πολύγωνον πρός τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ ὁμόλογος πλευρὰ πρὸς τὴν ΖΗ ὁμόλογον πλευράν.

Τὰ ἄρα ὅμοια πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ

to triangle LHK. Thus, the similar polygons ABCDE and FGHKL have been divided into equal numbers of similar triangles.

I also say that (the triangles) correspond (in proportion) to the wholes. That is to say, the triangles are proportional: ABE, EBC, and ECD are the leading (magnitudes), and their (associated) following (magnitudes are) FGL, LGH, and LHK (respectively). (I) also (say) that polygon ABCDE has a squared ratio to polygon FGHKL with respect to (that) a corresponding side (has) to a corresponding side—that is to say, (side) ABto FG.

For let AC and FH have been joined. And since angle ABC is equal to FGH, and as AB is to BC, so FG (is) to GH, on account of the similarity of the polygons, triangle ABC is equiangular to triangle FGH [Prop. 6.6]. Thus, angle BAC is equal to GFH, and (angle) BCA to GHF. And since angle BAM is equal to GFN, and (angle) ABM is also equal to FGN (see earlier), the remaining (angle) AMB is thus also equal to the remaining (angle) FNG [Prop. 1.32]. Thus, triangle ABM is equiangular to triangle FGN. So, similarly, we can show that triangle BMC is also equiangular to triangle GNH. Thus, proportionally, as AM is to MB, so FN (is) to NG, and as BM (is) to MC, so GN (is) to NH [Prop. 6.4]. Hence, also, via equality, as AM (is) to MC, so FN (is) to NH[Prop. 5.22]. But, as AM (is) to MC, so [triangle] ABMis to MBC, and AME to EMC. For they are to one another as their bases [Prop. 6.1]. And as one of the leading (magnitudes) is to one of the following (magnitudes), so (the sum of) all the leading (magnitudes) is to (the sum of) all the following (magnitudes) [Prop. 5.12]. Thus, as triangle AMB (is) to BMC, so (triangle) ABE (is) to CBE. But, as (triangle) AMB (is) to BMC, so AM (is) to MC. Thus, also, as AM (is) to MC, so triangle ABE(is) to triangle EBC. And so, for the same (reasons), as FN (is) to NH, so triangle FGL (is) to triangle GLH. And as AM is to MC, so FN (is) to NH. Thus, also, as triangle ABE (is) to triangle BEC, so triangle FGL (is) to triangle GLH, and, alternately, as triangle ABE (is) to triangle FGL, so triangle BEC (is) to triangle GLH[Prop. 5.16]. So, similarly, we can also show, by joining BD and GK, that as triangle BEC (is) to triangle LGH, so triangle ECD (is) to triangle LHK. And since as triangle ABE is to triangle FGL, so (triangle) EBC (is) to LGH, and, further, (triangle) ECD to LHK, and also as one of the leading (magnitudes is) to one of the following, so (the sum of) all the leading (magnitudes is) to (the sum of) all the following [Prop. 5.12], thus as triangle ABE is to triangle FGL, so polygon ABCDE (is) to polygon FGHKL. But, triangle ABE has a squared ratio πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν [ὅπερ ἔδει δεῖζαι].

Πόρισμα.

Ωσαύτως δὲ καὶ ἐπὶ τῶν [ὁμοίων] τετραπλεύρων δειχθήσεται, ὅτι ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἐδείχθη δὲ καὶ ἐπὶ τῶν τριγώνων· ὥστε καὶ καθόλου τὰ ὅμοια εὐθύγραμμα σχήματα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ὅπερ ἔδει δείζαι.

χα΄.

Τὰ τῷ αὐτῷ εὐθυγράμμῳ ὄμοια καὶ ἀλλήλοις ἐστὶν ὅμοια.



Έστω γὰρ ἑκάτερον τῶν Α, Β εὐθυγράμμων τῷ Γ ὅμοιον λέγω, ὅτι καὶ τὸ Α τῷ Β ἐστιν ὅμοιον.

Έπεὶ γὰρ ὅμοιόν ἐστι τὸ Α τῷ Γ, ἰσογώνιόν τέ ἐστιν αὐτῷ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. πάλιν, ἐπεὶ ὅμοιόν ἐστι τὸ Β τῷ Γ, ἰσογώνιόν τέ ἐστιν αὐτῷ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ἑκάτερον ἄρα τῶν Α, Β τῷ Γ ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει [ὥστε καὶ τὸ Α τῷ Β ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας to triangle FGL with respect to (that) the corresponding side AB (has) to the corresponding side FG. For, similar triangles are in the squared ratio of corresponding sides [Prop. 6.14]. Thus, polygon ABCDE also has a squared ratio to polygon FGHKL with respect to (that) the corresponding side AB (has) to the corresponding side FG.

Thus, similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side. [(Which is) the very thing it was required to show].

Corollary

And, in the same manner, it can also be shown for [similar] quadrilaterals that they are in the squared ratio of (their) corresponding sides. And it was also shown for triangles. Hence, in general, similar rectilinear figures are also to one another in the squared ratio of (their) corresponding sides. (Which is) the very thing it was required to show.

Proposition 21

(Rectilinear figures) similar to the same rectilinear figure are also similar to one another.



Let each of the rectilinear figures A and B be similar to (the rectilinear figure) C. I say that A is also similar to B.

For since A is similar to C, (A) is equiangular to (C), and has the sides about the equal angles proportional [Def. 6.1]. Again, since B is similar to C, (B) is equiangular to (C), and has the sides about the equal angles proportional [Def. 6.1]. Thus, A and B are each equiangular to C, and have the sides about the equal angles πλευρὰς ἀνάλογον ἔχει]. ὅμοιον ἄρα ἐστὶ τὸ A τῷ B· ὅπερ ἔδει δε
ῖξαι.

χβ'.

Έλν τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, καὶ τὰ ἀπ' ἀὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· κἂν τὰ ἀπ' ἀὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἦ, καὶ αὐτὰι αἱ εὐθεῖαι ἀνάλογον ἔσονται.



Έστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἰ AB, ΓΔ, ΕΖ, HΘ, ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν HΘ, καὶ ἀναγεγράφθωσαν ἀπὸ μὲν τῶν AB, ΓΔ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ KAB, ΛΓΔ, ἀπὸ δὲ τῶν ΕΖ, HΘ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ MZ, NΘ· λέγω, ὅτι ἐστὶν ὡς τὸ KAB πρὸς τὸ ΛΓΔ, οὕτως τὸ MZ πρὸς τὸ NΘ.

Eἰλήφθω γὰρ τῶν μὲν AB, ΓΔ τρίτη ἀνάλογον ή Ξ, τῶν δὲ EZ, HΘ τρίτη ἀνάλογον ή Ο. καὶ ἐπεί ἐστιν ὡς μὲν ή AB πρὸς τὴν ΓΔ, οὕτως ή EZ πρὸς τὴν HΘ, ὡς δὲ ή ΓΔ πρὸς τὴν Ξ, οὕτως ή HΘ πρὸς τὴν Ο, δι᾽ ἴσου ἄρα ἐστὶν ὡς ή AB πρὸς τὴν Ξ, οὕτως ή EZ πρὸς τὴν Ο. ἀλλ᾽ ὡς μὲν ή AB πρὸς τὴν Ξ, οὕτως [καὶ] τὸ KAB πρὸς τὸ ΛΓΔ, ὡς δὲ ή EZ πρὸς τὴν Ο, οὕτως τὸ MZ πρὸς τὸ NΘ· καὶ ὡς ἄρα τὸ KAB πρὸς τὸ ΛΓΔ, οὕτως τὸ MZ πρὸς τὸ NΘ.

Άλλὰ δὴ ἔστω ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ NΘ· λέγω, ὅτι ἐστὶ καὶ ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν HΘ. εἰ γὰρ μή ἐστιν, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν HΘ, ἔστω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, καὶ ἀναγεγράφθω ἀπὸ τῆς proportional [hence, A is also equiangular to B, and has the sides about the equal angles proportional]. Thus, A is similar to B [Def. 6.1]. (Which is) the very thing it was required to show.

Proposition 22

If four straight-lines are proportional then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional.



Let AB, CD, EF, and GH be four proportional straight-lines, (such that) as AB (is) to CD, so EF (is) to GH. And let the similar, and similarly laid out, rectilinear figures KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, rectilinear figures MF and NH on EF and GH(respectively). I say that as KAB is to LCD, so MF (is) to NH.

For let a third (straight-line) O have been taken (which is) proportional to AB and CD, and a third (straight-line) P proportional to EF and GH [Prop. 6.11]. And since as AB is to CD, so EF (is) to GH, and as CD(is) to O, so GH (is) to P, thus, via equality, as AB is to O, so EF (is) to P [Prop. 5.22]. But, as AB (is) to O, so [also] KAB (is) to LCD, and as EF (is) to P, so MF(is) to NH [Prop. 5.19 corr.]. And, thus, as KAB (is) to LCD, so MF (is) to NH.

And so let KAB be to LCD, as MF (is) to NH. I say also that as AB is to CD, so EF (is) to GH. For if as ABis to CD, so EF (is) not to GH, let AB be to CD, as EF ΠΡ ὑποτέρῳ τῶν ΜΖ, ΝΘ ὅμοιόν τε καὶ ὑμοίως κείμενον εὐθύγραμμον τὸ ΣΡ.

Έπεὶ οῦν ἐστιν ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ EZ πρὸς τὴν ΠΡ, καὶ ἀναγέγραπται ἀπὸ μὲν τῶν AB, ΓΔ ὅμοιά τε καὶ ὁμοίως κείμενα τὰ KAB, ΛΓΔ, ἀπὸ δὲ τῶν EZ, ΠΡ ὅμοιά τε καὶ ὁμοίως κείμενα τὰ MZ, ΣΡ, ἔστιν ἄρα ὡς τὸ KAB πρὸς τὸ ΛΓΔ, οὕτως τὸ MZ πρὸς τὸ ΣΡ. ὑπόκειται δὲ καὶ ὡς τὸ KAB πρὸς τὸ ΛΓΔ, οὕτως τὸ MZ πρὸς τὸ NΘ· καὶ ὡς ἄρα τὸ MZ πρὸς τὸ ΣΡ, οὕτως τὸ MZ πρὸς τὸ NΘ. τὸ MZ ἄρα πρὸς ἑκάτερον τῶν NΘ, ΣΡ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ NΘ τῷ ΣΡ. ἔστι δὲ αὐτῷ καὶ ὅμοιον καὶ ὁμοίως κείμενον· ἴση ἄρα ἡ HΘ τῷ ΠΡ. καὶ ἐπεί ἐστιν ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ EZ πρὸς τὴν ΠΡ, ἴση δὲ ἡ ΠΡ τῷ HΘ, ἔστιν ἄρα ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ EZ πρὸς τὴν HΘ.

Έλν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, καὶ τὰ ἀπ' αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· κἂν τὰ ἀπ' αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἢ, καὶ αὐτὰι αἱ εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι. (is) to QR [Prop. 6.12]. And let the rectilinear figure SR, similar, and similarly laid down, to either of MF or NH, have been described on QR [Props. 6.18, 6.21].

Therefore, since as AB is to CD, so EF (is) to QR, and the similar, and similarly laid out, (rectilinear figures) KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, (rectilinear figures) MF and SR on EF and QR (resespectively), thus as KAB is to LCD, so MF (is) to SR (see above). And it was also assumed that as KAB(is) to LCD, so MF (is) to NH. Thus, also, as MF (is) to SR, so MF (is) to NH [Prop. 5.11]. Thus, MF has the same ratio to each of NH and SR. Thus, NH is equal to SR [Prop. 5.9]. And it is also similar, and similarly laid out, to it. Thus, GH (is) equal to QR.[†] And since AB is to CD, as EF (is) to QR, and QR (is) equal to GH, thus as AB is to CD, so EF (is) to GH.

Thus, if four straight-lines are proportional, then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional. (Which is) the very thing it was required to show.

[†] Here, Euclid assumes, without proof, that if two similar figures are equal then any pair of corresponding sides is also equal.

×γ'.

Τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγχείμενον ἐχ τῶν πλευρῶν.

Έστω ἰσογώνια παραλληλόγραμμα τὰ ΑΓ, ΓΖ ἴσην ἔχοντα τὴν ὑπὸ ΒΓΔ γωνίαν τῆ ὑπὸ ΕΓΗ· λέγω, ὅτι τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον λόγον ἔχει τὸν συγχείμενον ἐχ τῶν πλευρῶν.

Κείσθω γὰρ ὥστε ἐπ' εὐθείας εἶναι τὴν BΓ τῆ ΓΗ· ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ Δ Γ τῆ ΓΕ. καὶ συμπεπληρώσθω τὸ Δ Η παραλληλόγραμμον, καὶ ἐκκείσθω τις εὐθεῖα ἡ Κ, καὶ γεγονέτω ὡς μὲν ἡ BΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, ὡς δὲ ἡ Δ Γ πρὸς τὴν ΓΕ, οὕτως ἡ Λ πρὸς τὴν Μ.

Οἱ ἄρα λόγοι τῆς τε Κ πρὸς τὴν Λ καὶ τῆς Λ πρὸς τὴν Μ οἱ αὐτοί εἰσι τοῖς λόγοις τῶν πλευρῶν, τῆς τε ΒΓ πρὸς τὴν ΓΗ καὶ τῆς ΔΓ πρὸς τὴν ΓΕ. ἀλλ' ὁ τῆς Κ πρὸς Μ λόγος σύγκειται ἕκ τε τοῦ τῆς Κ πρὸς Λ λόγου καὶ τοῦ τῆς Λ πρὸς Μ· ὥστε καὶ ἡ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΘ, ἀλλ' ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, καὶ ὡς ἄρα ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΘ. πάλιν, ἐπεί ἐστιν ὡς ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ, ἀλλ' ὡς ἡ ΔΓ πρὸς τὴν ΓΕ,

Proposition 23

Equiangular parallelograms have to one another the ratio compounded^{\dagger} out of (the ratios of) their sides.

Let AC and CF be equiangular parallelograms having angle BCD equal to ECG. I say that parallelogram AChas to parallelogram CF the ratio compounded out of (the ratios of) their sides.

For let BC be laid down so as to be straight-on to CG. Thus, DC is also straight-on to CE [Prop. 1.14]. And let the parallelogram DG have been completed. And let some straight-line K have been laid down. And let it be contrived that as BC (is) to CG, so K (is) to L, and as DC (is) to CE, so L (is) to M [Prop. 6.12].

Thus, the ratios of K to L and of L to M are the same as the ratios of the sides, (namely), BC to CG and DCto CE (respectively). But, the ratio of K to M is compounded out of the ratio of K to L and (the ratio) of Lto M. Hence, K also has to M the ratio compounded out of (the ratios of) the sides (of the parallelograms). And since as BC is to CG, so parallelogram AC (is) to CH [Prop. 6.1], but as BC (is) to CG, so K (is) to L, thus, also, as K (is) to L, so (parallelogram) AC (is) to CH. Again, since as DC (is) to CE, so parallelogram
οὕτως ἡ Λ πρὸς τὴν Μ, καὶ ὡς ἄρα ἡ Λ πρὸς τὴν Μ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον. ἐπεὶ οῦν ἐδείχθη, ὡς μὲν ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΘ παραλληλόγραμμον, ὡς δὲ ἡ Λ πρὸς τὴν Μ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον, δι' ἴσου ἄρα ἐστὶν ὡς ἡ Κ πρὸς τὴν Μ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΖ παραλληλόγραμμον. ἡ δὲ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν. καὶ τὸ ΑΓ ἄρα πρὸς τὸ ΓΖ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.



Τὰ ἄρα ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγχείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

CH (is) to *CF* [Prop. 6.1], but as *DC* (is) to *CE*, so *L* (is) to *M*, thus, also, as *L* (is) to *M*, so parallelogram *CH* (is) to parallelogram *CF*. Therefore, since it was shown that as *K* (is) to *L*, so parallelogram *AC* (is) to parallelogram *CH*, and as *L* (is) to *M*, so parallelogram *CH* (is) to parallelogram *CF*, thus, via equality, as *K* is to *M*, so (parallelogram) *AC* (is) to parallelogram *CF* [Prop. 5.22]. And *K* has to *M* the ratio compounded out of (the ratios of) the sides (of the parallelogram) *CF* the ratio compounded out of (the ratio of) the ratio of) their sides.



Thus, equiangular parallelograms have to one another the ratio compounded out of (the ratio of) their sides. (Which is) the very thing it was required to show.

[†] In modern terminology, if two ratios are "compounded" then they are multiplied together.

хδ'.

Παντός παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις.

Έστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα ἔστω τὰ ΕΗ, ΘΚ· λέγω, ὅτι ἑκάτερον τῶν ΕΗ, ΘΚ παραλληλογράμμων ὅμοιόν ἐστι ὅλῳ τῷ ΑΒΓΔ καὶ ἀλλήλοις.

Έπεὶ γὰρ τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἦχται ἡ ΕΖ, ἀνάλογόν ἐστιν ὡς ἡ ΒΕ πρὸς τὴν ΕΑ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΑ. πάλιν, ἐπεὶ τριγώνου τοῦ ΑΓΔ παρὰ μίαν τὴν ΓΔ ἦχται ἡ ΖΗ, ἀνάλογόν ἐστιν ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ. ἀλλ' ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ. ἀλλ' ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἐδείχθη καὶ ἡ ΒΕ πρὸς τὴν ΕΑ· καὶ ὡς ἄρα ἡ ΒΕ πρὸς τὴν ΕΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ, καὶ συνθέντι ἄρα ὡς ἡ ΒΑ πρὸς ΑΕ, οὕτως ἡ ΔΑ πρὸς ΑΗ, καὶ ἐναλλὰξ ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΗ. τῶν ἄρα ΑΒΓΔ, ΕΗ παραλληλογράμμων ἀνάλογόν εἰσιν αἱ πλευραὶ ἀ περὶ τὴν κοινὴν γωνίαν τὴν ὑπὸ ΒΑΔ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΗΖ τῇ ΔΓ, ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΖΗ γωνία τῇ ὑπὸ ΔΓΑ· καὶ κοινὴ τῶν δύο

Proposition 24

In any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another.

Let ABCD be a parallelogram, and AC its diagonal. And let EG and HK be parallelograms about AC. I say that the parallelograms EG and HK are each similar to the whole (parallelogram) ABCD, and to one another.

For since EF has been drawn parallel to one of the sides BC of triangle ABC, proportionally, as BE is to EA, so CF (is) to FA [Prop. 6.2]. Again, since FG has been drawn parallel to one (of the sides) CD of triangle ACD, proportionally, as CF is to FA, so DG (is) to GA [Prop. 6.2]. But, as CF (is) to FA, so it was also shown (is) BE to EA. And thus as BE (is) to EA, so DG (is) to AG (is) to GA. And, thus, compounding, as BA (is) to AE, so DA (is) to AG [Prop. 5.18]. And, alternately, as BA (is) to AD, so EA (is) to AG [Prop. 5.16]. Thus, in parallelograms ABCD and EG the sides about the common angle BAD are proportional. And since GF is parallel to DC, angle AFG is equal to DCA [Prop. 1.29].

τριγώνων τῶν ΑΔΓ, ΑΗΖ ἡ ὑπὸ ΔΑΓ γωνία ἰσογώνιον ἄρα ἐστὶ τὸ ΑΔΓ τρίγωνον τῷ AHZ τριγώνω. διὰ τὰ αὐτὰ δή και τὸ ΑΓΒ τρίγωνον ἰσογώνιόν ἐστι τῷ ΑΖΕ τριγώνω, καὶ ὅλον τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΗ παραλληλογράμμω ἰσογώνι
όν ἐστιν. ἀνάλογον ἄρα ἐστὶν ὡς ἡ Α Δ πρός τὴν ΔΓ, οὕτως ἡ ΑΗ πρός τὴν ΗΖ, ὡς δὲ ἡ ΔΓ πρός τὴν ΓΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΑΖ πρὸς τὴν ΖΕ, καὶ ἔτι ὡς ἡ ΓΒ πρὸς τὴν BA, οὕτως ἡ ΖΕ πρὸς τὴν ΕΑ. καὶ ἐπεὶ ἐδείχθη ὡς μὲν ή ΔΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ, ὡς δὲ ἡ ΑΓ πρός τὴν ΓΒ, οὕτως ἡ ΑΖ πρός τὴν ΖΕ, δι' ἴσου ἄρα έστιν ώς ή ΔΓ πρός την ΓΒ, οὕτως ή ΗΖ πρός την ΖΕ. τῶν ἄρα ΑΒΓΔ, ΕΗ παραλληλογράμμων ἀνάλογόν εἰσιν αί πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ὅμοιον ἄρα ἐστὶ τὸ ΑΒΓΔ παραλληλογράμμον τῷ ΕΗ παραλληλογράμμω. διὰ τὰ αὐτὰ δὴ τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τῷ ΚΘ παραλληλογράμμω ὄμοιόν ἐστιν ἑκάτερον ἄρα τῶν ΕΗ, ΘΚ παραλληλογράμμων τῷ ΑΒΓΔ [παραλληλογράμμω] ὄμοιόν έστιν. τὰ δὲ τῷ αὐτῷ εὐθυγράμμῳ ὄμοια καὶ ἀλλήλοις ἐστὶν δμοια· καὶ τὸ EH ἄρα παραλληλόγραμμον τῷ ΘK παραλληλογράμμω ὄμοιόν ἐστιν.



Παντὸς ἄρα παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὄμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις· ὅπερ ἔδει δεῖξαι.

×ε′**.**

Τῷ δοθέντι εὐθυγράμμῳ ὄμοιον καὶ ἄλλῳ τῷ δοθέντι ἴσον τὸ αὐτὸ συστήσασθαι. And angle DAC (is) common to the two triangles ADCand AGF. Thus, triangle ADC is equiangular to triangle AGF [Prop. 1.32]. So, for the same (reasons), triangle ACB is equiangular to triangle AFE, and the whole parallelogram ABCD is equiangular to parallelogram EG. Thus, proportionally, as AD (is) to DC, so AG (is) to GF, and as DC (is) to CA, so GF (is) to FA, and as AC(is) to CB, so AF (is) to FE, and, further, as CB (is) to BA, so FE (is) to EA [Prop. 6.4]. And since it was shown that as DC is to CA, so GF (is) to FA, and as AC (is) to CB, so AF (is) to FE, thus, via equality, as DC is to CB, so GF (is) to FE [Prop. 5.22]. Thus, in parallelograms ABCD and EG the sides about the equal angles are proportional. Thus, parallelogram ABCD is similar to parallelogram EG [Def. 6.1]. So, for the same (reasons), parallelogram ABCD is also similar to parallelogram KH. Thus, parallelograms EG and HK are each similar to [parallelogram] ABCD. And (rectilinear figures) similar to the same rectilinear figure are also similar to one another [Prop. 6.21]. Thus, parallelogram EG is also similar to parallelogram HK.



Thus, in any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another. (Which is) the very thing it was required to show.

Proposition 25

To construct a single (rectilinear figure) similar to a given rectilinear figure, and equal to a different given rectilinear figure.



Έστω τὸ μὲν δοθὲν εὐθύγραμμον, ῷ δεῖ ὅμοιον συστήσασθαι, τὸ $AB\Gamma$, ῷ δὲ δεῖ ἴσον, τὸ Δ · δεῖ δὴ τῷ μὲν $AB\Gamma$ ὅμοιον, τῷ δὲ Δ ἴσον τὸ αὐτὸ συστήσασθαι.

Παραβεβλήσθω γὰρ παρὰ μὲν τὴν ΒΓ τῷ ΑΒΓ τριγώνω ἴσον παραλληλόγραμμον τὸ ΒΕ, παρὰ δὲ τὴν ΓΕ τῷ Δ ἴσον παραλληλόγραμμον τὸ ΓΜ ἐν γωνία τῆ ὑπὸ ΖΓΕ, ἤ ἐστιν ἴση τῆ ὑπὸ ΓΒΛ. ἐπ' εὐθείας ἄρα ἐστὶν ἡ μὲν ΒΓ τῆ ΓΖ, ἡ δὲ ΛΕ τῆ ΕΜ. καὶ εἰλήφθω τῶν ΒΓ, ΓΖ μέση ἀνάλογον ἡ ΗΘ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΗΘ τῷ ΑΒΓ ὅμοιόν τε καὶ ὁμοίως κείμενον τὸ ΚΗΘ.

Καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΗΘ, οὕτως ἡ ΗΘ πρός την ΓΖ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔστιν ώς ή πρώτη πρός την τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης είδος πρός τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὑμοίως ἀναγραφόμενον, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΖ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΚΗΘ τρίγωνον. ἀλλὰ καὶ ὡς ἡ ΒΓ πρός την ΓΖ, ούτως τό ΒΕ παραλληλόγραμμον πρός τό ΕΖ παραλληλόγραμμον. καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΚΗΘ τρίγωνον, οὕτως τὸ ΒΕ παραλληλόγραμμον πρὸς τὸ ΕΖ παραλληλόγραμμον έναλλὰξ ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρός τό ΒΕ παραλληλόγραμμον, οὕτως τὸ ΚΗΘ τρίγωνον πρός τὸ ΕΖ παραλληλόγραμμον. ἴσον δὲ τὸ ΑΒΓ τρίγωνον τῷ ΒΕ παραλληλογράμμω. ἴσον ἄρα καὶ τὸ ΚΗΘ τρίγωνον τῷ ΕΖ παραλληλογράμμω. ἀλλὰ τὸ ΕΖ παραλληλόγραμμον τῷ Δ ἐστιν ἴσον· καὶ τὸ $KH\Theta$ ἄρα τῷ Δ ἐστιν ἴσον. ἔστι δὲ τὸ ΚΗΘ καὶ τῷ ΑΒΓ ὅμοιον.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ ABΓ ὅμοιον καὶ ἄλλῳ τῷ δοθέντι τῷ Δ ἴσον τὸ αὐτὸ συνέσταται τὸ KHΘ· ὅπερ ἔδει ποιῆσαι.

χς΄.

Έὰν ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῆ ὅμοιόν τε τῷ ὅλῷ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρόν ἐστι τῷ ὅλῷ.

Άπὸ γὰρ παραλληλογράμμου τοῦ ΑΒΓΔ παραλληλόγραμμον ἀφηρήσθω τὸ ΑΖ ὅμοιον τῷ ΑΒΓΔ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΔΑΒ· λέγω,



Let ABC be the given rectilinear figure to which it is required to construct a similar (rectilinear figure), and Dthe (rectilinear figure) to which (the constructed figure) is required (to be) equal. So it is required to construct a single (rectilinear figure) similar to ABC, and equal to D.

For let the parallelogram BE, equal to triangle ABC, have been applied to (the straight-line) BC [Prop. 1.44], and the parallelogram CM, equal to D, (have been applied) to (the straight-line) CE, in the angle FCE, which is equal to CBL [Prop. 1.45]. Thus, BC is straight-on to CF, and LE to EM [Prop. 1.14]. And let the mean proportion GH have been taken of BC and CF [Prop. 6.13]. And let KGH, similar, and similarly laid out, to ABC have been described on GH [Prop. 6.18].

And since as BC is to GH, so GH (is) to CF, and if three straight-lines are proportional then as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.], thus as BC is to CF, so triangle ABC(is) to triangle KGH. But, also, as BC (is) to CF, so parallelogram BE (is) to parallelogram EF [Prop. 6.1]. And, thus, as triangle ABC (is) to triangle KGH, so parallelogram BE (is) to parallelogram EF. Thus, alternately, as triangle ABC (is) to parallelogram BE, so triangle KGH (is) to parallelogram EF [Prop. 5.16]. And triangle ABC (is) equal to parallelogram EF. But, parallelogram EF is equal to D. Thus, KGH is also equal to D. And KGH is also similar to ABC.

Thus, a single (rectilinear figure) KGH has been constructed (which is) similar to the given rectilinear figure ABC, and equal to a different given (rectilinear figure) D. (Which is) the very thing it was required to do.

Proposition 26

If from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole.

For, from parallelogram *ABCD*, let (parallelogram)

ότι περὶ τὴν αὐτὴν διάμετρόν ἐστι τὸ ΑΒΓΔ τῷ ΑΖ.



Μή γάρ, ἀλλ' εἰ δυνατόν, ἔστω [αὐτῶν] διάμετρος ἡ ΑΘΓ, καὶ ἐκβληθεῖσα ἡ ΗΖ διήχθω ἐπὶ τὸ Θ, καὶ ἤχθω διὰ τοῦ Θ ὁπορέρα τῶν ΑΔ, ΒΓ παράλληλος ἡ ΘΚ.

Έπεὶ οὖν περὶ τὴν αὐτὴν διάμετρόν ἐστι τὸ ΑΒΓΔ τῷ KH, ἔστιν ἄρα ὡς ἡ ΔΑ πρὸς τὴν AB, οὕτως ἡ HA πρὸς τὴν AK. ἔστι δὲ καὶ διὰ τὴν ὁμοιότητα τῶν ΑΒΓΔ, ΕΗ καὶ ὡς ἡ ΔΑ πρὸς τὴν AB, οὕτως ἡ HA πρὸς τὴν AE· καὶ ὡς ἄρα ἡ HA πρὸς τὴν AK, οὕτως ἡ HA πρὸς τὴν AE· ἡ HA ἄρα πρὸς ἑκατέραν τῶν AK, AE τὸν αὐτὸν ἔχει λόγον. ἴση ἄρα ἐστὶν ἡ AE τῆ AK ἡ ἐλάττων τῆ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὕκ ἐστι περὶ τὴν αὐτὴν διάμετρον τὸ ABΓΔ τῷ AZ· περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὸ ABΓΔ παραλληλόγραμμον τῷ AZ παραλληλογράμμῳ.

Έλν ἄρα ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῆ ὅμοιόν τε τῷ ὅλῷ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρόν ἐστι τῷ ὅλῷ. ὅπερ ἔδει δεῖξαι.

хζ′.

Πάντων τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἶδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβαλλόμενον [παραλληλόγραμμον] ὅμοιον ὅν τῷ ἐλλείμμαντι.

Έστω εὐθεῖα ἡ AB καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ παραβεβλήσθω παρὰ τὴν AB εὐθεῖαν τὸ AΔ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμω τῷ ΔB ἀναγραφέντι ἀπὸ τῆς ἡμισείας τῆς AB, τουτέστι τῆς ΓB· λέγω, ὅτι πάντων τῶν παρὰ τὴν AB παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι [παραλληλογράμμοις] ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ΔB μέγιστόν ἐστι τὸ AF have been subtracted (which is) similar, and similarly laid out, to ABCD, having the common angle DAB with it. I say that ABCD is about the same diagonal as AF.



For (if) not, then, if possible, let AHC be [ABCD's] diagonal. And producing GF, let it have been drawn through to (point) H. And let HK have been drawn through (point) H, parallel to either of AD or BC [Prop. 1.31].

Therefore, since ABCD is about the same diagonal as KG, thus as DA is to AB, so GA (is) to AK [Prop. 6.24]. And, on account of the similarity of ABCD and EG, also, as DA (is) to AB, so GA (is) to AE. Thus, also, as GA (is) to AK, so GA (is) to AE. Thus, GA has the same ratio to each of AK and AE. Thus, AE is equal to AK [Prop. 5.9], the lesser to the greater. The very thing is impossible. Thus, ABCD is not not about the same diagonal as AF. Thus, parallelogram ABCD is about the same diagonal as parallelogram AF.

Thus, if from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole. (Which is) the very thing it was required to show.

Proposition 27

Of all the parallelograms applied to the same straightline, and falling short by parallelogrammic figures similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line) which (is) similar to (that parallelogram) by which it falls short.

Let AB be a straight-line, and let it have been cut in half at (point) C [Prop. 1.10]. And let the parallelogram AD have been applied to the straight-line AB, falling short by the parallelogrammic figure DB (which is) applied to half of AB—that is to say, CB. I say that of all the parallelograms applied to AB, and falling short by ΑΔ. παραβεβλήσθω γὰρ παρὰ τὴν AB εὐθεῖαν τὸ AZ παραλληλόγραμμον ἐλλεῖπον εἶδει παραλληλογράμμω τῷ ZB ὁμοίω τε καὶ ὁμοίως κειμένω τῷ ΔB· λέγω, ὅτι μεῖζόν ἐστι τὸ AΔ τοῦ AZ.



Έπεὶ γὰρ ὅμοιόν ἐστι τὸ ΔΒ παραλληλόγραμμον τῷ ΖΒ παραλληλογράμμω, περὶ τὴν αὐτήν εἰσι διάμετρον. ἤχθω αὐτῶν διάμετρος ἡ ΔΒ, καὶ καταγεγράφθω τὸ σχῆμα.

Έπει οὖν ἴσον ἐστι τὸ ΓΖ τῷ ΖΕ, Χοινὸν δὲ τὸ ΖΒ, ὅλον ἄρα τὸ ΓΘ ὅλῳ τῷ ΚΕ ἑστιν ἴσον. ἀλλὰ τὸ ΓΘ τῷ ΓΗ ἐστιν ἴσον, ἐπει Χαὶ ἡ ΑΓ τῆ ΓΒ. Χαὶ τὸ ΗΓ ἄρα τῷ ΕΚ ἐστιν ἴσον. Χοινὸν προσχείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ τῷ ΛΜΝ γνώμονί ἐστιν ἴσον· ὥστε τὸ ΔΒ παραλληλόγραμμον, τουτέστι τὸ ΑΔ, τοῦ ΑΖ παραλληλογράμμου μεῖζόν ἑστιν.

Πάντων ἄρα τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβληθέν· ὅπερ ἔδει δεῖξαι.

×η'.

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῷ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῷ ὑμοίῷ τῷ δοθέντι· δεῖ δὲ τὸ διδόμενον εὐθύγραμμον [ῷ δεῖ ἴσον παραβαλεῖν] μὴ μεῖζον εἶναι τοῦ ἀπὸ τῆς ἡμισείας ἀναγραφομένου ὑμοίου τῷ ἐλλείμματι [τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ῷ δεῖ ὅμοιον ἐλλείπειν].

Έστω ή μὲν δοθεῖσα εὐθεῖα ή AB, τὸ δὲ δοθὲν εὐθύγραμμον, ῷ δεῖ ἴσον παρὰ τὴν AB παραβαλεῖν, τὸ Γ μὴ μεῖζον [ὂν] τοῦ ἀπὸ τῆς ἡμισείας τῆς AB ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι, ῷ δὲ δεῖ ὅμοιον ἐλλείπειν, τὸ Δ· δεῖ δὴ

[parallelogrammic] figures similar, and similarly laid out, to DB, the greatest is AD. For let the parallelogram AF have been applied to the straight-line AB, falling short by the parallelogrammic figure FB (which is) similar, and similarly laid out, to DB. I say that AD is greater than AF.



For since parallelogram DB is similar to parallelogram FB, they are about the same diagonal [Prop. 6.26]. Let their (common) diagonal DB have been drawn, and let the (rest of the) figure have been described.

Therefore, since (complement) CF is equal to (complement) FE [Prop. 1.43], and (parallelogram) FB is common, the whole (parallelogram) CH is thus equal to the whole (parallelogram) KE. But, (parallelogram) CH is equal to CG, since AC (is) also (equal) to CB [Prop. 6.1]. Thus, (parallelogram) GC is also equal to EK. Let (parallelogram) CF have been added to both. Thus, the whole (parallelogram) AF is equal to the gnomon LMN. Hence, parallelogram DB—that is to say, AD—is greater than parallelogram AF.

Thus, for all parallelograms applied to the same straight-line, and falling short by a parallelogrammic figure similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line). (Which is) the very thing it was required to show.

Proposition 28[†]

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) falling short by a parallelogrammic figure similar to a given (parallelogram). It is necessary for the given rectilinear figure [to which it is required to apply an equal (parallelogram)] not to be greater than the (parallelogram) described on half (of the straight-line) and similar to the deficit.

Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to

παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ ὄντι τῷ Δ.



Τετμήσθω ή AB δίχα κατὰ τὸ Ε σημεῖον, καὶ ἀναγεγράφθω ἀπὸ τῆς EB τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ EBZH, καὶ συμπεπληρώσθω τὸ AH παραλληλόγραμμον.

Εἰ μὲν οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· παραβέβληται γὰρ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμω τῷ Γ ἴσον παραλληλόγραμμον τὸ ΑΗ ἐλλεῖπον εἴδει παραλληλογράμμω τῷ ΗΒ ὁμοίω ὄντι τῷ Δ . εἰ δὲ οὔ, μεῖζόν ἔστω τὸ ΘE τοῦ Γ. ἴσον δὲ τὸ ΘE τῷ ΗΒ· μεῖζον ἄρα καὶ τὸ ΗΒ τοῦ Γ. ῷ δὴ μεῖζόν ἐστι τὸ HB τοῦ Γ, ταύτη τῆ ὑπεροχῆ ἴσον, τῷ δ
ὲ Δ ὄμοιον καὶ όμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΚΛΜΝ. ἀλλὰ τὸ Δ τῷ ΗΒ [ἐστιν] ὄμοιον· καὶ τὸ ΚΜ ἄρα τῷ ΗΒ ἐστιν ὅμοιον. ἔστω οὖν ὑμόλογος ἡ μὲν ΚΛ τὴ ΗΕ, ἡ δὲ ΛΜ τῆ ΗΖ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ HB τοῖς Γ, KM, μεῖζον ἄρα ἐστὶ τὸ ΗΒ τοῦ ΚΜ· μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΗΕ τῆς ΚΛ, ἡ δὲ ΗΖ τῆς ΛΜ. κείσθω τῆ μὲν ΚΛ ἴση ἡ ΗΞ, τῆ δὲ ΛΜ ἴση ή ΗΟ, καὶ συμπεπληρώσθω τὸ ΞΗΟΠ παραλληλόγραμμον. ίσον ἄρα καὶ ὅμοιον ἐστι [τὸ ΗΠ] τῷ ΚΜ [ἀλλὰ τὸ ΚΜ τῷ ΗΒ ὅμοιόν ἐστιν]. καὶ τὸ ΗΠ ἄρα τῷ ΗΒ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΗΠ τῷ ΗΒ. ἔστω αὐτῶν διάμετρος ή ΗΠΒ, και καταγεγράφθω το σχήμα.

Έπεὶ οὖν ἴσον ἐστὶ τὸ BH τοῖς Γ, KM, ῶν τὸ HΠ τῷ KM ἐστιν ἴσον, λοιπὸς ἄρα ὁ ΥXΦ γνόμων λοιπῷ τῷ Γ ἴσος ἐστίν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ OP τῷ ΞΣ, κοινὸν προσκείσθω τὸ ΠB· ὅλον ἄρα τὸ OB ὅλῳ τῷ ΞΒ ἴσον ἐστίν. ἀλλὰ τὸ ΞΒ τῷ TE ἐστιν ἴσον, ἐπεὶ καὶ πλευρὰ ἡ AE πλευρᾶ τῆ EB ἐστιν ἴση· καὶ τὸ TE ἄρα τῷ OB ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΞΣ· ὅλον ἄρα τὸ TΣ ὅλῳ τῷ ΦXΥ γνώμονί ἐστιν ἴσον. ἀλλ' ὁ ΦXΥ γνώμων τῷ Γ ἐδείχθη ἴσος· καὶ τὸ TΣ ἄρα τῷ Γ ἐστιν ἴσον.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν AB τῷ δοθέντι εὐθυγράμμω τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΣΤ ἐλλεῖπον εἴδει παραλληλογράμμω τῷ ΠΒ ὑμοίω ὄντι

AB is required (to be) equal, [being] not greater than the (parallelogram) described on half of AB and similar to the deficit, and D the (parallelogram) to which the deficit is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C, to the straight-line AB, falling short by a parallelogrammic figure which is similar to D.



Let AB have been cut in half at point E [Prop. 1.10], and let (parallelogram) EBFG, (which is) similar, and similarly laid out, to (parallelogram) D, have been described on EB [Prop. 6.18]. And let parallelogram AG have been completed.

Therefore, if AG is equal to C then the thing prescribed has happened. For a parallelogram AG, equal to the given rectilinear figure C, has been applied to the given straight-line AB, falling short by a parallelogrammic figure GB which is similar to D. And if not, let HEbe greater than C. And HE (is) equal to GB [Prop. 6.1]. Thus, GB (is) also greater than C. So, let (parallelogram) KLMN have been constructed (so as to be) both similar, and similarly laid out, to D, and equal to the excess by which GB is greater than C [Prop. 6.25]. But, GB [is] similar to D. Thus, KM is also similar to GB[Prop. 6.21]. Therefore, let KL correspond to GE, and LM to GF. And since (parallelogram) GB is equal to (figure) C and (parallelogram) KM, GB is thus greater than KM. Thus, GE is also greater than KL, and GFthan LM. Let GO be made equal to KL, and GP to LM[Prop. 1.3]. And let the parallelogram *OGPQ* have been completed. Thus, [GQ] is equal and similar to KM [but, KM is similar to GB]. Thus, GQ is also similar to GB[Prop. 6.21]. Thus, GQ and GB are about the same diagonal [Prop. 6.26]. Let GQB be their (common) diagonal, and let the (remainder of the) figure have been described.

Therefore, since BG is equal to C and KM, of which GQ is equal to KM, the remaining gnomon UWV is thus equal to the remainder C. And since (the complement) PR is equal to (the complement) OS [Prop. 1.43], let (parallelogram) QB have been added to both. Thus, the whole (parallelogram) PB is equal to the parallelogram) PB is equal to the whole (parallelogram) PB is equal to the parallelogram) PB is equal to the parallelogram) PB is equal to the parallelogram PB is equal to th

τῷ Δ [ἐπειδήπερ τὸ ΠΒ τῷ ΗΠ
ὄμοιόν ἐστιν]· ὅπερ ἔδει ποιῆσαι.

allelogram) OB. But, OB is equal to TE, since side AE is equal to side EB [Prop. 6.1]. Thus, TE is also equal to PB. Let (parallelogram) OS have been added to both. Thus, the whole (parallelogram) TS is equal to the gnomon VWU. But, gnomon VWU was shown (to be) equal to C. Therefore, (parallelogram) TS is also equal to (figure) C.

Thus, the parallelogram ST, equal to the given rectilinear figure C, has been applied to the given straightline AB, falling short by the parallelogrammic figure QB, which is similar to D [inasmuch as QB is similar to GQ [Prop. 6.24]]. (Which is) the very thing it was required to do.

[†] This proposition is a geometric solution of the quadratic equation $x^2 - \alpha x + \beta = 0$. Here, x is the ratio of a side of the deficit to the corresponding side of figure D, α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the deficit running along AB, and β is the ratio of the areas of figures C and D. The constraint corresponds to the condition $\beta < \alpha^2/4$ for the equation to have real roots. Only the smaller root of the equation is found. The larger root can be found by a similar method.

хθ'.

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῷ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῷ ὁμοίῷ τῷ δοθέντι.



Έστω ή μὲν δοθεῖσα εὐθεῖα ή AB, τὸ δὲ δοθὲν εὐθύγραμμον, ῷ δεῖ ἴσον παρὰ τὴν AB παραβαλεῖν, τὸ Γ, ῷ δὲ δεῖ ὅμοιον ὑπερβάλλειν, τὸ Δ· δεῖ δὴ παρὰ τὴν AB εὐθεῖαν τῷ Γ εὐθυγράμμῷ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῷ ὁμοίῳ τῷ Δ.

Τετμήσθω ή AB δίχα κατὰ τὸ Ε, καὶ ἀναγεγράθω ἀπὸ τὴς EB τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον παραλληλόγραμμον τὸ BZ, καὶ συναμφοτέροις μὲν τοῖς BZ, Γ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ HΘ. ὁμόλογος δὲ ἔστω ἡ μὲν KΘ τῆ ZΛ, ἡ δὲ KH τῆ ZE. καὶ ἐπεὶ μεῖζόν ἐστι τὸ HΘ τοῦ ZB, μείζων ἄρα ἐστὶ καὶ ἡ μὲν KΘ τῆς ZΛ, ἡ δὲ KH τῆ ZE. ἐκβεβλήσθωσαν αἱ ZΛ, ZE, καὶ τῆ μὲν KΘ ἴση ἔστω ἡ ZΛM, τῆ δὲ KH ἴση ἡ ZEN, καὶ συμπεπληρώσθω τὸ MN· τὸ MN ἄρα τῷ HΘ ἴσον τέ ἐστι καὶ ὅμοιον. ἀλλὰ τὸ HΘ τῷ ΕΛ ἐστιν ὅμοιον·

Proposition 29[†]

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) overshooting by a parallelogrammic figure similar to a given (parallelogram).



Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to AB is required (to be) equal, and D the (parallelogram) to which the excess is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C, to the given straight-line AB, overshooting by a parallelogrammic figure similar to D.

Let AB have been cut in half at (point) E [Prop. 1.10], and let the parallelogram BF, (which is) similar, and similarly laid out, to D, have been described on EB[Prop. 6.18]. And let (parallelogram) GH have been constructed (so as to be) both similar, and similarly laid out, to D, and equal to the sum of BF and C [Prop. 6.25]. And let KH correspond to FL, and KG to FE. And since (parallelogram) GH is greater than (parallelogram) FB, καὶ τὸ MN ἄρα τῷ ΕΛ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΕΛ τῷ MN. ἦχθω αὐτῶν διάμετρος ἡ ΖΞ, καὶ καταγεγράφθω τὸ σχῆμα.

Έπεὶ ἴσον ἐστὶ τὸ HΘ τοῖς ΕΛ, Γ, ἀλλὰ τὸ HΘ τῷ MN ἴσον ἐστίν, καὶ τὸ MN ἄρα τοῖς ΕΛ, Γ ἴσον ἐστίν. κοινὸν ἀφηρήσθω τὸ ΕΛ· λοιπὸς ἄρα ὁ ΨΧΦ γνώμων τῷ Γ ἐστιν ἴσος. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῆ ΕΒ, ἴσον ἐστὶ καὶ τὸ ΑΝ τῷ NB, τουτέστι τῷ ΛΟ. κοινὸν προσκείσθω τὸ ΕΞ· ὅλον ἄρα τὸ ΑΞ ἴσον ἐστὶ τῷ ΦΧΨ γνώμονι. ἀλλὰ ὁ ΦΧΨ γνώμων τῷ Γ ἴσος ἐστίν· καὶ τὸ ΑΞ ἄρα τῷ Γ ἴσον ἐστίν.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν AB τῷ δοθέντι εὐθυγράμμω τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΑΞ ὑπερβάλλον εἴδει παραλληλογράμμω τῷ ΠΟ ὑμοίω ὄντι τῷ Δ, ἐπεὶ καὶ τῷ ΕΛ ἐστιν ὅμοιον τὸ ΟΠ· ὅπερ ἔδει ποιῆσαι. KH is thus also greater than FL, and KG than FE. Let FL and FE have been produced, and let FLM be (made) equal to KH, and FEN to KG [Prop. 1.3]. And let (parallelogram) MN have been completed. Thus, MN is equal and similar to GH. But, GH is similar to EL. Thus, MN is also similar to EL [Prop. 6.21]. EL is thus about the same diagonal as MN [Prop. 6.26]. Let their (common) diagonal FO have been drawn, and let the (remainder of the) figure have been described.

And since (parallelogram) GH is equal to (parallelogram) EL and (figure) C, but GH is equal to (parallelogram) MN, MN is thus also equal to EL and C. Let EL have been subtracted from both. Thus, the remaining gnomon XWV is equal to (figure) C. And since AE is equal to EB, (parallelogram) AN is also equal to (parallelogram) NB [Prop. 6.1], that is to say, (parallelogram) LP [Prop. 1.43]. Let (parallelogram) AO is equal to the gnomon VWX. But, the gnomon VWX is equal to (figure) C. Thus, (parallelogram) AO is also equal to (figure) C.

Thus, the parallelogram AO, equal to the given rectilinear figure C, has been applied to the given straightline AB, overshooting by the parallelogrammic figure QP which is similar to D, since PQ is also similar to EL [Prop. 6.24]. (Which is) the very thing it was required to do.

[†] This proposition is a geometric solution of the quadratic equation $x^2 + \alpha x - \beta = 0$. Here, x is the ratio of a side of the excess to the corresponding side of figure D, α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the excess running along AB, and β is the ratio of the areas of figures C and D. Only the positive root of the equation is found.

λ'.

Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην ἄκρον καὶ μέσον λόγον τεμεῖν.



Έστω ή δοθεῖσα εὐθεῖα πεπερασμένη ή AB· δεῖ δὴ τὴν AB εὐθεῖαν ἄχρον καὶ μέσον λόγον τεμεῖν. Proposition 30[†]

To cut a given finite straight-line in extreme and mean ratio.



Let AB be the given finite straight-line. So it is required to cut the straight-line AB in extreme and mean Άναγεγράφθω ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΒΓ, καὶ παραβεβλήσθω παρὰ τὴν ΑΓ τῷ ΒΓ ἴσον παραλληλόγραμμον τὸ ΓΔ ὑπερβάλλον εἶδει τῷ ΑΔ ὁμοίῳ τῷ ΒΓ.

Τετράγωνον δέ ἐστι τὸ BΓ· τετράγωνον ἄρα ἐστι καὶ τὸ AΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ BΓ τῷ ΓΔ, κοινὸν ἀφηρήσθω τὸ ΓΕ· λοιπὸν ἄρα τὸ BZ λοιπῷ τῷ AΔ ἐστιν ἴσον. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον· τῶν BZ, AΔ ἄρα ἀντιπεπόνθασιν αἰ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ἔστιν ἄρα ὡς ἡ ΖΕ πρὸς τὴν ΕΔ, οὕτως ἡ AE πρὸς τὴν EB. ἴση δὲ ἡ μὲν ΖΕ τῆ AB, ἡ δὲ ΕΔ τῆ AE. ἔστιν ἄρα ὡς ἡ BA πρὸς τὴν AE, οὕτως ἡ AE πρὸς τὴν EB. μείζων δὲ ἡ AB τῆς AE· μείζων ἄρα καὶ ἡ AE τῆς EB.

Ή ἄρα AB εὐθεῖα ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ε, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστι τὸ AE· ὅπερ ἔδει ποιῆσαι.

ratio.

Let the square BC have been described on AB [Prop. 1.46], and let the parallelogram CD, equal to BC, have been applied to AC, overshooting by the figure AD (which is) similar to BC [Prop. 6.29].

And BC is a square. Thus, AD is also a square. And since BC is equal to CD, let (rectangle) CE have been subtracted from both. Thus, the remaining (rectangle) BF is equal to the remaining (square) AD. And it is also equiangular to it. Thus, the sides of BF and AD about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as FE is to ED, so AE (is) to EB. And FE (is) equal to AB, and ED to AE. Thus, as BA is to AE, so AE (is) to EB. And AB (is) greater than AE. Thus, AE (is) also greater than EB [Prop. 5.14].

Thus, the straight-line AB has been cut in extreme and mean ratio at E, and AE is its greater piece. (Which is) the very thing it was required to do.

[†] This method of cutting a straight-line is sometimes called the "Golden Section"—see Prop. 2.11.

λα΄.

Έν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἶδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.



Έστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ εἶδεσι τοῖς ὑμοίοις τε καὶ ὑμοίως ἀναγραφομένοις.

"Ηχθω κάθετος ἡ ΑΔ.

Έπεὶ οῦν ἐν ὀρθογωνίω τριγώνω τῷ ABΓ ἀπὸ τῆς πρὸς τῷ A ὀρθῆς γωνίας ἐπὶ τὴν BΓ βάσιν κάθετος ῆκται ἡ AΔ, τὰ ABΔ, AΔΓ πρὸς τῆ καθέτω τρίγωνα ὅμοιά ἐστι τῷ τε ὅλω τῷ ABΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ABΓ τῷ ABΔ, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν BA, οὕτως ἡ AB πρὸς τὴν BΔ. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς

Proposition 31

In right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle.



Let ABC be a right-angled triangle having the angle BAC a right-angle. I say that the figure (drawn) on BC is equal to the (sum of the) similar, and similarly described, figures on BA and AC.

Let the perpendicular AD have been drawn [Prop. 1.12].

Therefore, since, in the right-angled triangle ABC, the (straight-line) AD has been drawn from the rightangle at A perpendicular to the base BC, the triangles ABD and ADC about the perpendicular are similar to the whole (triangle) ABC, and to one another [Prop. 6.8]. And since ABC is similar to ABD, thus τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ὡς ἄρα ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΓΒ εἶδος πρὸς τὸ ἀπὸ τῆς ΒΑ τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ ΒΓ πρὸς τὴν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὸ ἀπὸ τῆς ΓΑ. ὥστε καὶ ὡς ἡ ΒΓ πρὸς τὰς ΒΔ, ΔΓ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὰ ἀπὸ τῶν ΒΑ, ΑΓ τὰ ὅμοια καὶ ὁμοίως ἀναγραφόμενα. ἴση δὲ ἡ ΒΓ ταῖς ΒΔ, ΔΓ· ἴσον ἄρα καὶ τὸ ἄπὸ τῆς ΒΓ είδος τοῖς ἀπὸ τῶν ΒΑ, ΑΓ είδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Έν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς είδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν είδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις· ὅπερ ἔδει δεῖξαι.

λβ΄.

Έὰν δύο τρίγωνα συντεθῆ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὑμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ' εὐθείας ἔσονται.



Έστω δύο τρίγωνα τὰ ΑΒΓ, ΔΓΕ τὰς δύο πλευρὰς τὰς BA, ΑΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΓ, ΔΕ ἀνάλογον ἔχοντα, ὡς μὲν τὴν ΑΒ πρὸς τὴν ΑΓ, οὕτως τὴν ΔΓ πρὸς τὴν ΔΕ, παράλληλον δὲ τὴν μὲν ΑΒ τῆ ΔΓ, τὴν δὲ ΑΓ τῆ ΔΕ· λέγω, ὅτι ἐπ' εὐθείας ἐστὶν ἡ ΒΓ τῆ ΓΕ.

Έπει γὰρ παράλληλός ἐστιν ἡ AB τῆ ΔΓ, και εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ AΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ BAΓ, ΑΓΔ ἴσαι ἀλλήλαις εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΔΕ τῆ ὑπὸ ΑΓΔ ἴση ἐστίν. ὥστε καὶ ἡ ὑπὸ BAΓ τῆ ὑπὸ ΓΔΕ ἐστιν ἴση. καὶ ἐπεὶ δύο τρίγωνά ἐστι τὰ ABΓ, ΔΓΕ μίαν γωνίαν τὴν πρὸς τῷ Δ ἴσην ἔχοντα, περὶ

as CB is to BA, so AB (is) to BD [Def. 6.1]. And since three straight-lines are proportional, as the first is to the third, so the figure (drawn) on the first is to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. Thus, as CB (is) to BD, so the figure (drawn) on CB (is) to the similar, and similarly described, (figure) on BA. And so, for the same (reasons), as BC (is) to CD, so the figure (drawn) on BC (is) to the (figure) on CA. Hence, also, as BC (is) to BD and DC, so the figure (drawn) on BC (is) to the (sum of the) similar, and similarly described, (figures) on BA and AC[Prop. 5.24]. And BC is equal to BD and DC. Thus, the figure (drawn) on BC (is) also equal to the (sum of the) similar, and similarly described, figures on BA and AC[Prop. 5.9].

Thus, in right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle. (Which is) the very thing it was required to show.

Proposition 32

If two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another).



Let ABC and DCE be two triangles having the two sides BA and AC proportional to the two sides DC and DE—so that as AB (is) to AC, so DC (is) to DE—and (having side) AB parallel to DC, and AC to DE. I say that (side) BC is straight-on to CE.

For since AB is parallel to DC, and the straight-line AC has fallen across them, the alternate angles BAC and ACD are equal to one another [Prop. 1.29]. So, for the same (reasons), CDE is also equal to ACD. And, hence, BAC is equal to CDE. And since ABC and DCE are two triangles having the one angle at A equal to the one

δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὡς τὴν BA πρὸς τὴν AΓ, οὕτως τὴν ΓΔ πρὸς τὴν ΔΕ, ἰσογώνιον ἄρα ἐστὶ τὸ ABΓ τρίγωνον τῷ ΔΓΕ τριγώνῳ· ἴση ἄρα ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΓΕ. ἐδείχθη δὲ xaì ἡ ὑπὸ AΓΔ τῆ ὑπὸ BAΓ ἴση· ὅλη ἄρα ἡ ὑπὸ AΓΕ δυσὶ ταῖς ὑπὸ ABΓ, BAΓ ἴση ἐστίν. xοινὴ προσχείσθω ἡ ὑπὸ AΓΒ· αἰ ἄρα ὑπὸ AΓΕ, AΓΒ ταῖς ὑπὸ BAΓ, AΓB, ΓBA ἴσαι εἰσίν. ἀλλ' αἱ ὑπὸ BAΓ, ABΓ, AΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· xαὶ αἱ ὑπὸ AΓΕ, AΓΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. πρὸς δή τινι εὐθεία τῆ AΓ καὶ τῷ πρὸς αὐτῆ σημείω τῷ Γ δύο εὐθεῖαι αἱ BΓ, ΓΕ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνάις τὰς ὑπὸ AΓΕ, AΓΒ δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ BΓ τῆ ΓΕ.

Έὰν ἄρα δύο τρίγωνα συντεθῆ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ' εὐθείας ἔσονται· ὅπερ ἔδει δεῖξαι.

λγ΄.

Έν τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ' ῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ῶσι βεβηκυῖαι.



Έστωσαν ίσοι χύχλοι οἱ ABΓ, ΔΕΖ, καὶ πρὸς μὲν τοῖς κέντροις αὐτῶν τοῖς Η, Θ γωνίαι ἔστωσαν αἱ ὑπὸ BHΓ, EΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ BAΓ, ΕΔΖ· λέγω, ὅτι ἐστὶν ὡς ἡ BΓ περιφέρεια πρὸς τὴν ΕΖ περιφέρειαν, οὕτως ἤ τε ὑπὸ BHΓ γωνία πρὸς τὴν ὑπὸ EΘΖ καὶ ἡ ὑπὸ BAΓ πρὸς τὴν ὑπὸ EΔΖ.

Κείσθωσαν γὰρ τῆ μὲν ΒΓ περιφερεία ἴσαι κατὰ τὸ ἑξῆς ὑσαιδηποτοῦν αἱ ΓΚ, ΚΛ, τῆ δὲ ΕΖ περιφερεία ἴσαι ὑσαιδηποτοῦν αἱ ΖΜ, ΜΝ, καὶ ἐπεζεύχθωσαν αἱ ΗΚ, ΗΛ, ΘΜ, ΘΝ.

Έπεὶ οὕν ἴσαι εἰσὶν αἰ ΒΓ, ΓΚ, ΚΛ περιφέρειαι ἀλλήλαις, ἴσαι εἰσὶ καὶ αἱ ὑπὸ ΒΗΓ, ΓΗΚ, ΚΗΛ γωνίαι ἀλλήλαις· ὑσαπλασίων ἄρα ἐστὶν ἡ ΒΛ περιφέρεια τῆς ΒΓ, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΒΗΓ. διὰ τὰ angle at D, and the sides about the equal angles proportional, (so that) as BA (is) to AC, so CD (is) to DE, triangle ABC is thus equiangular to triangle DCE [Prop. 6.6]. Thus, angle ABC is equal to DCE. And (angle) ACD was also shown (to be) equal to BAC. Thus, the whole (angle) ACE is equal to the two (angles) ABC and BAC. Let ACB have been added to both. Thus, ACE and ACB are equal to BAC, ACB, and CBA. But, BAC, ABC, and ACB are equal to two right-angles [Prop. 1.32]. Thus, ACE and ACB are also equal to two right-angles. Thus, the two straight-lines BC and CE, not lying on the same side, make adjacent angles with some straight-line AC, at the point C on it. Thus, BC is straight-on to CE [Prop. 1.14].

Thus, if two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another). (Which is) the very thing it was required to show.

Proposition 33

In equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences.



Let ABC and DEF be equal circles, and let BGC and EHF be angles at their centers, G and H (respectively), and BAC and EDF (angles) at their circumferences. I say that as circumference BC is to circumference EF, so angle BGC (is) to EHF, and (angle) BAC to EDF.

For let any number whatsoever of consecutive (circumferences), CK and KL, be made equal to circumference BC, and any number whatsoever, FM and MN, to circumference EF. And let GK, GL, HM, and HN have been joined.

Therefore, since circumferences BC, CK, and KL are equal to one another, angles BGC, CGK, and KGL are also equal to one another [Prop. 3.27]. Thus, as many times as circumference BL is (divisible) by BC, so many

αὐτὰ δὴ χαὶ ὑσαπλασίων ἐστὶν ἡ ΝΕ περιφέρεια τῆς ΕΖ, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ ΝΘΕ γωνία τῆς ὑπὸ ΕΘΖ. εἰ ἄρα ἴση ἐστὶν ἡ BΛ περιφέρεια τῆ EN περιφερεία, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΗΛ τῆ ὑπὸ ΕΘΝ, καὶ εἰ μείζων ἐστὶν ἡ ΒΛ περιφέρεια τῆς ΕΝ περιφερείας, μείζων ἐστὶ καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΕΘΝ, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δή ὄντων μεγεθῶν, δύο μέν περιφερειῶν τῶν ΒΓ, ΕΖ, δύο δὲ γωνιῶν τῶν ὑπὸ ΒΗΓ, ΕΘΖ, εἴληπται τῆς μὲν ΒΓ περιφερείας καὶ τῆς ὑπὸ ΒΗΓ γωνίας ἰσάκις πολλαπλασίων ἤ τε ΒΛ περιφέρεια καὶ ἡ ὑπὸ ΒΗΛ γωνία, τῆς δὲ ΕΖ περιφερείας καὶ τῆς ὑπὸ ΕΘΖ γωνίας ἤ τε ΕΝ περιφέρια καὶ ἡ ὑπὸ ΕΘΝ γωνία. καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ ΒΛ περιφέρεια τῆς ΕΝ περιφερείας, ὑπερέχει καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπο ΕΘΝ γωνίας, καὶ εἰ ἴση, ἴση, καὶ εἰ ἐλάσσων, ἐλάσσων. έστιν άρα, ώς ή ΒΓ περιφέρεια πρός την ΕΖ, οὕτως ή ὑπὸ BHΓ γωνία πρός τὴν ὑπὸ ΕΘΖ. ἀλλ' ὡς ἡ ὑπὸ BHΓ γωνία πρός τὴν ὑπὸ ΕΘΖ, οὕτως ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ. διπλασία γὰρ ἑχατέρα ἑχατέρας. χαὶ ὡς ἄρα ἡ ΒΓ περιφέρεια πρός τὴν ΕΖ περιφέρειαν, οὕτως ἤ τε ὑπὸ ΒΗΓ γωνία πρός τὴν ὑπὸ ΕΘΖ καὶ ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ.

Έν ἄρα τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ' ῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ῶσι βεβηκυῖαι· ὅπερ ἔδει δεῖζαι.

times is angle BGL also (divisible) by BGC. And so, for the same (reasons), as many times as circumference NEis (divisible) by EF, so many times is angle NHE also (divisible) by EHF. Thus, if circumference BL is equal to circumference EN then angle BGL is also equal to EHN [Prop. 3.27], and if circumference BL is greater than circumference EN then angle BGL is also greater than EHN,[†] and if (BL is) less (than EN then BGL is also) less (than EHN). So there are four magnitudes, two circumferences BC and EF, and two angles BGCand EHF. And equal multiples have been taken of circumference BC and angle BGC, (namely) circumference BL and angle BGL, and of circumference EF and angle EHF, (namely) circumference EN and angle EHN. And it has been shown that if circumference BL exceeds circumference EN then angle BGL also exceeds angle EHN, and if (BL is) equal (to EN then BGL is also) equal (to EHN), and if (BL is) less (than EN then BGLis also) less (than EHN). Thus, as circumference BC(is) to EF, so angle BGC (is) to EHF [Def. 5.5]. But as angle BGC (is) to EHF, so (angle) BAC (is) to EDF[Prop. 5.15]. For the former (are) double the latter (respectively) [Prop. 3.20]. Thus, also, as circumference BC (is) to circumference EF, so angle BGC (is) to EHF, and BAC to EDF.

Thus, in equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences. (Which is) the very thing it was required to show.

[†] This is a straight-forward generalization of Prop. 3.27

ELEMENTS BOOK 7

Elementary Number Theory †

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

Οροι.

α'. Μονάς ἐστιν, καθ' ἢν ἕκαστον τῶν ὄντων ἕν λέγεται.

β΄. Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

γ΄. Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῆ τὸν μείζονα.

δ'. Μέρη δέ, ὄταν μὴ καταμετρῆ.

ε'. Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

γ΄. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

ζ΄. Περισσός δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

η'. Άρτιάχις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος χατὰ ἄρτιον ἀριθμόν.

θ΄. Ἄρτιάχις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

ι΄. Περισσάχις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος χατὰ περισσὸν ἀριθμόν.

ια'. Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.

ιβ΄. Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνη μετρούμενοι κοινῷ μέτρῳ.

ιγ΄. Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

ιδ'. Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

ιε΄. Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάχις συντεθῆ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.

ιτ΄. Όταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιζ΄. Όταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιη'. Τετράγωνος ἀριθμός ἐστιν ὁ ἰσάχις ἴσος ἢ [ἑ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

ιθ΄. Κύβος δὲ ὁ ἰσάχις ἴσος ἰσάχις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

x'. Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάχις ἢ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ῶσιν.

κα'. Όμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοί εἰσιν οἱ ανάλογον ἔχοντες τὰς πλευράς.

κβ΄. Τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν.

Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.

2. And a number (is) a multitude composed of units.^{\dagger}

3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.[‡]

4. But (the lesser is) parts (of the greater) when it does not measure it.[§]

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

6. An even number is one (which can be) divided in half.

7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.

8. An even-times-even number is one (which is) measured by an even number according to an even number.

9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.*

10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number.[§]

11. A prime^{||} number is one (which is) measured by a unit alone.

12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.

13. A composite number is one (which is) measured by some number.

14. And numbers composite to one another are those (which are) measured by some number as a common measure.

15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.

16. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.

17. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.

18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.

19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.

20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.

21. Similar plane and solid numbers are those having proportional sides.

22. A perfect number is that which is equal to its own parts. ††

 † In other words, a "number" is a positive integer greater than unity.

^{\ddagger} In other words, a number *a* is part of another number *b* if there exists some number *n* such that n a = b.

 § In other words, a number a is parts of another number b (where a < b) if there exist distinct numbers, m and n, such that n a = m b.

¶ In other words. an even-times-even number is the product of two even numbers.

* In other words, an even-times-odd number is the product of an even and an odd number.

^{\$} In other words, an odd-times-odd number is the product of two odd numbers.

[∥] Literally, "first".

^{††} In other words, a perfect number is equal to the sum of its own factors.

α΄.

Δύο ἀριθμῶν ἀνίσων ἐχχειμένων, ἀνθυφαιρουμένου δὲ ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε χαταμετρῆ τὸν πρὸ ἑαυτοῦ, ἕως οῦ λειφθῆ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλὴλους ἔσονται.



Δύο γὰρ [ἀνίσων] ἀριθμῶν τῶν AB, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος ὁ λειπόμενος μηδέποτε καταμετρείτω τὸν πρὸ ἑαυτοῦ, ἕως οῦ λειφθῆ μονάς· λέγω, ὅτι οἱ AB, ΓΔ πρῶτοι πρὸς ἀλλήλους εἰσίν, τουτέστιν ὅτι τοὺς AB, ΓΔ μονὰς μόνη μετρεῖ.

Εἰ γὰρ μή εἰσιν οἱ AB, ΓΔ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Ε΄ καὶ ὁ μὲν ΓΔ τὸν BZ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ZA, ὁ δὲ AZ τὸν ΔΗ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΗΓ, ὁ δὲ ΗΓ τὸν ΖΘ μετρῶν λειπέτω μονάδα τὴν ΘΑ.

Έπεὶ οῦν ὁ Ε τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΖ μετρεῖ, καὶ ὁ Ε ἄρα τὸν ΒΖ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΖ μετρήσει. ὁ δὲ ΑΖ τὸν ΔΗ μετρεῖ· καὶ ὁ Ε ἄρα τὸν ΔΗ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΗ μετρήσει. ὁ δὲ ΓΗ τὸν ΖΘ μετρεῖ·

Proposition 1

Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.



For two [unequal] numbers, AB and CD, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains. I say that AB and CD are prime to one another—that is to say, that a unit alone measures (both) AB and CD.

For if AB and CD are not prime to one another then some number will measure them. Let (some number) measure them, and let it be E. And let CD measuring BF leave FA less than itself, and let AF measuring DGleave GC less than itself, and let GC measuring FH leave a unit, HA.

In fact, since E measures CD, and CD measures BF, E thus also measures BF.[†] And (E) also measures the whole of BA. Thus, (E) will also measure the remainder

καὶ ὁ Ἐ ἄρα τὸν ΖΘ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΖΑ· καὶ λοιπὴν ἄρα τὴν ΑΘ μονάδα μετρήσει ἀριθμὸς ὤν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς μετρήσει τις ἀριθμός· οἱ ΑΒ, ΓΔ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι. AF.[‡] And AF measures DG. Thus, E also measures DG. And (E) also measures the whole of DC. Thus, (E) will also measure the remainder CG. And CG measures FH. Thus, E also measures FH. And (E) also measures the whole of FA. Thus, (E) will also measure the remaining unit AH, (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers AB and CD. Thus, AB and CD are prime to one another. (Which is) the very thing it was required to show.

^{\dagger} Here, use is made of the unstated common notion that if *a* measures *b*, and *b* measures *c*, then *a* also measures *c*, where all symbols denote numbers.

^{\ddagger} Here, use is made of the unstated common notion that if *a* measures *b*, and *a* measures part of *b*, then *a* also measures the remainder of *b*, where all symbols denote numbers.



Δύο ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν χοινὸν μέτρον εὑρεῖν.



Έστωσαν οἱ δοθέντες δύο ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ AB, ΓΔ. δεῖ δὴ τῶν AB, ΓΔ τὸ μέγιστον χοινὸν μέτρον εὑρεῖν.

Εἰ μὲν οὖν ὁ ΓΔ τὸν ΑΒ μετρεῖ, μετρεῖ δὲ καὶ ἑαυτόν, ὁ ΓΔ ἄρα τῶν ΓΔ, ΑΒ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· οὐδεἰς γὰρ μείζων τοῦ ΓΔ τὸν ΓΔ μετρήσει.

Εἰ δὲ οὐ μετρεῖ ὁ ΓΔ τὸν AB, τῶν AB, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος λειφθήσεταί τις ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. μονὰς μὲν γὰρ οὐ λειφθήσεται· εἰ δὲ μή, ἔσονται οἱ AB, ΓΔ πρῶτοι πρὸς ἀλλήλους· ὅπερ οὐχ ὑπόκειται. λειφθήσεταί τις ἄρα ἀριθμὸς, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. καὶ ὁ μὲν ΓΔ τὸν BE μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν EA, ὁ δὲ EA τὸν ΔΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ZΓ, ὁ δὲ ΓΖ τὸν AE μετρείτω. ἐπεὶ οῦν ὁ ΓΖ τὸν AE μετρεῖ, ὁ δὲ AE τὸν ΔΖ μετρεῖ, καὶ ὁ ΓΖ ἄρα τὸν ΔΖ μετρήσει. μετρεῖ δὲ καὶ ἑαυτόν· καὶ ὅλον ἄρα τὸν ΓΔ μετρήσει. ὑ δὲ ΓΔ τὸν BE μετρεῖ· καὶ ὁ ΓΖ ἄρα τὸν BE μετρεῖ· μετρεῖ δὲ καὶ τὸν EA· καὶ ὅλον ἄρα τὸν BA μετρήσει· μετρεῖ δὲ καὶ τὸν ΓΔ· ὁ ΓΖ ἄρα τοὺς AB, ΓΔ μετρεῖ. ὁ ΓΖ ἄρα τῶν AB, ΓΔ κοινὸν

Proposition 2

To find the greatest common measure of two given numbers (which are) not prime to one another.



Let AB and CD be the two given numbers (which are) not prime to one another. So it is required to find the greatest common measure of AB and CD.

In fact, if CD measures AB, CD is thus a common measure of CD and AB, (since CD) also measures itself. And (it is) manifest that (it is) also the greatest (common measure). For nothing greater than CD can measure CD.

But if CD does not measure AB then some number will remain from AB and CD, the lesser being continually subtracted, in turn, from the greater, which will measure the (number) preceding it. For a unit will not be left. But if not, AB and CD will be prime to one another [Prop. 7.1]. The very opposite thing was assumed. Thus, some number will remain which will measure the (number) preceding it. And let CD measuring BE leave EAless than itself, and let EA measuring DF leave FC less than itself, and let CF measure AE. Therefore, since CFmeasures AE, and AE measures DF, CF will thus also measure DF. And it also measures itself. Thus, it will μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ ΓΖ τῶν ΑΒ, ΓΔ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμὸς μείζων ఊν τοῦ ΓΖ. μετρείτω, καὶ ἔστω ὁ Η. καὶ ἐπεὶ ὁ Η τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΕ μετρεῖ, καὶ ὁ Η ἄρα τὸν ΒΕ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΕ μετρήσει. ἱ δὲ ΑΕ τὸν ΔΖ μετρεῖ· καὶ ὁ Η ἄρα τὸν ΔΖ μετρήσει. ὑ δὲ ΑΕ τὸν ΔΖ μετρεῖ· καὶ ὁ Η ἄρα τὸν ΔΖ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ఊν τοῦ ΓΖ· ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ μέγιστόν ἐστι κοινὸν μέτρον [ὅπερ ἔδει δεῖξαι].

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι ἐἀν ἀριθμὸς δύο ἀριθμοὺς μετρῆ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει· ὅπερ ἔδει δεῖξαι.

γ'.

Τριῶν ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν χοινὸν μέτρον εὐρεῖν.



Έστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Eἰλήφθω γὰρ δύο τῶν A, B τὸ μέγιστον κοινὸν μέτρον
ἑ $\Delta\cdot$ ὁ δὴ Δ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον·
μετρεῖ δέ καὶ τοὺς A, B· ὁ Δ ἄρα τοὺς A, B, Γ μετρεῖ· ὁ
Δ ἄρα τῶν A, B, Γ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ

also measure the whole of CD. And CD measures BE. Thus, CF also measures BE. And it also measures EA. Thus, it will also measure the whole of BA. And it also measures CD. Thus, CF measures (both) AB and CD. Thus, CF is a common measure of AB and CD. So I say that (it is) also the greatest (common measure). For if CF is not the greatest common measure of AB and CDthen some number which is greater than CF will measure the numbers AB and CD. Let it (so) measure (ABand CD), and let it be G. And since G measures CD, and CD measures BE, G thus also measures BE. And it also measures the whole of BA. Thus, it will also measure the remainder AE. And AE measures DF. Thus, Gwill also measure DF. And it also measures the whole of DC. Thus, it will also measure the remainder CF, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than CFcannot measure the numbers AB and CD. Thus, CF is the greatest common measure of AB and CD. [(Which is) the very thing it was required to show].

Corollary

So it is manifest, from this, that if a number measures two numbers then it will also measure their greatest common measure. (Which is) the very thing it was required to show.

Proposition 3

To find the greatest common measure of three given numbers (which are) not prime to one another.



Let A, B, and C be the three given numbers (which are) not prime to one another. So it is required to find the greatest common measure of A, B, and C.

For let the greatest common measure, D, of the two (numbers) A and B have been taken [Prop. 7.2]. So D either measures, or does not measure, C. First of all, let it measure (C). And it also measures A and B. Thus, D

μέγιστον. εἰ γὰρ μή ἐστιν ὁ Δ τῶν Α, Β, Γ μέγιστον χοινὸν μέτρον, μετρήσει τις τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμὸς μείζων ἂν τοῦ Δ. μετρείτω, καὶ ἔστω ὁ Ε. ἐπεὶ οῦν ὁ Ε τοὺς Α, Β, Γ μετρεῖ, καὶ τοὺς Α, Β ἄρα μετρήσει· καὶ τὸ τῶν Α, Β ἄρα μέγιστον χοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον χοινὸν μέτρον ἐστὶν ὁ Δ· ὁ Ε ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐχ ἄρα τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ἂν τοῦ Δ· ὁ Δ ἄρα τῶν Α, Β, Γ μέγιστόν ἐστι χοινὸν μέτρον.

Μὴ μετρείτω δὴ
 ὁ Δ τὸν Γ· λέγω πρῶτον, ὅτι οἱ Γ, Δ ούχ είσι πρῶτοι πρὸς ἀλλήλους. ἐπεὶ γὰρ οἱ Α, Β, Γ ούχ είσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. ὁ δή τούς Α, Β, Γ μετρῶν καὶ τούς Α, Β μετρήσει, καὶ τὸ τῶν Α, Β μέγιστον κοινόν μέτρον τόν Δ μετρήσει μετρεῖ δὲ καὶ τὸν Γ · τοὺς Δ , Γ ἄρα ἀριθμοὺς ἀριθμός τις μετρήσει· οί Δ , Γ ἄρα οὕκ εἰσι πρῶτοι πρὸς ἀλλήλους. εἰλήφθω οὕν αὐτῶν τὸ μέγιστον
 χοινὸν μέτρον ὁ Ε. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, ὁ δὲ Δ τοὺς A, B μετρεῖ, καὶ ὁ E ẳρα τοὺς A, B μετρεί μετρεί δε και τον Γ ό Ε άρα τους Α, Β, Γ μετρεί. ό Ε ἄρα τῶν Α, Β, Γ κοινόν ἐστι μέτρον. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ Ε τῶν Α, Β, Γ τὸ μέγιστον κοινόν μέτρον, μετρήσει τις τούς Α, Β, Γ ἀριθμούς ἀριθμός μείζων ών τοῦ Ε. μετρείτω, καὶ ἔστω ὁ Ζ. καὶ ἐπεὶ ὁ Ζ τοὺς A, B, Γ μετρεῖ, καὶ τοὺς A, B μετρεῖ· καὶ τὸ τῶν A, B ẳρα μέγιστον χοινόν μέτρον μετρήσει. τό δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶν
ἱ $\Delta\cdot$ ἱ Z ἄρα τὸν
Δ μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ· ὁ Ζ ẳρα τοὺς Δ, Γ μετρεĩ· καὶ τὸ τῶν Δ, Γ ẳρα μέγιστον χοινόν μέτρον μετρήσει. τό δὲ τῶν Δ, Γ μέγιστον χοινόν μέτρον ἐστὶν ὁ Ε· ὁ Ζ ἄρα τὸν Ε μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β, Γ άριθμούς άριθμός τις μετρήσει μείζων ών τοῦ Ε· ὁ Ε ἄρα τῶν Α, Β, Γ μέγιστόν ἐστι χοινὸν μέτρον ὅπερ ἔδει δεῖξαι.

δ'.

Άπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ῆτοι μέρος ἐστὶν ἢ μέρη.

Έστωσαν δύο ἀριθμοὶ οἱ Α, ΒΓ, καὶ ἔστω ἐλάσσων ὁ ΒΓ· λέγω, ὅτι ὁ ΒΓ τοῦ Α ἤτοι μέρος ἐστὶν ἢ μέρη. measures A, B, and C. Thus, D is a common measure of A, B, and C. So I say that (it is) also the greatest (common measure). For if D is not the greatest common measure of A, B, and C then some number greater than D will measure the numbers A, B, and C. Let it (so) measure (A, B, and C), and let it be E. Therefore, since E measures A, B, and C, it will thus also measure A and B. Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B. Thus, E measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than D cannot measure the numbers A, B, and C. Thus, D is the greatest common measure of A, B, and C.

So let D not measure C. I say, first of all, that Cand D are not prime to one another. For since A, B, Care not prime to one another, some number will measure them. So the (number) measuring A, B, and C will also measure A and B, and it will also measure the greatest common measure, D, of A and B [Prop. 7.2 corr.]. And it also measures C. Thus, some number will measure the numbers D and C. Thus, D and C are not prime to one another. Therefore, let their greatest common measure, E, have been taken [Prop. 7.2]. And since E measures D, and D measures A and B, E thus also measures Aand B. And it also measures C. Thus, E measures A, B, and C. Thus, E is a common measure of A, B, and C. So I say that (it is) also the greatest (common measure). For if E is not the greatest common measure of A, B, and C then some number greater than E will measure the numbers A, B, and C. Let it (so) measure (A, B, and C), and let it be F. And since F measures A, B, and C, it also measures A and B. Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B. Thus, Fmeasures D. And it also measures C. Thus, F measures D and C. Thus, it will also measure the greatest common measure of D and C [Prop. 7.2 corr.]. And E is the greatest common measure of D and C. Thus, F measures E, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than Edoes not measure the numbers A, B, and C. Thus, E is the greatest common measure of A, B, and C. (Which is) the very thing it was required to show.

Proposition 4

Any number is either part or parts of any (other) number, the lesser of the greater.

Let A and BC be two numbers, and let BC be the lesser. I say that BC is either part or parts of A.

Οἱ Α, ΒΓ γὰρ ἦτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὕ. ἔστωσαν πρότερον οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους. διαιρεθέντος δὴ τοῦ ΒΓ εἰς τὰς ἐν αὐτῷ μονάδας ἔσται ἑκάστη μονὰς τῶν ἐν τῷ ΒΓ μέρος τι τοῦ Α· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.



Μὴ ἔστωσαν δὴ οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους· ὁ δὴ ΒΓ τὸν Α ἤτοι μετρεῖ ἢ οὐ μετρεῖ. εἰ μὲν οῦν ὁ ΒΓ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ ΒΓ τοῦ Α. εἰ δὲ οὕ, εἰλήφθω τῶν Α, ΒΓ μέγιστον χοινὸν μέτρον ὁ Δ, χαὶ διῃρήσθω ὁ ΒΓ εἰς τοὺς τῷ Δ ἴσους τοὺς ΒΕ, ΕΖ, ΖΓ. χαὶ ἐπεὶ ὁ Δ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ Δ τοῦ Α· ἴσος δὲ ὁ Δ ἑχάστῷ τῶν ΒΕ, ΕΖ, ΖΓ· χαὶ ἔχαστος ἄρα τῶν ΒΕ, ΕΖ, ΖΓ τοῦ Α μέρος ἐστίν· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.

Άπας ἄρα ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη· ὅπερ ἔδει δεῖξαι.

ε΄.

Έὰν ἀριθμὸς ἀριθμοῦ μέρος ἢ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ἢ, καὶ συναμφότερος συναμφοτέρου τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ εῖς τοῦ ἑνός.



Άριθμός γὰρ ὁ Α [ἀριθμοῦ] τοῦ ΒΓ μέρος ἔστω, καὶ

For A and BC are either prime to one another, or not. Let A and BC, first of all, be prime to one another. So separating BC into its constituent units, each of the units in BC will be some part of A. Hence, BC is parts of A.



So let A and BC be not prime to one another. So BC either measures, or does not measure, A. Therefore, if BC measures A then BC is part of A. And if not, let the greatest common measure, D, of A and BC have been taken [Prop. 7.2], and let BC have been divided into BE, EF, and FC, equal to D. And since D measures A, D is a part of A. And D is equal to each of BE, EF, and FC. Thus, BE, EF, and FC are also each part of A. Hence, BC is parts of A.

Thus, any number is either part or parts of any (other) number, the lesser of the greater. (Which is) the very thing it was required to show.

Proposition 5[†]

If a number is part of a number, and another (number) is the same part of another, then the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.



For let a number A be part of a [number] BC, and

ἕτερος ὁ Δ ἑτέρου τοῦ ΕΖ τὸ αὐτὸ μέρος, ὅπερ ὁ Α τοῦ $B\Gamma$ · λέγω, ὅτι καὶ συναμφότερος ὁ Α, Δ συναμφοτέρου τοῦ $B\Gamma$, ΕΖ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ Α τοῦ $B\Gamma$.

Έπει γάρ, δ μέρος ἑστιν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἑστι καὶ ὁ Δ τοῦ ΕΖ, ὅσοι ἄρα εἰσιν ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διῆρήσθω ὁ μὲν ΒΓ εἰς τοὺς τῷ Α ἴσους τοὺς BH, ΗΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν BH, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσος ἑστιν ὁ μὲν BH τῷ Α, ὁ δὲ ΕΘ τῷ Δ, καὶ οἱ BH, ΕΘ ἄρα τοῖς Α, Δ ἴσοι. διὰ τὰ αὐτὰ δὴ καὶ οἱ ΗΓ, ΘΖ τοῖς Α, Δ. ὅσοι ἄρα [εἰσιν] ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τοῖς BΓ, ΕΖ ἴσοι τοῖς Α, Δ. ὁσαπλασίων ἄρα ἑστιν ὁ ΒΓ τοῦ Α, τοσαυταπλασίων ἐστὶ καὶ συναμφότερος ὁ ΒΓ, ΕΖ συναμφοτέρου τοῦ Α, Δ. ὃ ἄρα μέρος ἐστιν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ Α, Δ συναμφοτέρου τοῦ ΒΓ, ΕΖ· ὅπερ ἕδει δεῖξαι. another (number) D (be) the same part of another (number) EF that A (is) of BC. I say that the sum A, D is also the same part of the sum BC, EF that A (is) of BC.

For since which(ever) part A is of BC, D is the same part of EF, thus as many numbers as are in BC equal to A, so many numbers are also in EF equal to D. Let BC have been divided into BG and GC, equal to A, and EF into EH and HF, equal to D. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF. And since BG is equal to A, and EHto D, thus BG, EH (is) also equal to A, D. So, for the same (reasons), GC, HF (is) also (equal) to A, D. Thus, as many numbers as [are] in BC equal to A, so many are also in BC, EF equal to A, D. Thus, as many times as BC is (divisible) by A, so many times is the sum BC, EFalso (divisible) by the sum A, D. Thus, which(ever) part A is of BC, the sum A, D is also the same part of the sum BC, EF. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a = (1/n) b and c = (1/n) d then (a + c) = (1/n) (b + d), where all symbols denote numbers.

ኖ'.

Έὰν ἀριθμὸς ἀριθμοῦ μέρη ἢ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ μέρη ἢ, καὶ συναμφότερος συναμφοτέρου τὰ αὐτὰ μέρη ἔσται, ὅπερ ὁ εῖς τοῦ ἑνός.



Άριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔΕ ἑτέρου τοῦ Ζ τὰ αὐτὰ μέρη, ἄπερ ὁ AB τοῦ Γ· λέγω, ὅτι καὶ συναμφότερος ὁ AB, ΔΕ συναμφοτέρου τοῦ Γ, Ζ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὁ AB τοῦ Γ.

Έπεὶ γάρ, ἂ μέρη ἐστὶν ὁ AB τοῦ Γ, τὰ αὐτὰ μέρη καὶ
ἱ ΔE τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ AB μέρη τοῦ Γ, τοσαῦτά ἐστι καὶ ἐν τῷ ΔE μέρη τοῦ Ζ. διηρήσθω ὁ μὲν AB εἰς τὰ
 τοῦ Γ μέρη τὰ AH, HB, ὁ δὲ ΔE εἰς τὰ τοῦ Ζ μέρη τὰ
 $\Delta \Theta$, ΘΕ' ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH, HB τῷ πλήθει
 τῶν $\Delta \Theta$, ΘΕ. καὶ ἐπεί, ὁ μέρος ἐστὶν ὁ AH τοῦ Γ, τὸ

Proposition 6[†]

If a number is parts of a number, and another (number) is the same parts of another, then the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.



For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another (number) F that AB (is) of C. I say that the sum AB, DE is also the same parts of the sum C, F that AB (is) of C.

For since which (ever) parts AB is of C, DE (is) also the same parts of F, thus as many parts of C as are in AB, so many parts of F are also in DE. Let AB have been divided into the parts of C, AG and GB, and DE into the parts of F, DH and HE. So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH,

HE. And since which(ever) part AG is of C, DH is also the same part of F, thus which(ever) part AG is of C, the sum AG, DH is also the same part of the sum C, F[Prop. 7.5]. And so, for the same (reasons), which(ever) part GB is of C, the sum GB, HE is also the same part of the sum C, F. Thus, which(ever) parts AB is of C, the sum AB, DE is also the same parts of the sum C, F. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a = (m/n) b and c = (m/n) d then (a + c) = (m/n) (b + d), where all symbols denote numbers.

ζ.

Έὰν ἀριθμὸς ἀριθμοῦ μέρος ἦ, ὅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ ὅλος τοῦ ὅλου.



Άριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ ΓΔ μέρος ἔστω, ὅπερ ἀφαιρεθεὶς ὁ AE ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ EB λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ AB ὅλου τοῦ ΓΔ.

 $^{\circ}$ Ο γὰρ μέρος ἐστὶν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος ἔστω καὶ ὁ EB τοῦ ΓΗ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ EB τοῦ ΓΗ, ὃ ἄρα μέρος ἐστὶν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AB τοῦ ΗΖ. ὃ δὲ μέρος ἐστὶν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος ὑπόκειται καὶ ὁ AB τοῦ ΓΔ· ὃ ἄρα μέρος ἐστὶ καὶ ὁ AB τοῦ ΗΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ τοῦ ΓΔ· ἴσος ἄρα ἐστὶν ὁ HZ τῷ ΓΔ. κοινὸς ἀφηρήσθω ὁ ΓΖ· λοιπὸς ἄρα ὁ ΗΓ λοιπῷ τῷ ΖΔ ἐστιν ἴσος. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος [ἐστὶ] καὶ ὁ EB τοῦ ΗΓ, ἴσος δὲ ὁ ΗΓ τῷ ΖΔ, ὃ ἄρα μέρος ἐστὶν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ EB τοῦ ΖΔ. ἀλλὰ ὃ μέρος ἐστιν ὁ AE τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶν ἱ AB τοῦ ΓΔ· καὶ λοιπὸς ἄρα ὁ EB λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ AB ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.

Proposition 7[†]

If a number is that part of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same part of the remainder that the whole (is) of the whole.



For let a number AB be that part of a number CD that a (part) taken away AE (is) of a part taken away CF. I say that the remainder EB is also the same part of the remainder FD that the whole AB (is) of the whole CD.

For which (ever) part AE is of CF, let EB also be the same part of CG. And since which(ever) part AE is of CF, EB is also the same part of CG, thus which(ever) part AE is of CF, AB is also the same part of GF[Prop. 7.5]. And which(ever) part AE is of CF, AB is also assumed (to be) the same part of CD. Thus, also, which (ever) part AB is of GF, (AB) is also the same part of CD. Thus, GF is equal to CD. Let CF have been subtracted from both. Thus, the remainder GC is equal to the remainder FD. And since which(ever) part AE is of CF, EB [is] also the same part of GC, and GC (is) equal to FD, thus which(ever) part AE is of CF, EB is also the same part of FD. But, which(ever) part AE is of CF, AB is also the same part of CD. Thus, the remainder EB is also the same part of the remainder FD that the whole AB (is) of the whole CD. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a = (1/n) b and c = (1/n) d then (a - c) = (1/n) (b - d), where all symbols denote numbers.

η΄.

Έὰν ἀριθμὸς ἀριθμοῦ μέρη ἢ, ἄπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὰ αὐτὰ μέρη ἔσται, ἄπερ ὁ ὅλος τοῦ ὅλου.

Proposition 8[†]

If a number is those parts of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same parts of the remainder that the



Άριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ ΓΔ μέρη ἔστω, ἄπερ ἀφαιρεθεὶς ὁ AE ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ EB λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ AB ὅλου τοῦ ΓΔ.

Κείσθω γὰρ τῷ ΑΒ ἴσος ὁ ΗΘ, ἁ ἄρα μέρη ἐστὶν ὁ ΗΘ τοῦ ΓΔ, τὰ αὐτὰ μέρη ἐστὶ χαὶ ὁ ΑΕ τοῦ ΓΖ. διηρήσθω ὁ μèν HΘ εἰς τὰ τοῦ ΓΔ μέρη τὰ HK, KΘ,
ὁ δὲ AE εἰς τὰ τοῦ ΓZ μέρη τὰ AA, AE· ἔσται δὴ ἴσον τὸ πλῆθος τῶν HK, KΘ τῷ πλήθει τῶν ΑΛ, ΛΕ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΗΚ τοῦ $\Gamma\Delta$, τὸ αὐτὸ μέρος ἐστὶ χαὶ ὁ ΑΛ τοῦ Γ Ζ, μείζων δὲ ὁ $\Gamma\Delta$ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΗΚ τοῦ ΑΛ. κείσθω τῷ ΑΛ ἴσος ό ΗΜ. ὃ ἄρα μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ HM τοῦ ΓΖ· καὶ λοιπὸς ẳρα ὁ MK λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ. πάλιν ἐπεί, ὃ μέρος ἐστὶν ὁ ΚΘ τοῦ Γ Δ , τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΘΚ τοῦ ΕΛ. κείσθω τῷ ΕΛ ἴσος ὁ ΚΝ. ὃ ἄρα μέρος ἐστὶν ὁ ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ἑ KN τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΝΘ λοιποῦ τοῦ Ζ Δ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΚΘ ὅλου τοῦ ΓΔ. ἐδείχϑη δὲ καὶ λοιπὸς ὁ ΜΚ λοιποῦ τοῦ $Z\Delta$ τὸ αὐτὸ μέρος ὤν, ὅπερ ὅλος ὁ HK ὅλου τοῦ ΓΔ· καὶ συναμφότερος
 άρα ὁ MK, NΘ τοῦ ΔZ τὰ αὐτὰ μέρη ἐστίν, **ἄπερ ὅλος ὁ ΘΗ ὅλου τοῦ ΓΔ. ἴσος δὲ συναμφότερος μὲν** ό ΜΚ, ΝΘ τῷ ΕΒ, ὁ δὲ ΘΗ τῷ ΒΑ· καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ Z Δ τὰ αὐτὰ μέρη ἐστίν, ἄπερ ὅλος ὁ AB ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.



For let a number AB be those parts of a number CD that a (part) taken away AE (is) of a (part) taken away CF. I say that the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD.

For let GH be laid down equal to AB. Thus, which(ever) parts GH is of CD, AE is also the same parts of CF. Let GH have been divided into the parts of CD, GK and KH, and AE into the part of CF, ALand LE. So the multitude of (divisions) GK, KH will be equal to the multitude of (divisions) AL, LE. And since which (ever) part GK is of CD, AL is also the same part of CF, and CD (is) greater than CF, GK (is) thus also greater than AL. Let GM be made equal to AL. Thus, which (ever) part GK is of CD, GM is also the same part of CF. Thus, the remainder MK is also the same part of the remainder FD that the whole GK (is) of the whole CD [Prop. 7.5]. Again, since which(ever) part KH is of CD, EL is also the same part of CF, and CD (is) greater than CF, HK (is) thus also greater than EL. Let KN be made equal to EL. Thus, which(ever) part KH (is) of CD, KN is also the same part of CF. Thus, the remainder NH is also the same part of the remainder FD that the whole KH (is) of the whole CD [Prop. 7.5]. And the remainder MK was also shown to be the same part of the remainder FD that the whole GK (is) of the whole CD. Thus, the sum MK, NH is the same parts of DFthat the whole HG (is) of the whole CD. And the sum MK, NH (is) equal to EB, and HG to BA. Thus, the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a = (m/n) b and c = (m/n) d then (a - c) = (m/n) (b - d), where all symbols denote numbers.

θ'.

Έὰν ἀριθμὸς ἀριθμοῦ μέρος ἦ, καὶ ἔτερος ἑτέρου τὸ αὐτὸ μέρος ἦ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ἢ μέρη ὁ πρῶτος τοῦ τρίτου, τὸ αὐτὸ μέρος ἔσται ἢ τὰ αὐτὰ μέρη καὶ ὁ δεύτερος τοῦ τετάρτου.

Proposition 9[†]

If a number is part of a number, and another (number) is the same part of another, also, alternately, which(ever) part, or parts, the first (number) is of the third, the second (number) will also be the same part, or



Άριθμὸς γὰρ ὁ Α ἀριθμοῦ τοῦ ΒΓ μέρος ἔστω, καὶ ἕτερος ὁ Δ ἑτέρου τοῦ ΕΖ τὸ αὐτὸ μέρος, ὅπερ ὁ Α τοῦ ΒΓ· λέγω, ὅτι καὶ ἐναλλάξ, ὅ μέρος ἐστιν ὁ Α τοῦ Δ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΒΓ τοῦ ΕΖ ἢ μέρη.

Έπει γὰρ δ μέρος ἐστὶν ὁ A τοῦ BΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Δ τοῦ EZ, ὅσοι ἄρα εἰσὶν ἐν τῷ BΓ ἀριθμοὶ ἴσοι τῷ A, τοσοῦτοί εἰσι καὶ ἐν τῷ EZ ἴσοι τῷ Δ. διηρήσθω ὁ μὲν BΓ εἰς τοὺς τῷ A ἴσους τοὺς BH, HΓ, ὁ δὲ EZ εἰς τοὺς τῷ Δ ἴσους τοὺς EΘ, ΘΖ. ἔσται δὴ ἴσον τὸ πλῆθος τῶν BH, HΓ τῷ πλήθει τῶν EΘ, ΘΖ.

Καὶ ἑπεὶ ἴσοι εἰσὶν οἱ BH, ΗΓ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ EΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν BH, ΗΓ τῷ πλήθει τῶν EΘ, ΘΖ, ὃ ἄρα μέρος ἐστὶν ὁ BH τοῦ EΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΓ τοῦ ΘΖ ἢ τὰ αὐτὰ μέρη· ὥστε καὶ ὃ μέρος ἐστὶν ὁ BH τοῦ EΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ BΓ συναμφοτέρου τοῦ EZ ἢ τὰ αὐτὰ μέρη. ἴσος δὲ ὁ μὲν BH τῷ A, ὁ δὲ EΘ τῷ Δ· ὃ ἄρα μέρος ἐστὶν ὁ A τοῦ Δ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ BΓ τοῦ EZ ἢ τὰ αὐτὰ μέρη· ὅπερ ἕδει δεῖξαι.

the same parts, of the fourth.



For let a number A be part of a number BC, and another (number) D (be) the same part of another EF that A (is) of BC. I say that, also, alternately, which(ever) part, or parts, A is of D, BC is also the same part, or parts, of EF.

For since which(ever) part A is of BC, D is also the same part of EF, thus as many numbers as are in BC equal to A, so many are also in EF equal to D. Let BC have been divided into BG and GC, equal to A, and EF into EH and HF, equal to D. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF.

And since the numbers BG and GC are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) BG, GCis equal to the multitude of (divisions) EH, HC, thus which(ever) part, or parts, BG is of EH, GC is also the same part, or the same parts, of HF. And hence, which(ever) part, or parts, BG is of EH, the sum BCis also the same part, or the same parts, of the sum EF[Props. 7.5, 7.6]. And BG (is) equal to A, and EH to D. Thus, which(ever) part, or parts, A is of D, BC is also the same part, or the same parts, of EF. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a = (1/n) b and c = (1/n) d then if a = (k/l) c then b = (k/l) d, where all symbols denote numbers.

ι΄.

Έὰν ἀριθμὸς ἀριθμοῦ μέρη ἢ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ μέρη ἢ, καὶ ἐναλλάξ, ἂ μέρη ἐστὶν ὁ πρῶτος τοῦ τρίτου ἢ μέρος, τὰ αὐτὰ μέρη ἔσται καὶ ὁ δεύτερος τοῦ τετάρτου ἢ τὸ αὐτὸ μέρος.

Ἀριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔΕ ἑτέρου τοῦ Ζ τὰ αὐτὰ μέρη· λέγω, ὅτι καὶ ἐναλλάξ, ἂ μέρη ἐστὶν ὁ AB τοῦ ΔΕ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὸ αὐτὸ μέρος.

Proposition 10[†]

If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another F. I say that, also, alternately, which(ever) parts, or part,



Έπει γάρ, & μέρη ἐστιν ὁ ΑΒ τοῦ Γ, τὰ αὐτὰ μέρη ἐστι καὶ ὁ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ AB μέρη τοῦ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ μέρη τοῦ Ζ. διῃρήσθω ὁ μὲν ΑΒ εἰς τὰ τοῦ Γ μέρη τὰ AH, HB, ὁ δὲ ΔE εἰς τὰ τοῦ Z μέρη τὰ $\Delta\Theta$, ΘE · ἕσται δὴ ἴσον τὸ πλῆθος τῶν AH, HB τῷ πλήθει τῶν $\Delta\Theta$, ΘE . καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ AH τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΔΘ τοῦ Ζ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ὁ AH τοῦ $\Delta\Theta$ η μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ
ἑ Γ τοῦ Z η τὰ αὐτὰ μέρη. διὰ τὰ αὐτὰ δὴ καί, ὃ μέρος ἐστὶν ὁ ΗΒ τοῦ ΘΕ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ χαὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη· ὥστε καί [ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΒ τοῦ ΘΕ ἢ τὰ αὐτὰ μέρη. καὶ ὃ ἄρα μέρος ἐστὶν ὁ AH τοῦ $\Delta \Theta$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AB τοῦ ΔE ἢ τὰ αὐτὰ μέρη· ἀλλ' ὃ μέρος ἐστὶν ὁ AH τοῦ $\Delta\Theta$ η μέρη, τὸ αὐτὸ μέρος ἐδείχθη καὶ ὁ Γ τοῦ Z η τὰ αὐτὰ μέρη, καὶ] ἂ [ἄρα] μέρη ἐστὶν ὁ ΑΒ τοῦ ΔΕ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὸ αὐτὸ μέρος. ὅπερ ἔδει δεĩξαι.

AB is of DE, C is also the same parts, or the same part, of F.



For since which(ever) parts AB is of C, DE is also the same parts of F, thus as many parts of C as are in AB, so many parts of F (are) also in DE. Let AB have been divided into the parts of C, AG and GB, and DEinto the parts of F, DH and HE. So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH, HE. And since which(ever) part AG is of C, DH is also the same part of F, also, alternately, which (ever) part, or parts, AG is of DH, C is also the same part, or the same parts, of F [Prop. 7.9]. And so, for the same (reasons), which(ever) part, or parts, GB is of HE, C is also the same part, or the same parts, of F [Prop. 7.9]. And so [which(ever) part, or parts, AG is of DH, GB is also the same part, or the same parts, of HE. And thus, which (ever) part, or parts, AG is of DH, AB is also the same part, or the same parts, of *DE* [Props. 7.5, 7.6]. But, which(ever) part, or parts, AG is of DH, Cwas also shown (to be) the same part, or the same parts, of F. And, thus] which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a = (m/n) b and c = (m/n) d then if a = (k/l) c then b = (k/l) d, where all symbols denote numbers.

ια΄.

Έαν ἢ ὡς ὅλος πρὸς ὅλον, οὕτως ἀφαιρεθεὶς πρὸς ἀφαιρεθέντα, καὶ ὁ λοιπὸς πρὸς τὸν λοιπὸν ἔσται, ὡς ὅλος πρὸς ὅλον.

Έστω ὡς ὅλος ὁ AB πρὸς ὅλον τὸν ΓΔ, οὕτως ἀφαιρεθεὶς ὁ AE πρὸς ἀφαιρεθέντα τὸν ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ EB πρὸς λοιπὸν τὸν ΖΔ ἐστιν, ὡς ὅλος ὁ AB πρὸς ὅλον τὸν ΓΔ.

Proposition 11

If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, then the remainder will also be to the remainder as the whole (is) to the whole.

Let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF. I say that the remainder EB is to the remainder FD as the whole AB (is) to the whole CD.



Έπεί ἐστιν ὡς ὁ AB πρὸς τὸν ΓΔ, οὕτως ὁ AE πρὸς τὸν ΓΖ, ὃ ἄρα μέρος ἐστὶν ὁ AB τοῦ ΓΔ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AE τοῦ ΓΖ ἢ τὰ αὐτὰ μέρη. καὶ λοιπὸς ӑρα ὁ EB λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστὶν ἢ μέρη, ἄπερ ἱ AB τοῦ ΓΔ. ἔστιν ἄρα ὡς ὁ EB πρὸς τὸν ΖΔ, οὕτως ἱ AB πρὸς τὸν ΓΔ· ὅπερ ἔδει δεῖξαι.



(For) since as AB is to CD, so AE (is) to CF, thus which(ever) part, or parts, AB is of CD, AE is also the same part, or the same parts, of CF [Def. 7.20]. Thus, the remainder EB is also the same part, or parts, of the remainder FD that AB (is) of CD [Props. 7.7, 7.8]. Thus, as EB is to FD, so AB (is) to CD [Def. 7.20]. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a:b::c:d then a:b::a-c:b-d, where all symbols denote numbers.

ıβ'.

Έὰν ῶσιν ὑποσοιοῦν ἀριθμοὶ ἀνάλογον, ἔσται ὡς εἶς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἄπαντες οἱ ἡγούμενοι πρὸς ἄπαντας τοὺς ἑπομένους.



Έστωσαν ὑποσοιοῦν ἀριθμοὶ ἀνάλογον οἱ A, B, Γ, Δ,
ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Γ πρὸς τὸν Δ· λέγω, ὅτι ἐστὶν
ὡς ὁ A πρὸς τὸν B, οὕτως οἱ A, Γ πρὸς τοὺς B, Δ.

Έπεὶ γάρ ἐστιν ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Γ πρὸς τὸν Δ, ὃ ἄρα μέρος ἐστιν ὁ A τοῦ B ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ μέρη. καὶ συναμφότερος ἄρα ὁ A, Γ συναμφοτέρου τοῦ B, Δ τὸ αὐτὸ μέρος ἐστὶν ἢ τὰ αὐτὰ μέρη, ἄπερ ὁ A τοῦ B. ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B, οὕτως οἱ A, Γ πρὸς τοὺς B, Δ· ὅπερ ἔδει δεῖξαι.

Proposition 12[†]

If any multitude whatsoever of numbers are proportional then as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.



Let any multitude whatsoever of numbers, A, B, C, D, be proportional, (such that) as A (is) to B, so C (is) to D. I say that as A is to B, so A, C (is) to B, D.

For since as A is to B, so C (is) to D, thus which(ever) part, or parts, A is of B, C is also the same part, or parts, of D [Def. 7.20]. Thus, the sum A, C is also the same part, or the same parts, of the sum B, D that A (is) of B [Props. 7.5, 7.6]. Thus, as A is to B, so A, C (is) to B, D [Def. 7.20]. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a:b::c:d then a:b::a+c:b+d, where all symbols denote numbers.

ιγ'.

Έὰν τέσσαρες ἀριθμοὶ ἀνάλογον ῶσιν, καὶ ἐναλλὰξ ἀνάλογον ἔσονται.



Έστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ A, B, Γ , Δ, ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Γ πρὸς τὸν Δ· λέγω, ὅτι καὶ ἐναλλὰξ ἀνάλογον ἔσονται, ὡς ὁ A πρὸς τὸν Γ , οὕτως ὁ B πρὸς τὸν Δ.

Έπεὶ γάρ ἐστιν ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Γ πρὸς τὸν Δ, ὅ ἄρα μέρος ἐστὶν ὁ A τοῦ B ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἐναλλὰξ ἄρα, ὃ μέρος ἐστὶν ὁ A τοῦ Γ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ B τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἔστιν ἄρα ὡς ὁ A πρὸς τὸν Γ, οὕτως ὁ B πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

Proposition 13[†]

If four numbers are proportional then they will also be proportional alternately.



Let the four numbers A, B, C, and D be proportional, (such that) as A (is) to B, so C (is) to D. I say that they will also be proportional alternately, (such that) as A (is) to C, so B (is) to D.

For since as A is to B, so C (is) to D, thus which(ever) part, or parts, A is of B, C is also the same part, or the same parts, of D [Def. 7.20]. Thus, alterately, which(ever) part, or parts, A is of C, B is also the same part, or the same parts, of D [Props. 7.9, 7.10]. Thus, as A is to C, so B (is) to D [Def. 7.20]. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a : b :: c : d then a : c :: b : d, where all symbols denote numbers.

ιδ'.

Έὰν ῶσιν ὁποσοιοῦν ἀριθμοὶ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῷ ἔσονται.



Έστωσαν ὑποσοιοῦν ἀριθμοὶ οἱ A, B, Γ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι ἐν τῷ αὐτῷ λόγῳ οἱ Δ, E, Z, ὡς μὲν ὁ A πρὸς τὸν B, οὕτως ὁ Δ πρὸς τὸν E, ὡς δὲ ὁ B πρὸς τὸν Γ, οὕτως ὁ Ε πρὸς τὸν Ζ· λέγω, ὅτι καὶ δι° ἴσου ἐστὶν ὡς ὁ A πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, οὕτως ὁ Β πρὸς τὸν Ε. πάλιν, ἐπεί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ

Proposition 14[†]

If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.



Let there be any multitude of numbers whatsoever, A, B, C, and (some) other (numbers), D, E, F, of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that also, via equality, as A is to C, so D (is) to F.

For since as A is to B, so D (is) to E, thus, alternately, as A is to D, so B (is) to E [Prop. 7.13]. Again, since as B is to C, so E (is) to F, thus, alternately, as B is

Ε πρός τόν Z, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Β πρός τὸν Ε, οὕτως to E, so C (is) to F [Prop. 7.13]. And as B (is) to E, ό Γ πρὸς τὸν Ζ. ὡς δὲ ὁ Β πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Δ · καὶ ὡς ἄρα ὁ Α πρὸς τὸν Δ , οὕτως ὁ Γ πρὸς τὸν Ζ· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ. ὅπερ ἔδει δεῖξαι.

so A (is) to D. Thus, also, as A (is) to D, so C (is) to F. Thus, alternately, as A is to C, so D (is) to F [Prop. 7.13]. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if a:b::d:e and b:c::e:f then a:c::d:f, where all symbols denote numbers.

ιε΄.

Ἐὰν μονὰς ἀριθμόν τινα μετρῆ, ἰσακις δὲ ἕτερος ἀριθμὸς άλλον τινὰ ἀριθμὸν μετρῆ, καὶ ἐναλλὰξ ἰσάκις ἡ μονὰς τὸν τρίτον ἀριθμόν μετρήσει καὶ ὁ δεύτερος τὸν τέταρτον.



Μονὰς γὰρ ἡ Α ἀριθμόν τινα τὸν ΒΓ μετρείτω, ἰσάχις δὲ ἕτερος ἀριθμὸς ὁ Δ ἄλλον τινὰ ἀριθμὸν τὸν ΕΖ μετρείτω· λέγω, ὅτι καὶ ἐναλλὰξ ἰσάκις ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ.

Ἐπεὶ γὰρ ἰσάχις ἡ Α μονὰς τὸν ΒΓ ἀριθμὸν μετρεῖ χαὶ ὁ Δ τὸν EZ, ὅσαι ἄρα εἰσὶν ἐν τῷ BΓ μονάδες, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τὰς έν ἑαυτῷ μονάδας τὰς BH, HΘ, ΘΓ, ὁ δὲ EZ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΚ, ΚΛ, ΛΖ. ἔσται δὴ ἴσον τὸ πλῆθος τῶν BH, ΗΘ, ΘΓ τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ. καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΒΗ, ΗΘ, ΘΓ μονάδες ἀλλήλαις, εἰσὶ δὲ καὶ οἱ ΕΚ, ΚΛ, ΛΖ άριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ μονάδων τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ ἀριθμῶν, έσται άρα ώς ή BH μονάς πρός τὸν EK ἀριθμόν, οὕτως ή ΗΘ μονὰς πρὸς τὸν ΚΛ ἀριθμὸν καὶ ἡ ΘΓ μονὰς πρὸς τὸν ΛZ ἀριθμόν. ἕσται ἄρα καὶ ὡς εῖς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἄπαντες οἱ ἡγούμενοι πρὸς ἄπαντας τοὺς ἑπομένους· ἔστιν ἄρα ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ άριθμόν, οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἴση δὲ ἡ ΒΗ μονὰς τῆ Α μονάδι, ὁ δὲ ΕΚ ἀριθμὸς τῷ Δ ἀριθμῷ. ἔστιν ἄρα ὡς ἡ Α μονὰς πρὸς τὸν Δ ἀριθμόν, οὕτως ὁ $B\Gamma$ πρὸς τὸν EZ. ἰσάχις ἄρα ἡ A μονὰς τὸν Δ ἀριθμὸν μετρεῖ χαὶ
ἑ ${\rm B}\Gamma$ τὸν ΕΖ· ὅπερ ἔδει δεῖξαι.

Proposition 15

If a unit measures some number, and another number measures some other number as many times, then, also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.

For let a unit A measure some number BC, and let another number D measure some other number EF as many times. I say that, also, alternately, the unit A also measures the number D as many times as BC (measures) EF.

For since the unit A measures the number BC as many times as D (measures) EF, thus as many units as are in BC, so many numbers are also in EF equal to D. Let BC have been divided into its constituent units, BG, GH, and HC, and EF into the (divisions) EK, KL, and LF, equal to D. So the multitude of (units) BG, GH, HC will be equal to the multitude of (divisions) EK, KL, LF. And since the units BG, GH, and HCare equal to one another, and the numbers EK, KL, and LF are also equal to one another, and the multitude of the (units) BG, GH, HC is equal to the multitude of the numbers EK, KL, LF, thus as the unit BG (is) to the number EK, so the unit GH will be to the number KL, and the unit HC to the number LF. And thus, as one of the leading (numbers is) to one of the following, so (the sum of) all of the leading will be to (the sum of) all of the following [Prop. 7.12]. Thus, as the unit BG (is) to the number EK, so BC (is) to EF. And the unit BG (is) equal to the unit A, and the number EK to the number D. Thus, as the unit A is to the number D, so BC (is) to EF. Thus, the unit A measures the number D as many times as BC (measures) EF [Def. 7.20]. (Which is) the very thing it was required to show.

[†] This proposition is a special case of Prop. 7.9.

ເຈ່.

Εὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινας, οἱ γενόμενοι ἐξ αὐτῶν ἴσοι ἀλλήλοις ἔσονται.



Έστωσαν δύο ἀριθμοὶ οἱ A, B, καὶ ὁ μὲν A τὸν B πολλαπλασιάσας τὸν Γ ποιείτω, ὁ δὲ B τὸν A πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Γ τῷ Δ.

Έπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Α ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Γ. ἐναλλὰξ ἄρα ἰσάκις ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Γ. πάλιν, ἐπεὶ ὁ Β τὸν Α πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Α ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Β μονάδας· μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Β κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Δ. ἰσάκις ἄρα ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Δ. ἰσάκις ἄρα ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Γ· ἰσάκις ἄρα ὁ Α ἑκάτερον τῶν Γ, Δ μετρεῖ. ἴσος ἄρα ἐστὶν ὁ Γ τῷ Δ· ὅπερ ἔδει δεῖξαι.

Proposition 16[†]

If two numbers multiplying one another make some (numbers) then the (numbers) generated from them will be equal to one another.



Let A and B be two numbers. And let A make C (by) multiplying B, and let B make D (by) multiplying A. I say that C is equal to D.

For since A has made C (by) multiplying B, B thus measures C according to the units in A [Def. 7.15]. And the unit E also measures the number A according to the units in it. Thus, the unit E measures the number A as many times as B (measures) C. Thus, alternately, the unit E measures the number B as many times as A (measures) C [Prop. 7.15]. Again, since B has made D (by) multiplying A, A thus measures D according to the units in B [Def. 7.15]. And the unit E also measures B according to the units in it. Thus, the unit E measures the number B as many times as A (measures) D. And the unit E was measuring the number B as many times as A (measures) C. Thus, A measures each of C and D an equal number of times. Thus, C is equal to D. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that a b = b a, where all symbols denote numbers.

ιζ΄.

Έλν ἀριθμὸς δύο ἀριθμοὺς πολλαπλασιάσας ποιῆ τινας, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἔξουσι λόγον τοῖς πολλαπλασιασθεῖσιν.



Ἀριθμὸς γὰρ ὁ Α δύο ἀριθμοὺς τοὺς Β, Γ πολλαπλασιάσας τοὺς Δ, Ε ποιείτω· λέγω, ὅτι ἐστὶν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ε.

Έπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Β ẳρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ

Proposition 17[†]

If a number multiplying two numbers makes some (numbers) then the (numbers) generated from them will have the same ratio as the multiplied (numbers).



For let the number A make (the numbers) D and E (by) multiplying the two numbers B and C (respectively). I say that as B is to C, so D (is) to E.

For since A has made D (by) multiplying B, B thus measures D according to the units in A [Def. 7.15]. And

δὲ καὶ ἡ Z μονὰς τὸν A ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Z μονὰς τὸν A ἀριθμὸν μετρεῖ καὶ ὁ B τὸν Δ. ἔστιν ἄρα ὡς ἡ Z μονὰς πρὸς τὸν A ἀριθμόν, οὕτως ὁ B πρὸς τὸν Δ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Z μονὰς πρὸς τὸν A ἀριθμόν, οὕτως ὁ Γ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ B πρὸς τὸν Δ, οὕτως ὁ Γ πρὸς τὸν Ε. ἐναλλὰξ ἄρα ἑστὶν ὡς ὁ B πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ε· ὅπερ ἔδει δεῖξαι. the unit F also measures the number A according to the units in it. Thus, the unit F measures the number A as many times as B (measures) D. Thus, as the unit F is to the number A, so B (is) to D [Def. 7.20]. And so, for the same (reasons), as the unit F (is) to the number A, so C (is) to E. And thus, as B (is) to D, so C (is) to E. Thus, alternately, as B is to C, so D (is) to E [Prop. 7.13]. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if d = a b and e = a c then d : e :: b : c, where all symbols denote numbers.

*ι*η'.

Έὰν δύο ἀριθμοὶ ἀριθμόν τινα πολλαπλασιάσαντες ποιῶσί τινας, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἕξουσι λόγον τοῖς πολλαπλασιάσασιν.



 Δ ύο γὰρ ἀριθμοὶ οἱ A, B ἀριθμόν τινα τὸν Γ πολλαπλασιάσαντες τοὺς Δ, Ε ποιείτωσαν λέγω, ὅτι ἐστὶν ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Δ πρὸς τὸν Ε.

Ἐπεὶ γὰρ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηχεν, καὶ ὁ Γ ἄρα τὸν Α πολλαπλασιάσας τὸν Δ πεποίηχεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Β πολλαπλασιάσας τὸν Ε πεποίηχεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Α, Β πολλαπλασιάσας τοὺς Δ, Ε πεποίηχεν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· ὅπερ ἔδει δεῖξαι.

Proposition 18[†]

If two numbers multiplying some number make some (other numbers) then the (numbers) generated from them will have the same ratio as the multiplying (numbers).



For let the two numbers A and B make (the numbers) D and E (respectively, by) multiplying some number C. I say that as A is to B, so D (is) to E.

For since A has made D (by) multiplying C, C has thus also made D (by) multiplying A [Prop. 7.16]. So, for the same (reasons), C has also made E (by) multiplying B. So the number C has made D and E (by) multiplying the two numbers A and B (respectively). Thus, as A is to B, so D (is) to E [Prop. 7.17]. (Which is) the very thing it was required to show.

[†] In modern notation, this propositions states that if a c = d and b c = e then a : b :: d : e, where all symbols denote numbers.

ıϑ'.

Έὰν τέσσαρες ἀριθμοὶ ἀνάλογον ῶσιν, ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ἔσται τῷ ἐκ δευτέρου καὶ τρίτου γενομένῷ ἀριθμῷ· καὶ ἐὰν ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ἢ τῷ ἐκ δευτέρου καὶ τρίτου, οἱ τέσσασρες ἀριθμοὶ ἀνάλογον ἔσονται.

Έστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ A, B, Γ , Δ, ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Γ πρὸς τὸν Δ, καὶ ὁ μὲν A τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ B τὸν Γ πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Ε τῷ Ζ.

Proposition 19[†]

If four number are proportional then the number created from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third then the four numbers will be proportional.

Let A, B, C, and D be four proportional numbers, (such that) as A (is) to B, so C (is) to D. And let A make E (by) multiplying D, and let B make F (by) multiplying C. I say that E is equal to F.



Ό γὰρ Α τὸν Γ πολλαπλασιάσας τὸν Η ποιείτω. ἐπεὶ οῦν ὁ Α τὸν Γ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Δ πολλαπλασιάσας τὸν Ε πεποίηκεν, ἀριθμὸς δὴ ὁ Α δύο ἀριθμοὺς τοὺς Γ, Δ πολλαπλασιάσας τούς Η, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Η πρὸς τὸν Ε. ἀλλʾ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ε. πάλιν, ἐπεὶ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Η πεποίηκεν, ἀλλὰ μὴν καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Ζ πεποίηκεν, ἀλλὰ μὴν καὶ ὁ α, Β ἀριθμόν τινα τὸν Γ πολλαπλασιάσαντες τοὺς Η, Ζ πεποιήκασιν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ζ. ἀλλὰ μὴν καὶ ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Ε, οῦτως ὁ Η πρὸς τὸν Ζ. ὁ Η ἄρα πρὸς ἑκάτερον τῶν Ε, Ζ τὸν αὐτὸν ἔχει λόγον· ἴσος ἄρα ἐστὶν ὁ Ε τῷ Ζ.

Έστω δη πάλιν ίσος ὁ Ε τῷ Ζ· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴσος ἐστὶν ὁ Ε τῷ Ζ, ἔστιν ἄρα ὡς ὁ Η πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Ζ. ἀλλ' ὡς μὲν ὁ Η πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ, ὡς δὲ ὁ Η πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.



For let *A* make *G* (by) multiplying *C*. Therefore, since *A* has made *G* (by) multiplying *C*, and has made *E* (by) multiplying *D*, the number *A* has made *G* and *E* by multiplying the two numbers *C* and *D* (respectively). Thus, as *C* is to *D*, so *G* (is) to *E* [Prop. 7.17]. But, as *C* (is) to *D*, so *A* (is) to *B*. Thus, also, as *A* (is) to *B*, so *G* (is) to *E*. Again, since *A* has made *G* (by) multiplying *C*, but, in fact, *B* has also made *F* (by) multiplying *C*, the two numbers *A* and *B* have made *G* and *F* (respectively, by) multiplying some number *C*. Thus, as *A* is to *B*, so *G* (is) to *E*. And thus, as *G* (is) to *E*, so *G* (is) to *F*. Thus, as *G* (is) to *F*. Thus, as *G* (is) to *E*. And thus, as *G* (is) to *E* and *F*. Thus, *E* is equal to *F* [Prop. 5.9].

So, again, let E be equal to F. I say that as A is to B, so C (is) to D.

For, with the same construction, since E is equal to F, thus as G is to E, so G (is) to F [Prop. 5.7]. But, as G (is) to E, so C (is) to D [Prop. 7.17]. And as G (is) to F, so A (is) to B [Prop. 7.18]. And, thus, as A (is) to B, so C (is) to D. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if a : b :: c : d then a d = b c, and vice versa, where all symbols denote numbers.

κ΄.

Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα.

Έστωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β οἱ ΓΔ, ΕΖ· λέγω, ὅτι ἰσάχις ὁ ΓΔ τὸν Α μετρεῖ χαὶ ὁ ΕΖ τὸν Β.

Proposition 20

The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

For let CD and EF be the least numbers having the same ratio as A and B (respectively). I say that CD measures A the same number of times as EF (measures) B.



Ὁ ΓΔ γὰρ τοῦ Α οὖκ ἐστι μέρη. εἰ γὰρ δυνατόν, ἔστω· καὶ ὁ ΕΖ ἄρα τοῦ Β τὰ αὐτὰ μέρη ἐστίν, ἄπερ ὁ ΓΔ τοῦ A. ὅσα ἄρα ἐστὶν ἐν τῷ ΓΔ μέρη τοῦ Α, τοσαῦτά ἐστι καὶ έν τῷ EZ μέρη τοῦ B. διηρήσθω ὁ μὲν Γ Δ εἰς τὰ τοῦ A μέρη τὰ ΓΗ, ΗΔ,
ὁ δὲ ΕΖ εἰς τὰ τοῦ Β μέρη τὰ ΕΘ, ΘΖ· έσται δὴ ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλήθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΓΗ, ΗΔ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΕΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πληθος τῶν ΓΗ, ΗΔ τῷ πλήθει τῶν ΕΘ, ΘΖ, ἔστιν ἄρα ὡς ό ΓΗ πρός τὸν ΕΘ, οὕτως ὁ ΗΔ πρὸς τὸν ΘΖ. ἔσται ἄρα καὶ ὡς εἶς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως άπαντες οἱ ἡγούμενοι πρὸς ἄπαντας τοὺς ἑπομένους. ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΓΔ πρὸς τὸν ΕΖ· οἱ $\Gamma H, E\Theta$ ἄρα τοῖς $\Gamma\Delta, EZ$ ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες όντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον· ὑπόκεινται γὰρ οἱ ΓΔ, ΕΖ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οὐχ άρα μέρη ἐστὶν
ὁ ΓΔ τοῦ Α· μέρος ἄρα. καὶ ὁ ΕΖ τοῦ Β τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ ΓΔ τοῦ Α· ἰσάχις ἄρα ὁ ΓΔ τὸν Α μετρεῖ καὶ ὁ ΕΖ τὸν Β· ὅπερ ἔδει δεῖξαι.

κα'.

Οἱ πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Έστωσαν πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔσονταί τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῷ ὄντες τοῖς Α, Β. ἔστωσαν οἱ Γ, Δ.





For CD is not parts of A. For, if possible, let it be (parts of A). Thus, EF is also the same parts of B that CD (is) of A [Def. 7.20, Prop. 7.13]. Thus, as many parts of A as are in CD, so many parts of B are also in EF. Let *CD* have been divided into the parts of *A*, *CG* and *GD*, and EF into the parts of B, EH and HF. So the multitude of (divisions) CG, GD will be equal to the multitude of (divisions) EH, HF. And since the numbers CG and GD are equal to one another, and the numbers EH and *HF* are also equal to one another, and the multitude of (divisions) CG, GD is equal to the multitude of (divisions) EH, HF, thus as CG is to EH, so GD (is) to HF. Thus, as one of the leading (numbers is) to one of the following, so will (the sum of) all of the leading (numbers) be to (the sum of) all of the following [Prop. 7.12]. Thus, as CG is to EH, so CD (is) to EF. Thus, CGand EH are in the same ratio as CD and EF, being less than them. The very thing is impossible. For CD and *EF* were assumed (to be) the least of those (numbers) having the same ratio as them. Thus, CD is not parts of A. Thus, (it is) a part (of A) [Prop. 7.4]. And EF is the same part of B that CD (is) of A [Def. 7.20, Prop 7.13]. Thus, CD measures A the same number of times that EF(measures) B. (Which is) the very thing it was required to show.

Proposition 21

Numbers prime to one another are the least of those (numbers) having the same ratio as them.

Let A and B be numbers prime to one another. I say that A and B are the least of those (numbers) having the same ratio as them.

For if not then there will be some numbers less than A and B which are in the same ratio as A and B. Let them be C and D.



Έπεὶ οὖν οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάχις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάττων τὸν ἐλάττονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ἰσάχις ἄρα ὁ Γ τὸν Α μετρεῖ καὶ ὁ Δ τὸν Β. ἱσάχις δὴ ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. καὶ ὁ Δ ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. καὶ ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, καί ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, καί ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, ό Ε ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστιν ἀδύνατον. οὐχ ἄρα ἔσονταί τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. οἱ Α, Β ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.



Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς πρῶτοι πρὸς ἀλλήλους εἰσίν.



Έστωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β· λέγω, ὅτι οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ὁσάκις μὲν ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ,



Therefore, since the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following—C thus measures A the same number of times that D (measures) B[Prop. 7.20]. So as many times as C measures A, so many units let there be in E. Thus, D also measures B according to the units in E. And since C measures A according to the units in E, E thus also measures A according to the units in C [Prop. 7.16]. So, for the same (reasons), E also measures B according to the units in D [Prop. 7.16]. Thus, E measures A and B, which are prime to one another. The very thing is impossible. Thus, there cannot be any numbers less than A and B which are in the same ratio as A and B. Thus, A and B are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

Proposition 22

The least numbers of those (numbers) having the same ratio as them are prime to one another.



Let A and B be the least numbers of those (numbers) having the same ratio as them. I say that A and B are prime to one another.

For if they are not prime to one another then some number will measure them. Let it (so measure them), and let it be C. And as many times as C measures A, so

όσάχις δὲ ὁ Γ τὸν Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

Έπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, ὁ Γ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ε πολλαπλασιάσας τὸν Β πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Δ, Ε πολλαπλασιάσας τοὺς Α, Β πεποίηκεν· ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Β· οἱ Δ, Ε ἄρα τοῖς Α, Β ἐν τῷ αὐτῷ λόγῷ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

χγ'.

Έὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ῶσιν, ὁ τὸν ἕνα αὐτῶν μετρῶν ἀριθμὸς πρὸς τὸν λοιπὸν πρῶτος ἔσται.



Έστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ A, B, τὸν δὲ A μετρείτω τις ἀριθμὸς ὁ Γ · λέγω, ὅτι καὶ οἱ Γ , B πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Β ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. ἐπεὶ ὁ Δ τὸν Γ μετρεῖ, ὁ δὲ Γ τὸν Α μετρεῖ, καὶ ὁ Δ ἄρα τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Β· ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Β ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

χδ'.

Έὰν δύο ἀριθμοὶ πρός τινα ἀριθμὸν πρῶτοι ῶσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν αὐτὸν πρῶτος ἔσται. many units let there be in D. And as many times as C measures B, so many units let there be in E.

Since *C* measures *A* according to the units in *D*, *C* has thus made *A* (by) multiplying *D* [Def. 7.15]. So, for the same (reasons), *C* has also made *B* (by) multiplying *E*. So the number *C* has made *A* and *B* (by) multiplying the two numbers *D* and *E* (respectively). Thus, as *D* is to *E*, so *A* (is) to *B* [Prop. 7.17]. Thus, *D* and *E* are in the same ratio as *A* and *B*, being less than them. The very thing is impossible. Thus, some number does not measure the numbers *A* and *B*. Thus, *A* and *B* are prime to one another. (Which is) the very thing it was required to show.

Proposition 23

If two numbers are prime to one another then a number measuring one of them will be prime to the remaining (one).



Let A and B be two numbers (which are) prime to one another, and let some number C measure A. I say that C and B are also prime to one another.

For if C and B are not prime to one another then [some] number will measure C and B. Let it (so) measure (them), and let it be D. Since D measures C, and Cmeasures A, D thus also measures A. And (D) also measures B. Thus, D measures A and B, which are prime to one another. The very thing is impossible. Thus, some number does not measure the numbers C and B. Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

Proposition 24

If two numbers are prime to some number then the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).



 Δ ύο γὰρ ἀριθμοὶ οἱ A, B πρός τινα ἀριθμὸν τὸν Γ πρῶτοι ἔστωσαν, καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Δ ποιείτω·λέγω, ὅτι οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Δ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Ε. καὶ ἐπεὶ οί Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν, τὸν δὲ Γ μετρεῖ τις άριθμός ὁ Ε, οἱ Α, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. όσάχις δη ό Ε τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Z· καὶ ὁ Z ẳρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ E μονάδας. ό Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μήν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος άρα ἐστὶν ὁ ἑκ τῶν Ε, Ζ τῷ ἐκ τῶν Α, Β. ἐὰν δὲ ὁ ὑπὸ τῶν ἄχρων ἴσος ἢ τῷ ὑπὸ τῶν μέσων, οἱ τέσσαρες ἀριθμοὶ άνάλογόν εἰσιν ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Α, οὕτως ὁ Β πρός τόν Ζ. οί δὲ Α, Ε πρῶτοι, οί δὲ πρῶτοι καὶ ἐλάχιστοι, οί δὲ ἐλάγιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐγόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάχις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ήγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. ὁ Ε ἄρα τὸν B μετρεĩ. μετρεῖ δὲ καὶ τὸν Γ ὁ E ἄρα τοὺς B, Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους. ὅπερ ἐστὶν ἀδύνατον. ούχ ἄρα τοὺς Γ , Δ ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Γ , Δ άρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ὅπερ ἔδει δεῖξαι.

×ε′**.**

Έὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὥσιν, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτος ἔσται.

Έστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι



For let A and B be two numbers (which are both) prime to some number C. And let A make D (by) multiplying B. I say that C and D are prime to one another.

For if C and D are not prime to one another then [some] number will measure C and D. Let it (so) measure them, and let it be E. And since C and A are prime to one another, and some number E measures C, A and E are thus prime to one another [Prop. 7.23]. So as many times as E measures D, so many units let there be in F. Thus, F also measures D according to the units in E [Prop. 7.16]. Thus, E has made D (by) multiplying F [Def. 7.15]. But, in fact, A has also made D (by) multiplying B. Thus, the (number created) from (multiplying) E and F is equal to the (number created) from (multiplying) A and B. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four numbers are proportional [Prop. 6.15]. Thus, as E is to A, so B (is) to F. And A and E (are) prime (to one another). And (numbers) prime (to one another) are also the least (of those numbers having the same ratio) [Prop. 7.21]. And the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures B. And it also measures C. Thus, E measures B and C, which are prime to one another. The very thing is impossible. Thus, some number cannot measure the numbers C and D. Thus, C and D are prime to one another. (Which is) the very thing it was required to show.

Proposition 25

If two numbers are prime to one another then the number created from (squaring) one of them will be prime to the remaining (number).

Let A and B be two numbers (which are) prime to

οί Β, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν.



Κείσθω γὰρ τῷ A ἴσος ὁ Δ. ἐπεὶ οἱ A, B πρῶτοι πρὸς ἀλλήλους εἰσίν, ἴσος δὲ ὁ A τῷ Δ, καί οἱ Δ, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ἐκάτερος ἄρα τῶν Δ, A πρὸς τὸν B πρῶτός ἐστιν· καὶ ὁ ἐκ τῶν Δ, A ἄρα γενόμενος πρὸς τὸν B πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Δ, A γενόμενος ἀριθμός ἐστιν ὁ Γ. οἱ Γ, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

x໌.

Έὰν δύο ἀριθμοὶ πρὸς δύο ἀριθμοὺς ἀμφότεροι πρὸς ἑκάτερον πρῶτοι ῶσιν, καὶ οἱ ἐξ αὐτῶν γενόμενοι πρῶτοι πρὸς ἀλλήλους ἔσονται.



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς δύο ἀριθμοὺς τοὺς Γ, Δ ἀμφότεροι πρὸς ἑκάτερον πρῶτοι ἔστωσαν, καὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι οἱ Ε, Ζ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Έπεὶ γὰρ ἑxάτερος τῶν Α, Β πρὸς τὸν Γ πρῶτός ἐστιν, xaì ὁ ἐx τῶν Α, Β ἄρα γενόμενος πρὸς τὸν Γ πρῶτος ἔσται. ὁ δὲ ἐx τῶν Α, Β γενόμενός ἑστιν ὁ E· οἱ Ε, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ xaì οἱ Ε, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν. ἑxάτερος ἄρα τῶν Γ, Δ πρὸς τὸν Ε πρῶτός ἐστιν. xaì ὁ ἐx τῶν Γ, Δ ἄρα γενόμενος πρὸς τὸν Ε πρῶτος ἔσται. ὁ δὲ ἐx τῶν Γ, Δ γενόμενος ἐστιν ὁ Ζ. οἱ Ε, Ζ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι. one another. And let A make C (by) multiplying itself. I say that B and C are prime to one another.



For let D be made equal to A. Since A and B are prime to one another, and A (is) equal to D, D and B are thus also prime to one another. Thus, D and A are each prime to B. Thus, the (number) created from (multilying) D and A will also be prime to B [Prop. 7.24]. And C is the number created from (multiplying) D and A. Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

Proposition 26

If two numbers are both prime to each of two numbers then the (numbers) created from (multiplying) them will also be prime to one another.



For let two numbers, A and B, both be prime to each of two numbers, C and D. And let A make E (by) multiplying B, and let C make F (by) multiplying D. I say that E and F are prime to one another.

For since A and B are each prime to C, the (number) created from (multiplying) A and B will thus also be prime to C [Prop. 7.24]. And E is the (number) created from (multiplying) A and B. Thus, E and C are prime to one another. So, for the same (reasons), E and D are also prime to one another. Thus, C and D are each prime to E. Thus, the (number) created from (multiplying) C and D will also be prime to E [Prop. 7.24]. And F is the (number) created from (multiplying) C and D will also be prime to E [Prop. 7.24]. And F is the (number) created from (multiplying) C and D. Thus, E and F are prime to one another. (Which is) the very thing it was required to show.

χζ΄.

Έὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ῶσιν, καὶ πολλαπλασιάσας ἐκάτερος ἑαυτὸν ποιῆ τινα, οἱ γενόμενοι ἐξ αὐτῶν πρῶτοι πρὸς ἀλλήλους ἔσονται, κἂν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσί τινας, κἀκεῖνοι πρῶτοι πρὸς ἀλλήλους ἔσονται [καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].



Έστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ ποιείτω, τὸν δὲ Γ πολλαπλασιάσας τὸν Δ ποιείτω, ὁ δὲ Β ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιείτω, τὸν δὲ Ε πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι οἴ τε Γ, Ε καὶ οἱ Δ, Ζ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Έπει γὰρ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν, οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὕν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Γ, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. πάλιν, ἑπεὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὑ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Α, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἀριθμοὺς τοὺς Β, Ε ἀμφότεροι πρὸς ἑκάτερον πρῶτοί εἰσιν, καὶ ὁ ἐκ τῶν Α, Γ ἄρα γενόμενος πρὸς τὸν ἐκ τῶν Β, Ε πρῶτός ἑστιν. καί ἐστιν ὁ μὲν ἐκ τῶν Α, Γ ὁ Δ, ὁ δὲ ἐκ τῶν Β, Ε ὁ Ζ. οἱ Δ, Ζ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν.

Proposition 27[†]

If two numbers are prime to one another and each makes some (number by) multiplying itself then the numbers created from them will be prime to one another, and if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be prime to one another [and this always happens with the extremes].



Let A and B be two numbers prime to one another, and let A make C (by) multiplying itself, and let it make D (by) multiplying C. And let B make E (by) multiplying itself, and let it make F by multiplying E. I say that C and E, and D and F, are prime to one another.

For since A and B are prime to one another, and A has made C (by) multiplying itself, C and B are thus prime to one another [Prop. 7.25]. Therefore, since C and Bare prime to one another, and B has made E (by) multiplying itself, C and E are thus prime to one another [Prop. 7.25]. Again, since A and B are prime to one another, and B has made E (by) multiplying itself, A and *E* are thus prime to one another [Prop. 7.25]. Therefore, since the two numbers A and C are both prime to each of the two numbers B and E, the (number) created from (multiplying) A and C is thus prime to the (number created) from (multiplying) B and E [Prop. 7.26]. And D is the (number created) from (multiplying) A and C, and F the (number created) from (multiplying) B and E. Thus, D and F are prime to one another. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if *a* is prime to *b*, then a^2 is also prime to b^2 , as well as a^3 to b^3 , *etc.*, where all symbols denote numbers.

×η'.

Έὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ῶσιν, καὶ συναμφότερος πρὸς ἑκάτερον αὐτῶν πρῶτος ἔσται· καὶ ἐὰν συναμφότερος πρὸς ἕνα τινὰ αὐτῶν πρῶτος ἦ, καὶ οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.

Proposition 28

If two numbers are prime to one another then their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them then the original numbers will also be prime to one another.


Συγκείσθωσαν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ AB, BΓ· λέγω, ὅτι καὶ συναμφότερος ὁ AΓ πρὸς ἑκάτερον τῶν AB, BΓ πρῶτός ἐστιν.

Εἰ γὰρ μή εἰσιν οἱ ΓΑ, ΑΒ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς ΓΑ, ΑΒ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. ἐπεὶ οῦν ὁ Δ τοὺς ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸν ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸν ΒΑ· ὁ Δ ἄρα τοὺς ΑΒ, ΒΓ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΓΑ, ΑΒ ἀριθμοὺς ἀριθμός τις μετρήσει· οἱ ΓΑ, ΑΒ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ ΑΓ, ΓΒ πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁ ΓΑ ἄρα πρὸς ἑκάτερον τῶν ΑΒ, ΒΓ πρῶτός ἐστιν.

Έστωσαν δὴ πάλιν οἱ ΓΑ, ΑΒ πρῶτοι πρὸς ἀλλήλους· λέγω, ὅτι καὶ οἱ ΑΒ, ΒΓ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ AB, BΓ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς AB, BΓ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. καὶ ἐπεὶ ὁ Δ ἑκάτερον τῶν AB, BΓ μετρεῖ, καὶ ὅλον ἄρα τὸν ΓΑ μετρήσει. μετρεῖ δὲ καὶ τὸν AB· ὁ Δ ἄρα τοὺς ΓΑ, AB μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς AB, BΓ ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ AB, BΓ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

χθ'.

Άπας πρῶτος ἀριθμὸς πρὸς ἄπαντα ἀριθμόν, ὃν μὴ μετρεῖ, πρῶτός ἐστιν.



Έστω πρῶτος ἀριθμὸς ὁ Α καὶ τὸν Β μὴ μετρείτω· λέγω, ὅτι οἱ Β, Α πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ Β, Α πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω ὁ Γ. ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Α τὸν Β οὐ μετρεῖ, ὁ Γ ἄρα τῷ Α οὕχ ἐστιν ὁ αὐτός. καὶ ἐπεὶ ὁ Γ τοὺς Β, Α μετρεῖ, καὶ τὸν Α ἄρα μετρεῖ πρῶτον ὄντα μὴ ἂν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐχ ἄρα τοὺς Β, Α μετρήσει τις ἀριθμός. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.



For let the two numbers, AB and BC, (which are) prime to one another, be laid down together. I say that their sum AC is also prime to each of AB and BC.

For if CA and AB are not prime to one another then some number will measure CA and AB. Let it (so) measure (them), and let it be D. Therefore, since D measures CA and AB, it will thus also measure the remainder BC. And it also measures BA. Thus, D measures AB and BC, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers CA and AB. Thus, CA and AB are prime to one another. So, for the same (reasons), AC and CBare also prime to one another. Thus, CA is prime to each of AB and BC.

So, again, let CA and AB be prime to one another. I say that AB and BC are also prime to one another.

For if AB and BC are not prime to one another then some number will measure AB and BC. Let it (so) measure (them), and let it be D. And since D measures each of AB and BC, it will thus also measure the whole of CA. And it also measures AB. Thus, D measures CAand AB, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers AB and BC. Thus, AB and BC are prime to one another. (Which is) the very thing it was required to show.

Proposition 29

Every prime number is prime to every number which it does not measure.



Let A be a prime number, and let it not measure B. I say that B and A are prime to one another. For if B and A are not prime to one another then some number will measure them. Let C measure (them). Since C measures B, and A does not measure B, C is thus not the same as A. And since C measures B and A, it thus also measures A, which is prime, (despite) not being the same as it. The very thing is impossible. Thus, some number cannot measure (both) B and A. Thus, A and B are prime to one another. (Which is) the very thing it was required to

show.

λ'.

Ἐἀν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἐξ αὐτῶν μετρῆ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει.



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πολλαπλασιάσαντες ἀλλήλους τὸν Γ ποιείτωσαν, τὸν δὲ Γ μετρείτω τις πρῶτος ἀριθμὸς ὁ Δ · λέγω, ὅτι ὁ Δ ἕνα τῶν A, B μετρεĩ.

Τὸν γὰρ A μὴ μετρείτω· καί ἐστι πρῶτος ὁ Δ · οἱ A, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ۂσάκις
ἑ Δ τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. ἐπεὶ οὖν ὁ Δ τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, ὁ Δ ẳρα τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηχεν. ἀλλὰ μὴν χαὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηχεν ἴσος ἄρα ἐστὶν ὁ ἐχ τῶν Δ , E τῷ ἐκ τῶν A, B. ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν A, οὕτως ὁ Β πρὸς τὸν Ε. οἱ δὲ Δ, Α πρῶτοι, οἱ δὲ πρῶτοι καὶ έλάχιστοι, οί δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον έγοντας ἰσάχις ὅ τε μείζων τὸν μείζονα χαὶ ὁ ἐλάσσων τὸν έλάσσονα, τουτέστιν ὄ τε ήγούμενος τὸν ἡγούμενον καὶ ὁ έπόμενος τὸν ἑπόμενον·
ἑ Δ ἄρα τὸν B μετρεĩ. ἑμοίως δὴ δείξομεν, ὄτι καὶ ἐἀν τὸν Β μὴ μετρῆ, τὸν Α μετρήσει. ὁ Δ άρα ἕνα τῶν Α, Β μετρεῖ· ὅπερ ἔδει δεῖξαι.

λα'.

Άπας σύνθεντος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Έστω σύνθεντος ἀριθμὸς ὁ Α· λέγω, ὅτι ὁ Α ὑπὸ πρώτου τινός ἀριθμοῦ μετρεῖται.

Proposition 30

If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).



For let two numbers A and B make C (by) multiplying one another, and let some prime number D measure C. I say that D measures one of A and B.

For let it not measure A. And since D is prime, Aand D are thus prime to one another [Prop. 7.29]. And as many times as D measures C, so many units let there be in E. Therefore, since D measures C according to the units E, D has thus made C (by) multiplying E [Def. 7.15]. But, in fact, A has also made C (by) multiplying B. Thus, the (number created) from (multiplying) D and E is equal to the (number created) from (multiplying) A and B. Thus, as D is to A, so B (is) to E [Prop. 7.19]. And D and A (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, D measures B. So, similarly, we can also show that if (D) does not measure B then it will measure A. Thus, D measures one of A and B. (Which is) the very thing it was required to show.

Proposition 31

Every composite number is measured by some prime number.

Let A be a composite number. I say that A is measured by some prime number.

For since A is composite, some number will measure Έπει γαρ σύνθετός έστιν ό A, μετρήσει τις αὐτὸν it. Let it (so) measure (A), and let it be B. And if B

ΣΤΟΙΧΕΙΩΝ ζ΄.

ἀριθμός. μετρείτω, καὶ ἔστω ὁ Β. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Β, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Β τὸν Α μετρεῖ, καὶ ὁ Γ ἄρα τὸν Α μετρεῖ. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. τοιαύτης δὴ γινομένης ἐπισκέψεως ληφθήσεται τις πρῶτος ἀριθμός, ὃς μετρήσει. εἰ γὰρ οὐ ληφθήσεται, μετρήσουσι τὸν Α ἀριθμὸν ἄπειροι ἀριθμοί, ῶν ἕτερος ἑτέρου ἐλάσσων ἐστίν· ὅπερ ἐστὶν ἀδύνατον ἐν ἀριθμοῖς. ληφθήσεταί τις ἄρα πρῶτος ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ, ὃς καὶ τὸν Α μετρήσει.



Άπας ἄρα σύνθεντος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

λβ΄.

Άπας ἀριθμὸς ἦτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.



Έστω ἀριθμὸς ὁ Α· λέγω, ὅτι ὁ Α ἦτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

El μέν οὖν πρῶτός ἐστιν ὁ Α, γεγονὸς ἂν εἴη τό ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν πρῶτος ἀριθμός.

Άπας ἄρα ἀριθμὸς ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

λγ΄.

Άριθμῶν δοθέντων ὁποσωνοῦν εὑρεῖν τοὺς ἐλαχίστους τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Έστωσαν οἱ δοθέντες ὁποσοιοῦν ἀριθμοὶ οἱ A, B, $\Gamma \cdot$ δεĩ δὴ εὑρεῖν τοὺς ἐλαχίστους τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B, $\Gamma.$

Οἱ A, B, Γ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσιν ἢ οὕ. εἰ μὲν οῦν οἱ A, B, Γ πρῶτοι πρὸς ἀλλήλους εἰσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. is prime then that which was prescribed has happened. And if (B is) composite then some number will measure it. Let it (so) measure (B), and let it be C. And since C measures B, and B measures A, C thus also measures A. And if C is prime then that which was prescribed has happened. And if (C is) composite then some number will measure it. So, in this manner of continued investigation, some prime number will be found which will measure (the number preceding it, which will also measure A). And if (such a number) cannot be found then an infinite (series of) numbers, each of which is less than the preceding, will measure the number A. The very thing is impossible for numbers. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure A.



Thus, every composite number is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 32

Every number is either prime or is measured by some prime number.



Let A be a number. I say that A is either prime or is measured by some prime number.

In fact, if A is prime then that which was prescribed has happened. And if (it is) composite then some prime number will measure it [Prop. 7.31].

Thus, every number is either prime or is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 33

To find the least of those (numbers) having the same ratio as any given multitude of numbers.

Let A, B, and C be any given multitude of numbers. So it is required to find the least of those (numbers) having the same ratio as A, B, and C.

For A, B, and C are either prime to one another, or not. In fact, if A, B, and C are prime to one another then they are the least of those (numbers) having the same ratio as them [Prop. 7.22].

ELEMENTS BOOK 7

А	В	Γ	Δ	Е	Ζ	Н	Θ	Κ	Λ	Μ	
			I							Ι	

Εἰ δὲ οὔ, εἰλήφθω τῶν Α, Β, Γ τὸ μέγιστον χοινὸν μέτρον ὁ Δ, καὶ ὁσάκις ὁ Δ ἕκαστον τῶν Α, Β, Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν ἑχάστω τῶν Ε, Ζ, Η. καὶ έκαστος ἄρα τῶν Ε, Ζ, Η ἕκαστον τῶν Α, Β, Γ μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. οἱ E, Z, H ἄρα τοὺς A, B, Γ ἰσάχις μετροῦσιν οἱ Ε, Ζ, Η ἄρα τοῖς Α, Β, Γ ἐν τῷ αὐτῷ λόγῳ εἰσίν. λέγω δή, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μή εἰσιν οἱ Ε, Ζ, Η ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ, έσονται [τινες] τῶν Ε, Ζ, Η ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγω ὄντες τοῖς A, B, Γ. ἔστωσαν οἱ Θ, K, Λ· ἰσάχις ἄρα ὁ Θ τὸν Α μετρεῖ καὶ ἑκάτερος τῶν Κ, Λ ἑκάτερον τῶν Β, Γ. όσάχις δὲ ὁ Θ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ M· καὶ ἑκάτερος ἄρα τῶν K, Λ ἑκάτερον τῶν B, Γ μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας. καὶ ἐπεὶ ὁ Θ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας, καὶ ὁ Μ ẳρα τὸν Α μετρεῖ κατὰ τὰς έν τῷ Θ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ Μ καὶ ἑκάτερον τῶν Β, Γ μετρεῖ κατὰ τὰς ἐν ἑκατέρω τῶν Κ, Λ μονάδας· ὁ Μ ẳρα τούς Α, Β, Γ μετρεῖ. καὶ ἐπεὶ ὁ Θ τὸν Α μετρεῖ κατὰ τὰς έν τῷ Μ μονάδας, ὁ Θ ἄρα τὸν Μ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ
 ὁ Ε τὸν Δ πολλαπλασιάσας τὸν Α πεποίηχεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Ε, Δ τῷ ἐκ τῶν Θ, Μ. ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Μ πρὸς τὸν Δ . μείζων δὲ ὁ Ε τοῦ Θ· μείζων ἄρα καὶ ὁ M τοῦ Δ . καὶ μετρεῖ τοὺς Α, Β, Γ· ὅπερ ἐστὶν ἀδύνατον· ὑπόκειται γὰρ ὁ Δ τῶν Α, Β, Γ τὸ μέγιστον χοινὸν μέτρον. οὐχ ἄρα έσονταί τινες τῶν Ε, Ζ, Η ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγω ὄντες τοῖς Α, Β, Γ. οἱ Ε, Ζ, Η ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ· ὅπερ ἔδει δεῖξαι.

λδ'.

Δύο ἀριθμῶν δοθέντων εὑρεῖν, δν ἐλάχιστον μετροῦσιν ἀριθμόν.

Έστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β· δεῖ δὴ εὑρεῖν,

And if not, let the greatest common measure, D, of A, B, and C have be taken [Prop. 7.3]. And as many times as D measures A, B, C, so many units let there be in E, F, G, respectively. And thus E, F, G measure A, B, C, respectively, according to the units in D [Prop. 7.15]. Thus, E, F, G measure A, B, C (respectively) an equal number of times. Thus, E, F, G are in the same ratio as A, B, C (respectively) [Def. 7.20]. So I say that (they are) also the least (of those numbers having the same ratio as A, B, C). For if E, F, G are not the least of those (numbers) having the same ratio as A, B, C (respectively), then there will be [some] numbers less than E, F, G which are in the same ratio as A, B, C(respectively). Let them be H, K, L. Thus, H measures A the same number of times that K, L also measure B, C, respectively. And as many times as H measures A, so many units let there be in M. Thus, K, L measure B, C, respectively, according to the units in M. And since H measures A according to the units in M, M thus also measures A according to the units in H [Prop. 7.15]. So, for the same (reasons), M also measures B, C according to the units in K, L, respectively. Thus, M measures A, B, and C. And since H measures A according to the units in M, H has thus made A (by) multiplying M. So, for the same (reasons), E has also made A (by) multiplying D. Thus, the (number created) from (multiplying) E and D is equal to the (number created) from (multiplying) H and M. Thus, as E (is) to H, so M (is) to D [Prop. 7.19]. And E (is) greater than H. Thus, M(is) also greater than D [Prop. 5.13]. And (M) measures A, B, and C. The very thing is impossible. For D was assumed (to be) the greatest common measure of A, B, and C. Thus, there cannot be any numbers less than E, F, G which are in the same ratio as A, B, C (respectively). Thus, E, F, G are the least of (those numbers) having the same ratio as A, B, C (respectively). (Which is) the very thing it was required to show.

Proposition 34

To find the least number which two given numbers (both) measure.

Let A and B be the two given numbers. So it is re-

ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.



Οί Α, Β γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὕ. έστωσαν πρότερον οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Α πολλαπλασιάσας τὸν Γ πεποίηχεν. οἱ Α, Β ἄρα τὸν Γ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρείτωσαν τὸν Δ. καὶ ὁσάχις ὁ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες έστωσαν έν τῷ Ε, ὁσάχις δὲ ὁ Β τὸν Δ μετρεῖ, τοσαῦται μονάδες ἕστωσαν ἐν τῷ Ζ. ὁ μὲν Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηχεν, ὁ δὲ B τὸν Z πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν A, E τῷ ἐκ τῶν B, Z. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Ε. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ έλάγιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔγοντας ἰσάχις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ Β ἄρα τὸν E μετρεĩ, ὡς ἑπόμενος ἑπόμενον. καὶ ἐπεὶ ὁ Α τοὺς B, Ε πολλαπλασιάσας τοὺς Γ, Δ πεποίηκεν, ἔστιν ἄρα ὡς ό B πρός τὸν E, οὕτως ὁ Γ πρὸς τὸν Δ . μετρεῖ δὲ ὁ B τὸν Ε· μετρεῖ ἄρα καὶ
ὑ Γ τὸν Δ ὁ μείζων τὸν ἐλάσσονα· ὅπερ έστιν άδύνατον. ούκ άρα οι Α, Β μετροῦσί τινα ἀριθμὸν έλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάγιστος ὢν ὑπὸ τῶν Α, Β μετρεῖται.



Μή ἔστωσαν δή οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, χαὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον let the least numbers, F and E, have been taken having έχόντων τοῖς A, B ol Z, E ἴσος ἄρα ἐστὶν ὁ ἐx τῶν A, E τῷ the same ratio as A and B (respectively) [Prop. 7.33].

quired to find the least number which they (both) measure.



For A and B are either prime to one another, or not. Let them, first of all, be prime to one another. And let Amake C (by) multiplying B. Thus, B has also made C(by) multiplying A [Prop. 7.16]. Thus, A and B (both) measure C. So I say that (C) is also the least (number which they both measure). For if not, A and B will (both) measure some (other) number which is less than C. Let them (both) measure D (which is less than C). And as many times as A measures D, so many units let there be in E. And as many times as B measures D, so many units let there be in F. Thus, A has made D(by) multiplying E, and B has made D (by) multiplying F. Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F. Thus, as A (is) to B, so F (is) to E [Prop. 7.19]. And A and B are prime (to one another), and prime (numbers) are the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, B measures E, as the following (number measuring) the following. And since A has made C and D (by) multiplying B and E (respectively), thus as B is to E, so C(is) to D [Prop. 7.17]. And B measures E. Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, A and B do not (both) measure some number which is less than C. Thus, C is the least (number) which is measured by (both) A and B.



So let A and B be not prime to one another. And

έκ τῶν Β, Ζ. καὶ ὁ Α τὸν Ε πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Ζ πολλαπλασιάσας τὸν Γ πεποίηκεν· οἱ Α, Β ἄρα τὸν Γ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρείτωσαν τὸν Δ. καὶ ۂσάκις μὲν ۂ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Η, ὑσάκις δὲ ὁ B τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Θ. ὁ μὲν Α ἄρα τὸν Η πολλαπλασιάσας τὸν Δ
 πεποίηχεν, ὁ δὲ Β τὸν Θ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν A, Η τῷ ἐχ τῶν B, Θ · ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B, οὕτως ό Θ πρός τὸν Η. ὡς δὲ ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Ζ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Η. οἱ δὲ Ζ, Ε ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάχις ὅ τε μείζων τὸν μείζονα χαὶ ὁ έλάσσων τὸν ἐλάσσονα· ὁ Ε ἄρα τὸν Η μετρεĩ. καὶ ἐπεὶ ὁ Α τούς Ε, Η πολλαπλασιάσας τούς Γ, Δ πεποίηχεν, ἕστιν ἄρα ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ δὲ Ε τὸν Η μετρεί και ό Γ άρα τὸν Δ μετρεί ὁ μείζων τὸν ἐλάσσονα. ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β μετρήσουσί τινα άριθμὸν ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάγιστος ὢν ὑπὸ τῶν Α, Β μετρεῖται· ὅπερ ἔπει δεῖξαι.

λε΄.

Έὰν δύο ἀριθμοὶ ἀριθμόν τινα μετρῶσιν, καὶ ὁ ἐλάχιστος ὑπ' αὐτῶν μετρούμενος τὸν αὐτὸν μετρήσει.



Δύο γὰρ ἀριθμοὶ οἱ A, B ἀριθμόν τινα τὸν ΓΔ μετρείτωσαν, ἐλάχιστον δὲ τὸν Ε· λέγω, ὅτι καὶ ὁ Ε τὸν ΓΔ μετρεĩ.

Εἰ γὰρ οὐ μετρεῖ ὁ Ε τὸν ΓΔ, ὁ Ε τὸν ΔΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΓΖ. καὶ ἐπεὶ οἱ Α, Β τὸν Ε μετροῦσιν, ὁ δὲ Ε τὸν ΔΖ μετρεῖ, καὶ οἱ Α, Β ἄρα τὸν ΔΖ μετρήσουσιν. μετροῦσι δὲ καὶ ὅλον τὸν ΓΔ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσουσιν ἐλάσσονα ὄντα τοῦ Ε· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὐ μετρεῖ ὁ Ε τὸν ΓΔ· μετρεῖ ἄρα· ὅπερ ἔδει δεῖξαι. Thus, the (number created) from (multiplying) A and Eis equal to the (number created) from (multiplying) Band F [Prop. 7.19]. And let A make C (by) multiplying E. Thus, B has also made C (by) multiplying F. Thus, A and B (both) measure C. So I say that (C) is also the least (number which they both measure). For if not, Aand B will (both) measure some number which is less than C. Let them (both) measure D (which is less than C). And as many times as A measures D, so many units let there be in G. And as many times as B measures D, so many units let there be in H. Thus, A has made D(by) multiplying G, and B has made D (by) multiplying H. Thus, the (number created) from (multiplying) A and G is equal to the (number created) from (multiplying) Band H. Thus, as A is to B, so H (is) to G [Prop. 7.19]. And as A (is) to B, so F (is) to E. Thus, also, as F (is) to E, so H (is) to G. And F and E are the least (numbers having the same ratio as A and B), and the least (numbers) measure those (numbers) having the same ratio an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, E measures G. And since A has made C and D (by) multiplying E and G (respectively), thus as E is to G, so C(is) to D [Prop. 7.17]. And E measures G. Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, A and B do not (both) measure some (number) which is less than C. Thus, C (is) the least (number) which is measured by (both) A and B. (Which is) the very thing it was required to show.

Proposition 35

If two numbers (both) measure some number then the least (number) measured by them will also measure the same (number).



For let two numbers, A and B, (both) measure some number CD, and (let) E (be the) least (number measured by both A and B). I say that E also measures CD.

For if E does not measure CD then let E leave CF less than itself (in) measuring DF. And since A and B (both) measure E, and E measures DF, A and B will thus also measure DF. And (A and B) also measure the whole of CD. Thus, they will also measure the remainder CF, which is less than E. The very thing is impossible. Thus, E cannot not measure CD. Thus, (E) measures

λς'.

Τριῶν ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Έστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ Α, Β, Γ· δεῖ δὴ εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.



Εἰλήφθω γὰρ ὑπὸ δύο τῶν Α, Β ἐλάχιστος μετρούμενος ὁ Δ. ὁ δὴ Γ τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον. μετροῦσι δὲ xaì oἱ Α, Β τὸν Δ· οἱ Α, Β, Γ ἄρα τὸν Δ μετροῦσιν. λέγω δή, ὅτι xaì ἐλάχιστον. εἰ γὰρ μή, μετρήσουσιν [τινα] ἀριθμὸν οἱ Α, Β, Γ ἑλάσσονα ὄντα τοῦ Δ. μετρείτωσαν τὸν Ε. ἐπεὶ οἱ Α, Β, Γ τὸν Ε μετροῦσιν, xaì οἱ Α, Β ἄρα τὸν Ε μετροῦσιν. xaì ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Α, Β μετρούμενος [τὸν Ε] μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Α, Β μετρούμενος [τὸν Ε] μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Α, Β μετρούμενος ἔστιν ὁ Δ· ὁ Δ ἄρα τὸν Ε μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐx ἄρα οἱ Α, Β, Γ μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Δ· οἱ Α, Β, Γ ἅρα ἐλάχιστον τὸν Δ μετροῦσιν.

Μὴ μετρείτω δὴ πάλιν ὁ Γ τὸν Δ, καὶ εἰλήφθω ὑπὸ τῶν Γ , Δ ἐλάγιστος μετρούμενος ἀριθμὸς ὁ Ε. ἐπεὶ οἱ Α, Β τὸν Δ μετροῦσιν, ὁ δ
ὲ Δ τὸν Ε μετρεῖ, καὶ οἱ Α, Β ẳρα τὸν Ε μετροῦσιν. μετρεῖ δὲ καὶ ὁ Γ [τὸν Ε· καὶ] οἱ A, B, Γ ἄρα τὸν Ε μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα οἱ Α, Β, Γ ἐλάσσονα ὄντα τοῦ Ε. μετρείτωσαν τὸν Ζ. ἐπεὶ οἱ Α, Β, Γ τὸν Ζ μετροῦσιν, καὶ οἱ A, B ἄρα τὸν Z μετροῦσιν· καὶ ὁ ἐλάγιστος ἄρα ὑπὸ τῶν A, B μετρούμενος τὸν Z μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν A, B μετρούμενός ἐστιν ὁ Δ · ὁ Δ ἄρα τὸν Z μετρεĩ. μετρεĩ δ
ἑ καὶ ὁ Γ τὸν Ζ· οἱ Δ, Γ ẳρα τὸν Ζ μετροῦσιν· ὥστε καὶ ὁ έλάχιστος ὑπὸ τῶν Δ, Γ μετρούμενος τὸν Ζ μετρήσει. ὁ δὲ ἐλάχιστος ὑπὸ τῶν Γ, Δ μετρούμενός ἐστιν
ἑ ${\rm E}^{\cdot}$ ἑ Ε ἄρα τὸν Ζ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. ούκ ἄρα οἱ Α, Β, Γ μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Ε. ὁ Ε ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β, Γ μετρεῖται· ὄπερ ἔδει δεῖξαι.

(CD). (Which is) the very thing it was required to show.

Proposition 36

To find the least number which three given numbers (all) measure.

Let A, B, and C be the three given numbers. So it is required to find the least number which they (all) measure.



For let the least (number), D, measured by the two (numbers) A and B have been taken [Prop. 7.34]. So C either measures, or does not measure, D. Let it, first of all, measure (D). And A and B also measure D. Thus, A, B, and C (all) measure D. So I say that (D is) also the least (number measured by A, B, and C). For if not, A, B, and C will (all) measure [some] number which is less than D. Let them measure E (which is less than D). Since A, B, and C (all) measure E then A and B thus also measure E. Thus, the least (number) measured by A and B will also measure [E] [Prop. 7.35]. And D is the least (number) measured by A and B. Thus, Dwill measure E, the greater (measuring) the lesser. The very thing is impossible. Thus, A, B, and C cannot (all) measure some number which is less than D. Thus, A, B, and C (all) measure the least (number) D.

So, again, let C not measure D. And let the least number, E, measured by C and D have been taken [Prop. 7.34]. Since A and B measure D, and D measures E, A and B thus also measure E. And C also measures [E]. Thus, A, B, and C [also] measure E. So I say that (E is) also the least (number measured by A, B, and C). For if not, A, B, and C will (all) measure some (number) which is less than E. Let them measure F (which is less than E). Since A, B, and C (all) measure F, A and Bthus also measure F. Thus, the least (number) measured by A and B will also measure F [Prop. 7.35]. And Dis the least (number) measured by A and B. Thus, Dmeasures F. And C also measures F. Thus, D and C(both) measure F. Hence, the least (number) measured by D and C will also measure F [Prop. 7.35]. And E

is the least (number) measured by C and D. Thus, E measures F, the greater (measuring) the lesser. The very thing is impossible. Thus, A, B, and C cannot measure some number which is less than E. Thus, E (is) the least (number) which is measured by A, B, and C. (Which is) the very thing it was required to show.

Proposition 37

If a number is measured by some number then the (number) measured will have a part called the same as the measuring (number).



Άριθμός γάρ ὁ Α ὑπό τινος ἀριθμοῦ τοῦ Β μετρείσθω·
For le λέγω, ὅτι ὁ Α ὑμώνυμον μέρος ἔχει τῷ Β.
B. I say t

Όσάχις γὰρ ὁ B τὸν A μετρεĩ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Γ. ἑπεὶ ὁ B τὸν A μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, μετρεῖ δὲ καὶ ἡ Δ μονὰς τὸν Γ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ B τὸν A. ἐναλλὰξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν B ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν A· δ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ B ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ A. ἡ δὲ Δ μονὰς τοῦ B ἀριθμοῦ μέρος ἐστὶν ὁμώνυμον αὐτῷ καὶ ὁ Γ ἄρα τοῦ A μέρος ἐστὶν ὁμώνυμον τῷ B. ὥστε ὁ A μέρος ἔχει τὸν Γ ὑμώνυμον ὄντα τῷ B· ὅπερ ἔδει δεῖξαι.

λζ'.

όμώνυμον μέρος ἕξει τῷ μετροῦντι.

B⊢−−−−−

A⊢

 $\Lambda \vdash$

Έὰν ἀριθμὸς ὑπό τινος ἀριθμοῦ μετρῆται, ὁ μετρούμενος

λη'.

Έὰν ἀριθμος μέρος ἔχῃ ὅτιοῦν, ὑπὸ ὁμωνύμου ἀριθμοῦ μετρηθήσεται τῷ μέρει.



Ἀριθμὸς γὰρ ὁ A μέρος ἐχέτω ὁτιοῦν τὸν B, καὶ τῷ B μέρει ὁμώνυμος ἔστω [ἀριθμὸς] ὁ Γ · λέγω, ὅτι ὁ Γ τὸν A μετρεĩ.

Έπεὶ γὰρ ὁ B τοῦ A μέρος ἐστὶν ὁμώνυμον τῷ Γ, ἔστι δὲ καὶ ἡ Δ μονὰς τοῦ Γ μέρος ὁμώνυμον αὐτῷ, ὃ ẳρα μέρος

For let the number A be measured by some number B. I say that A has a part called the same as B.

For as many times as B measures A, so many units let there be in C. Since B measures A according to the units in C, and the unit D also measures C according to the units in it, the unit D thus measures the number C as many times as B (measures) A. Thus, alternately, the unit D measures the number B as many times as C(measures) A [Prop. 7.15]. Thus, which(ever) part the unit D is of the number B, C is also the same part of A. And the unit D is a part of the number B called the same as it (*i.e.*, a Bth part). Thus, C is also a part of A called the same as B (*i.e.*, C is the Bth part of A). Hence, A has a part C which is called the same as B (*i.e.*, A has a Bth part). (Which is) the very thing it was required to show.

Proposition 38

If a number has any part whatever then it will be measured by a number called the same as the part.



For let the number A have any part whatever, B. And let the [number] C be called the same as the part B (*i.e.*, B is the Cth part of A). I say that C measures A.

For since B is a part of A called the same as C, and the unit D is also a part of C called the same as it (*i.e.*,

ἐστὶν ἡ Δ μονὰς τοῦ Γ ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ xaì ὁ B τοῦ A· ἰσάχις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ xaì ὁ B τὸν A. ἐναλλὰξ ἄρα ἰσάχις ἡ Δ μονὰς τὸν B ἀριθμὸν μετρεῖ χαὶ ὁ Γ τὸν A. ὁ Γ ἄρα τὸν Α μετρεῖ· ὅπερ ἔδει δεἰξαι.

λθ'.

Αριθμόν εὐρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ δοθέντα μέρη.



Έστω τὰ δοθέντα μέρη τὰ A, B, Γ δεῖ δὴ ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ A, B, Γ μέρη.

Έστωσαν γὰρ τοῖς A, B, Γ μέρεσιν ὑμώνυμοι ἀριθμοὶ οἱ Δ , E, Z, καὶ εἰλήφθω ὑπὸ τῶν Δ , E, Z ἐλάχιστος μετρούμενος ἀριθμὸς ὁ H.

Ο Η ἄρα ὑμώνυμα μέρη ἔχει τοῖς Δ, Ε, Ζ. τοῖς δὲ Δ, Ε, Ζ ὑμώνυμα μέρη ἐστὶ τὰ Α, Β, Γ· ὁ Η ἄρα ἔχει τὰ Α, Β, Γ μέρη. λέγω δή, ὅτι καὶ ἐλάχιστος ὤν, εἰ γὰρ μή, ἔσται τις τοῦ Η ἐλάσσων ἀριθμός, ὅς ἕξει τὰ Α, Β, Γ μέρη. ἔστω ὑ Θ. ἐπεὶ ὁ Θ ἔχει τὰ Α, Β, Γ μέρη, ὁ Θ ἄρα ὑπὸ ὑμωνύμων ἀριθμῶν μετρηθήσεται τοῖς Α, Β, Γ μέρεσιν. τοῖς δὲ Α, Β, Γ μέρεσιν ὑμώνυμοι ἀριθμοί εἰσιν οἱ Δ, Ε, Ζ· ὁ Θ ἄρα ὑπὸ τῶν Δ, Ε, Ζ μετρεῖται. καί ἐστιν ἐλάσσων τοῦ Η· ὅπερ ἐστὶν ἀδύνατον. οὐχ ἄρα ἔσται τις τοῦ Η ἐλάσσων ἀριθμός, ὅς ἕξει τὰ Α, Β, Γ μέρη· ὅπερ ἔδει δεῖξαι. *D* is the *C*th part of *C*), thus which(ever) part the unit *D* is of the number *C*, *B* is also the same part of *A*. Thus, the unit *D* measures the number *C* as many times as *B* (measures) *A*. Thus, alternately, the unit *D* measures the number *B* as many times as *C* (measures) *A* [Prop. 7.15]. Thus, *C* measures *A*. (Which is) the very thing it was required to show.

Proposition 39

To find the least number that will have given parts.



Let A, B, and C be the given parts. So it is required to find the least number which will have the parts A, B, and C (*i.e.*, an Ath part, a Bth part, and a Cth part).

For let D, E, and F be numbers having the same names as the parts A, B, and C (respectively). And let the least number, G, measured by D, E, and F, have been taken [Prop. 7.36].

Thus, G has parts called the same as D, E, and F [Prop. 7.37]. And A, B, and C are parts called the same as D, E, and F (respectively). Thus, G has the parts A, B, and C. So I say that (G) is also the least (number having the parts A, B, and C). For if not, there will be some number less than G which will have the parts A, B, and C. Let it be H. Since H has the parts A, B, and C, H will thus be measured by numbers called the same as the parts A, B, and C [Prop. 7.38]. And D, E, and F are numbers called the same as the parts A, B, and C (respectively). Thus, H is measured by D, E, and F. And (H) is less than G. The very thing is impossible. Thus, there cannot be some number less than G which will have the parts A, B, and C. (Which is) the very thing it was required to show.

ELEMENTS BOOK 8

Continued Proportion †

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

α΄.

Έὰν ὥσιν ὑσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄχροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὥσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.



Έστωσαν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, οἱ δὲ ἄχροι αὐτῶν οἱ Α, Δ, πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οἱ Α, Β, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔστωσαν ἐλάττονες τῶν Α, Β, Γ, Δ οἱ Ε, Ζ, Η, Θ ἐν τῷ αὐτῷ λόγῳ ὄντες αὐτοῖς. καὶ ἐπεὶ οἱ Α, Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς Ε, Ζ, Η, Θ, καί ἐστιν ἴσον τὸ πλῆθος [τῶν Α, Β, Γ, Δ] τῷ πλήθει [τῶν Ε, Ζ, Η, Θ], δι' ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Θ. οἱ δὲ Α, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Ε ὁ μείζων τὸν ἐλάσσονα[°] ὅπερ ἐστὶν ἀδύνατον. οὐx ἄρα οἱ Ε, Ζ, Η, Θ ἐλάσσονες ὄντες τῶν Α, Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶν αὐτοῖς. οἱ Α, Β, Γ, Δ ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἑχόντων αὐτοῖς[°] ὅπερ ἕδει δεῖξαι.

β'.

Αριθμούς εύρεῖν ἑξῆς ἀνάλογον ἐλαχίστους, ὅσους ἂν ἐπιτάξῃ τις, ἐν τῷ δοθέντι λόγῳ.

Έστω ὁ δοθεὶς λόγος ἐν ἐλάχίστοις ἀριθμοῖς ὁ τοῦ Α πρὸς τὸν Β· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους, ὅσους ἄν τις ἐπιτάξῃ, ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ.

Ἐπιτετάχθωσαν δὴ τέσσαρες, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω, τὸν δὲ Β πολλαπλασιάσας τὸν Δ ποιείτω, καὶ ἔτι ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιείτω, καὶ ἕτι ὁ Α τοὺς Γ, Δ, Ε πολλαπλασιάσας τοὺς Ζ, Η, Θ ποιείτω, ὁ δὲ Β τὸν Ε πολλαπλασιάσας τὸν Κ ποιείτω.

Proposition 1

If there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D, be prime to one another. I say that A, B, C, D are the least of those (numbers) having the same ratio as them.

For if not, let E, F, G, H be less than A, B, C, D (respectively), being in the same ratio as them. And since A, B, C, D are in the same ratio as E, F, G, H, and the multitude [of A, B, C, D] is equal to the multitude [of E, F, G, H], thus, via equality, as A is to D, (so) E (is) to H [Prop. 7.14]. And A and D (are) prime (to one another). And prime (numbers are) also the least of those (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures E, the greater (measuring) the lesser. The very thing is impossible. Thus, E, F, G, H, being less than A, B, C, D, are not in the same ratio as them. Thus, A, B, C, D are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

Proposition 2

To find the least numbers, as many as may be prescribed, (which are) continuously proportional in a given ratio.

Let the given ratio, (expressed) in the least numbers, be that of A to B. So it is required to find the least numbers, as many as may be prescribed, (which are) in the ratio of A to B.

Let four (numbers) have been prescribed. And let A make C (by) multiplying itself, and let it make D (by) multiplying B. And, further, let B make E (by) multiplying itself. And, further, let A make F, G, H (by) multiplying C, D, E. And let B make K (by) multiplying E.

ΣΤΟΙΧΕΙΩΝ η'.



Καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ πεποίηχεν, τὸν δὲ Β πολλαπλασιάσας τὸν Δ πεποίηχεν, έστιν ἄρα ώς ὁ Α πρὸς τὸν Β, [οὕτως] ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ό δὲ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηχεν, ἑχάτερος άρα τῶν Α, Β τὸν Β πολλαπλασιάσας ἑκάτερον τῶν Δ, Ε πεποίηκεν.
 ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε. ἀλλ' ὡς ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Δ · καὶ ὡς ἄρα ὁ Γ πρὸς τὸν $\Delta,$ ὁ Δ πρὸς τὸν Ε. καὶ ἐπεὶ ὁ Α τοὺς Γ, Δ πολλαπλασιάσας τοὺς Z, H πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , [οὕτως] ὁ Ζ πρὸς τὸν Η. ὡς δὲ ὁ Γ πρὸς τὸν Δ , οὕτως ἦν ἑ A πρὸς τὸν B· καὶ ὡς ἄρα ἑ A πρὸς τὸν B, ἑ Ζ πρός τὸν Η. πάλιν, ἐπεὶ ὁ Α τοὺς Δ, Ε πολλαπλασιάσας τοὺς Η, Θ πεποίηχεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, ὁ Η πρὸς τὸν Θ . ἀλλ' ὡς ὁ Δ πρὸς τὸν E, ὁ A πρὸς τὸν B. καὶ ώς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Θ. καὶ ἐπεὶ οί Α, Β τὸν Ε πολλαπλασιάσαντες τοὺς Θ, Κ πεποιήκασιν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Κ. ἀλλỉ ώς ὁ Α πρὸς τὸν Β, οὕτως ὅ τε Ζ πρὸς τὸν Η καὶ ὁ Η πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Ζ πρὸς τὸν Η, οὕτως ὅ τε Η πρὸς τόν Θ καὶ ὁ Θ πρὸς τὸν K· οἱ Γ , Δ , E ἄρα καὶ οἱ Z, H, Θ, Κ ἀνάλογόν εἰσιν ἐν τῷ τοῦ Α πρὸς τὸν Β λόγω. λέγω δή, ὅτι καὶ ἐλάχιστοι. ἐπεὶ γὰρ οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ δὲ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων πρῶτοι πρὸς ἀλλήλους εἰσίν, οἱ Α, Β άρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἑκάτερος μὲν τῶν Α, Β ἑαυτὸν πολλαπλασιάσας ἑκάτερον τῶν Γ, Ε πεποίηκεν, έκάτερον δὲ τῶν Γ, Ε πολλαπλασιάσας ἑκάτερον τῶν Ζ, Κ πεποίηκεν οί Γ, Ε άρα και οί Ζ, Κ πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ ῶσιν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄχροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οἱ Γ, Δ, Ε ἄρα καὶ οἱ Ζ, Η, Θ, Κ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β. ὅπερ ἔδει δεῖξαι.



And since A has made C (by) multiplying itself, and has made D (by) multiplying B, thus as A is to B, [so] C (is) to D [Prop. 7.17]. Again, since A has made D (by) multiplying B, and B has made E (by) multiplying itself, A, B have thus made D, E, respectively, (by) multiplying B. Thus, as A is to B, so D (is) to E [Prop. 7.18]. But, as A (is) to B, (so) C (is) to D. And thus as C (is) to D, (so) D (is) to E. And since A has made F, G (by) multiplying C, D, thus as C is to D, [so] F (is) to G [Prop. 7.17]. And as C (is) to D, so A was to B. And thus as A (is) to B, (so) F (is) to G. Again, since A has made G, H(by) multiplying D, E, thus as D is to E, (so) G (is) to H [Prop. 7.17]. But, as D (is) to E, (so) A (is) to B. And thus as A (is) to B, so G (is) to H. And since A, B have made H, K (by) multiplying E, thus as A is to B, so H (is) to K. But, as A (is) to B, so F (is) to G, and G to H. And thus as F (is) to G, so G (is) to H, and Hto K. Thus, C, D, E and F, G, H, K are (both continuously) proportional in the ratio of A to B. So I say that (they are) also the least (sets of numbers continuously proportional in that ratio). For since A and B are the least of those (numbers) having the same ratio as them, and the least of those (numbers) having the same ratio are prime to one another [Prop. 7.22], A and B are thus prime to one another. And A, B have made C, E, respectively, (by) multiplying themselves, and have made F, Kby multiplying C, E, respectively. Thus, C, E and F, Kare prime to one another [Prop. 7.27]. And if there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them [Prop. 8.1]. Thus, C, D, E and F, G, H, K are the least of those (continuously proportional sets of numbers) having the same ratio as Aand B. (Which is) the very thing it was required to show.

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι ἐἀν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι ῶσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄχρον αὐτῶν τετράγωνοί εἰσιν, ἐἀν δὲ τέσσαρες, χύβοι.

γ'.

Έὰν ῶσιν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄχροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.



Έστωσαν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β, Γ, Δ· λέγω, ὅτι οἱ ἄχροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν A, B, Γ, Δ λόγῷ οἱ E, Z, τρεῖς δὲ οἱ H, Θ, K, καὶ ἑξῆς ἑνὶ πλείους, ἕως τὸ λαμβανόμενον πλῆθος ἴσον γένηται τῷ πλήθει τῶν A, B, Γ, Δ. εἰλήφθωσαν καὶ ἔστωσαν οἱ Λ, M, N, Ξ.

Καὶ ἑπεὶ οἱ Ε, Ζ ἑλάχιστοἱ εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ἑκάτερος τῶν Ε, Ζ ἑαυτὸν μὲν πολλαπλασιάσας ἑκάτερον τῶν Η, Κ πεποίηκεν, ἑκάτερον δὲ τῶν Η, Κ πολλαπλασιάσας ἑκάτερον τῶν Λ, Ξ πεποίηκεν, καὶ οἱ Η, Κ ἄρα καὶ οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ Α, Β, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, εἰσὶ δὲ καὶ οἱ Λ, Μ, Ν, Ξ ἐλάχιστοι ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ, Δ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Β, Γ, Δ τῷ πλήθει τῶν Λ, Μ, Ν, Ξ ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν Α τῷ Λ, ὁ δὲ Δ τῷ Ξ. καί εἰσιν οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους. καὶ οἱ Α, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Corollary

So it is clear, from this, that if three continuously proportional numbers are the least of those (numbers) having the same ratio as them then the outermost of them are square, and, if four (numbers), cube.

Proposition 3

If there are any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them then the outermost of them are prime to one another.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that the outermost of them, A and D, are prime to one another.

For let the two least (numbers) E, F (which are) in the same ratio as A, B, C, D have been taken [Prop. 7.33]. And the three (least numbers) G, H, K [Prop. 8.2]. And (so on), successively increasing by one, until the multitude of (numbers) taken is made equal to the multitude of A, B, C, D. Let them have been taken, and let them be L, M, N, O.

And since E and F are the least of those (numbers) having the same ratio as them they are prime to one another [Prop. 7.22]. And since E, F have made G, K, respectively, (by) multiplying themselves [Prop. 8.2 corr.], and have made L, O (by) multiplying G, K, respectively, G, K and L, O are thus also prime to one another [Prop. 7.27]. And since A, B, C, D are the least of those (numbers) having the same ratio as them, and L, M, N, O are also the least (of those numbers having the same ratio as them), being in the same ratio as A, B, C, D, and the multitude of A, B, C, D is equal to the multitude of L, M, N, O, thus A, B, C, D are equal to L, M, N, O, respectively. Thus, A is equal to L, and D to O. And Land O are prime to one another. Thus, A and D are also prime to one another. (Which is) the very thing it was

required to show.

δ'.

Λόγων δοθέντων ὑποσωνοῦν ἐν ἐλαχίστοις ἀριθμοῖς ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους ἐν τοῖς δοθεῖσι λόγοις.



Έστωσαν οἱ δοθέντες λόγοι ἐν ἐλαχίστοις ἀριθμοῖς ὅ τε τοῦ Α πρὸς τὸν Β καὶ ὁ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ὁ τοῦ Ε πρὸς τὸν Ζ· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους ἕν τε τῷ τοῦ Α πρὸς τὸν Β λόγῳ καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Ζ.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν Β, Γ ἐλάχιστος μετρούμενος άριθμὸς ὁ Η. καὶ ὁσάκις μὲν ὁ Β τὸν Η μετρεῖ, τοσαυτάκις καὶ ὁ Α τὸν Θ μετρείτω, ὁσάκις δὲ ὁ Γ τὸν Η μετρεῖ, τοσαυτάχις χαὶ
ὑ Δ τὸν Κ μετρείτω.
ὑ δὲ Ε τὸν Κ ἤτοι μετρεῖ η ού μετρεῖ. μετρείτω πρότερον. καὶ ὑσάκις ὁ Ε τὸν Κ μετρεῖ, τοσαυτάχις χαὶ ὁ Z τὸν Λ μετρείτω.
 χαὶ ἐπεὶ ἰσάχις ὁ Α τὸν Θ μετρεῖ καὶ ὁ Β τὸν Η, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν B, οὕτως ὁ Θ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ Η πρὸς τὸν Κ, καὶ ἔτι ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ · οἱ Θ, Η, Κ, Λ ἄρα ἑξῆς ἀνάλογόν εἰσιν ἕν τε τῷ τοῦ Α πρὸς τὸν Β καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ
ἔτι ἐν τῷ τοῦ Ε πρὸς τὸν Ζ λόγ
ῳ. λέγω δή, ὅτι καὶ έλάχιστοι. εἰ γὰρ μή εἰσιν οἱ Θ , Η, Κ, Λ ἑξῆς ἀνάλογον έλάγιστοι ἕν τε τοῖς τοῦ Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἐν τῷ τοῦ Ε πρὸς τὸν Ζ λόγοις, ἔστωσαν οἱ Ν, Ξ, Μ, Ο. καὶ ἐπεί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάχις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ὁ Β ἄρα τὸν Ξ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ
 ὁ Γ τὸν Ξ μετρεῖ· οἱ Β, Γ άρα τὸν Ξ μετροῦσιν· καὶ ὁ ἐλάγιστος ἄρα ὑπὸ τῶν Β, Γ μετρούμενος τὸν Ξ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρεῖται
ὑ Η· ὑ Η ἄρα τὸν Ξ μετρεῖ ὑ μείζων τὸν ἐλάσσονα· όπερ ἐστὶν ἀδύντατον. οὐκ ἄρα ἔσονταί τινες τῶν Θ , H, K, Λ ἑλάσσονες ἀριθμοὶ ἑξῆς ἔν τε τῷ τοῦ Α πρὸς τὸν Β καὶ τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ E πρὸς τὸν Z λόγῷ.

Proposition 4

For any multitude whatsoever of given ratios, (ex-



Let the given ratios, (expressed) in the least numbers, be the (ratios) of A to B, and of C to D, and, further, of E to F. So it is required to find the least numbers continuously proportional in the ratio of A to B, and of C to B, and, further, of E to F.

For let the least number, G, measured by (both) B and C have be taken [Prop. 7.34]. And as many times as B measures G, so many times let A also measure H. And as many times as C measures G, so many times let D also measure K. And E either measures, or does not measure, K. Let it, first of all, measure (K). And as many times as E measures K, so many times let F also measure L. And since A measures H the same number of times that B also (measures) G, thus as A is to B, so H (is) to G [Def. 7.20, Prop. 7.13]. And so, for the same (reasons), as C (is) to D, so G (is) to K, and, further, as E (is) to F, so K (is) to L. Thus, H, G, K, L are continuously proportional in the ratio of A to B, and of C to D, and, further, of E to F. So I say that (they are) also the least (numbers continuously proportional in these ratios). For if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B, and of C to D, and of E to F, let N, O, M, P be (the least such numbers). And since as A is to B, so N (is) to O, and A and B are the least (numbers which have the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures O. So, for the same (reasons), C also measures O. Thus, B and C (both) measure O. Thus, the least number measured by (both) B and C will also measure O [Prop. 7.35]. And G (is) the least number measured by (both) B and C.



Thus, G measures O, the greater (measuring) the lesser. The very thing is impossible. Thus, there cannot be any numbers less than H, G, K, L (which are) continuously (proportional) in the ratio of A to B, and of C to D, and, further, of E to F.



Μή μετρείτω δή ὁ Ε τὸν Κ, καὶ εἰλήφθω ὑπὸ τῶν Ε, Κ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Μ. καὶ ὑσάκις μὲν ό Κ τὸν Μ μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Η ἑκάτερον τῶν Ν, Ξ μετρείτω, ὑσά
ακις δὲ ὁ Ε τὸν Μ μετρεῖ, τοσαυτάχις χαὶ ὁ Ζ τὸν Ο μετρείτω. ἐπεὶ ἰσάχις ὁ Θ τὸν Ν μετρεῖ καὶ ὁ Η τὸν Ξ, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Ν πρὸς τὸν Ξ. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ό Α πρός τὸν Β΄ καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ό Ξ πρός τὸν Μ. πάλιν, ἐπεὶ ἰσάχις ὁ Ε τὸν Μ μετρεῖ καὶ ό Ζ τὸν Ο, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Μ πρὸς τὸν O· o
ἱ N, Ξ, Μ, O ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τοῖς τοῦ τε Α πρός τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις. λέγω δή, ὅτι καὶ ἐλάχιστοι ἐν τοῖς Α Β, Γ Δ , E Z λόγοις. εἰ γὰρ μή, ἔσονταί τινες τῶν N, Ξ, M, O έλάσσονες ἀριθμοὶ ἑξῆς ἀνάλογον ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. ἔστωσαν οἱ Π, Ρ, Σ, Τ. καὶ ἐπεί ἐστιν ὡς ἑ Π πρὸς τὸν Ρ, οὕτως ὁ Α πρὸς τὸν Β, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ έλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ίσάχις ὄ τε ἡγούμενος τὸν ἡγούμενον χαὶ ὁ ἑπόμενος τὸν έπόμενον, ὁ Β ẳρα τὸν Ρ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ρ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ρ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετούμενος τὸν P μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρούμενος ἐστιν ὁ Η· ὁ Η ἄρα τὸν Ρ μετρεῖ. χαί ἐστιν ὡς ὁ Η πρὸς τὸν Ρ, οὕτως ὁ Κ πρὸς τὸν Σ · καὶ ὁ K ẳρα τὸν Σ μετρεῖ. μετρεῖ δὲ καὶ ὁ Ε τὸν Σ· οἱ Ε, Κ άρα τὸν Σ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν E, K μετρούμενος τὸν Σ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Ε, Κ μετρούμενός ἐστιν
 ὁ M· ὁ M ἄρα τὸν Σ μετρεῖ ὁ μείζων τὸν έλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν

So let E not measure K. And let the least number, M, measured by (both) E and K have been taken [Prop. 7.34]. And as many times as K measures M, so many times let H, G also measure N, O, respectively. And as many times as E measures M, so many times let F also measure P. Since H measures N the same number of times as G (measures) O, thus as H is to G, so N (is) to O [Def. 7.20, Prop. 7.13]. And as H (is) to G, so A (is) to B. And thus as A (is) to B, so N (is) to O. And so, for the same (reasons), as C (is) to D, so O (is) to M. Again, since E measures M the same number of times as F (measures) P, thus as E is to F, so M (is) to P [Def. 7.20, Prop. 7.13]. Thus, N, O, M, P are continuously proportional in the ratios of A to B, and of C to D, and, further, of E to F. So I say that (they are) also the least (numbers) in the ratios of A B, C D, E F. For if not, then there will be some numbers less than N, O, M, P (which are) continuously proportional in the ratios of A B, C D, E F. Let them be Q, R, S, T. And since as Q is to R, so A (is) to B, and A and B(are) the least (numbers having the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures R. So, for the same (reasons), C also measures R. Thus, B and C(both) measure R. Thus, the least (number) measured by (both) B and C will also measure R [Prop. 7.35]. And Gis the least number measured by (both) B and C. Thus, G measures R. And as G is to R, so K (is) to S. Thus, N, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἑξῆς ἀνάλογον ἕν τε τοῖς τοῦ Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις· οἱ Ν, Ξ, Μ, Ο ἄρα ἑξῆς ἀνάλογον ἐλάχιστοί εἰσιν ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις· ὅπερ ἔδει δεῖξαι.

ε΄.

Οἱ ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσι τὸν συγχείμενον ἐχ τῶν πλευρῶν.



Έστωσαν ἐπίπεδοι ἀριθμοὶ οἱ A, B, καὶ τοῦ μὲν A πλευραὶ ἔστωσαν οἱ Γ , Δ ἀριθμοί, τοῦ δὲ B οἱ E, Z· λέγω, ὅτι ὁ A πρὸς τὸν B λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Λόγων γὰρ δοθέντων τοῦ τε ὃν ἔχει ὁ Γ πρὸς τὸν Ε καὶ
ἱ Δ πρὸς τὸν Ζ εἰλήφθωσαν ἀριθμοὶ ἑξῆς ἐλάχιστοι ἐν τοῖς
Γ Ε, Δ Ζ λόγοις, οἱ Η, Θ, Κ, ὥστε εἶναι ὡς μὲν τὸν Γ πρὸς τὸν Ε, οὕτως τὸν Η πρὸς τὸν Θ, ὡς δὲ τὸν Δ πρὸς τὸν Ζ,
οὕτως τὸν Θ πρὸς τὸν Κ. καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Λ ποιείτω.

Καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Α πεποίηχεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Λ πεποίηχεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Λ. ὡς δὲ ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Λ. πάλιν, ἐπεὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν Λ πεποίηχεν, ἀλλὰ μὴν καὶ τὸν Ζ πολλαπλασιάσας τὸν Β πεποίηχεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Λ πρὸς τὸν Β. ἀλλ᾽ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Κ, οὕτως ὁ Λ πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Η πρὸς τὸν Θ, οὕτως ἱ Α πρὸς τὸν Λ· δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Η πρὸς τὸν Κ, [οῦτως] ὁ Α πρὸς τὸν Β. ὁ δὲ Η πρὸς τὸν Κ λόγον ἔχει *K* also measures *S* [Def. 7.20]. And *E* also measures *S* [Prop. 7.20]. Thus, *E* and *K* (both) measure *S*. Thus, the least (number) measured by (both) *E* and *K* will also measure *S* [Prop. 7.35]. And *M* is the least (number) measured by (both) *E* and *K*. Thus, *M* measures *S*, the greater (measuring) the lesser. The very thing is impossible. Thus there cannot be any numbers less than *N*, *O*, *M*, *P* (which are) continuously proportional in the ratios of *A* to *B*, and of *C* to *D*, and, further, of *E* to *F*. Thus, *N*, *O*, *M*, *P* are the least (numbers) continuously proportional in the ratios of *A* be and *B*, *C D*, *E F*. (Which is) the very thing it was required to show.

Proposition 5

Plane numbers have to one another the ratio compounded^{\dagger} out of (the ratios of) their sides.



Let A and B be plane numbers, and let the numbers C, D be the sides of A, and (the numbers) E, F (the sides) of B. I say that A has to B the ratio compounded out of (the ratios of) their sides.

For given the ratios which C has to E, and D (has) to F, let the least numbers, G, H, K, continuously proportional in the ratios C E, D F have been taken [Prop. 8.4], so that as C is to E, so G (is) to H, and as D (is) to F, so H (is) to K. And let D make L (by) multiplying E.

And since D has made A (by) multiplying C, and has made L (by) multiplying E, thus as C is to E, so A (is) to L [Prop. 7.17]. And as C (is) to E, so G (is) to H. And thus as G (is) to H, so A (is) to L. Again, since E has made L (by) multiplying D [Prop. 7.16], but, in fact, has also made B (by) multiplying F, thus as D is to F, so L(is) to B [Prop. 7.17]. But, as D (is) to F, so H (is) to K. And thus as H (is) to K, so L (is) to B. And it was also shown that as G (is) to H, so A (is) to L. Thus, via equality, as G is to K, [so] A (is) to B [Prop. 7.14]. And G has to K the ratio compounded out of (the ratios of) the sides (of A and B). Thus, A also has to B the ratio compounded out of (the ratios of) the sides (of A and B). τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ ὁ Α ἄρα πρὸς τὸν (Which is) the very thing it was required to show. Β λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

† *i.e.*, multiplied.

ኖ'.

Έὰν ῶσιν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, ὑ δὲ πρῶτος τὸν δεύτερον μὴ μετρῆ, οὐδὲ ἄλλος οὐδεἰς οὐδένα μετρήσει.



Έστωσαν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ A, B, Γ , Δ , E, ὁ δὲ A τὸν B μὴ μετρείτω· λέγω, ὅτι οὐδὲ ἄλλος οὐδεἰς οὐδένα μετρήσει.

Ότι μὲν οῦν οἱ Α, Β, Γ, Δ, Ε ἑξῆς ἀλλήλους οὐ μετροῦσιν, φανερόν οὐδὲ γὰρ ὁ Α τὸν Β μετρεῖ. λέγω δή, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει. εἰ γὰρ δυνατόν, μετρείτω ὁ Α τὸν Γ. καὶ ὅσοι εἰσὶν οἱ Α, Β, Γ, τοσοῦτοι εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ οἱ Ζ, Η, Θ. καὶ ἐπεὶ οἱ Ζ, Η, Θ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς Α, Β, Γ, καί ἐστιν ἴσον τὸ πλήθος τῶν Α, Β, Γ τῷ πλήθει τῶν Ζ, Η, Θ, δι' ἴσου ἄρα έστιν ώς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Ζ πρὸς τὸν Θ. καὶ ἐπεί έστιν ώς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Η, οὐ μετρεῖ δὲ ὁ Α τὸν Β, οὐ μετρεῖ ἄρα οὐδὲ ὁ Ζ τὸν Η· οὐκ ἄρα μονάς έστιν ὁ Ζ· ἡ γὰρ μονὰς πάντα ἀριθμὸν μετρεῖ. καί εἰσιν οἱ Ζ, Θ πρῶτοι πρὸς ἀλλήλους [οὐδὲ ὁ Ζ ἄρα τὸν Θ μετρεῖ]. καί ἐστιν ὡς ὁ Ζ πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Γ· οὐδὲ ό Α ἄρα τὸν Γ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλος ούδεις ούδένα μετρήσει· ὅπερ ἔδει δεῖξαι.

ζ'.

Έὰν ῶσιν ὑποσοιοῦν ἀριθμοὶ [ἑξῆς] ἀνάλογον, ὁ δὲ πρῶτος τὸν ἔσχατον μετρῆ, καὶ τὸν δεύτερον μετρήσει.

Proposition 6

If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other (number) will measure any other (number) either.



Let A, B, C, D, E be any multitude whatsoever of continuously proportional numbers, and let A not measure B. I say that no other (number) will measure any other (number) either.

Now, (it is) clear that A, B, C, D, E do not successively measure one another. For A does not even measure B. So I say that no other (number) will measure any other (number) either. For, if possible, let A measure C. And as many (numbers) as are A, B, C, let so many of the least numbers, F, G, H, have been taken of those (numbers) having the same ratio as A, B, C [Prop. 7.33]. And since F, G, H are in the same ratio as A, B, C, and the multitude of A, B, C is equal to the multitude of F, G, H, thus, via equality, as A is to C, so F (is) to H [Prop. 7.14]. And since as A is to B, so F (is) to G, and A does not measure B, F does not measure G either [Def. 7.20]. Thus, F is not a unit. For a unit measures all numbers. And F and H are prime to one another [Prop. 8.3] [and thus F does not measure H]. And as F is to H, so A (is) to C. And thus A does not measure C either [Def. 7.20]. So, similarly, we can show that no other (number) can measure any other (number) either. (Which is) the very thing it was required to show.

Proposition 7

If there are any multitude whatsoever of [continuously] proportional numbers, and the first measures the



Έστωσαν ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ Α τὸν Δ μετρείτω· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ.

Eἰ γὰρ οὐ μετρεῖ ὁ Α τὸν Β, οὐδὲ ἄλλος οὐδεἰς οὐδένα μετρήσει· μετρεῖ δὲ ὁ Α τὸν Δ. μετρεῖ ἄρα καὶ ὁ Α τὸν Β· ὅπερ ἔδει δεῖξαι.

η'.

Έὰν δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνόλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας [αὐτοῖς] μεταξὺ κατὰ τὸ συνὲχες ἀνάλογον ἑμπεσοῦνται



Δύο γὰρ ἀριθμῶν τῶν Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἱ Γ, Δ, καὶ πεποιήσθω ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ε πρὸς τὸν Ζ· λέγω, ὅτι ὅσοι εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Ε, Ζ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἑμπεσοῦνται.

⁶Οσοι γάρ εἰσι τῷ πλήθει οἱ A, B, Γ, Δ, τοσοῦτοι εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, Γ, Δ, B οἱ H, Θ, K, Λ· οἱ ἄρα ἄχροι αὐτῶν οἱ H, Λ πρῶτοι πρὸς ἀλλήλους εἰσίν. xαὶ ἐπεὶ οἱ A, Γ, Δ, B τοῖς H, Θ, K, Λ ἐν τῷ αὐτῷ λόγῳ εἰσίν, xαί ἐστιν ἴσον τὸ πλῆθος τῶν A, Γ, Δ, B τῷ πλήθει τῶν H, Θ, K, Λ, δι ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν B, οὕτως ὁ H πρὸς τὸν Λ. ὡς δὲ ὁ Α πρὸς τὸν B, οὕτως ὁ Ε πρὸς τὸν Ζ· xαὶ last, then (the first) will also measure the second.



Let A, B, C, D be any number whatsoever of continuously proportional numbers. And let A measure D. I say that A also measures B.

For if A does not measure B then no other (number) will measure any other (number) either [Prop. 8.6]. But A measures D. Thus, A also measures B. (Which is) the very thing it was required to show.

Proposition 8

If between two numbers there fall (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall in between (any two numbers) having the same ratio [as them] in continued proportion.



For let the numbers, C and D, fall between two numbers, A and B, in continued proportion, and let it have been contrived (that) as A (is) to B, so E (is) to F. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall in between E and F in continued proportion.

For as many as A, B, C, D are in multitude, let so many of the least numbers, G, H, K, L, having the same ratio as A, B, C, D, have been taken [Prop. 7.33]. Thus, the outermost of them, G and L, are prime to one another [Prop. 8.3]. And since A, B, C, D are in the same ratio as G, H, K, L, and the multitude of A, B, C, D is equal to the multitude of G, H, K, L, thus, via equality, as A is to B, so G (is) to L [Prop. 7.14]. And as A (is) to B, so ώς ἄρα ὁ Η πρὸς τὸν Λ, οὕτως ὁ Ε πρὸς τὸν Ζ. οἱ δὲ Η, Λ E (is πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ G ar μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. ἰσάκις ἄρα ὁ Η τὸν Ε μετρεῖ καὶ ὁ Λ τὸν Ζ. ὁσάκις δὴ ὁ Η (as t τὸν Ε μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Κ ἑκάτερον ing) τῶν Μ, Ν μετρείτω· οἱ Η, Θ, Κ, Λ ἄρα τοὺς Ε, Μ, Ν, Ζ the Ì ἰσάκις μετροῦσιν. οἱ Η, Θ, Κ, Λ τοῖς Α, Γ, Δ, Β ἐν τῷ αὐτῷ λόγῳ εἰσίν· καὶ οἱ Α, Γ, Δ, Β ἄρα τοῖς Ε, Μ, Ν, Ζ ἐν G m τῶ αὐτῷ λόγῳ εἰσίν. οἱ δὲ Α, Γ, Δ, Β ἑξῆς ἀνάλογόν εἰσιν· Ν, r

και οἱ Ε, Μ, Ν, Ζ ἄρα ἑξῆς ἀνάλογόν εἰσιν. ὅσοι ἄρα εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Ε, Ζ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

θ'.

Έὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ῶσιν, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἑμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.



Έστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν οἱ Γ, Δ, καὶ ἐκκείσθω ἡ Ε μονάς· λέγω, ὅτι ὅσοι εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν Α, Β καὶ τῆς μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν Α, Γ, Δ, Β λόγῳ ὄντες οἱ Ζ, Η, τρεῖς δὲ οἱ Θ, Κ, Λ, καὶ ἀεὶ

E (is) to F. And thus as G (is) to L, so E (is) to F. And G and L (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, G measures E the same number of times as L (measures) F. So as many times as G measures E, so many times let H, K also measure M, N, respectively. Thus, G, H, K, L measure E, M, N, F (respectively) an equal number of times. Thus, G, H, K, L are in the same ratio as E, M, N, F [Def. 7.20]. But, G, H, K, L are in the same ratio as A, C, D, B. Thus, A, C, D, B are also in the same ratio as E, M, N, F. And A, C, D, B are continuously proportional. Thus, E, M, N, F are also continuously proportional. Thus, as many numbers as have fallen in between A and B in continued proportion, so many numbers have also fallen in between E and F in continued proportion. (Which is) the very thing it was required to show.

Proposition 9

If two numbers are prime to one another and there fall in between them (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall between each of them and a unit in continued proportion.



Let A and B be two numbers (which are) prime to one another, and let the (numbers) C and D fall in between them in continued proportion. And let the unit Ebe set out. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall between each of A and B and the unit in continued proportion.

For let the least two numbers, F and G, which are in the ratio of A, C, D, B, have been taken [Prop. 7.33].

έξῆς ἑνὶ πλείους, ἕως ἂν ἴσον γένηται τὸ πλῆθος αὐτῶν τῷ πλήθει τῶν Α, Γ, Δ, Β. εἰλήφθωσαν, καὶ ἔστωσαν οἱ Μ, Ν, Ξ, Ο. φανερὸν δή, ὅτι ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, τὸν δὲ Θ πολλαπλασιάσας τὸν Μ πεποίηκεν, καὶ ὁ Η ἑαυτὸν μὲν πολλαπλασιάσας τὸν Λ πεποίηκεν, τὸν δὲ Λ πολλαπλασιάσας τὸν Ο πεποίηκεν. καὶ ἐπεὶ οἱ Μ, Ν, Ξ, Ο ἐλάγιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐγόντων τοῖς Ζ, Η, εἰσὶ δὲ καὶ οἱ Α, Γ, Δ, Β ἐλάχιστοι τῶν τὸν αὐτὸν λόγον έγόντων τοῖς Ζ, Η, καί ἐστιν ἴσον τὸ πλῆθος τῶν Μ, Ν, Ξ, Ο τῷ πλήθει τῶν Α, Γ, Δ, Β, ἕκαστος ἄρα τῶν Μ, Ν, Ξ, Ο έκάστω τῶν Α, Γ, Δ, Β ἴσος ἐστίν ἴσος ἄρα ἐστὶν ὁ μὲν Μ τῷ Α, ὁ δὲ Ο τῷ Β. καὶ ἐπεὶ ὁ Ζ ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν,
ἑ Z ẳρα τὸν Θ μετρεῖ κατὰ τὰς ἐν τῷ Z μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Ζ κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάχις ἄρα
ἡEμονὰς τὸνZἀριθμὸν μετρεĩ
 χαὶ ὁZτὸν Θ.
ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Z ἀριθμόν, οὕτως ό Ζ πρός τὸν Θ. πάλιν, ἐπεὶ ὁ Ζ τὸν Θ πολλαπλασιάσας τὸν M πεποίηκεν,
ἱ Θ ਕ́ρα τὸν M μετρεῖ κατὰ τὰς ἐν τῷ Z μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Ζ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας. ἰσάχις ἄρα ἡ Ε μονὰς τὸν Ζ ἀριθμὸν μετρεῖ καὶ ὁ Θ τὸν Μ. ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Θ πρὸς τὸν Μ. ἐδείχϑη δὲ καὶ ὡς ἡ Ε μονὰς πρὸς τὸν Z ἀριθμόν, οὕτως
ὁ Z πρὸς τὸν Θ · καὶ ὡς ἄρα ἡ Ε μονὰς πρός τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν M. ἴσος δὲ ὁ M τῷ A· ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Α. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Ε μονὰς πρὸς τὸν Η ἀριθμόν, οὕτως ὁ Η πρὸς τὸν Λ χαὶ ὁ Λ πρὸς τὸν B. ὅσοι ἄρα εἰς τοὺς A, Bμεταξύ κατά τὸ συνεγὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν Α, Β καὶ μονάδος τῆς Ε μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί. ὅπερ ἔδει δεĩξαι.

Έάν δύο ἀριθμῶν ἑκατέρου καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

ι'.

Δύο γὰρ ἀριθμῶν τῶν Α, Β καὶ μονάδος τῆς Γ μεταξύ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἴ τε Δ, Ε καὶ οἱ Ζ, Η· λέγω, ὅτι ὅσοι ἑκατέρου τῶν Α, Β καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἑμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἑμπεσοῦνται. And the (least) three (numbers), H, K, L. And so on, successively increasing by one, until the multitude of the (least numbers taken) is made equal to the multitude of A, C, D, B [Prop. 8.2]. Let them have been taken, and let them be M, N, O, P. So (it is) clear that F has made H (by) multiplying itself, and has made M (by) multiplying H. And G has made L (by) multiplying itself, and has made P (by) multiplying L [Prop. 8.2 corr.]. And since M, N, O, P are the least of those (numbers) having the same ratio as F, G, and A, C, D, B are also the least of those (numbers) having the same ratio as F, G[Prop. 8.2], and the multitude of M, N, O, P is equal to the multitude of A, C, D, B, thus M, N, O, P are equal to A, C, D, B, respectively. Thus, M is equal to A, and P to B. And since F has made H (by) multiplying itself, F thus measures H according to the units in F[Def. 7.15]. And the unit E also measures F according to the units in it. Thus, the unit E measures the number Fas many times as F (measures) H. Thus, as the unit E is to the number F, so F (is) to H [Def. 7.20]. Again, since F has made M (by) multiplying H, H thus measures Maccording to the units in F [Def. 7.15]. And the unit Ealso measures the number F according to the units in it. Thus, the unit E measures the number F as many times as H (measures) M. Thus, as the unit E is to the number F, so H (is) to M [Prop. 7.20]. And it was shown that as the unit E (is) to the number F, so F (is) to H. And thus as the unit E (is) to the number F, so F (is) to H, and H (is) to M. And M (is) equal to A. Thus, as the unit E is to the number F, so F (is) to H, and H to A. And so, for the same (reasons), as the unit E (is) to the number G, so G (is) to L, and L to B. Thus, as many (numbers) as have fallen in between A and B in continued proportion, so many numbers have also fallen between each of A and B and the unit E in continued proportion. (Which is) the very thing it was required to show.

Proposition 10

If (some) numbers fall between each of two numbers and a unit in continued proportion then, as many (numbers) as fall between each of the (two numbers) and the unit in continued proportion, so many (numbers) will also fall in between the (two numbers) themselves in continued proportion.

For let the numbers D, E and F, G fall between the numbers A and B (respectively) and the unit C in continued proportion. I say that, as many numbers as have fallen between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion.



Ο Δ γὰρ τὸν Ζ πολλαπλασιάσας τὸν Θ ποιείτω, ἑκάτερος δὲ τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ ποιείτω.

Καὶ
ἐπεί ἐστιν ὡς ἡ Γ μονὰς πρὸς τὸν Δ ἀριθμόν, οὕτως ό Δ πρός τὸν E, ἰσάχις ἄρα ή Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ
ὑ Δ τὸν Ε. ἡ δὲ Γ μονὰς τὸν
 Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Δ ἄρα ἀριθμὸς τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν. πάλιν, ἐπεί ἐστιν ὡς ἡ Γ [μονὰς] πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Ε πρὸς τὸν Α, ἰσάχις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ χαὶ ὁ Ε τὸν Α. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Ε ἄρα τὸν A μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ẳρα τὸν Eπολλαπλασιάσας τὸν Α πεποίηχεν. διὰ τὰ αὐτὰ δὴ χαὶ ὁ μέν Ζ έαυτὸν πολλαπλασιάσας τὸν Η πεποίηχεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηχεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Θ πεποίηχεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Z, οὕτως ὁ E πρὸς τὸν Θ . διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Z, οὕτως ὁ Θ πρός τὸν H. καὶ ὡς ἄρα ὁ Ε πρὸς τὸν Θ , οὕτως ὁ Θ πρὸς τὸν Η. πάλιν, ἐπεὶ
ὁ Δ ἑκάτερον τῶν Ε, Θ πολλαπλασιάσας έκάτερον τῶν Α, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Κ. ἀλ
λ
՝ ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Δ πρὸς τὸν Z· χαὶ ὡς ἄρα ὁ Δ πρὸς τὸν Z, οὕτως ὁ A πρὸς τὸν Κ. πάλιν, ἐπεὶ ἑκάτερος τῶν Δ , Ζ τὸν Θ πολλαπλασιάσας έκάτερον τῶν Κ,
 Λ πεποίηκεν, ἕστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως
ὑ Κ πρὸς τὸν Λ. ἀλλ' ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ. ἔτι ἐπεὶ ὁ Ζ ἑκάτερον τῶν Θ, Η πολλαπλασιάσας έκάτερον τῶν Λ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως
 ὁ Λ πρὸς τὸν Β. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ό Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Λ πρὸς τὸν B. ἐδείχθη δὲ χαὶ ὡς ὁ Δ πρὸς τὸν Z, οὕτως ὅ τε Α πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Λ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. οἱ Α, Κ, Λ, Β ἄρα κατὰ τὸ συνεχὲς ἑξῆς εἰσιν ἀνάλογον. ὅσοι άρα ἑκατέρου τῶν Α, Β καὶ τῆς Γ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἐμπεσοῦνται· ὅπερ ἔδει



For let D make H (by) multiplying F. And let D, F make K, L, respectively, by multiplying H.

As since as the unit C is to the number D, so D (is) to E, the unit C thus measures the number D as many times as D (measures) E [Def. 7.20]. And the unit C measures the number D according to the units in D. Thus, the number D also measures E according to the units in D. Thus, D has made E (by) multiplying itself. Again, since as the [unit] C is to the number D, so E (is) to A, the unit C thus measures the number D as many times as E(measures) A [Def. 7.20]. And the unit C measures the number D according to the units in D. Thus, E also measures A according to the units in D. Thus, D has made A (by) multiplying E. And so, for the same (reasons), Fhas made G (by) multiplying itself, and has made B (by) multiplying G. And since D has made E (by) multiplying itself, and has made H (by) multiplying F, thus as D is to F, so E (is) to H [Prop 7.17]. And so, for the same reasons, as D (is) to F, so H (is) to G [Prop. 7.18]. And thus as E (is) to H, so H (is) to G. Again, since D has made A, K (by) multiplying E, H, respectively, thus as E is to H, so A (is) to K [Prop 7.17]. But, as E (is) to H, so D (is) to F. And thus as D (is) to F, so A (is) to K. Again, since D, F have made K, L, respectively, (by) multiplying H, thus as D is to F, so K (is) to L [Prop. 7.18]. But, as D (is) to F, so A (is) to K. And thus as A (is) to K, so K (is) to L. Further, since F has made L, B (by) multiplying H, G, respectively, thus as H is to G, so L (is) to B [Prop 7.17]. And as H (is) to G, so D (is) to F. And thus as D (is) to F, so L (is) to B. And it was also shown that as D (is) to F, so A (is) to K, and K to L. And thus as A (is) to K, so K (is) to L, and L to B. Thus, A, K, L, B are successively in continued proportion. Thus, as many numbers as fall between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion. (Which is) the very thing it was required to show.

δεĩξαι.

ια΄.

Δύο τετραγώνων ἀριθμῶν εἶς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν.



Έστωσαν τετράγωνοι ἀριθμοὶ οἱ A, B, καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ, τοῦ δὲ B ὁ Δ· λέγω, ὅτι τῶν A, B εἶς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ.

Ό Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω. καὶ ἑπεὶ τετράγωνός ἐστιν ὁ Α, πλευρὰ δὲ αὐτοῦ ἐστιν ὁ Γ, ὁ Γ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν. ἐπεὶ οῦν ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Α, Ε πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Ε. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Ε, οῦτως ὁ Ε πρὸς τὸν Β. τῶν Α, Β ἄρα εἴς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τρεῖς ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Ε, Β, ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Α πρὸς τὸν Ε. ὡς δὲ ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ἡ Γ πλευρὰ πρὸς τὴν Δ. ὅπερ ἔδει δεῖξαι.

Proposition 11

There exists one number in mean proportion to two (given) square numbers.[†] And (one) square (number) has to the (other) square (number) a squared[‡] ratio with respect to (that) the side (of the former has) to the side (of the latter).



Let A and B be square numbers, and let C be the side of A, and D (the side) of B. I say that there exists one number in mean proportion to A and B, and that A has to B a squared ratio with respect to (that) C (has) to D.

For let C make E (by) multiplying D. And since A is square, and C is its side, C has thus made A (by) multiplying itself. And so, for the same (reasons), D has made B (by) multiplying itself. Therefore, since C has made A, E (by) multiplying C, D, respectively, thus as C is to D, so A (is) to E [Prop. 7.17]. And so, for the same (reasons), as C (is) to D, so E (is) to B [Prop. 7.18]. And thus as A (is) to E, so E (is) to B. Thus, one number (namely, E) is in mean proportion to A and B.

So I say that A also has to B a squared ratio with respect to (that) C (has) to D. For since A, E, B are three (continuously) proportional numbers, A thus has to B a squared ratio with respect to (that) A (has) to E[Def. 5.9]. And as A (is) to E, so C (is) to D. Thus, A has to B a squared ratio with respect to (that) side C (has) to (side) D. (Which is) the very thing it was required to show.

[†] In other words, between two given square numbers there exists a number in continued proportion. [‡] Literally, "double".

ιβ΄.

Δύο χύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ χύβος πρὸς τὸν χύβον τριπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Έστωσαν χύβοι ἀριθμοὶ οἱ Α, Β καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἦπερ ὁ Γ πρὸς τὸν Δ.

Proposition 12

There exist two numbers in mean proportion to two (given) cube numbers.[†] And (one) cube (number) has to the (other) cube (number) a cubed[‡] ratio with respect to (that) the side (of the former has) to the side (of the latter).

Let A and B be cube numbers, and let C be the side of A, and D (the side) of B. I say that there exist two numbers in mean proportion to A and B, and that A has



Ο γὰρ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιείτω, τὸν δὲ Δ πολλαπλασιάσας τὸν Ζ ποιείτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιείτω, ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑχάτερον τῶν Θ, Κ ποιείτω.

Καὶ ἐπεὶ χύβος ἐστὶν ὁ Α, πλευρὰ δὲ αὐτοῦ ὁ Γ, καὶ ὁ Γ έαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηχεν, ὁ Γ ἄρα έαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Α πεποίηχεν. διὰ τὰ αὐτὰ δὴ χαὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Ε, Ζ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. διὰ τὰ αὐτὰ δὴ χαὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Γ ἑκάτερον τῶν Ε, Ζ πολλαπλασιάσας έκάτερον τῶν Α, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Z, οὕτως ὁ A πρὸς τὸν Θ . ὡς δὲ ὁ Ε πρὸς τὸν Z, οὕτως ὁ Γ πρὸς τὸν Δ · καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ , οὕτως ὁ Α πρὸς τὸν Θ. πάλιν, ἐπεὶ ἑκάτερος τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας έκάτερον τῶν Θ , K πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας ἑκάτερον τῶν Κ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Κ πρὸς τὸν Β. ὡς δὲ ὁ Ζ πρός τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ · καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὄ τε Α πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ καὶ ό Κ πρός τὸν Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν oi Θ , K.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ . ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν οἱ A, Θ , K, B, ὁ A ẳρα πρὸς τὸν B τριπλασίονα λόγον έχει ήπερ ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν
Δ· καὶ ὁ Α [ắρα] πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ
ὑ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

to B a cubed ratio with respect to (that) C (has) to D.



For let C make E (by) multiplying itself, and let it make F (by) multiplying D. And let D make G (by) multiplying itself, and let C, D make H, K, respectively, (by) multiplying F.

And since A is cube, and C (is) its side, and C has made E (by) multiplying itself, C has thus made E (by) multiplying itself, and has made A (by) multiplying E. And so, for the same (reasons), D has made G (by) multiplying itself, and has made B (by) multiplying G. And since C has made E, F (by) multiplying C, D, respectively, thus as C is to D, so E (is) to F [Prop. 7.17]. And so, for the same (reasons), as C (is) to D, so F (is) to G [Prop. 7.18]. Again, since C has made A, H (by) multiplying E, F, respectively, thus as E is to F, so A (is) to H [Prop. 7.17]. And as E (is) to F, so C (is) to D. And thus as C (is) to D, so A (is) to H. Again, since C, D have made H, K, respectively, (by) multiplying F, thus as C is to D, so H (is) to K [Prop. 7.18]. Again, since D has made K, B (by) multiplying F, G, respectively, thus as F is to G, so K (is) to B [Prop. 7.17]. And as F (is) to G, so C (is) to D. And thus as C (is) to D, so A (is) to H, and H to K, and K to B. Thus, H and K are two (numbers) in mean proportion to A and B.

So I say that A also has to B a cubed ratio with respect to (that) C (has) to D. For since A, H, K, B are four (continuously) proportional numbers, A thus has to B a cubed ratio with respect to (that) A (has) to H [Def. 5.10]. And as A (is) to H, so C (is) to D. And [thus] A has to B a cubed ratio with respect to (that) C (has) to D. (Which is) the very thing it was required to show.

[†] In other words, between two given cube numbers there exist two numbers in continued proportion.

[‡] Literally, "triple".

ιγ'.

Έὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, καὶ πολλαπλασιάσας ἕχαστος ἑαυτὸν ποιῆ τινα, οἱ γενόμενοι έξ αὐτῶν ἀνάλογον ἔσονται· καὶ ἐὰν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσί τινας, καὶ αὐτοὶ άνάλογον ἕσονται [καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].

Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some (number by) multiplying itself, then the (numbers) created from them will (also) be (continuously) proportional. And if the original (numbers) make some (more numbers by) Έστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ Α, Β, multiplying the created (numbers) then these will also

Γ, ὡς ὁ A πρὸς τὸν B, οὕτως ὁ B πρὸς τὸν Γ, καὶ οἱ A, B, Γ ἑαυτοὺς μὲν πολλαπλασιάσαντες τοὺς Δ, E, Z ποιείτωσαν, τοὺς δὲ Δ, E, Z πολλαπλασιάσαντες τοὺς H, Θ, K ποιείτωσαν· λέγω, ὅτι οἴ τε Δ, E, Z καὶ οἱ H, Θ, K ἑξῆς ἀνάλογον εἰσιν.



Ο μὲν γὰρ Α τὸν Β πολλαπλασιάσας τὸν Λ ποιείτω, ἑκάτερος δὲ τῶν Α, Β τὸν Λ πολλαπλασιάσας ἑκάτερον τῶν Μ, Ν ποιείτω. καὶ πάλιν ὁ μὲν Β τὸν Γ πολλαπλασιάσας τὸν Ξ ποιείτω, ἐκάτερος δὲ τῶν Β, Γ τὸν Ξ πολλαπλασιάσας ἑκάτερον τῶν Ο, Π ποιείτω.

Όμοίως δὴ τοῖς ἐπάνω δεῖξομεν, ὅτι οἱ Δ, Λ, Ε καὶ οἱ Η, Μ, Ν, Θ ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ, καὶ ἔτι οἱ Ε, Ξ, Ζ καὶ οἱ Θ, Ο, Π, Κ ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Β πρὸς τὸν Γ λόγῳ. καί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· καὶ οἱ Δ, Λ, Ε ἄρα τοῖς Ε, Ξ, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσὶ καὶ ἔτι οἱ Η, Μ, Ν, Θ τοῖς Θ, Ο, Π, Κ. καί ἐστιν ἴσον τὸ μὲν τῶν Δ, Λ, Ε πλῆθος τῷ τῶν Ε, Ξ, Ζ πλήθει, τὸ δὲ τῶν Η, Μ, Ν, Θ τῷ τῶν Θ, Ο, Π, Κ· δι' ἴσου ἄρα ἐστὶν ὡς μὲν ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Ζ, ὡς δὲ ὁ Η πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Κ· ὅπερ ἕδει δεῖξαι.

ιδ'.

Έὰν τετράγωνος τετράγωνον μετρῆ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῆ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.

Έστωσαν τετράγωνοι ἀριθμοὶ οἱ A, B, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ , Δ , ὁ δὲ A τὸν B μετρείτω· λέγω, ὅτι καὶ ὁ Γ τὸν Δ μετρεῖ. be (continuously) proportional [and this always happens with the extremes].

Let A, B, C be any multitude whatsoever of continuously proportional numbers, (such that) as A (is) to B, so B (is) to C. And let A, B, C make D, E, F (by) multiplying themselves, and let them make G, H, K (by) multiplying D, E, F. I say that D, E, F and G, H, K are continuously proportional.



For let A make L (by) multiplying B. And let A, B make M, N, respectively, (by) multiplying L. And, again, let B make O (by) multiplying C. And let B, C make P, Q, respectively, (by) multiplying O.

So, similarly to the above, we can show that D, L, E and G, M, N, H are continuously proportional in the ratio of A to B, and, further, (that) E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C. And as A is to B, so B (is) to C. And thus D, L, E are in the same ratio as E, O, F, and, further, G, M, N, H (are in the same ratio) as H, P, Q, K. And the multitude of D, L, E is equal to the multitude of E, O, F, and that of G, M, N, H to that of H, P, Q, K. Thus, via equality, as D is to E, so E (is) to F, and as G (is) to H, so H (is) to K [Prop. 7.14]. (Which is) the very thing it was required to show.

Proposition 14

If a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number).

Let A and B be square numbers, and let C and D be their sides (respectively). And let A measure B. I say that C also measures D.



 $^\circ$ Ο Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω· οἱ A, E, B ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ A, E, B ἐξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ A τὸν B, μετρεῖ ἄρα καὶ ὁ A τὸν E. καί ἐστιν ὡς ὁ A πρὸς τὸν E, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Πάλιν δὴ
 ὁ Γ τὸν Δ μετρείτω· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεĩ.

Tῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι οἱ A, E, B ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγω. καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ A πρὸς τὸν E, μετρεĩ δὲ ὁ Γ τὸν Δ, μετρεĩ ἄρα καὶ ὁ A τὸν E. καί εἰσιν οἱ A, E, B ἑξῆς ἀνάλογον· μετρεĩ ἄρα καὶ ὁ A τὸν B.

Έὰν ἄρα τετράγωνος τετράγωνον μετρῆ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῆ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει· ὅπερ ἔδει δείξαι.

ιε΄.

Έὰν χύβος ἀριθμὸς χύβον ἀριθμὸν μετρῆ, xaì ἡ πλευρὰ τὴν πλευρὰν μετρήσει· xaì ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῆ, xaì ὁ χύβος τὸν χύβον μετρήσει.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον τὸν Β μετρείτω, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ μετρεĩ.



 $^\circ O$ Γ γὰρ ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιείτω,
ἑ δὲ Δ



For let *C* make *E* (by) multiplying *D*. Thus, *A*, *E*, *B* are continuously proportional in the ratio of *C* to *D* [Prop. 8.11]. And since *A*, *E*, *B* are continuously proportional, and *A* measures *B*, *A* thus also measures *E* [Prop. 8.7]. And as *A* is to *E*, so *C* (is) to *D*. Thus, *C* also measures *D* [Def. 7.20].

So, again, let C measure D. I say that A also measures B.

For similarly, with the same construction, we can show that A, E, B are continuously proportional in the ratio of C to D. And since as C is to D, so A (is) to E, and C measures D, A thus also measures E [Def. 7.20]. And A, E, B are continuously proportional. Thus, A also measures B.

Thus, if a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number). (Which is) the very thing it was required to show.

Proposition 15

If a cube number measures a(nother) cube number then the side (of the former) will also measure the side (of the latter). And if the side (of a cube number) measures the side (of another cube number) then the (former) cube (number) will also measure the (latter) cube (number).

For let the cube number A measure the cube (number) B, and let C be the side of A, and D (the side) of B. I say that C measures D.



For let C make E (by) multiplying itself. And let

ἑαυτὸν πολλαπλασιάσας τὸν Η ποιείτω, καὶ
ἔτι ὁ Γ τὸν Δ
 πολλαπλασιάσας τὸν Ζ [ποιείτω], ἑκάτερος δὲ τῶν Γ, Δ τὸν
 Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιείτω. φανερὸν
 ὅή, ὅτι οἱ Ε, Ζ, Η καὶ οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν
 ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Θ, Κ, Β ἑξῆς
 ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ τὸν
 Θ. καί ἐστιν ὡς ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ.

Άλλὰ δὴ μετρείτω
ὑ Γ τὸν Δ· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρήσει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δὴ δείξομεν, ὅτι οἱ A, Θ, K, B ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ὁ Γ τὸν Δ μετρεῖ, καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ A πρὸς τὸν Θ, καὶ ὁ A ἄρα τὸν Θ μετρεῖ· ὥστε καὶ τὸν B μετρεῖ ὁ A· ὅπερ ἔδει δείξαι.

ເຈ′.

Έὰν τετράγωνος ἀριθμὸς τετράγωνον ἀριθμὸν μὴ μετρῆ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· κἂν ἡ πλευρὰ τὴν πλευρὰν μὴ μετρῆ, οὐδὲ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.



Έστωσαν τετράγωνοι άριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, καὶ μὴ μετρείτω ὁ Α τὸν Β· λὲγω, ὅτι οὐδὲ ὁ Γ τὸν Δ μετρεῖ.

Eỉ γὰρ μετρεῖ
 ὁ Γ τὸν Δ, μετρήσει καὶ ὁ Α τὸν Β. οὐ μετρ
εῖ δὲ ὁ Α τὸν Β· οὐδὲ ἄρα ὁ Γ τὸν Δ μετρήσει.

Μὴ μετρείτω [δὴ] πάλιν
 ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Eỉ γὰρ μετρεῖ ὁ A τὸν B, μετρήσει καὶ ὁ Γ τὸν Δ. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ' ἄρα ὁ A τὸν B μετρήσει· ὅπερ ἔδει δεῖξαι.

ιζ΄.

Έὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μὴ μετρῆ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· κἂν ἡ πλευρὰ τὴν πλευρὰν μὴ μετρῆ, οὐδὲ ὁ κύβος τὸν κύβον μετρήσει. *D* make *G* (by) multiplying itself. And, further, [let] *C* [make] *F* (by) multiplying *D*, and let *C*, *D* make *H*, *K*, respectively, (by) multiplying *F*. So it is clear that *E*, *F*, *G* and *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D* [Prop. 8.12]. And since *A*, *H*, *K*, *B* are continuously proportional, and *A* measures *B*, (*A*) thus also measures *H* [Prop. 8.7]. And as *A* is to *H*, so *C* (is) to *D*. Thus, *C* also measures *D* [Def. 7.20].

And so let C measure D. I say that A will also measure B.

For similarly, with the same construction, we can show that A, H, K, B are continuously proportional in the ratio of C to D. And since C measures D, and as C is to D, so A (is) to H, A thus also measures H [Def. 7.20]. Hence, A also measures B. (Which is) the very thing it was required to show.

Proposition 16

If a square number does not measure a(nother) square number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a square number) does not measure the side (of another square number) then the (former) square (number) will not measure the (latter) square (number) either.



Let A and B be square numbers, and let C and D be their sides (respectively). And let A not measure B. I say that C does not measure D either.

For if C measures D then A will also measure B [Prop. 8.14]. And A does not measure B. Thus, C will not measure D either.

[So], again, let C not measure D. I say that A will not measure B either.

For if A measures B then C will also measure D [Prop. 8.14]. And C does not measure D. Thus, A will not measure B either. (Which is) the very thing it was required to show.

Proposition 17

If a cube number does not measure a(nother) cube number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a cube number) does not measure the side (of another cube number) then the (former) cube (number) will not measure the (latter) cube (number) either.



Κύβος γὰρ ἀριθμὸς ὁ Α κύβον ἀριθμὸν τὸν Β μὴ μετρείτω, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ οὐ μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, καὶ ὁ Α τὸν Β μετρήσει. οὐ μετρεῖ δὲ ὁ A τὸν B· οὐδ' ἄρα ὁ Γ τὸν Δ
 μετρεῖ.

Άλλὰ δὴ μὴ μετρείτω ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ ὁ Α τὸν Β μετρεῖ, χαὶ ὁ Γ τὸν Δ μετρήσει. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ' ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ έδει δεῖξαι.

ιη'.

Δύο ὁμοίων ἐπιπέδων ἀριθμῶν εἶς μέσος ἀνάλογόν ἐστιν ἀριθμός· καὶ ὁ ἐπίπεδος πρὸς τὸν ἐπίπεδον διπλασίονα λόγον έχει ήπερ ή δμόλογος πλευρὰ πρὸς τὴν δμόλογον πλευράν.



Έστωσαν δύο ὄμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μέν Α πλευραί ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ Β οἱ Ε, Ζ. καὶ ἐπεὶ ὅμοιοι ἐπίπεδοί εἰσιν οἱ ἀνάλογον ἔγοντες τὰς πλευράς, ἕστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. λέγω οὖν, ὅτι τῶν Α, Β εἶς μέσος ἀνάλογόν ἐστιν άριθμός, καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Ε η̈̀ ὁ Δ πρὸς τὸν Ζ, τουτέστιν ἦπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὅμόλογον [πλευράν].

Καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ε, ὁ Δ πρὸς τὸν Ζ. καὶ ἐπεὶ ἐπίπεδός ἐστιν ὁ Α, πλευραὶ δὲ αὐτοῦ οἱ Γ , Δ, ὁ Δ άρα τὸν Γ πολλαπλασιάσας τὸν Α πεποίηχεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε τὸν Ζ πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Δ δὴ τὸν Ε πολλαπλασιάσας τὸν Η ποιείτω.
 καὶ ἐπεὶ ὁ Δ τὸν μέν Γ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Η πεποίηχεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Η. ἀλλ' ὡς ὁ Γ πρὸς τὸν Ε, [οὕτως] ό Δ πρὸς τὸν Ζ' καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Ε τὸν μὲν Δ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ώς
ὑ Δ πρὸς τὸν Ζ, οὕτως ὁ Η πρὸς τὸν Β.



For let the cube number A not measure the cube number B. And let C be the side of A, and D (the side) of B. I say that C will not measure D.

For if C measures D then A will also measure B[Prop. 8.15]. And A does not measure B. Thus, C does not measure D either.

And so let C not measure D. I say that A will not measure B either.

For if A measures B then C will also measure D[Prop. 8.15]. And C does not measure D. Thus, A will not measure B either. (Which is) the very thing it was required to show.

Proposition 18

There exists one number in mean proportion to two similar plane numbers. And (one) plane (number) has to the (other) plane (number) a squared^{\dagger} ratio with respect to (that) a corresponding side (of the former has) to a corresponding side (of the latter).



Let A and B be two similar plane numbers. And let the numbers C, D be the sides of A, and E, F (the sides) of B. And since similar numbers are those having proportional sides [Def. 7.21], thus as C is to D, so E (is) to F. Therefore, I say that there exists one number in mean proportion to A and B, and that A has to B a squared ratio with respect to that C (has) to E, or D to F—that is to say, with respect to (that) a corresponding side (has) to a corresponding [side].

For since as C is to D, so E (is) to F, thus, alternately, as C is to E, so D (is) to F [Prop. 7.13]. And since A is plane, and C, D its sides, D has thus made A (by) multiplying C. And so, for the same (reasons), E has made B (by) multiplying F. So let D make G (by) multiplying E. And since D has made A (by) multiplying C, and has made G (by) multiplying E, thus as C is to E, so A (is) to G [Prop. 7.17]. But as C (is) to E, [so] D (is) to F. And thus as D (is) to F, so A (is) to G. Again, since E has made G (by) multiplying D, and has made B (by) multiplying F, thus as D is to F, so G (is) to B [Prop. 7.17]. And it was also shown that as D (is) to F, so A (is) to G. έδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν And thus as A (is) to G, so G (is) to B. Thus, A, G, B are H· καὶ ὡς ἄρα ὁ A πρὸς τὸν H, οὕτως ὁ H πρὸς τὸν B. οἱ A, H, B ἄρα ἑξῆς ἀνάλογόν εἰσιν. τῶν A, B ἄρα εἶς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ. ἐπεὶ γὰρ οἱ Α, Η, Β ἑξῆς ἀνάλογόν εἰσιν, ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ πρὸς τὸν Η. καί ἐστιν ὡς ὁ Α πρὸς τὸν Η, οὕτως ὅ τε Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ. καὶ ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ε ἢ

† Literally, "double".

ıθ'.

Δύο ὁμοίων στερεῶν ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· καὶ ὁ στερεὸς πρὸς τὸν ὅμοιον στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.



Έστωσαν δύο ὅμοιοι στερεοὶ οἱ A, B, καὶ τοῦ μὲν A πλευραὶ ἔστωσαν οἱ Γ, Δ, E, τοῦ δὲ B οἱ Z, H, Θ. καὶ ἐπεὶ ὅμοιοι στερεοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς μὲν ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν H, ὡς δὲ ὁ Δ πρὸς τὸν E, οὕτως ὁ Η πρὸς τὸν Θ. λέγω, ὅτι τῶν A, B δύο μέσοι ἀνάλογόν ἐμπίπτουσιν ἀριθμοί, καὶ ὁ A πρὸς τὸν B τριπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ.

Ό Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Κ ποιείτω, ὁ δὲ Ζ τὸν Η πολλαπλασιάσας τὸν Λ ποιείτω. καὶ ἐπεὶ οἱ Γ, Δ τοὶς Ζ, Η ἐν τῷ αὐτῷ λόγῳ εἰσίν, καὶ ἐκ μὲν τῶν Γ, Δ ἐστιν ὁ Κ, ἐκ δὲ τῶν Ζ, Η ὁ Λ, οἱ Κ, Λ [ἄρα] ὄμοιοι ἐπίπεδοί εἰσιν ἀριθμοί· τῶν Κ, Λ ἄρα εἶς μέσος ἀνάλογόν ἐστιν ἀριθμός. ἔστω ὁ Μ. ὁ Μ ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ζ, ὡς ἐν τῷ πρὸ τούτου θεωρήματι ἐδείχθη. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Κ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Μ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Μ. ἀλλ' ὡς ὁ Κ πρὸς τὸν Μ, ὁ Μ πρὸς τὸν Λ. οἱ Κ, Μ, Λ ἄρα ἑξῆς εἰσιν ἀνάλογον ἐν continuously proportional. Thus, there exists one number (namely, G) in mean proportion to A and B.

So I say that A also has to B a squared ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) C(has) to E, or D to F. For since A, G, B are continuously proportional, A has to B a squared ratio with respect to (that A has) to G [Prop. 5.9]. And as A is to G, so C (is) to E, and D to F. And thus A has to B a squared ratio with respect to (that) C (has) to E, or D to F. (Which is) the very thing it was required to show.

Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed[†] ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let *A* and *B* be two similar solid numbers, and let *C*, *D*, *E* be the sides of *A*, and *F*, *G*, *H* (the sides) of *B*. And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as *C* is to *D*, so *F* (is) to *G*, and as *D* (is) to *E*, so *G* (is) to *H*. I say that two numbers fall (between) *A* and *B* in mean proportion, and (that) *A* has to *B* a cubed ratio with respect to (that) *C* (has) to *F*, and *D* to *G*, and, further, *E* to *H*.

For let *C* make *K* (by) multiplying *D*, and let *F* make *L* (by) multiplying *G*. And since *C*, *D* are in the same ratio as *F*, *G*, and *K* is the (number created) from (multiplying) *C*, *D*, and *L* the (number created) from (multiplying) *F*, *G*, [thus] *K* and *L* are similar plane numbers [Def. 7.21]. Thus, there exits one number in mean proportion to *K* and *L* [Prop. 8.18]. Let it be *M*. Thus, *M* is the (number created) from (multiplying) *D*, *F*, as shown in the theorem before this (one). And since *D* has made *K* (by) multiplying *C*, and has made *M* (by) multiplying *F*, thus as *C* is to *F*, so *K* (is) to *M* [Prop. 7.17]. But, as

τῷ τοῦ Γ πρὸς τὸν Ζ λόγῷ. καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ Ζ πρὸς τὸν Η, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως
ἱ Δ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς
ἱ Δ πρὸς τὸν Η, οὕτως ὁ Ε πρὸς τὸν Θ. οἱ Κ, Μ, Λ ẳρα ἑξῆς είσιν άνάλογον ἕν τε τῷ τοῦ Γ πρὸς τὸν Ζ λόγω καὶ τῷ τοῦ Δ πρὸς τὸν Η καὶ
ἔτι τῷ τοῦ Ε πρὸς τὸν Θ. ἑκατερος δή τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑχάτερον τῶν Ν, Ξ ποιείτω. και ἐπει στερεός ἐστιν ὁ Α, πλευραι δὲ αὐτοῦ εἰσιν οί Γ, Δ, Ε,
ό Ε ἄρα τὸν ἐκ τῶν Γ, Δ πολλαπλασιάσας τὸν Α πεποίηχεν.
 ὁ δὲ ἐχ τῶν Γ, Δ ἐστιν ὁ Κ· ὁ Ε ἄρα τὸν Κ πολλαπλασιάσας τὸν Α πεποίηχεν. διὰ τὰ αὐτὰ δὴ χαὶ ὁ Θ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Ε τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Μ πολλαπλασιάσας τὸν Ν πεποίηχεν, ἔστιν ἄρα ὡς ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Α πρὸς τὸν Ν. ὡς δὲ ὁ Κ πρὸς τὸν Μ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ
ἔτι ὁ Ε πρὸς τὸν Θ · καὶ ὡς ẳρα ὁ Γ πρὸς τὸν Z καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ , οὕτως ὁ Α πρὸς τὸν Ν. πάλιν, ἐπεὶ έκάτερος τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ πεποίηχεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Ν πρὸς τὸν Ξ. ἀλλ
 ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὅ τε Γ πρὸς τὸν Z καὶ
ὁ Δ πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Z καὶ ὁ
 Δ πρὸς τὸν Η
 καὶ
ἑ Ε πρὸς τὸν Θ, οὕτως ὅ τε Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. πάλιν, ἐπεὶ ὁ Θ τὸν Μ πολλαπλασιάσας τὸν Ξ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηχεν,
 ἔστιν ἄρα ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Ξ πρὸς τὸν B. ἀλλ' ὡς ὁ M πρὸς τὸν Λ, οὕτως ὅ τε Γ πρὸς τὸν Z καὶ
ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Γ πρός τὸν Z καὶ ὁ Δ πρὸς τὸν H καὶ ὁ E πρὸς τὸν Θ, οὕτως ού μόνον ὁ Ξ πρὸς τὸν Β, ἀλλὰ καὶ ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. οἱ Α, Ν, Ξ, Β ἄρα ἑξῆς εἰσιν ἀνάλογον ἐν τοῖς εἰρημένοις τῶν πλευρῶν λόγοις.

Λέγω, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ ἢ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογόν εἰσιν οἱ Α, Ν, Ξ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Α πρὸς τὸν Ν. ἀλλ' ὡς ὁ Α πρὸς τὸν Ν, οὕτως ἐδείχθη ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ἡ ομόλογος πλευρὰ πρὸς τὴν ὑμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ ἀριθμὸς πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ἡ ομόλογος πλευρὰ πρὸς τὴν ὑμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· ὅπερ ἔδει δείξαι.

[†] Literally, "triple".

κ.

Έὰν δύο ἀριθμῶν εἶς μέσος ἀνάλογον ἐμπίπτῆ ἀριθμός, ὅμοιοι ἐπίπεδοι ἔσονται οἱ ἀριθμοί.

K (is) to M, (so) M (is) to L. Thus, K, M, L are continuously proportional in the ratio of C to F. And since as C is to D, so F (is) to G, thus, alternately, as C is to F, so D (is) to G [Prop. 7.13]. And so, for the same (reasons), as D (is) to G, so E (is) to H. Thus, K, M, L are continuously proportional in the ratio of C to F, and of D to G, and, further, of E to H. So let E, H make N, O, respectively, (by) multiplying M. And since A is solid, and C, D, E are its sides, E has thus made A (by) multiplying the (number created) from (multiplying) C, D. And Kis the (number created) from (multiplying) C, D. Thus, E has made A (by) multiplying K. And so, for the same (reasons), H has made B (by) multiplying L. And since E has made A (by) multiplying K, but has, in fact, also made N (by) multiplying M, thus as K is to M, so A (is) to N [Prop. 7.17]. And as K (is) to M, so C (is) to F, and D to G, and, further, E to H. And thus as C (is) to F, and D to G, and E to H, so A (is) to N. Again, since E, H have made N, O, respectively, (by) multiplying M, thus as E is to H, so N (is) to O [Prop. 7.18]. But, as E (is) to H, so C (is) to F, and D to G. And thus as C(is) to F, and D to G, and E to H, so (is) A to N, and N to O. Again, since H has made O (by) multiplying M, but has, in fact, also made B (by) multiplying L, thus as M (is) to L, so O (is) to B [Prop. 7.17]. But, as M (is) to L, so C (is) to F, and D to G, and E to H. And thus as C (is) to F, and D to G, and E to H, so not only (is) O to B, but also A to N, and N to O. Thus, A, N, O, B are continuously proportional in the aforementioned ratios of the sides.

So I say that A also has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C(has) to F, or D to G, and, further, E to H. For since A, N, O, B are four continuously proportional numbers, Athus has to B a cubed ratio with respect to (that) A (has) to N [Def. 5.10]. But, as A (is) to N, so it was shown (is) C to F, and D to G, and, further, E to H. And thus A has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, and D to G, and, further, E to H. (Which is) the very thing it was required to show.

Proposition 20

If one number falls between two numbers in mean proportion then the numbers will be similar plane (num-

Δύο γὰρ ἀριθμῶν τῶν Α, Β εἶς μέσος ἀνάλογον ἐμπιπτέτω ἀριθμὸς ὁ Γ· λέγω, ὅτι οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.



Εἰλήφθωσαν [γὰρ] ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐγόντων τοῖς Α, Γ οἱ Δ, Ε· ἰσάχις ẳρα
 ὁ Δ τὸν Α μετρεῖ καὶ
ἑ Ε τὸν Γ. ἑσάκις δὴ ἑ Δ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ·
 ὁ Ζ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηχεν. ὥστε ὁ Α ἐπίπεδός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ Δ, Ζ. πάλιν, ἐπεὶ οἱ Δ, Ε ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, Β, ἰσάχις ἄρα
 ὁ Δ τὸν Γ μετρεῖ καὶ ὁ Ε τὸν Β. ὁσάχις δὴ ὁ Ε τὸν Β μετρεῖ, τοσαῦται μονάδες ἕστωσαν ἐν τῷ Η. ὁ Ε ἄρα τὸν Β μετρεῖ κατὰ τὰς έν τῷ Η μονάδας. ὁ Η ẳρα τὸν Ε πολλαπλασιάσας τὸν Β πεποίηχεν. ὁ Β ἄρα ἐπίπεδος ἐστι, πλευραὶ δὲ αὐτοῦ εἰσιν οί Ε, Η. οί Α, Β ἄρα ἐπίπεδοί εἰσιν ἀριθμοί. λέγω δή, ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ ὁ Ζ τὸν μὲν Δ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Γ πρὸς τὸν Β. πάλιν, ἐπεὶ ὁ Ε ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας το
ὺς Γ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρός τὸν Η, οὕτως ὁ Γ πρὸς τὸν Β. ὡς δὲ ὁ Γ πρὸς τὸν Β, οὕτως
ὑ Δ πρὸς τὸν Ε· καὶ ὡς ἄρα ὑ Δ πρὸς τὸν Ε, οὕτως ό Z πρός τὸν Η· καὶ ἐναλλὰξ ὡς
ὑ Δ πρὸς τὸν Z, οὕτως ὁ Ε πρός τὸν Η. οἱ Α, Β ἄρα ὄμοιοι ἐπίπεδοι ἀριθμοί εἰσιν· αἱ γὰρ πλευραὶ αὐτῶν ἀνάλογόν εἰσιν. ὅπερ ἔδει δεῖξαι.

bers).

For let one number C fall between the two numbers A and B in mean proportion. I say that A and B are similar plane numbers.



[For] let the least numbers, D and E, having the same ratio as A and C have been taken [Prop. 7.33]. Thus, D measures A as many times as E (measures) C[Prop. 7.20]. So as many times as D measures A, so many units let there be in F. Thus, F has made A (by) multiplying D [Def. 7.15]. Hence, A is plane, and D, F (are) its sides. Again, since D and E are the least of those (numbers) having the same ratio as C and B, D thus measures C as many times as E (measures) B[Prop. 7.20]. So as many times as E measures B, so many units let there be in G. Thus, E measures B according to the units in G. Thus, G has made B (by) multiplying E [Def. 7.15]. Thus, B is plane, and E, G are its sides. Thus, A and B are (both) plane numbers. So I say that (they are) also similar. For since F has made A(by) multiplying D, and has made C (by) multiplying E, thus as D is to E, so A (is) to C—that is to say, C to B [Prop. 7.17].[†] Again, since E has made C, B (by) multiplying F, G, respectively, thus as F is to G, so C (is) to B [Prop. 7.17]. And as C (is) to B, so D (is) to E. And thus as D (is) to E, so F (is) to G. And, alternately, as D (is) to F, so E (is) to G [Prop. 7.13]. Thus, A and B are similar plane numbers. For their sides are proportional [Def. 7.21]. (Which is) the very thing it was required to show.

[†] This part of the proof is defective, since it is not demonstrated that $F \times E = C$. Furthermore, it is not necessary to show that D : E :: A : C, because this is true by hypothesis.

κα'.

Έὰν δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅμοιοι στερεοί εἰσιν οἱ ἀριθμοί.

 Δ ύο γὰρ ἀριθμῶν τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἱ Γ, Δ· λέγω, ὅτι οἱ Α, Β ὅμοιοι στερεοί εἰσιν.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ τρεῖς οἱ Ε, Ζ, Η· οἱ ἄρα ἄχροι αὐτῶν οἱ Ε, Η πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ τῶν Ε, Η εῖς μέσος ἀνάλογον ἐμπέπτωχεν ἀριθμὸς ὁ Ζ, οἱ Ε, Η ἄρα ἀριθμοὶ ὅμοιοι ἐπίπεδοί εἰσιν. ἔστωσαν οὕν τοῦ μὲν

Proposition 21

If two numbers fall between two numbers in mean proportion then the (latter) are similar solid (numbers).

For let the two numbers C and D fall between the two numbers A and B in mean proportion. I say that A and B are similar solid (numbers).

For let the three least numbers E, F, G having the same ratio as A, C, D have been taken [Prop. 8.2]. Thus, the outermost of them, E and G, are prime to one another [Prop. 8.3]. And since one number, F, has fallen (between) E and G in mean proportion, E and G are

Ε πλευραί οἱ Θ , K, τοῦ δὲ H οἱ Λ, M. φανερὸν ἄρα ἐστὶν έκ τοῦ πρὸ τούτου, ὅτι οἱ Ε, Ζ, Η ἑξῆς εἰσιν ἀνάλογον ἕν τε τῷ τοῦ Θ πρὸς τὸν Λ λόγ ω καὶ τῷ τοῦ K πρὸς τὸν M. καὶ ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον έχόντων τοῖς Α, Γ, Δ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Ε, Ζ, Η τῷ πλήθει τῶν Α, Γ, Δ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν Η, ο
ὕτως ὁ Α πρὸς τὸν Δ. οἱ δὲ Ε, Η πρῶτοι, οἱ δὲ πρῶτοι χαὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάχις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. ἰσάκις ἄρα ό Ε τὸν Α μετρεῖ καὶ ὁ Η τὸν Δ. ὁσάκις δὴ ὁ Ε τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ν. ὁ Ν ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ Ε ἐστιν ὁ ἐκ τῶν Θ, K· ὁ N ἄρα τὸν ἐκ τῶν Θ, Κ πολλαπλασιάσας τὸν Α πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσιν οί Θ, Κ, Ν. πάλιν, ἐπεὶ οἱ Ε, Ζ, Η ἐλάγιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐγόντων τοῖς Γ, Δ, Β, ἰσάχις ἄρα
ἑ Ε τὸν Γ μετρεῖ καὶ ὁ Η τὸν Β. ὁσάκις δὴ ὁ Ε τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ξ. ὁ Η ἄρα τὸν Β μετρεῖ κατὰ τὰς έν τῷ Ξ μονάδας·
ό Ξ ἄρα τὸν Η πολλαπλασιάσας τὸν Β πεποίη
κεν.
 ὁ δὲ Η ἐστιν ὁ ἐκ τῶν Λ, Μ· ὁ Ξ ἄρα τὸν ἐκ τῶν Λ, Μ πολλαπλασιάσας τὸν Β πεποίηκεν. στερεὸς ẳρα έστιν ό B, πλευραί δὲ αὐτοῦ εἰσιν οἱ A, M, Ξ· οἱ A, B ẳρα στερεοί εἰσιν.



Λέγω [δή], ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ οἱ Ν, Ξ τὸν Ε πολλαπλασιάσαντες τοὺς Α, Γ πεποιήκασιν, ἔστιν ἄρα ὡς ὁ Ν πρὸς τὸν Ξ, ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Ε πρὸς τὸν Ζ. ἀλλỉ ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Θ πρὸς τὸν Λ καὶ ὁ Κ πρὸς τὸν Μ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Λ, οὕτως ὁ Κ πρὸς τὸν Μ καὶ ὁ Ν πρὸς τὸν Ξ. καί εἰσιν οἱ μὲν Θ, Κ, Ν πλευραὶ τοῦ Α,

thus similar plane numbers [Prop. 8.20]. Therefore, let H, K be the sides of E, and L, M (the sides) of G. Thus, it is clear from the (proposition) before this (one) that E, F, G are continuously proportional in the ratio of H to L, and of K to M. And since E, F, G are the least (numbers) having the same ratio as A, C, D, and the multitude of E, F, G is equal to the multitude of A, C, D, thus, via equality, as E is to G, so A (is) to D [Prop. 7.14]. And E and G (are) prime (to one another), and prime (numbers) are also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A the same number of times as G (measures) D. So as many times as E measures A, so many units let there be in N. Thus, N has made A (by) multiplying E [Def. 7.15]. And E is the (number created) from (multiplying) H and K. Thus, Nhas made A (by) multiplying the (number created) from (multiplying) H and K. Thus, A is solid, and its sides are H, K, N. Again, since E, F, G are the least (numbers) having the same ratio as C, D, B, thus E measures C the same number of times as G (measures) B [Prop. 7.20]. So as many times as E measures C, so many units let there be in O. Thus, G measures B according to the units in O. Thus, O has made B (by) multiplying G. And G is the (number created) from (multiplying) L and M. Thus, O has made B (by) multiplying the (number created) from (multiplying) L and M. Thus, B is solid, and its sides are L, M, O. Thus, A and B are (both) solid.



[So] I say that (they are) also similar. For since N, O have made A, C (by) multiplying E, thus as N is to O, so A (is) to C—that is to say, E to F [Prop. 7.18]. But, as E (is) to F, so H (is) to L, and K to M. And thus as H (is) to L, so K (is) to M, and N to O. And H, K, N are the sides of A, and L, M, O the sides of B. Thus, A and

στερεοί εἰσιν ὅπερ ἔδει δεῖξαι.

^{\dagger} The Greek text has "O, L, M", which is obviously a mistake.

xβ'.

Έαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος τετράγωνος ή, και ό τρίτος τετράγωνος έσται.



Έστωσαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, ὁ δὲ πρῶτος ὁ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ τρίτος ὁ Γ τετράγωνός ἐστιν.

Ἐπεὶ γὰρ τῶν Α, Γ εἶς μέσος ἀνάλογόν ἐστιν ἀριθμὸς ό B, οἱ A, Γ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τετράγωνος δὲ ὁ Α· τετράγωνος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

χγ'.

Ἐἀν τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογον ὥσιν, ὁ δὲ πρῶτος κύβος ἢ, καὶ ὁ τέταρτος κύβος ἔσται.



Έστωσαν τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ , ὁ δὲ A κύβος ἔστω· λέγω, ὅτι καὶ ὁ Δ κύβος ἐστίν.

Έπεὶ γὰρ τῶν Α, Δ δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοὶ οί Β, Γ, οί Α, Δ ἄρα ὄμοιοί εἰσι στερεοὶ ἀριθμοί. κύβος δὲ ό Α· κύβος ἄρα καὶ ὁ Δ· ὅπερ ἔδει δεῖξαι.

χδ'.

Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ πρῶτος τετράγωνος ἤ, καὶ ὁ δεύτερος τετράγωνος ἔσται.



οί δὲ Ξ, Λ, Μ πλευραὶ τοῦ B. oí A, B ἄρα ἀριθμοὶ ὅμοιοι B are similar solid numbers [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.



Let A, B, C be three continuously proportional numbers, and let the first A be square. I say that the third Cis also square.

For since one number, B, is in mean proportion to A and C, A and C are thus similar plane (numbers) [Prop. 8.20]. And A is square. Thus, C is also square [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.



Let A, B, C, D be four continuously proportional numbers, and let A be cube. I say that D is also cube.

For since two numbers, B and C, are in mean proportion to A and D, A and D are thus similar solid numbers [Prop. 8.21]. And A (is) cube. Thus, D (is) also cube [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 24

If two numbers have to one another the ratio which a square number (has) to a(nother) square number, and the first is square, then the second will also be square.



For let two numbers, A and B, have to one another

έχέτωσαν, ὃν τετράγωνος ἀριθμὸς ὁ Γ πρὸς τετράγωνον ἀριθμὸν τὸν Δ, ὁ δὲ Α τετράγωνος ἔστω· λ
έγω, ὅτι καὶ ὁ Β τετράγωνός ἐστιν.

Έπεὶ γὰρ οἱ Γ, Δ τετράγωνοί εἰσιν, οἱ Γ, Δ ἄρα ὄμοιοι ἐπίπεδοί εἰσιν. τῶν Γ , Δ ἄρα εἶς μέσος ἀνάλογον ἐμπίπτει άριθμός. καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἶς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί έστιν ὁ Α τετράγωνος· καὶ ὁ Β ἄρα τετράγωνός ἐστιν· ὅπερ έδει δεῖξαι.

χε'.

Έαν δύο αριθμοί πρός αλλήλους λόγον έχωσιν, ὃν χύβος ἀριθμὸς πρὸς χύβον ἀριθμόν, ὁ δὲ πρῶτος χύβος ἤ, καὶ ὁ δεύτερος κύβος ἔσται.



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον έχέτωσαν, ὃν χύβος ἀριθμὸς ὁ Γ πρὸς χύβον ἀριθμὸν τὸν Δ, κύβος δὲ ἔστω ὁ Α· λέγω [δή], ὅτι καὶ ὁ Β κύβος ἐστίν.

Έπει γάρ οι Γ, Δ κύβοι είσιν, οι Γ, Δ όμοιοι στερεοί εἰσιν· τῶν Γ, Δ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν άριθμοί. ὄσοι δὲ εἰς τοὺς Γ, Δ μεταξὺ κατὰ τὸ συνεχὲς άνάλογον έμπίπτουσιν, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς· ὥστε καὶ τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Ε, Ζ. ἐπεὶ ούν τέσσαρες ἀριθμοὶ οἱ Α, Ε, Ζ, Β ἑξῆς ἀνάλογόν εἰσιν, καί ἐστι κύβος ὁ Α, κύβος ἄρα καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

xς'.

Οἱ ὅμοιοι ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.



Έστωσαν ὄμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β΄ λέγω, ὅτι ό A πρός τὸν B λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς has to B the ratio which (some) square number (has) to

the ratio which the square number C (has) to the square number D. And let A be square. I say that B is also square.

For since C and D are square, C and D are thus similar plane (numbers). Thus, one number falls (between) C and D in mean proportion [Prop. 8.18]. And as C is to D, (so) A (is) to B. Thus, one number also falls (between) A and B in mean proportion [Prop. 8.8]. And A is square. Thus, B is also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 25

If two numbers have to one another the ratio which a cube number (has) to a (nother) cube number, and the first is cube, then the second will also be cube.



For let two numbers, A and B, have to one another the ratio which the cube number C (has) to the cube number D. And let A be cube. [So] I say that B is also cube.

For since C and D are cube (numbers), C and D are (thus) similar solid (numbers). Thus, two numbers fall (between) C and D in mean proportion [Prop. 8.19]. And as many (numbers) as fall in between C and D in continued proportion, so many also (fall) in (between) those (numbers) having the same ratio as them (in continued proportion) [Prop. 8.8]. And hence two numbers fall (between) A and B in mean proportion. Let E and F (so) fall. Therefore, since the four numbers A, E, F, Bare continuously proportional, and A is cube, B (is) thus also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 26

Similar plane numbers have to one another the ratio which (some) square number (has) to a(nother) square number.



Let A and B be similar plane numbers. I say that A

Έπει γὰρ οἱ Α, Β ὄμοιοι ἐπίπεδοί εἰσιν, τῶν Α, Β ἄρα εἴς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἑμπιπτέτω καὶ ἔστω ὁ Γ, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Β οἱ Δ, Ε, Ζ· οἱ ἄρα ἄκροι αὐτῶν οἱ Δ, Ζ τετράγωνοί εἰσιν. καὶ ἐπεί ἐστιν ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Β, καί εἰσιν οἱ Δ, Ζ τετράγωνοι, ὁ Α ἄρα πρὸς τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

χζ΄.

Οἱ ὅμοιοι στερεοὶ ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.



Έστωσαν δμοιοι στερεοὶ ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.

Έπει γὰρ οἱ A, B ὅμοιοι στερεοί εἰσιν, τῶν A, B ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Γ, Δ, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, Γ, Δ, B ἴσοι αὐτοῖς τὸ πλῆθος οἱ Ε, Ζ, Η, Θ· οἱ ἄρα ἄκροι αὐτῶν οἱ Ε, Θ κύβοι εἰσίν. καί ἐστιν ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ A πρὸς τὸν B· καὶ ὁ A ἄρα πρὸς τὸν B λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν· ὅπερ ἔδει δεῖξαι. For since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. Let it (so) fall, and let it be C. And let the least numbers, D, E, F, having the same ratio as A, C, B have been taken [Prop. 8.2]. The outermost of them, D and F, are thus square [Prop. 8.2 corr.]. And since as D is to F, so A (is) to B, and D and F are square, A thus has to B the ratio which (some) square number (has) to a(nother) square number. (Which is) the very thing it was required to show.

Proposition 27

Similar solid numbers have to one another the ratio which (some) cube number (has) to a(nother) cube number.



Let A and B be similar solid numbers. I say that A has to B the ratio which (some) cube number (has) to a(nother) cube number.

For since A and B are similar solid (numbers), two numbers thus fall (between) A and B in mean proportion [Prop. 8.19]. Let C and D have (so) fallen. And let the least numbers, E, F, G, H, having the same ratio as A, C, D, B, (and) equal in multitude to them, have been taken [Prop. 8.2]. Thus, the outermost of them, E and H, are cube [Prop. 8.2 corr.]. And as E is to H, so A (is) to B. And thus A has to B the ratio which (some) cube number (has) to a(nother) cube number. (Which is) the very thing it was required to show.
ELEMENTS BOOK 9

Applications of Number Theory †

 $^{^\}dagger The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.$

α΄.

Έὰν δύο ὄμοιοι ἐπίπεδοι ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος τετράγωνος ἔσται.



Έστωσαν δύο ὄμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ τετράγωνός ἐστιν.

Ο γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. ὁ Δ ӑρα τετράγωνός ἑστιν. ἑπεὶ οῦν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Γ. καὶ ἑπεὶ οἱ Α, Β ὄμοιοι ἐπίπεδοί εἰσιν ἀριθμοί, τῶν Α, Β ἄρα εἶς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐὰν δὲ δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς ἐμπίπτουσι, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας· ὥστε καὶ τῶν Δ, Γ εῖς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστι τετράγωνος ὁ Δ· τετράγωνος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

β'.

Έὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσι τετράγωνον, ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.



Έστωσαν δύο ἀριθμοὶ οἱ A, B, καὶ ὁ A τὸν B πολλαπλασιάσας τετράγωνον τὸν Γ ποιείτω· λέγω, ὅτι οἱ A, B ὄμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.

Ό γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα τετράγωνός ἑστιν. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ ὁ Δ τετράγωνός ἐστιν, ἀλλὰ καὶ ὁ Γ, οἱ Δ, Γ ἅρα ὅμοιοι ἑπίπεδοί εἰσιν. τῶν Δ, Γ ἄρα εἴς μέσος ἀνάλογον

Proposition 1

If two similar plane numbers make some (number by) multiplying one another then the created (number) will be square.



Let A and B be two similar plane numbers, and let A make C (by) multiplying B. I say that C is square.

For let A make D (by) multiplying itself. D is thus square. Therefore, since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as Ais to B, so D (is) to C [Prop. 7.17]. And since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. And if (some) numbers fall between two numbers in continued proportion then, as many (numbers) as fall in (between) them (in continued proportion), so many also (fall) in (between numbers) having the same ratio (as them in continued proportion) [Prop. 8.8]. And hence one number falls (between) D and C in mean proportion. And Dis square. Thus, C (is) also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 2

If two numbers make a square (number by) multiplying one another then they are similar plane numbers.



Let A and B be two numbers, and let A make the square (number) C (by) multiplying B. I say that A and B are similar plane numbers.

For let A make D (by) multiplying itself. Thus, D is square. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D (is) to C [Prop. 7.17]. And since D is square, and C (is) also, D and C are thus similar plane numbers. Thus, one (number) falls (between) D and C in mean proporἐμπίπτει. Χαί ἐστιν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἶς μέσος ἀνάλογον ἐμπίπτει. ἐἀν δὲ δύο ἀριθμῶν εἶς μέσος ἀνάλογον ἐμπίπτη, ὅμοιοι ἐπίπεδοί εἰσιν [oi] ἀριθμοί· οἱ ἄρα Α, Β ὅμοιοί εἰσιν ἐπίπεδοι· ὅπερ ἕδει δεῖξαι. tion [Prop. 8.18]. And as D is to C, so A (is) to B. Thus, one (number) also falls (between) A and B in mean proportion [Prop. 8.8]. And if one (number) falls (between) two numbers in mean proportion then [the] numbers are similar plane (numbers) [Prop. 8.20]. Thus, A and B are similar plane (numbers). (Which is) the very thing it was required to show.

γ'.

Έαν κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῆ τινα, ὅ γενόμενος κύβος ἔσται.



Κύβος γὰρ ἀριθμὸς ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β ποιείτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Εἰλήφθω γὰρ τοῦ Α πλευρὰ ὁ Γ, καὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. φανερὸν δή ἐστιν, ὅτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ πεποίηκεν,
ὑ Γ ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν αὑτῷ μονάδας. ἀλλὰ μὴν καὶ ἡ μονὰς τὸν Γ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν,
ἱ Δ ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Γ κατὰ τὰς ἐν αὐτῷ μονάδας.
 ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Δ πρὸς τὸν A. ἀλλ' ὡς ἡ μονὰς πρὸς τὸν Γ, ὅ Γ πρὸς τὸν Δ · καὶ ὡς ἄρα ή μονὰς πρὸς τὸν Γ, οὕτως ὁ Γ πρὸς τὸν Δ καὶ ὁ Δ πρὸς τὸν Α. τῆς ἄρα μονάδος καὶ τοῦ Α ἀριθμοῦ δύο μέσοι άνάλογον κατά τὸ συνεχὲς ἐμπεπτώκασιν ἀριθμοὶ οἱ Γ, Δ. πάλιν, ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ό Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ή μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β. τῆς δὲ μονάδος καὶ τοῦ Α δύο μέσοι ἀνάλογον ἐμπεπτώχασιν ἀριθμοί· καὶ τῶν A, B ἄρα δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. ἐὰν δὲ δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν, ὁ δὲ πρῶτος κύβος ἤ, καὶ ὁ δεύτερος κύβος ἔσται. καί ἐστιν ὁ Α κύβος. καὶ ὁ Β ἄρα κύβος ἐστίν· ὅπερ ἔδει δεῖξαι.

δ'.

Έὰν κύβος ἀριθμὸς κύβον ἀριθμὸν πολλαπλασιάσας ποιῆ τινα, ὁ γενόμενος κύβος ἔσται.

Proposition 3

If a cube number makes some (number by) multiplying itself then the created (number) will be cube.



For let the cube number A make B (by) multiplying itself. I say that B is cube.

For let the side C of A have been taken. And let Cmake D by multiplying itself. So it is clear that C has made A (by) multiplying D. And since C has made D(by) multiplying itself, C thus measures D according to the units in it [Def. 7.15]. But, in fact, a unit also measures C according to the units in it [Def. 7.20]. Thus, as a unit is to C, so C (is) to D. Again, since C has made A (by) multiplying D, D thus measures A according to the units in C. And a unit also measures C according to the units in it. Thus, as a unit is to C, so D (is) to A. But, as a unit (is) to C, so C (is) to D. And thus as a unit (is) to C, so C (is) to D, and D to A. Thus, two numbers, Cand D, have fallen (between) a unit and the number Ain continued mean proportion. Again, since A has made B (by) multiplying itself, A thus measures B according to the units in it. And a unit also measures A according to the units in it. Thus, as a unit is to A, so A (is) to B. And two numbers have fallen (between) a unit and A in mean proportion. Thus two numbers will also fall (between) A and B in mean proportion [Prop. 8.8]. And if two (numbers) fall (between) two numbers in mean proportion, and the first (number) is cube, then the second will also be cube [Prop. 8.23]. And A is cube. Thus, Bis also cube. (Which is) the very thing it was required to show.

Proposition 4

If a cube number makes some (number by) multiplying a(nother) cube number then the created (number)



Κύβος γὰρ ἀριθμὸς ὁ Α κύβον ἀριθμὸν τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ κύβος ἐστίν.

Ο γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Α, Β κύβοι εἰσίν, ὄμοιοι στερεοί εἰσιν οἱ Α, Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· ὥστε καὶ τῶν Δ, Γ δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. καί ἐστι κύβος ὁ Δ· κύβος ἄρα καὶ ὁ Γ· ὅπερ ἕδει δεῖξαι.

ε΄.

Έὰν κύβος ἀριθμὸς ἀριθμόν τινα πολλαπλασιάσας κύβον ποιῆ, καὶ ὁ πολλαπλασιασθεὶς κύβος ἔσται.



Κύβος γὰρ ἀριθμὸς ὁ Α ἀριθμόν τινα τὸν Β πολλαπλασιάσας κύβον τὸν Γ ποιείτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Ο γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· κύβος ӑρα ἐστίν ὁ Δ. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἀρα ὡς ὁ Α πρὸς τὸν Β, ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Δ, Γ κύβοι εἰσίν, ὄμοιοι στερεοί εἰσιν. τῶν Δ, Γ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καί ἐστιν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καί ἐστι κύβος ὁ Α· κύβος ἄρα ἐστὶ καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

ج'.

Έαν αριθμός έαυτόν πολλαπλασιάσας κύβον ποιῆ, καὶ

will be cube.



For let the cube number A make C (by) multiplying the cube number B. I say that C is cube.

For let A make D (by) multiplying itself. Thus, D is cube [Prop. 9.3]. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D (is) to C [Prop. 7.17]. And since A and B are cube, A and B are similar solid (numbers). Thus, two numbers fall (between) A and B in mean proportion [Prop. 8.19]. Hence, two numbers will also fall (between) D and C in mean proportion [Prop. 8.8]. And D is cube. Thus, C (is) also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 5

If a cube number makes a(nother) cube number (by) multiplying some (number) then the (number) multiplied will also be cube.



For let the cube number A make the cube (number) C (by) multiplying some number B. I say that B is cube.

For let A make D (by) multiplying itself. D is thus cube [Prop. 9.3]. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as Ais to B, so D (is) to C [Prop. 7.17]. And since D and Care (both) cube, they are similar solid (numbers). Thus, two numbers fall (between) D and C in mean proportion [Prop. 8.19]. And as D is to C, so A (is) to B. Thus, two numbers also fall (between) A and B in mean proportion [Prop. 8.8]. And A is cube. Thus, B is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 6

If a number makes a cube (number by) multiplying



᾿Αριθμός γὰρ ὁ Α ἑαυτὸν πολλαπλασιάσας κύβον τὸν Β ποιείτω· λέγω, ὅτι καὶ ὁ Α κύβος ἐστίν.

Ο γὰρ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω. ἐπεὶ οὖν ό Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Β πεποίηχεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὅ Γ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ὁ A ἄρα τὸν B μετρεῖ κατὰ τὰς ἐν αὑτῷ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ώς ή μονάς πρός τὸν Α, οὕτως ὁ Α πρὸς τὸν Β. καὶ ἐπεὶ ό Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηχεν, ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεὶ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Γ. ἀλλ᾽ ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, ό Β πρός τὸν Γ. καὶ ἐπεὶ οἱ Β, Γ κύβοι εἰσίν, ὅμοιοι στερεοί είσιν. τῶν Β, Γ ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καί έστιν ώς
ό B πρὸς τὸν Γ,
ό A πρὸς τὸν B.
 καὶ τῶν A, B άρα δύο μέσοι άνάλογόν εἰσιν ἀριθμοί. καί ἐστιν κύβος ὁ Β· κύβος ἄρα ἐστὶ καὶ ὁ Α· ὅπερ ἔδει δεὶξαι.

ζ.

Έὰν σύνθετος ἀριθμὸς ἀριθμόν τινα πολλαπλασιάσας ποιῆ τινα, ὁ γενόμενος στερεὸς ἔσται.



Σύνθετος γὰρ ἀριθμὸς ὁ Α ἀριθμόν τινα τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ στερεός ἐστιν.

Έπεὶ γὰρ ὁ Α σύνθετός ἐστιν, ὑπὸ ἀριθμοῦ τινος μετρηθήσεται. μετρείσθω ὑπὸ τοῦ Δ, καὶ ὁσάκις ὁ Δ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. ἐπεὶ οὕν ὁ Δ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, ὁ Ε ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ δὲ Α ἐστιν ὁ ἐκ τῶν Δ, Ε, ὁ ἄρα ἐκ τῶν Δ, Ε τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ Γ ǎρα στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Δ, Ε, Β· itself then it itself will also be cube.



For let the number A make the cube (number) B (by) multiplying itself. I say that A is also cube.

For let A make C (by) multiplying B. Therefore, since A has made B (by) multiplying itself, and has made C(by) multiplying B, C is thus cube. And since A has made B (by) multiplying itself, A thus measures B according to the units in (A). And a unit also measures A according to the units in it. Thus, as a unit is to A, so A (is) to B. And since A has made C (by) multiplying B, B thus measures C according to the units in A. And a unit also measures A according to the units in it. Thus, as a unit is to A, so B (is) to C. But, as a unit (is) to A, so A (is) to B. And thus as A (is) to B, (so) B (is) to C. And since B and C are cube, they are similar solid (numbers). Thus, there exist two numbers in mean proportion (between) B and C [Prop. 8.19]. And as B is to C, (so) A (is) to B. Thus, there also exist two numbers in mean proportion (between) A and B [Prop. 8.8]. And B is cube. Thus, A is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 7

If a composite number makes some (number by) multiplying some (other) number then the created (number) will be solid.



For let the composite number A make C (by) multiplying some number B. I say that C is solid.

For since A is a composite (number), it will be measured by some number. Let it be measured by D. And, as many times as D measures A, so many units let there be in E. Therefore, since D measures A according to the units in E, E has thus made A (by) multiplying D [Def. 7.15]. And since A has made C (by) multiplying B, and A is the (number created) from (multiplying) D, E, the (number created) from (multiplying) D, E has thus ὄπερ ἔδει δεῖξαι.

η'.

Έὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὥσιν, ὁ μὲν τρίτος ἀπὸ τῆς μονάδος τετράγωνος ἔσται καὶ οἱ ἕνα διαλείποντες, ὁ δὲ τέταρτος κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἔβδομος κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες.



Έστωσαν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, Ε, Ζ· λέγω, ὅτι ὁ μὲν τρίτος ἀπὸ τῆς μονάδος ὁ Β τετράγωνός ἐστι καὶ οἱ ἕνα διαλείποντες πάντες, ὁ δὲ τέταρτος ὁ Γ κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος ὁ Ζ κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες πάντες.

Έπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρός τόν Β, ἰσάκις ἄρα ἡ μονὰς τόν Α ἀριθμόν μετρεῖ καὶ ό Α τὸν Β. ἡ δὲ μονὰς τὸν Α ἀριθμὸν μετρεῖ κατὰ τὰς έν αὐτῷ μονάδας· καὶ
ὁ A ẳρα τὸν B μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηχεν· τετράγωνος ἄρα ἐστιν ὁ B. χαὶ ἐπεὶ οἱ B, Γ , Δ ἑξῆς ἀνάλογόν εἰσιν, ὁ δ
ἑ ${\rm B}$ τετράγωνός ἐστιν, καὶ ὁ Δ ἄρα τετράγωνός ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ζ τετράγωνός έστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ ἕνα διαλείποντες πάντες τετράγωνοί εἰσιν. λέγω δή, ὅτι καὶ ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες. έπει γάρ έστιν ώς ή μονάς πρός τὸν Α, οὕτως ὁ Β πρὸς τὸν Γ, ἰσάχις ἄρα ἡ μονὰς τὸν Α ἀριθμὸν μετρεῖ χαὶ ὁ Β τὸν Γ. ἡ δὲ μονὰς τὸν Α ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. καὶ ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας· ὁ Α άρα τὸν Β πολλαπλασιάσας τὸν Γ πεποίηχεν. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Β πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηχεν, χύβος ἄρα ἐστὶν ὁ Γ. χαὶ ἐπεὶ οἱ Γ, Δ, Ε, Ζ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ Γ κύβος ἐστίν,

made C (by) multiplying B. Thus, C is solid, and its sides are D, E, B. (Which is) the very thing it was required to show.

Proposition 8

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one (number), and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two (numbers), and the seventh (will be) both cube and square, and (all) those (numbers after that) which leave an interval of five (numbers).



Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit. I say that the third from the unit, B, is square, and all those (numbers after that) which leave an interval of one (number). And the fourth (from the unit), C, (is) cube, and all those (numbers after that) which leave an interval of two (numbers). And the seventh (from the unit), F, (is) both cube and square, and all those (numbers after that) which leave an interval of five (numbers).

For since as the unit is to A, so A (is) to B, the unit thus measures the number A the same number of times as A (measures) B [Def. 7.20]. And the unit measures the number A according to the units in it. Thus, A also measures B according to the units in A. A has thus made B (by) multiplying itself [Def. 7.15]. Thus, B is square. And since B, C, D are continuously proportional, and Bis square, D is thus also square [Prop. 8.22]. So, for the same (reasons), F is also square. So, similarly, we can also show that all those (numbers after that) which leave an interval of one (number) are square. So I also say that the fourth (number) from the unit, C, is cube, and all those (numbers after that) which leave an interval of two (numbers). For since as the unit is to A, so B (is) to C, the unit thus measures the number A the same number of times that B (measures) C. And the unit measures the καὶ ὁ Ζ ἄρα κύβος ἐστίν. ἐδείχθη δὲ καὶ τετράγωνος· ὁ ἄρα ἕβδομος ἀπὸ τῆς μονάδος κύβος τέ ἐστι καὶ τετράγωνος. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ πέντε διαλείποντες πάντες κύβοι τέ εἰσι καὶ τετράγωνοι· ὅπερ ἔδει δεῖξαι.

θ'.

Έὰν ἀπὸ μονάδος ὑποσοιοῦν ἑξῆς κατὰ τὸ συνεχὲς ἀριθμοὶ ἀνάλογον ὥσιν, ὁ δὲ μετὰ τὴν μονάδα τετράγωνος ἢ, καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος ἦ, καὶ οἱ λοιποὶ πάντες κύβοι ἔσονται.



Έστωσαν ἀπὸ μονάδος ἑξῆς ἀνάλογον ὁσοιδηποτοῦν ἀριθμοὶ οἱ Α, Β, Γ, Δ, Ε, Ζ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται.

Ότι μὲν οῦν ὁ τρίτος ἀπὸ τῆς μονάδος ὁ Β τετράγωνός ἐστι καὶ οἱ ἕνα διαπλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν. ἐπεὶ γὰρ οἱ Α, Β, Γ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Α τετράγωνος, καὶ ὁ Γ [ἄρα] τετράγωνος ἐστιν. πάλιν, ἐπεὶ [καὶ] οἱ Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Β τετράγωνος, καὶ ὁ Δ [ἄρα] τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν.

Άλλὰ δὴ ἔστω ὁ Α κύβος· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν.

Ότι μὲν οῦν ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν. ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β, ἰσάκις ἀρα ἡ μονὰς τὸν Α μετρεῖ καὶ ὁ Α τὸν Β. ἡ δὲ μονὰς τὸν Α μετρεῖ κατὰ τὰς ἐν number A according to the units in A. And thus B measures C according to the units in A. A has thus made C (by) multiplying B. Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B, C is thus cube. And since C, D, E, F are continuously proportional, and C is cube, F is thus also cube [Prop. 8.23]. And it was also shown (to be) square. Thus, the seventh (number) from the unit is (both) cube and square. So, similarly, we can show that all those (numbers after that) which leave an interval of five (numbers) are (both) cube and square. (Which is) the very thing it was required to show.

Proposition 9

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is square, then all the remaining (numbers) will also be square. And if the (number) after the unit is cube, then all the remaining (numbers) will also be cube.



Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, be square. I say that all the remaining (numbers) will also be square.

In fact, it has (already) been shown that the third (number) from the unit, B, is square, and all those (numbers after that) which leave an interval of one (number) [Prop. 9.8]. [So] I say that all the remaining (numbers) are also square. For since A, B, C are continuously proportional, and A (is) square, C is [thus] also square [Prop. 8.22]. Again, since B, C, D are [also] continuously proportional, and B is square, D is [thus] also square [Prop. 8.22]. So, similarly, we can show that all the remaining (numbers) are also square.

And so let A be cube. I say that all the remaining (numbers) are also cube.

In fact, it has (already) been shown that the fourth (number) from the unit, *C*, is cube, and all those (numbers after that) which leave an interval of two (numbers)

αὐτῷ μονάδας· καὶ ὁ A ἄρα τὸν B μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ A ἄρα ἑαυτὸν πολλαπλασιάσας τὸν B πεποίηκεν. καί ἐστιν ὁ A κύβος. ἐὰν δὲ κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῆ τινα, ὁ γενόμενος κύβος ἐστίν· καὶ ὁ B ἄρα κύβος ἐστίν. καὶ ἐπεὶ τέσσαρες ἀριθμοὶ οἱ A, B, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ A κύβος, καὶ ὁ Δ ἄρα κύβος ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε κύβος ἐστίν, καὶ ὁμοίως οἱ λοιποὶ πάντες κύβοι εἰσίν· ὅπερ ἔδει δεῖξαι.

ι΄.

Έὰν ἀπὸ μονάδος ὅποσοιοῦν ἀριθμοὶ [ἑξῆς] ἀνάλογον ῶσιν, ὁ δὲ μετὰ τὴν μονάδα μὴ ἢ τετράγωνος, οὐδ° ἄλλος οὐδεἰς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἕνα διαλειπόντων πάντων. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος μὴ ἢ, οὐδὲ ἄλλος οὐδεἰς κύβος ἔσται χωρἰς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων πάντων.



Έστωσαν ἀπὸ μονάδος ἑξῆς ἀνάλογον ὁσοιδηποτοῦν ἀριθμοὶ οἱ Α, Β, Γ, Δ, Ε, Ζ, ὁ μετὰ τὴν μονάδα ὁ Α μὴ ἔστω τετράγωνος· λέγω, ὅτι οὐδὲ ἄλλος οὐδεἰς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τὴς μονάδος [καὶ τῶν ἕνα διαλειπόντων].

Εἰ γὰρ δυνατόν, ἔστω ὁ Γ τετράγωνος. ἔστι δὲ καὶ ὁ Β τετράγωνος· οἱ Β, Γ ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὂν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ὁ Α πρὸς τὸν Β· οἱ Α, Β ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὥστε οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν. καί ἐστι τετράγωνος ὁ Β· τετράγωνος ἄρα ἐστὶ καὶ ὁ Α· ὅπερ οὐχ ὑπέκειτο. οὐκ ἄρα ὁ Γ τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλος οὐδεἰς τετράγωνός ἐστι χωρὶς [Prop. 9.8]. [So] I say that all the remaining (numbers) are also cube. For since as the unit is to A, so A (is) to B, the unit thus measures A the same number of times as A (measures) B. And the unit measures A according to the units in it. Thus, A also measures B according to the units in (A). A has thus made B (by) multiplying itself. And A is cube. And if a cube number makes some (number by) multiplying itself then the created (number) is cube [Prop. 9.3]. Thus, B is also cube. And since the four numbers A, B, C, D are continuously proportional, and A is cube, D is thus also cube, and, similarly, all the remaining (numbers) are cube. (Which is) the very thing it was required to show.

Proposition 10

If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the (number) after the unit is not square, then no other (number) will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one (number). And if the (number) after the unit is not cube, then no other (number) will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two (numbers).



Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, not be square. I say that no other (number) will be square either, apart from the third from the unit [and (all) those (numbers after that) which leave an interval of one (number)].

For, if possible, let C be square. And B is also square [Prop. 9.8]. Thus, B and C have to one another (the) ratio which (some) square number (has) to (some other) square number. And as B is to C, (so) A (is) to B. Thus, A and B have to one another (the) ratio which (some) square number has to (some other) square number. Hence, A and B are similar plane (numbers) τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἕνα διαλειπόντων.

Άλλὰ δὴ μὴ ἔστω ὁ Α κύβος. λέγω, ὅτι οὐδ' ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων.

Εἰ γὰρ δυνατόν, ἔστω ὁ Δ κύβος. ἔστι δὲ καὶ ὁ Γ κύβος· τέταρτος γάρ ἐστιν ἀπὸ τῆς μονάδος. καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Β πρὸς τὸν Γ· καὶ ὁ Β ἄρα πρὸς τὸν Γ λόγον ἔχει, ὃν κύβος πρὸς κύβον. καί ἐστιν ὁ Γ κύβος· καὶ ὁ Β ἄρα κύβος ἐστίν. καὶ ἐπεί ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β, ἡ δὲ μονὰς τὸν Α μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας, καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας κύβον τὸν Β πεποίηκεν. ἐὰν δὲ ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῆ, καὶ αὐτὸς κύβος ἐσται. κύβος ἄρα καὶ ὁ Α· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ Δ κύβος ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ° ἄλλος οὐδεὶς κύβος ἐστὶ χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων· ὅπερ ἔδει δείζαι.

ıα'.

Έὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ῶσιν, ὁ ἐλάττων τὸν μείζονα μετρεῖ κατά τινα τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.



Έστωσαν ἀπὸ μονάδος τῆς Α ὑποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ B, Γ, Δ, Ε· λέγω, ὅτι τῶν B, Γ, Δ, Ε ὑ ἑλάχιστος ὁ B τὸν Ε μετρεῖ κατά τινα τῶν Γ, Δ.

Έπεὶ γάρ ἐστιν ὡς ἡ A μονὰς πρὸς τὸν B, οὕτως ὁ Δ πρὸς τὸν E, ἰσάχις ἄρα ἡ A μονὰς τὸν B ἀριθμὸν μετρεῖ χαὶ ὁ Δ τὸν E· ἐναλλὰζ ἄρα ἰσάχις ἡ A μονὰς τὸν Δ μετρεῖ χαὶ ὁ B τὸν E. ἡ δὲ A μονὰς τὸν Δ μετρεῖ χατὰ τὰς ἐν [Prop. 8.26]. And B is square. Thus, A is also square. The very opposite thing was assumed. C is thus not square. So, similarly, we can show that no other (number is) square either, apart from the third from the unit, and (all) those (numbers after that) which leave an interval of one (number).

And so let *A* not be cube. I say that no other (number) will be cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers).

For, if possible, let D be cube. And C is also cube [Prop. 9.8]. For it is the fourth (number) from the unit. And as C is to D, (so) B (is) to C. And B thus has to C the ratio which (some) cube (number has) to (some other) cube (number). And C is cube. Thus, B is also cube [Props. 7.13, 8.25]. And since as the unit is to A, (so) A (is) to B, and the unit measures A according to the units in it, A thus also measures B according to the units in (A). Thus, A has made the cube (number) B (by) multiplying itself. And if a number makes a cube (number by) multiplying itself then it itself will be cube [Prop. 9.6]. Thus, A (is) also cube. The very opposite thing was assumed. Thus, D is not cube. So, similarly, we can show that no other (number) is cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers). (Which is) the very thing it was required to show.

Proposition 11

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser (number) measures a greater according to some existing (number) among the proportional numbers.



Let any multitude whatsoever of numbers, B, C, D, E, be continuously proportional, (starting) from the unit A. I say that, for B, C, D, E, the least (number), B, measures E according to some (one) of C, D.

For since as the unit A is to B, so D (is) to E, the unit A thus measures the number B the same number of times as D (measures) E. Thus, alternately, the unit A

αὐτῷ μονάδας· καὶ ὁ Β ἄρα τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὥστε ὁ ἐλάσσων ὁ Β τὸν μείζονα τὸν Ε μετρεῖ κατά τινα ἀριθμὸν τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Πόρισμα.

Καὶ φανερόν, ὅτι ἡν ἔχει τάξιν ὁ μετρῶν ἀπὸ μονάδος, τὴν αὐτὴν ἔχει καὶ ὁ καθ᾽ ὃν μετρεῖ ἀπὸ τοῦ μετρουμένου ἐπὶ τὸ πρὸ αὐτοῦ. ὅπερ ἔδει δεῖξαι.

ıβ'.

Έὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ῶσιν, ὑφ' ὅσων ἂν ὁ ἔσχατος πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ παρὰ τὴν μονάδα μετρηθήσεται.



Έστωσαν ἀπὸ μονάδος ὑποσοιδηποτοῦν ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ· λέγω, ὅτι ὑφ' ὅσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ Α μετρηθήσεται.

Μετρείσθω γὰρ
 ὁ Δ ὑπό τινος πρώτου ἀριθμοῦ τοῦ Ε· λέγω, ὅτι ὁ Ε τὸν Α μετρεῖ. μὴ γάρ· καί ἐστιν ὁ Ε πρῶτος, άπας δὲ πρῶτος ἀριθμὸς πρὸς ἄπαντα, ὃν μὴ μετρεῖ, πρῶτός έστιν οι Ε, Α άρα πρῶτοι πρὸς ἀλλήλους εἰσίν. και ἐπεὶ ό Ε τὸν Δ μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ·
ἑ Ε ẳρα τὸν Z πολλαπλασιάσας τὸν
 Δ πεποίηκεν. πάλιν, ἐπεὶ ὁ A τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, ὁ Α ἄρα τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ
ἑ Ε τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν ὁ ἄρα ἐκ τῶν Α, Γ ἴσος έστι τῷ ἐχ τῶν Ε, Ζ. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Ε, ὁ Ζ πρός τόν Γ. οί δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οί δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ίσάχις ὅ τε ἡγούμενος τὸν ἡγούμενον χαὶ ὁ ἑπόμενος τὸν έπόμενον μετρεῖ ἄρα ὁ Ε τὸν Γ. μετρείτω αὐτὸν κατὰ τὸν Η· ὁ Ε ἄρα τὸν Η πολλαπλασιάσας τὸν Γ πεποίηχεν. ἀλλὰ μήν διὰ τὸ πρὸ τούτου καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν.
 ὁ ἄρα ἐκ τῶν Α, Β ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Η. έστιν ἄρα ὡς ὁ Α πρὸς τὸν Ε, ὁ Η πρὸς τὸν Β. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι χαὶ ἐλάγιστοι, οἱ δὲ ἐλάγιστοι ἀριθμοὶ

measures D the same number of times as B (measures) E [Prop. 7.15]. And the unit A measures D according to the units in it. Thus, B also measures E according to the units in D. Hence, the lesser (number) B measures the greater E according to some existing number among the proportional numbers (namely, D).

Corollary

And (it is) clear that what(ever relative) place the measuring (number) has from the unit, the (number) according to which it measures has the same (relative) place from the measured (number), in (the direction of the number) before it. (Which is) the very thing it was required to show.

Proposition 12

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then however many prime numbers the last (number) is measured by, the (number) next to the unit will also be measured by the same (prime numbers).



Let any multitude whatsoever of numbers, A, B, C, D, be (continuously) proportional, (starting) from a unit. I say that however many prime numbers D is measured by, A will also be measured by the same (prime numbers).

For let D be measured by some prime number E. I say that E measures A. For (suppose it does) not. E is prime, and every prime number is prime to every number which it does not measure [Prop. 7.29]. Thus, E and A are prime to one another. And since E measures D, let it measure it according to F. Thus, E has made D (by) multiplying F. Again, since A measures D according to the units in C [Prop. 9.11 corr.], A has thus made D (by) multiplying C. But, in fact, E has also made D (by) multiplying F. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F. Thus, as A is to E, (so) F (is) to C [Prop. 7.19]. And A and E (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leadμετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάχις ὅ τε ήγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον μετρε
ĩ ਕρα δ Ε τὸν Β. μετρείτω αὐτὸν κατὰ τὸν Θ·
 δ Ε ਕρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α έαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν ὁ ἄρα ἐκ τῶν Ε, Θ ἴσος ἐστὶ τῷ ἀπὸ τοῦ A. ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν A, ὁ A πρός τόν Θ. οί δὲ Α, Ε πρῶτοι, οί δὲ πρῶτοι καὶ ἐλάχιστοι, οί δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάχις ὄ ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Ε τὸν Α ὡς ἡγούμενος ἡγούμενον. ἀλλὰ μὴν καὶ οὐ μετρεῖ· ὅπερ ἀδύνατον. οὐκ ἄρα οἱ Ε, Α πρῶτοι πρὸς άλλήλους εἰσίν. σύνθετοι ἄρα. οἱ δὲ σύνθετοι ὑπὸ [πρώτου] άριθμοῦ τινος μετροῦνται. καὶ ἐπεὶ ὁ Ε πρῶτος ὑπόκειται, ὁ δὲ πρῶτος ὑπὸ ἑτέρου ἀριθμοῦ οὐ μετρεῖται ἢ ὑφ' ἑαυτοῦ, ὁ Ε ἄρα τοὺς Α, Ε μετρεῖ· ὥστε ὁ Ε τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Δ ·
ἑ E ẳρα τοὺς A, Δ μετρεῖ.
 ὑμοίως δὴ δείξομεν, ότι ὑφ' ὄσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ Α μετρηθήσεται· ὅπερ ἔδει δεῖξαι.

ιγ'.

Έαν από μονάδος όποσοιοῦν αριθμοὶ ἑξῆς ανάλογον ῶσιν, ὁ δὲ μετὰ τὴν μονάδα πρῶτος ἦ, ὁ μέγιστος ὑπ ούδενὸς [ἄλλου] μετρηθήσεται παρὲξ τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

οί Α, Β, Γ, Δ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτος ἔστω· λέγω, ὅτι ὁ μέγιστος αὐτῶν ὁ Δ ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρέξ τῶν Α, Β, Γ.

ing, and the following the following [Prop. 7.20]. Thus, E measures C. Let it measure it according to G. Thus, E has made C (by) multiplying G. But, in fact, via the (proposition) before this, A has also made C (by) multiplying B [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, B is equal to the (number created) from (multiplying) E, G. Thus, as A is to E, (so) G(is) to B [Prop. 7.19]. And A and E (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, Emeasures B. Let it measure it according to H. Thus, E has made B (by) multiplying H. But, in fact, A has also made B (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) E, H is equal to the (square) on A. Thus, as E is to A, (so) A (is) to H [Prop. 7.19]. And A and E are prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, Emeasures A, as the leading (measuring the) leading. But, in fact, (E) also does not measure (A). The very thing (is) impossible. Thus, E and A are not prime to one another. Thus, (they are) composite (to one another). And (numbers) composite (to one another) are (both) measured by some [prime] number [Def. 7.14]. And since E is assumed (to be) prime, and a prime (number) is not measured by another number (other) than itself [Def. 7.11], E thus measures (both) A and E. Hence, E measures A. And it also measures D. Thus, E measures (both) A and D. So, similarly, we can show that however many prime numbers D is measured by, A will also be measured by the same (prime numbers). (Which is) the very thing it was required to show.

Proposition 13

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is prime, then the greatest (number) will be measured by no [other] (numbers) except (num- \tilde{z} Εστωσαν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον bers) existing among the proportional numbers.

> Let any multitude whatsoever of numbers, A, B, C, D, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, be prime. I say



τῶν Α, Β, Γ ἔστω ὁ αὐτός. φανερὸν δή, ὅτι ὁ Ε πρῶτος οὔχ ἐστιν. εἰ γὰρ ὁ Ε πρῶτός ἐστι χαὶ μετρεῖ τὸν Δ , χαὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν άδύνατον. ούκ ἄρα ὁ Ε πρῶτός ἐστιν. σύνθετος ἄρα. πᾶς δὲ σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. ό Ε ἄρα ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δή, ὅτι ύπ' ούδενὸς ἄλλου πρώτου μετρηθήσεται πλὴν τοῦ Α. εἰ γὰρ ύφ' ἑτέρου μετρεῖται ὁ Ε, ὁ δὲ Ε τὸν Δ μετρεῖ, κἀκεῖνος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ών αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ Α ἄρα τὸν Ε μετρεῖ.
 καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ. λέγω, ὅτι ὁ Ζ οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ ὁ Z ἑνὶ τῶν A, B, Γ ἐστιν ὁ αὐτὸς καὶ μετρεῖ τὸν Δ κατὰ τὸν Ε, καὶ εἶς ἄρα τῶν Α, Β, Γ τὸν Δ μετρεῖ κατά τὸν Ε. ἀλλὰ ε
ἶς τῶν Α, Β, Γ τὸν Δ μετρεῖ κατά τινα τῶν Α, Β, Γ· καὶ ὁ Ε ἄρα ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός· ὅπερ οὐχ ὑπόκειται. ούκ ἄρα ὁ Ζ ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. ὁμοίως δὴ δείξομεν, ὄτι μετρεῖται ὁ Ζ ὑπὸ τοῦ Α, δειχνύντες πάλιν, ότι ὁ Z οὖχ ἐστι πρῶτος. εἰ γὰρ, καὶ μετρεῖ τὸν Δ, καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν άδύνατον· ούκ ἄρα πρῶτός ἐστιν ὁ Ζ· σύνθετος ἄρα. ἄπας δὲ σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὁ Ζ ἄρα ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δή, ὅτι ὑφ᾽ έτέρου πρώτου οὐ μετρηθήσεται πλὴν τοῦ Α. εἰ γὰρ ἕτερός τις πρῶτος τὸν Z μετρεῖ, ὁ δὲ Z τὸν Δ μετρεῖ, κἀχεῖνος άρα τὸν Δ μετρήσει· ὥστε καὶ τὸν ${
m A}$ μετρήσει πρῶτον ὄντα μή ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ Α ἄρα τὸν Ζ μετρεῖ.
 καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ κατὰ τὸν Ζ, ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν ὁ ẳρα ἐκ τῶν Α, Γ ἴσος ἐστὶ τῷ ἐϰ τῶν Ε, Ζ. ἀνάλογον ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Ε, οὕτως
 ὁ Ζ πρὸς τὸν Γ. ὁ δὲ Α τὸν Ε μετρεĩ καὶ ὁ Ζ άρα τὸν Γ μετρεῖ. μετρείτω αὐτὸν κατὰ τὸν Η. ὁμοίως δὴ δείξομεν, ὅτι ὁ Η οὐδενὶ τῶν Α, Β ἐστιν ὁ αὐτός, καὶ ὅτι μετρεῖται ὑπὸ τοῦ Α. καὶ ἐπεὶ ὁ Ζ τὸν Γ μετρεῖ κατὰ τὸν Η, ό Ζ ἄρα τὸν Η πολλαπλασιάσας τὸν Γ πεποίηχεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ ẳρα ἐκ τῶν A, B ἴσος ἐστὶ τῷ ἐ
κ τῶν Z, H. ἀν
άλογον ẳρα ὡς ὁ A πρός τόν Ζ, ὁ Η πρός τὸν Β. μετρεῖ δὲ ὁ Α τὸν Ζ· μετρεῖ ἄρα καὶ ὁ Η τὸν Β. μετρείτω αὐτὸν κατὰ τὸν Θ. ὁμοίως δὴ δείξομεν, ὅτι ὁ Θ τῷ Α οὐ
κ ἔστιν ὁ αὐτός. καὶ ἐπεὶ ὁ Η τὸν that the greatest of them, D, will be measured by no other (numbers) except A, B, C.



For, if possible, let it be measured by E, and let E not be the same as one of A, B, C. So it is clear that E is not prime. For if E is prime, and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, E is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, E is measured by some prime number. So I say that it will be measured by no other prime number than A. For if E is measured by another (prime number), and E measures D, then this (prime number) will thus also measure D. Hence, it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures E. And since E measures D, let it measure it according to F. I say that F is not the same as one of A, B, C. For if F is the same as one of A, B, C, and measures D according to E, then one of A, B, C thus also measures D according to E. But one of A, B, C (only) measures D according to some (one) of A, B, C [Prop. 9.11]. And thus E is the same as one of A, B, C. The very opposite thing was assumed. Thus, F is not the same as one of A, B, C. Similarly, we can show that F is measured by A, (by) again showing that F is not prime. For if (Fis prime), and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, F is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, F is measured by some prime number. So I say that it will be measured by no other prime number than A. For if some other prime (number) measures F, and F measures D, then this (prime number) will thus also measure D. Hence, it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures F. And since E measures D according to F, E has thus made D (by) multiplying F. But, in fact, A has also made D (by) multiplying C [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F. Thus, proportionally, as A is to E, so F (is) to C [Prop. 7.19]. And A measures Β μετρεϊ κατὰ τὸν Θ, ὁ Η ἄρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· ὁ ἄρα ὑπὸ Θ, Η ἴσος ἐστὶ τῷ ἀπὸ τοῦ Α τετραγώνῳ· ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Α, ὁ Α πρὸς τὸν Η. μετρεῖ δὲ ὁ Α τὸν Η· μετρεῖ ἄρα καὶ ὁ Θ τὸν Α πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἄτοπον. οὐκ ἄρα ὁ μέγιστος ὁ Δ ὑπὸ ἑτέρου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ· ὅπερ ἔδει δεῖξαι.

ιδ'.

Έὰν ἐλάχιστος ἀριθμὸς ὑπὸ πρώτων ἀριθμῶν μετρῆται, ὑπ' οὑδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν ἐξ ἀρχῆς μετρούντων.



Ἐλάχιστος γὰρ ἀριθμὸς ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν B, Γ, Δ μετρείσθω· λέγω, ὅτι ὁ Α ὑπ' οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν B, Γ, Δ.

Εἰ γὰρ δυνατόν, μετρείσθω ὑπὸ πρώτου τοῦ Ε, καὶ ὁ Ε μηδενὶ τῶν Β, Γ, Δ ἔστω ὁ αὐτός. καὶ ἐπεὶ ὁ Ε τὸν Α μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ· ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ μετρεῖται ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ. ἐἀν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἑξ αὐτῶν μετρῆ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει· οἱ Β, Γ, Δ ἄρα ἕνα τῶν Ε, Ζ μετρήσουσιν. τὸν μὲν οῦν Ε οὐ μετρήσουσιν· ὁ γὰρ Ε πρῶτός ἐστι καὶ οὐδενὶ τῶν Β, Γ, Δ ὁ αὐτός. τὸν Ζ ἄρα μετροῦσιν ἐλάσσονα ὄντα τοῦ Α· ὅπερ ἀδύνατον. ὁ γὰρ Α ὑπόκειται ἐλάχιστος ὑπὸ τῶν Β, Γ, Δ μετρούμενος. οὐx ἄρα τὸν Α μετρήσει πρῶτος ἀριθμὸς παρὲξ τῶν Β, Γ, Δ· ὅπερ ἕδει δεῖξαι.

E. Thus, F also measures C. Let it measure it according to G. So, similarly, we can show that G is not the same as one of A, B, and that it is measured by A. And since F measures C according to G, F has thus made C (by) multiplying G. But, in fact, A has also made C (by) multiplying B [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, B is equal to the (number created) from (multiplying) F, G. Thus, proportionally, as A (is) to F, so G (is) to B [Prop. 7.19]. And A measures F. Thus, G also measures B. Let it measure it according to H. So, similarly, we can show that H is not the same as A. And since G measures B according to H, G has thus made B (by) multiplying H. But, in fact, A has also made B (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) H, G is equal to the square on A. Thus, as H is to A, (so) A (is) to G [Prop. 7.19]. And A measures G. Thus, H also measures A, (despite A) being prime (and) not being the same as it. The very thing (is) absurd. Thus, the greatest (number) D cannot be measured by another (number) except (one of) A, B, C. (Which is) the very thing it was required to show.

Proposition 14

If a least number is measured by (some) prime numbers then it will not be measured by any other prime number except (one of) the original measuring (numbers).



For let A be the least number measured by the prime numbers B, C, D. I say that A will not be measured by any other prime number except (one of) B, C, D.

For, if possible, let it be measured by the prime (number) E. And let E not be the same as one of B, C, D. And since E measures A, let it measure it according to F. Thus, E has made A (by) multiplying F. And A is measured by the prime numbers B, C, D. And if two numbers make some (number by) multiplying one another, and some prime number measures the number created from them, then (the prime number) will also measure one of the original (numbers) [Prop. 7.30]. Thus, B, C, D will measure one of E, F. In fact, they do not measure E. For E is prime, and not the same as one of B, C, D. Thus, they (all) measure F, which is less than A. The very thing (is) impossible. For A was assumed (to be) the least (number) measured by B, C, D. Thus, no prime

ιε΄.

Έὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ῶσιν ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, δύο ὑποιοιοῦν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοί εἰσιν.



Έστωσαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β, Γ· λέγω, ὅτι τῶν Α, Β, Γ δύο ὁποιοιοῦν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοι εἰσιν, οἱ μὲν Α, Β πρὸς τὸν Γ, οἱ δὲ Β, Γ πρὸς τὸν Α καὶ ἔτι οἱ Α, Γ πρὸς τὸν Β.

Εἰλήφθωσαν γὰρ ἐλάγιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ δύο οἱ ΔΕ, ΕΖ. φανερὸν δή, ὅτι ὁ μὲν ΔΕ ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ ΕΖ πολλαπλασιάσας τὸν Β πεποίηκεν, καὶ ἔτι ὁ ΕΖ έαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν. καὶ ἐπεὶ οἱ ΔΕ, ΕΖ ἐλάχιστοί εἰσιν, πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφότερος πρός ἑχάτερον πρῶτός ἐστιν· χαὶ ὁ ΔΖ ἄρα πρὸς ἑχάτερον τῶν ΔΕ, ΕΖ πρῶτός ἐστιν. ἀλλὰ μὴν καὶ ὁ ΔΕ πρὸς τὸν EZ πρῶτός ἐστιν· οἱ ΔZ , ΔE ἄρα πρὸς τὸν EZ πρῶτοί εἰσιν. έὰν δὲ δύο ἀριθμοὶ πρός τινα ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ έξ αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν. ὥστε ό ἐκ τῶν ΖΔ, ΔΕ πρὸς τὸν ΕΖ πρῶτός ἐστιν
· ὥστε καὶ ὁ έκ τῶν $Z\Delta$, ΔE πρὸς τὸν ἀπὸ τοῦ EZ πρῶτός ἐστιν. [ἐὰν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὥσιν, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν]. ἀλλ' ὁ ἐχ τῶν $Z\Delta$, ΔE ὁ ἀπὸ τοῦ ΔE ἐστι μετὰ τοῦ ἐχ τῶν ΔE , EZ·
 ὁ ẳρα ἀπὸ τοῦ ΔE μετὰ τοῦ ἐκ τῶν ΔE , EZ πρὸς τὸν ἀπὸ τοῦ EZ πρῶτός ἐστιν. καί ἐστιν ὁ μὲν ἀπὸ τοῦ ΔE ό A, ό δὲ ἐχ τῶν ΔE , EZ ὁ B, ὁ δὲ ἀπὸ τοῦ EZ ὁ Γ· οἱ A, B ἄρα συντεθέντες πρός τὸν Γ πρῶτοί εἰσιν. ὑμοίως δὴ δείξομεν, ὅτι καὶ οἱ Β, Γ πρὸς τὸν Α πρῶτοί εἰσιν. λέγω δή, ὅτι καὶ οἱ Α, Γ πρὸς τὸν Β πρῶτοί εἰσιν. ἐπεὶ γὰρ
ὁ ΔZ πρός ἑκάτερον τῶν ΔΕ, ΕΖ πρῶτός ἐστιν, καὶ ὁ ἀπὸ τοῦ ΔZ πρὸς τὸν ἐχ τῶν ΔE , EZ πρῶτός ἐστιν. ἀλλὰ τῷ ἀπὸ τοῦ ΔZ ἴσοι εἰσὶν οἱ ἀπὸ τῶν ΔE , EZ μετὰ τοῦ δὶς ἐκ τῶν $\Delta E, EZ$ · καὶ οἱ ἀπὸ τῶν $\Delta E, EZ$ ἄρα μετὰ τοῦ δὶς ὑπὸ τῶν ΔE , EZ πρὸς τὸν ὑπὸ τῶν ΔE , EZ πρῶτοί [εἰσι]. διελόντι οἱ ἀπὸ τῶν ΔE , EZ μετὰ τοῦ ἄπαξ ὑπὸ ΔE , EZ πρὸς τὸν ύπο ΔΕ, ΕΖ πρῶτοί εἰσιν. ἔτι διελόντι οἱ ἀπὸ τῶν ΔΕ, ΕΖ άρα πρός τὸν ὑπὸ ΔΕ, ΕΖ πρῶτοί εἰσιν. καί ἐστιν ὁ μὲν

number can measure A except (one of) B, C, D. (Which is) the very thing it was required to show.

Proposition 15

If three continuously proportional numbers are the least of those (numbers) having the same ratio as them then two (of them) added together in any way are prime to the remaining (one).



Let A, B, C be three continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that two of A, B, C added together in any way are prime to the remaining (one), (that is) A and B (prime) to C, B and C to A, and, further, A and C to B.

Let the two least numbers, DE and EF, having the same ratio as A, B, C, have been taken [Prop. 8.2]. So it is clear that DE has made A (by) multiplying itself, and has made B (by) multiplying EF, and, further, EF has made C (by) multiplying itself [Prop. 8.2]. And since DE, EF are the least (of those numbers having the same ratio as them), they are prime to one another [Prop. 7.22]. And if two numbers are prime to one another then the sum (of them) is also prime to each [Prop. 7.28]. Thus, DF is also prime to each of DE, EF. But, in fact, DE is also prime to EF. Thus, DF, DEare (both) prime to EF. And if two numbers are (both) prime to some number then the (number) created from (multiplying) them is also prime to the remaining (number) [Prop. 7.24]. Hence, the (number created) from (multiplying) FD, DE is prime to EF. Hence, the (number created) from (multiplying) FD, DE is also prime to the (square) on EF [Prop. 7.25]. [For if two numbers are prime to one another then the (number) created from (squaring) one of them is prime to the remaining (number).] But the (number created) from (multiplying) FD, DE is the (square) on DE plus the (number created) from (multiplying) DE, EF [Prop. 2.3]. Thus, the (square) on DE plus the (number created) from (multiplying) DE, EF is prime to the (square) on EF. And the (square) on *DE* is *A*, and the (number created) from (multiplying) DE, EF (is) B, and the (square) on EF(is) C. Thus, A, B summed is prime to C. So, similarly, we can show that B, C (summed) is also prime to A. So I say that A, C (summed) is also prime to B. For since ảπὸ τοῦ ΔE ὁ A, ὁ δὲ ὑπὸ τῶν ΔE , EZ ὁ B, ὁ δὲ ἀπὸ τοῦ EZ ὁ Γ . οἱ A, Γ ἄρα συντεθέντες πρὸς τὸν B πρῶτοί εἰσιν· ὅπερ ἔδει δεῖξαι. DF is prime to each of DE, EF then the (square) on DF is also prime to the (number created) from (multiplying) DE, EF [Prop. 7.25]. But, the (sum of the squares) on DE, EF plus twice the (number created) from (multiplying) DE, EF is equal to the (square) on DF [Prop. 2.4]. And thus the (sum of the squares) on *DE*, *EF* plus twice the (rectangle contained) by DE, EF [is] prime to the (rectangle contained) by DE, EF. By separation, the (sum of the squares) on DE, EF plus once the (rectangle contained) by DE, EF is prime to the (rectangle contained) by DE, EF.[†] Again, by separation, the (sum of the squares) on DE, EF is prime to the (rectangle contained) by DE, EF. And the (square) on DE is A, and the (rectangle contained) by DE, EF (is) B, and the (square) on EF (is) C. Thus, A, C summed is prime to *B*. (Which is) the very thing it was required to show.

[†] Since if $\alpha \beta$ measures $\alpha^2 + \beta^2 + 2 \alpha \beta$ then it also measures $\alpha^2 + \beta^2 + \alpha \beta$, and vice versa.

เร'.

Έὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὥσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ δεύτερος πρὸς ἀλλον τινά.



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς ἄλλον τινά.

Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Α τὸν Β ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν· ὁ Α ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· ὅπερ ἔδει δεῖξαι.

ιζ΄.

Έὰν ὥσιν ὑσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄχροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὥσιν, οὐχ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ ἔσχατος πρὸς ἄλλον

Proposition 16

If two numbers are prime to one another then as the first is to the second, so the second (will) not (be) to some other (number).



For let the two numbers A and B be prime to one another. I say that as A is to B, so B is not to some other (number).

For, if possible, let it be that as A (is) to B, (so) B (is) to C. And A and B (are) prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures B, as the leading (measuring) the leading. And (A) also measures itself. Thus, A measures A and B, which are prime to one another. The very thing (is) absurd. Thus, as A (is) to B, so B cannot be to C. (Which is) the very thing it was required to show.

Proposition 17

If any multitude whatsoever of numbers is continuously proportional, and the outermost of them are prime to one another, then as the first (is) to the second, so the τινά.

Έστωσαν ὑσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, B, Γ, Δ, οἱ δὲ ἄχροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐχ ἔστιν ὡς ὁ Α πρὸς τὸν B, οὕτως ὁ Δ πρὸς ἄλλον τινά.



Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, ὁ Β πρὸς τὸν Ε. οἱ δὲ Α, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεĩ ἄρα ὁ Α τὸν Β. καί ἐστιν ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. καὶ ὁ Β ἄρα τὸν Γ μετρεῖ΄· ὥστε καὶ ὁ Α τὸν Γ μετρεῖ. καὶ ἐπεί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ἱ Γ πρὸς τὸν Δ, μετρεῖ δὲ ὁ Β τὸν Γ, μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ. ἀλλ' ὁ Α τὸν Γ ἐμέτρει· ὥστε ὁ Α καὶ τὸν Δ μετρεῖ. μετρεῖ δὲ καὶ ἑαυτόν. ὁ Α ἄρα τοὺς Α, Δ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς ἄλλον τινά· ὅπερ ἔδει δεῖξαι.

*ι*η'.

Δύο ἀριθμῶν δοθέντων ἐπισκέψασθαι, εἰ δυνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.



Έστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β, καὶ δέον ἔστω ἐπισκέψασθαι, εἰ δυνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Οἱ δὴ A, B ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὕ. καὶ εἰ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Άλλὰ δὴ μὴ ἔστωσαν οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Β ἑαυτον πολλαπλασιάσας τὸν Γ ποιείτω. ὁ Α δὴ τὸν Γ ῆτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον κατὰ τὸν Δ· ὁ Α ἄρα τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλα μὴν καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ ἄρα last will not be to some other (number).

Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D, be prime to one another. I say that as A is to B, so D (is) not to some other (number).



For, if possible, let it be that as A (is) to B, so D(is) to E. Thus, alternately, as A is to D, (so) B (is) to E [Prop. 7.13]. And A and D are prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures B. And as A is to B, (so) B (is) to C. Thus, Balso measures C. And hence A measures C [Def. 7.20]. And since as B is to C, (so) C (is) to D, and B measures C, C thus also measures D [Def. 7.20]. But, A was (found to be) measuring C. And hence A also measures D. And (A) also measures itself. Thus, A measures A and D, which are prime to one another. The very thing is impossible. Thus, as A (is) to B, so D cannot be to some other (number). (Which is) the very thing it was required to show.

Proposition 18

For two given numbers, to investigate whether it is possible to find a third (number) proportional to them.



Let A and B be the two given numbers. And let it be required to investigate whether it is possible to find a third (number) proportional to them.

So A and B are either prime to one another, or not. And if they are prime to one another then it has (already) been show that it is impossible to find a third (number) proportional to them [Prop. 9.16].

And so let A and B not be prime to one another. And let B make C (by) multiplying itself. So A either measures, or does not measure, C. Let it first of all measure (C) according to D. Thus, A has made C (by) multiplyἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Β. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Δ· τοῖς Α, Β ἄρα τρίτος ἀριθμὸς ἀνάλογον προσηύρηται ὁ Δ.

Άλλὰ δὴ μὴ μετρείτω ὁ Α τὸν Γ· λέγω, ὅτι τοῖς Α, Β ἀδύνατόν ἐστι τρίτον ἀνάλογον προσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσηυρήσθω ὁ Δ. ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Β. ὁ δὲ ἀπὸ τοῦ Β ἐστιν ὁ Γ· ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ Γ. ὥστε ὁ Α τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ Α ἄρα τὸν Γ μετρεῖ κατὰ τὸν Δ. ἀλλα μὴν ὑπόκειται καὶ μὴ μετρῶν· ὅπερ ἄτοπον. οὐκ ἄρα δυνατόν ἐστι τοῖς Α, Β τρίτον ἀνάλογον προσευρεῖν ἀριθμὸν, ὅταν ὁ Α τὸν Γ μὴ μετρῆ· ὅπερ ἔδει δεῖξαι.

ıθ'.

Τριῶν ἀριθμῶν δοθέντων ἐπισκέψασθαι, πότε δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν.



Έστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ Α, Β, Γ, καὶ δέον ἔστω επισκέψασθαι, πότε δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν.

Ήτοι ούν ούχ εἰσιν ἑξῆς ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οὕχ εἰσι πρῶτοι πρὸς ἀλλήλους, ἢ οὕτε ἑξῆς εἰσιν ἀνάλογον, οὕτε οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ καὶ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ μὲν οὕν οἱ Α, Β, Γ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οἱ Α, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν. μὴ ἔστωσαν δὴ οἱ Α, Β, Γ ἑξῆς ἀνάλογον τῶν ἀκρῶν πάλιν ὄντων πρώτων πρὸς ἀλλήλους. λέγω, ὅτι καὶ οὕτως ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν. εἰ γὰρ δυνατόν, προσευρήσθω ὁ Δ, ὥστε εἶναι ὡς τὸν Α πρὸς τὸν Β, τὸν Γ πρὸς τὸν Δ, καὶ γεγονέτω ὡς ὁ Β πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Ε. καὶ ἐπεί ἐστιν ὡς μὲν ὁ Α πρὸς τὸν Β, ὅ Γ πρὸς τὸν Δ, ὡς δὲ ὁ Β πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Ε, δι' ἴσου ἄρα ὡς ὁ Α πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Ε. οἰ δὲ Α, Γ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι

ing *D*. But, in fact, *B* has also made *C* (by) multiplying itself. Thus, the (number created) from (multiplying) *A*, *D* is equal to the (square) on *B*. Thus, as *A* is to *B*, (so) *B* (is) to *D* [Prop. 7.19]. Thus, a third number has been found proportional to *A*, *B*, (namely) *D*.

And so let A not measure C. I say that it is impossible to find a third number proportional to A, B. For, if possible, let it have been found, (and let it be) D. Thus, the (number created) from (multiplying) A, D is equal to the (square) on B [Prop. 7.19]. And the (square) on B is C. Thus, the (number created) from (multiplying) A, D is equal to C. Hence, A has made C (by) multiplying D. Thus, A measures C according to D. But (A) was, in fact, also assumed (to be) not measuring (C). The very thing (is) absurd. Thus, it is not possible to find a third number proportional to A, B when A does not measure C. (Which is) the very thing it was required to show.

Proposition 19[†]

For three given numbers, to investigate when it is possible to find a fourth (number) proportional to them.



Let A, B, C be the three given numbers. And let it be required to investigate when it is possible to find a fourth (number) proportional to them.

In fact, (A, B, C) are either not continuously proportional and the outermost of them are prime to one another, or are continuously proportional and the outermost of them are not prime to one another, or are neither continuously proportional nor are the outermost of them prime to one another, or are continuously proportional and the outermost of them are prime to one another.

In fact, if A, B, C are continuously proportional, and the outermost of them, A and C, are prime to one another, (then) it has (already) been shown that it is impossible to find a fourth number proportional to them [Prop. 9.17]. So let A, B, C not be continuously proportional, (with) the outermost of them again being prime to one another. I say that, in this case, it is also impossible to find a fourth (number) proportional to them. For, if possible, let it have been found, (and let it be) D. Hence, it will be that as A (is) to B, (so) C (is) to D. And let it be contrived that as B (is) to C, (so) D (is) to E. And since μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Γ ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν ὁ Α ἄρα τοὺς Α, Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοῖς Α, Β, Γ δυνατόν ἐστι τέταρτον ἀνάλογον προσευρεῖν.

Άλλά δὴ πάλιν ἕστωσαν οἱ Α, Β, Γ ἑξῆς ἀνάλογον, οἱ δὲ Α, Γ μὴ ἔστωσαν πρῶτοι πρὸς ἀλλήλους. λέγω, ὅτι δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν. ὁ γὰρ Β τὸν Γ πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Α ἄρα τὸν Δ ἤτοι μετρεĩ ἢ οὐ μετρεῖ. μετρείτω αὐτὸν πρότερον κατὰ τὸν Ε· ὁ Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Β, Γ. ἀνάλογον ἄρα [ἐστὶν] ὡς ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Ε· τοἰς Α, Β, Γ ἄρα τέταρτος ἀνάλογον προσηύρηται ὁ Ε.

Άλλὰ δὴ μὴ μετρείτω ὁ Α τὸν Δ· λέγω, ὅτι ἀδύνατόν ἐστι τοῖς Α, Β, Γ τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσευρήσθω ὁ Ε· ὁ ἄρα ἐκ τῶν Α, Ε ἰσος ἐστὶ τῷ ἐκ τῶν Β, Γ. ἀλλὰ ὁ ἑκ τῶν Β, Γ ἐστιν ὁ Δ· καὶ ὁ ἐκ τῶν Α, Ε ἄσα ἰσος ἐστὶ τῷ Δ. ὁ Α ἄρα τὸν Ε πολλα-πλασιάσας τὸν Δ πεποίηκεν· ὁ Α ἄρα τὸν Δ μετρεῖ κατὰ τὸν Ε· ὥστε μετρεῖ ὁ Α τὸν Δ. ἀλλὰ καὶ οὐ μετρεῖ· ὅπερ ἄτοπον. οὐκ ἄρα δυνάτον ἐστι τοῖς Α, Β, Γ τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν.

as A is to B, (so) C (is) to D, and as B (is) to C, (so) D (is) to E, thus, via equality, as A (is) to C, (so) C (is) to E [Prop. 7.14]. And A and C (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least (numbers) measure those numbers having the same ratio as them (the same number of times), the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures C, (as) the leading (measuring) the leading. And it also measures itself. Thus, A measures A and C, which are prime to one another. The very thing is impossible. Thus, it is not possible to find a fourth (number) proportional to A, B, C.

And so let A, B, C again be continuously proportional, and let A and C not be prime to one another. I say that it is possible to find a fourth (number) proportional to them. For let B make D (by) multiplying C. Thus, A either measures or does not measure D. Let it, first of all, measure (D) according to E. Thus, A has made D (by) multiplying E. But, in fact, B has also made D (by) multiplying C. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C. Thus, proportionally, as A [is] to B, (so) C (is) to E [Prop. 7.19]. Thus, a fourth (number) proportional to A, B, C has been found, (namely) E.

And so let A not measure D. I say that it is impossible to find a fourth number proportional to A, B, C. For, if possible, let it have been found, (and let it be) E. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C. But, the (number created) from (multiplying) B, C is D. And thus the (number created) from (multiplying) A, E is equal to D. Thus, A has made D (by) multiplying E. Thus, A measures D according to E. Hence, A measures D. But, it also does not measure (D). The very thing (is) absurd. Thus, it is not possible to find a fourth number proportional to A, B, C when A does not measure D. And so (let) A, B, C (be) neither continuously proportional, nor (let) the outermost of them (be) prime to one another. And let B make D (by) multiplying C. So, similarly, it can be show that if A measures D then it is possible to find a fourth (number) proportional to (A, B, C), and impossible if (A) does not measure (D). (Which is) the very thing it was required to show.

[†] The proof of this proposition is incorrect. There are, in fact, only two cases. Either *A*, *B*, *C* are continuously proportional, with *A* and *C* prime to one another, or not. In the first case, it is impossible to find a fourth proportional number. In the second case, it is possible to find a fourth proportional number provided that *A* measures *B* times *C*. Of the four cases considered by Euclid, the proof given in the second case is incorrect, since it only demonstrates that if A : B :: C : D then a number *E* cannot be found such that B : C :: D : E. The proofs given in the other three

cases are correct.

χ΄.

Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.



Έστωσαν οἱ προτεθέντες πρῶτοι ἀριθμοὶ οἱ Α, Β, Γ· λέγω, ὅτι τῶν Α, Β, Γ πλείους εἰσὶ πρῶτοι ἀριθμοί.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν A, B, Γ ἐλάχιστος μετρούμενος καὶ ἔστω ΔΕ, καὶ προσκείσθω τῷ ΔΕ μονὰς ἡ ΔΖ. ὁ δὴ ΕΖ ἤτοι πρῶτός ἐστιν ἢ οὖ. ἔστω πρότερον πρῶτος· εὐρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ οἱ A, B, Γ, ΕΖ πλείους τῶν A, B, Γ.

Άλλὰ δὴ μὴ ἔστω ὁ ΕΖ πρῶτος· ὑπὸ πρώτου ἄρα τινὸς ἀριθμοῦ μετρεῖται. μετρείσθω ὑπὸ πρώτου τοῦ Η· λέγω, ὅτι ὁ Η οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ δυνατόν, ἔστω. οἱ δὲ Α, Β, Γ τὸν ΔΕ μετροῦσιν· καὶ ὁ Η ἄρα τὸν ΔΕ μετρήσει. μετρεῖ δὲ καὶ τὸν ΕΖ· καὶ λοιπὴν τὴν ΔΖ μονάδα μετρήσει ὁ Η ἀριθμὸς ὥν· ὅπερ ἄτοπον. οὐκ ἄρα ὁ Η ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. καὶ ὑπόκειται πρῶτος. εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ πλείους τοῦ προτεθέντος πλήθους τῶν Α, Β, Γ οἱ Α, Β, Γ, Η· ὅπερ ἔδει δεῖξαι.

хα'.

Έὰν ἄρτιοι ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, ὁ ὅλος ἀρτιός ἐστιν.



Συγχείσθωσαν γὰρ ἄρτιοι ἀριθμοὶ ὁποσοιοῦν οἱ AB, BΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὅλος ὁ ΑΕ ἄρτιός ἐστιν.

Έπει γὰρ ἕχαστος τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ ἄρτιός ἐστιν, ἔχει μέρος ἥμισυ· ὥστε χαι ὅλος ὁ ΑΕ ἔχει μέρος ἤμισυ. ἄρτιος δὲ ἀριθμός ἐστιν ὁ δίχα διαιρούμενος· ἄρτιος ἄρα ἐστιν ὁ ΑΕ· ὅπερ ἔδει δεῖξαι.

ELEMENTS BOOK 9

Proposition 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.



Let A, B, C be the assigned prime numbers. I say that the (set of all) primes numbers is more numerous than A, B, C.

For let the least number measured by A, B, C have been taken, and let it be DE [Prop. 7.36]. And let the unit DF have been added to DE. So EF is either prime, or not. Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF, (which is) more numerous than A, B, C, has been found.

And so let EF not be prime. Thus, it is measured by some prime number [Prop. 7.31]. Let it be measured by the prime (number) G. I say that G is not the same as any of A, B, C. For, if possible, let it be (the same). And A, B, C (all) measure DE. Thus, G will also measure DE. And it also measures EF. (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C. And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Proposition 21

If any multitude whatsoever of even numbers is added together then the whole is even.



For let any multitude whatsoever of even numbers, AB, BC, CD, DE, lie together. I say that the whole, AE, is even.

For since everyone of AB, BC, CD, DE is even, it has a half part [Def. 7.6]. And hence the whole AE has a half part. And an even number is one (which can be) divided in half [Def. 7.6]. Thus, AE is even. (Which is)

the very thing it was required to show.

Proposition 22

If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.



For let any even multitude whatsoever of odd numbers, AB, BC, CD, DE, lie together. I say that the whole, AE, is even.

For since everyone of AB, BC, CD, DE is odd then, a unit being subtracted from each, everyone of the remainders will be (made) even [Def. 7.7]. And hence the sum of them will be even [Prop. 9.21]. And the multitude of the units is even. Thus, the whole AE is also even [Prop. 9.21]. (Which is) the very thing it was required to show.

Proposition 23

If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.

For let any multitude whatsoever of odd numbers, AB, BC, CD, lie together, and let the multitude of them be odd. I say that the whole, AD, is also odd.

For let the unit DE have been subtracted from CD. The remainder CE is thus even [Def. 7.7]. And CA is also even [Prop. 9.22]. Thus, the whole AE is also even [Prop. 9.21]. And DE is a unit. Thus, AD is odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 24

If an even (number) is subtracted from an(other) even number then the remainder will be even.



For let the even (number) BC have been subtracted from the even number AB. I say that the remainder CAis even.

For since AB is even, it has a half part [Def. 7.6]. So, for the same (reasons), BC also has a half part. And hence the remainder [CA has a half part]. [Thus,] AC is even. (Which is) the very thing it was required to show.

хβ'.

Έαν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν ἀρτιον ῇ, ὁ ὅλος ἀρτιος ἔσται.



Συγκείσθωσαν γὰρ περισσοὶ ἀριθμοὶ ὁσοιδηποτοῦν ἄρτιοι τὸ πλῆθος οἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὅλος ὁ ΑΕ ἄρτιός ἐστιν.

Έπει γὰρ ἕκαστος τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ περιττός ἐστιν, ἀφαιρεθείσης μονάδος ἀφ' ἑκάστου ἕκαστος τῶν λοιπῶν ἀρτιος ἔσται· ὥστε καὶ ὁ συγκείμενος ἐξ αὐτῶν ἄρτιος ἔσται. ἔστι δὲ καὶ τὸ πλῆθος τῶν μονάδων ἄρτιον. καὶ ὅλος ἄρα ὁ ΑΕ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

хγ′.

Έὰν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν περισσὸν ἢ, καὶ ὁ ὅλος περισσὸς ἔσται.

$$A \quad B \quad \Gamma \qquad E \quad \Delta$$

Συγκείσθωσαν γὰρ ὑποσοιοῦν περισσοὶ ἀριθμοί, ῶν τὸ πλῆθος περισσὸν ἔστω, οἱ ΑΒ, ΒΓ, ΓΔ· λέγω, ὅτι καὶ ὅλος ὁ ΑΔ περισσός ἐστιν.

Άφηρήσθω ἀπὸ τοῦ ΓΔ μονὰς ἡ ΔΕ· λοιπὸς ἄρα ὁ ΓΕ ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΓΑ ἄρτιος· καὶ ὅλος ἄρα ὁ ΑΕ ἄρτιός ἐστιν. καί ἐστι μονὰς ἡ ΔΕ. περισσὸς ἄρα ἐστὶν ὁ ΑΔ· ὅπερ ἔδει δεῖξαι.

χδ'.

Έὰν ἀπὸ ἀρτίου ἀριθμοῦ ἄρτιος ἀφαιρεθῆ, ὁ λοιπὸς ἄρτιος ἔσται.



Άπὸ γὰρ ἀρτίου τοῦ ΑΒ ἄρτιος ἀφηρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Έπεὶ γὰρ ὁ ΑΒ ἄρτιός ἐστιν, ἔχει μέρος ἥμισυ. διὰ τὰ αὐτὰ δὴ καὶ ὁ ΒΓ ἔχει μέρος ἥμισυ· ὥστε καὶ λοιπὸς [ὁ ΓΑ ἔχει μέρος ἥμισυ] ἄρτιος [ἄρα] ἐστὶν ὁ ΑΓ· ὅπερ ἔδει δεῖξαι.

κε΄.

Έλν ἀπὸ ἀρτίου ἀριθμοῦ περισσὸς ἀφαιρεθῆ, ὁ λοιπὸς περισσὸς ἔσται.



Άπὸ γὰρ ἀρτίου τοῦ ΑΒ περισσὸς ἀφηρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσός ἐστιν.

Άφηρήσθω γὰρ ἀπὸ τοῦ ΒΓ μονὰς ἡ ΓΔ· ὁ ΔΒ ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΑΒ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. καί ἐστι μονὰς ἡ ΓΔ· ὁ ΓΑ ἄρα περισσός ἐστιν· ὅπερ ἔδει δεῖξαι.

xร'**.**

Έὰν ἀπὸ περισσοῦ ἀριθμοῦ περισσὸς ἀφαιρεθῆ, ὁ λοιπὸς ẳρτιος ἔσται.

ἀπὸ γὰρ περισσοῦ τοῦ AB περισσὸς ἀφηρήσθω ὁ BΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Έπεὶ γὰρ ὁ AB περισσός ἐστιν, ἀφηρήσθω μονὰς ἡ BΔ· λοιπὸς ἄρα ὁ AΔ ἄρτιός ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ὁ ΓΔ ἄρτιός ἐστιν· ὥστε καὶ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

хζ'.

Έαν ἀπὸ περισσοῦ ἀριθμοῦ ἀρτιος ἀφαιρεθῆ, ὁ λοιπὸς περισσὸς ἔσται.



Άπὸ γὰρ περισσοῦ τοῦ AB ἄρτιος ἀφηρήσθω ὁ BΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσός ἐστιν.

Άφηρήσθω [γὰρ] μονὰς ἡ ΑΔ· ὁ ΔΒ ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΒΓ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΓΔ ἄρτιός ἐστιν. περισσὸς ἄρα ὁ ΓΑ· ὅπερ ἔδει δεῖξαι.

×η'.

Έὰν περισσὸς ἀριθμὸς ἄρτιον πολλαπλασιάσας ποιῆ τινα, ὁ γενόμενος ἄρτιος ἔσται.

Proposition 25

If an odd (number) is subtracted from an even number then the remainder will be odd.



For let the odd (number) BC have been subtracted from the even number AB. I say that the remainder CA is odd.

For let the unit CD have been subtracted from BC. DB is thus even [Def. 7.7]. And AB is also even. And thus the remainder AD is even [Prop. 9.24]. And CD is a unit. Thus, CA is odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 26

If an odd (number) is subtracted from an odd number then the remainder will be even.



For let the odd (number) BC have been subtracted from the odd (number) AB. I say that the remainder CAis even.

For since AB is odd, let the unit BD have been subtracted (from it). Thus, the remainder AD is even [Def. 7.7]. So, for the same (reasons), CD is also even. And hence the remainder CA is even [Prop. 9.24]. (Which is) the very thing it was required to show.

Proposition 27

If an even (number) is subtracted from an odd number then the remainder will be odd.



For let the even (number) BC have been subtracted from the odd (number) AB. I say that the remainder CAis odd.

[For] let the unit AD have been subtracted (from AB). DB is thus even [Def. 7.7]. And BC is also even. Thus, the remainder CD is also even [Prop. 9.24]. CA (is) thus odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 28

If an odd number makes some (number by) multiplying an even (number) then the created (number) will be even.



Περισσός γὰρ ἀριθμός ὁ Α ἄρτιον τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ ἄρτιός ἐστιν.

Έπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ό Γ ἄρα σύγκειται έκ τοσούτων ἴσων τῷ Β, ὄσαι εἰσὶν ἐν τῷ Α μονάδες. καί ἐστιν ὁ Β ἄρτιος ὁ Γ ἄρα σύγκειται έξ ἀρτίων. ἐὰν δὲ ἄρτιοι ἀριθμοὶ ὑποσοιοῦν συντεθῶσιν, ὑ όλος ἄρτιός ἐστιν. ἄρτιος ἄρα ἐστὶν ὁ Γ· ὅπερ ἔδει δεῖξαι.

×ϑ'.

Έὰν περισσὸς ἀριθμὸς περισσὸν ἀριθμὸν πολλαπλασιάςας ποιῆ τινα, ὁ γενόμενος περισσὸς ἔσται.



Περισσός γὰρ ἀριθμός ὁ Α περισσόν τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ περισσός ἐστιν.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηχεν, ό Γ ἄρα σύγκειται έκ τοσούτων ἴσων τῷ Β, ὅσαι εἰσὶν ἐν τῷ Α μονάδες. καί ἐστιν ἑκάτερος τῶν Α, Β περισσός ὁ Γ ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ῶν τὸ πλῆθος περισσόν έστιν. ὥστε ὁ Γ περισσός ἐστιν· ὅπερ ἔδει δεῖξαι.

λ'.

Έὰν περισσὸς ἀριθμὸς ἄρτιον ἀριθμὸν μετρῆ, καὶ τὸν ημισυν αὐτοῦ μετρήσει.



Περισσός γὰρ ἀριθμὸς ὁ Α ἄρτιον τὸν Β μετρείτω· λέγω, ότι καὶ τὸν ἤμισυν αὐτοῦ μετρήσει.

Έπεὶ γὰρ ὁ Α τὸν Β μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Γ · λέγω, ὅτι ὁ Γ οὐχ ἔστι περισσός. εἰ γὰρ δυνατόν, ἔστω. καὶ ἐπεὶ ὁ Α τὸν Β μετρεῖ κατὰ τὸν Γ, ὁ Α ἄρα τὸν Γ πολλαπλασιάσας τὸν Β πεποίηχεν. ὁ Β ἄρα σύγχειται ἐχ περισσῶν ἀριθμῶν, ῶν τὸ πλῆθος περισσόν ἐστιν. ὁ B ἄρα numbers, (and) the multitude of them is odd. B is thus



For let the odd number A make C (by) multiplying the even (number) B. I say that C is even.

For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A [Def. 7.15]. And B is even. Thus, C is composed out of even (numbers). And if any multitude whatsoever of even numbers is added together then the whole is even [Prop. 9.21]. Thus, C is even. (Which is) the very thing it was required to show.

Proposition 29

If an odd number makes some (number by) multiplying an odd (number) then the created (number) will be odd.



For let the odd number A make C (by) multiplying the odd (number) B. I say that C is odd.

For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A [Def. 7.15]. And each of A, B is odd. Thus, C is composed out of odd (numbers), (and) the multitude of them is odd. Hence C is odd [Prop. 9.23]. (Which is) the very thing it was required to show.

Proposition 30

If an odd number measures an even number then it will also measure (one) half of it.



For let the odd number A measure the even (number) B. I say that (A) will also measure (one) half of (B).

For since A measures B, let it measure it according to C. I say that C is not odd. For, if possible, let it be (odd). And since A measures B according to C, A has thus made B (by) multiplying C. Thus, B is composed out of odd περισσός ἐστιν· ὅπερ ἄτοπον· ὑπόκειται γὰρ ἄρτιος. οὐκ ἄρα ὁ Γ περισσός ἐστιν· ἄρτιος ἄρα ἐστὶν ὁ Γ. ὥστε ὁ Α τὸν Β μετρεῖ ἀρτιάκις. διὰ δὴ τοῦτο καὶ τὸν ἤμισυν αὐτοῦ μετρήσει· ὅπερ ἔδει δεῖξαι.

λα΄.

Έὰν περισσὸς ἀριθμὸς πρός τινα ἀριθμὸν πρῶτος ἦ, καὶ πρὸς τὸν διπλασίονα αὐτοῦ πρῶτος ἔσται.



Περισσὸς γὰρ ἀριθμὸς ὁ Α πρός τινα ἀριθμὸν τὸν Β πρῶτος ἔστω, τοῦ δὲ Β διπλασίων ἔστω ὁ Γ· λέγω, ὅτι ὁ Α [καὶ] πρὸς τὸν Γ πρῶτός ἐστιν.

Εἰ γὰρ μή εἰσιν [οἱ Α, Γ] πρῶτοι, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. καί ἐστιν ὁ Α περισσός περισσὸς ἄρα καὶ ὁ Δ. καὶ ἐπεὶ ὁ Δ περισσὸς ῶν τὸν Γ μετρεῖ, καί ἐστιν ὁ Γ ἄρτιος, καὶ τὸν ἤμισυν ἄρα τοῦ Γ μετρήσει [ὁ Δ]. τοῦ δὲ Γ ἤμισύ ἐστιν ὁ Β· ὁ Δ ἄρα τὸν Β μετρεῖ. μετρεῖ δὲ καὶ τὸν Α. ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ Α πρὸς τὸν Γ πρῶτος οὕκ ἐστιν. οἱ Α, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

λβ'.

Τῶν ἀπὸ δύαδος διπλασιαζομένων ἀριθμων ἕκαστος ἀρτιάκις ἄρτιός ἐστι μόνον.



Ἀπὸ γὰρ δύαδος τῆς Α δεδιπλασιάσθωσαν ὑσοιδηποτοῦν ἀριθμοὶ οἱ Β, Γ, Δ· λέγω, ὅτι οἱ Β, Γ, Δ ἀρτιάχις ἀρτιοί εἰσι μόνον.

Ότι μὲν οὖν ἕκαστος [τῶν B, Γ, Δ] ἀρτιάκις ἄρτιός ἐστιν, φανερόν· ἀπὸ γὰρ δυάδος ἐστὶ διπλασιασθείς. λέγω, ὅτι καὶ μόνον. ἐκκείσθω γὰρ μονάς. ἐπεὶ οὖν ἀπὸ μονάδος ὅποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτός ἐστιν, ὁ μέγιστος τῶν Α, Β, Γ, Δ ὁ odd [Prop. 9.23]. The very thing (is) absurd. For (B) was assumed (to be) even. Thus, C is not odd. Thus, C is even. Hence, A measures B an even number of times. So, on account of this, (A) will also measure (one) half of (B). (Which is) the very thing it was required to show.

Proposition 31

If an odd number is prime to some number then it will also be prime to its double.



For let the odd number A be prime to some number B. And let C be double B. I say that A is [also] prime to C.

For if [A and C] are not prime (to one another) then some number will measure them. Let it measure (them), and let it be D. And A is odd. Thus, D (is) also odd. And since D, which is odd, measures C, and C is even, [D] will thus also measure half of C [Prop. 9.30]. And Bis half of C. Thus, D measures B. And it also measures A. Thus, D measures (both) A and B, (despite) them being prime to one another. The very thing is impossible. Thus, A is not unprime to C. Thus, A and C are prime to one another. (Which is) the very thing it was required to show.

Proposition 32

Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even (number) only.



For let any multitude of numbers whatsoever, B, C, D, have been (continually) doubled, (starting) from the dyad A. I say that B, C, D are even-times-even (numbers) only.

In fact, (it is) clear that each [of B, C, D] is an even-times-even (number). For it is doubled from a dyad [Def. 7.8]. I also say that (they are even-times-even numbers) only. For let a unit be laid down. Therefore, since

Δ ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καί ἐστιν ἕκαστος τῶν Α, Β, Γ ἄρτιος· ὁ Δ ἄρα ἀρτιάκις ἄρτιός ἐστι μόνον. ὁμοίως δὴ δείξομεν, ὅτι [καὶ] ἑκάτερος τῶν Β, Γ ἀρτιάκις ἄρτιός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

λγ΄.

Έὰν ἀριθμὸς τὸν ἥμισυν ἔχῃ περισσόν, ἀρτιάχις περισσός ἐστι μόνον.

Άριθμὸς γὰρ ὁ Α τὸν ἥμισυν ἐχέτω περισσόν λέγω, ὅτι ὁ Α ἀρτιάχις περισσός ἐστι μόνον.

Ότι μὲν οὖν ἀρτιάχις περισσός ἐστιν, φανερόν· ὁ γὰρ ἥμισυς αὐτοῦ περισσὸς ὢν μετρεῖ αὐτὸν ἀρτιάχις, λέγω δή, ὅτι χαὶ μόνον. εἰ γὰρ ἔσται ὁ Α χαὶ ἀρτιάχις ἄρτιος, μετρηθήσεται ὑπὸ ἀρτίου χατὰ ἄρτιον ἀριθμόν· ὥστε χαὶ ὁ ἥμισυς αὐτοῦ μετρηθήσεται ὑπὸ ἀρτίου ἀριθμοῦ περισσὸς ὥν· ὅπερ ἐστὶν ἄτοπον. ὁ Α ἄρα ἀρτιάχις περισσός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

λδ΄.

Έὰν ἀριθμὸς μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ἢ, μήτε τὸν ἥμισυν ἔχῃ περισσόν, ἀρτιάχις τε ἄρτιός ἐστι καὶ ἀρτιάχις περισσός.

Άριθμὸς γὰρ ὁ Α μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ἔστω μήτε τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάχις τέ ἐστιν ἄρτιος καὶ ἀρτιάχις περισσός.

Ότι μὲν οῦν ὁ A ἀρτιάχις ἐστὶν ἄρτιος, φανερόν· τὸν γὰρ ἥμισυν οὐχ ἔχει περισσόν. λέγω δή, ὅτι καὶ ἀρτιάχις περισσός ἐστιν. ἐὰν γὰρ τὸν A τέμνωμεν δίχα καὶ τὸν ἥμισυν αὐτοῦ δίχα καὶ τοῦτο ἀεὶ ποιῶμεν, καταντήσομεν εἴς τινα ἀριθμὸν περισσόν, ὃς μετρήσει τὸν A κατὰ ἄρτιον ἀριθμόν. εἰ γὰρ οὕ, καταντήσομεν εἰς δυάδα, καὶ ἔσται ὁ A τῶν ἀπὸ δυάδος διπλασιαζομένων· ὅπερ οὐχ ὑπόχειται. ὥστε ὁ A ἀρτιάχις περισσόν ἐστιν. ἐδείχθη δὲ καὶ ἀρτιάχις ἄρτιος. ὁ A ἄρα ἀρτιάχις τε ἄρτιός ἐστι καὶ ἀρτιάχις περισσός· ὅπερ ἔδει δεῖξαι. any multitude of numbers whatsoever are continuously proportional, starting from a unit, and the (number) A after the unit is prime, the greatest of A, B, C, D, (namely) D, will not be measured by any other (numbers) except A, B, C [Prop. 9.13]. And each of A, B, C is even. Thus, D is an even-time-even (number) only [Def. 7.8]. So, similarly, we can show that each of B, C is [also] an even-time-even (number) only. (Which is) the very thing it was required to show.

Proposition 33

If a number has an odd half then it is an even-timeodd (number) only.

For let the number *A* have an odd half. I say that *A* is an even-times-odd (number) only.

In fact, (it is) clear that (A) is an even-times-odd (number). For its half, being odd, measures it an even number of times [Def. 7.9]. So I also say that (it is an even-times-odd number) only. For if A is also an even-times-even (number) then it will be measured by an even (number) according to an even number [Def. 7.8]. Hence, its half will also be measured by an even number, (despite) being odd. The very thing is absurd. Thus, A is an even-times-odd (number) only. (Which is) the very thing it was required to show.

Proposition 34

If a number is neither (one) of the (numbers) doubled from a dyad, nor has an odd half, then it is (both) an even-times-even and an even-times-odd (number).

For let the number A neither be (one) of the (numbers) doubled from a dyad, nor let it have an odd half. I say that A is (both) an even-times-even and an even-times-odd (number).

In fact, (it is) clear that A is an even-times-even (number) [Def. 7.8]. For it does not have an odd half. So I say that it is also an even-times-odd (number). For if we cut A in half, and (then cut) its half in half, and we do this continually, then we will arrive at some odd number which will measure A according to an even number. For if not, we will arrive at a dyad, and A will be (one) of the (numbers) doubled from a dyad. The very opposite thing (was) assumed. Hence, A is an even-times-odd (number) [Def. 7.9]. And it was also shown (to be) an even-times-even (number). Thus, A is (both) an even-times-even and an even-times-odd (number). (Which is)

the very thing it was required to show.

λε΄.

Έὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, ἀφαιρεθῶσι δὲ ἀπό τε τοῦ δευτέρου καὶ τοῦ ἐσχάτου ἴσοι τῷ πρώτω, ἔσται ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ή τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας.



Έστωσαν ὁποσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, BΓ, Δ, ΕΖ ἀφχόμενοι ἀπὸ ἐλαχίστου τοῦ Α, καὶ ἀφηρήσθω ἀπὸ τοῦ ΒΓ καὶ τοῦ ΕΖ τῷ Α ἴσος ἑκάτερος τῶν ΒΗ, ΖΘ· λέγω, ὅτι ἐστὶν ὡς ὁ ΗΓ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τούς A, BΓ, Δ .

Κείσθω γὰρ τῶ μὲν $B\Gamma$ ἴσος ὁ ZK, τῶ δὲ Δ ἴσος ὁ ZΛ. καὶ ἐπεὶ ὁ ΖΚ τῷ ΒΓ ἴσος ἐστίν, ῶν ὁ ΖΘ τῷ ΒΗ ἴσος ἐστίν, λοιπὸς ἄρα ὁ ΘΚ λοιπῷ τῷ ΗΓ ἐστιν ἴσος. καὶ ἐπεί ἐστιν ὡς ό EZ πρὸς τὸν Δ, οὕτως
ὁ Δ πρὸς τὸν BΓ καὶ ὁ BΓ πρὸς τὸν A, ἴσος δὲ ὁ μὲν Δ τῷ ZA, ὁ δὲ BΓ τῷ ZK, ὁ δὲ A τῷ $Z\Theta$, ἔστιν ἄρα ὡς ὁ EZ πρὸς τὸν $Z\Lambda$, οὕτως ὁ ΛZ πρὸς τὸν ΖΚ καὶ ὁ ΖΚ πρὸς τὸν ΖΘ. διελόντι, ὡς ὁ ΕΛ πρὸς τὸν ΛΖ, οὕτως
ὑ ΛΚ πρὸς τὸν ΖΚ καὶ ὑ ΚΘ πρὸς τὸν ΖΘ. ἔστιν ἄρα καὶ ὡς εἶς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως άπαντες οἱ ἡγούμενοι πρὸς ἄπαντας τοὺς ἑπομένους. ἔστιν ἄρα ὡς ὁ ΚΘ πρὸς τὸν ΖΘ, οὕτως οἱ ΕΛ, ΛΚ, ΚΘ πρὸς τούς ΛΖ, ΖΚ, ΘΖ. ἴσος δὲ ὁ μὲν ΚΘ τῷ ΓΗ, ὁ δὲ ΖΘ τῷ A, où bè ΛZ , ZK, ΘZ toùc Δ , BF, A[.] ἔστιν ἄρα ὡς ὁ FH πρὸς τὸν Α, οὕτως
ἑ ΕΘ πρὸς τοὺς Δ, ΒΓ, Α. ἔστιν ἄρα ώς ή τοῦ δευτέρου ὑπεροχή πρὸς τὸν πρῶτον, οὕτως ή τοῦ έσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας. ὅπερ ἔδει δεῖξαι.

Proposition 35[†]

If there is any multitude whatsoever of continually proportional numbers, and (numbers) equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those (numbers) before it.



Let A, BC, D, EF be any multitude whatsoever of continuously proportional numbers, beginning from the least A. And let BG and FH, each equal to A, have been subtracted from BC and EF (respectively). I say that as GC is to A, so EH is to A, BC, D.

For let FK be made equal to BC, and FL to D. And since FK is equal to BC, of which FH is equal to BG, the remainder HK is thus equal to the remainder GC. And since as EF is to D, so D (is) to BC, and BC to A [Prop. 7.13], and D (is) equal to FL, and BC to FK, and A to FH, thus as EF is to FL, so LF (is) to FK, and FK to FH. By separation, as EL (is) to LF, so LK (is) to FK, and KH to FH [Props. 7.11, 7.13]. And thus as one of the leading (numbers) is to one of the following, so (the sum of) all of the leading (numbers is) to (the sum of) all of the following [Prop. 7.12]. Thus, as KHis to FH, so EL, LK, KH (are) to LF, FK, HF. And KH (is) equal to CG, and FH to A, and LF, FK, HF to D, BC, A. Thus, as CG is to A, so EH (is) to D, BC, A. Thus, as the excess of the second (number) is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it. (Which is) the very thing it was required to show.

[†] This proposition allows us to sum a geometric series of the form a, ar, ar^2 , ar^3 , $\cdots ar^{n-1}$. According to Euclid, the sum S_n satisfies $(ar-a)/a = (ar^n - a)/S_n$. Hence, $S_n = a(r^n - 1)/(r - 1)$.

λς'.

Έὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἐκτεθῶσιν ἐν

Proposition 36[†]

If any multitude whatsoever of numbers is set out conτῆ διπλασίονι ἀναλογία, ἔως οὖ ὁ σύμπας συντεθεὶς πρῶτος tinuously in a double proportion, (starting) from a unit, γένηται, και ὁ σύμπας ἐπὶ τὸν ἔσχατον πολλαπλασιασθεὶς until the whole sum added together becomes prime, and ποιῆ τινα, ὁ γενόμενος τέλειος ἔσται.

Άπὸ γὰρ μονάδος ἐχχείσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἐν τῆ διπλασίονι ἀναλογία, ἕως οῦ ὁ σύμπας συντεθεὶς πρῶτος γένηται, οἱ Α, Β, Γ, Δ, χαὶ τῷ σύμπαντι ἴσος ἔστω ὁ Ε, χαὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ ποιείτω. λέγω, ὅτι ὁ ΖΗ τέλειός ἐστιν.



τοῦ Ε εἰλήφθωσαν ἐν τῆ διπλασίονι ἀναλογία οἱ Ε, ΘΚ, Λ, M^{\cdot} δι' ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Μ. ὁ ἄρα ἐκ τῶν Ε, Δ ἴσος ἐστὶ τῷ ἐκ τῶν Α, Μ. καί έστιν ὁ ἐκ τῶν Ε, Δ ὁ ZH· καὶ ὁ ἐκ τῶν Α, Μ ἄρα ἐστὶν ὁ ZH. ὁ A ẳρα τὸν Μ πολλαπλασιάσας τὸν ZH πεποίηκεν· ὁ Μ ἄρα τὸν ΖΗ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. καί ἐστι δυὰς ὁ Α· διπλάσιος ἄρα ἐστὶν ὁ ΖΗ τοῦ Μ. εἰσὶ δὲ καὶ οἱ Μ, Λ , ΘK, E ἑξῆς διπλάσιοι ἀλλήλων· οἱ E, ΘK, Λ , M, ZH ẳρα έξῆς ἀνάλογόν εἰσιν ἐν τῆ διπλασίονι ἀναλογία. ἀφηρήσθω δή ἀπὸ τοῦ δευτέρου τοῦ ΘΚ καὶ τοῦ ἐσχάτου τοῦ ΖΗ τῷ πρώτω τῷ Ε ἴσος ἑχάτερος τῶν ΘΝ, ΖΞ· ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ἀριθμοῦ ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας. ἔστιν ἄρα ὡς ὁ NK πρὸς τὸν Ε, οὕτως ὁ ΞΗ πρὸς τοὺς Μ, Λ, ΚΘ, Ε. καί ἐστιν ὁ ΝΚ ἴσος τῷ Ε· καὶ ὁ ΞΗ ἄρα ἴσος ἐστὶ τοῖς Μ, Λ, ΘΚ, Ε. ἔστι δὲ καὶ ὁ ΖΞ τῷ Ε ἴσος, ὁ δὲ Ε τοῖς Α, Β, Γ, Δ καὶ τῇ μονάδι. ὅλος ἄρα ὁ ΖΗ ἴσος ἐστὶ τοῖς τε E, Θ K, Λ, M καὶ τοῖς A, B, Γ, Δ καὶ τῆ μονάδι· καὶ μετρεῖται ὑπ' αὐτῶν. λέγω, ὅτι καὶ ὁ ΖΗ ὐπ' οὐδενὸς άλλου μετρηθήσεται παρ
έξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος. εἰ γὰρ δυνατόν, μετρείτω τις τὸν ΖΗ ὁ Ο, καὶ ὁ Ο μηδενὶ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ ἔστω ὁ αὐτός. καὶ ὅσάκις ὁ Ο τὸν ΖΗ μετρεῖ, τοσαῦται μονάδες the sum multiplied into the last (number) makes some (number), then the (number so) created will be perfect.

For let any multitude of numbers, A, B, C, D, be set out (continuouly) in a double proportion, until the whole sum added together is made prime. And let E be equal to the sum. And let E make FG (by) multiplying D. I say that FG is a perfect (number).



For as many as is the multitude of A, B, C, D, let so many (numbers), E, HK, L, M, have been taken in a double proportion, (starting) from E. Thus, via equality, as A is to D, so E (is) to M [Prop. 7.14]. Thus, the (number created) from (multiplying) E, D is equal to the (number created) from (multiplying) A, M. And FG is the (number created) from (multiplying) E, D. Thus, FG is also the (number created) from (multiplying) A, M [Prop. 7.19]. Thus, A has made FG (by) multiplying M. Thus, M measures FG according to the units in A. And A is a dyad. Thus, FG is double M. And M, L, HK, E are also continuously double one another. Thus, E, HK, L, M, FG are continuously proportional in a double proportion. So let HN and FO, each equal to the first (number) E, have been subtracted from the second (number) HK and the last FG (respectively). Thus, as the excess of the second number is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it [Prop. 9.35]. Thus, as NK is to E, so OG (is) to M, L, KH, E. And NK is equal to E. And thus OGis equal to M, L, HK, E. And FO is also equal to E, and E to A, B, C, D, and a unit. Thus, the whole of FGis equal to E, HK, L, M, and A, B, C, D, and a unit. And it is measured by them. I also say that FG will be ἔστωσαν ἐν τῷ Π· ὁ Π ἄρα τὸν Ο πολλαπλασιάσας τὸν ZH πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ πεποίηχεν ἕστιν ἄρα ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ. καὶ ἐπεὶ ἀπὸ μονάδος ἑξῆς ἀνάλογόν εἰσιν οἱ Α, Β, Γ, Δ, ὁ Δ ἄρα ὑπ' οὐδενὸς ἄλλου ἀριθμοῦ μετρηθήσεται παρέξ τῶν Α, Β, Γ. καὶ ὑπόκειται ὁ Ο οὐδενὶ τῶν Α, Β, Γ ὁ αὐτός· ούχ ἄρα μετρήσει ὁ Ο τὸν Δ . ἀλλ' ὡς ὁ Ο πρὸς τὸν Δ , ὁ Ε πρὸς τὸν Π· οὐδὲ
ἑ Ε ἄρα τὸν Π μετρεĩ. καί ἐστιν
ἑ Ε πρῶτος· πᾶς δὲ πρῶτος ἀριθμὸς πρὸς ἄπαντα, ὃν μὴ μετρεῖ, πρῶτός [ἐστιν]. οἱ Ε, Π ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. οἱ δὲ πρῶτοι καὶ ἐλάγιστοι, οἱ δὲ ἐλάγιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάχις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον καί ἐστιν ὡς ὁ Ε πρὸς τὸν Π, ό Ο πρὸς τὸν Δ. ἰσάχις ẳρα
ὁ Ε τὸν Ο μετρεῖ χαὶ ὁ Π τὸν Δ . ὁ δ
ἑ Δ ὑπ' οὐδενὸς ἄλλου μετρεῖται παρ
ἑξ τῶν A, B, Γ· ό Π ἄρα ἑνὶ τῶν A, B, Γ ἐστιν ὁ αὐτός. ἔστω τῷ B ὁ αὐτός. καὶ ὅσοι εἰσὶν οἱ Β, Γ, Δ τῷ πλήθει τοσοῦτοι εἰλήφθωσαν άπὸ τοῦ Ε οἱ Ε, ΘΚ, Λ. καί εἰσιν οἱ Ε, ΘΚ, Λ τοῖς Β, Γ, Δ ἐν τῷ αὐτῷ λόγω· δι' ἴσου ἄρα ἐστὶν ὡς ὁ ${
m B}$ πρὸς τὸν ${
m \Delta},$ ὁ Ε πρὸς τὸν Λ. ὁ ἄρα ἐχ τῶν Β, Λ ἴσος ἐστὶ τῷ ἐχ τῶν Δ , E^{\cdot} ἀλλ' ὁ ἐκ τῶν Δ, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Π, Ο· καὶ ὁ ἐκ τῶν Π, Ο ἀρα ἴσος ἐστὶ τῷ ἐ
κ τῶν Β, Λ. ἔστιν ἀρα ὡς ὁ Π πρὸς τὸν B,
ὑ Λ πρὸς τὸν Ο.
 χαί ἐστιν ὑ Π τῷ B ὑ αὐτός: καὶ ὁ Λ ẳρα τω Ο ἐστιν ὁ αὐτός· ὅπερ ἀδύνατον· ὁ γὰρ Ο ύπόχειται μηδενί τῶν ἐχχειμένων ὁ αὐτός· οὐχ ἄρα τὸν ΖΗ μετρήσει τις ἀριθμὸς παρὲξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος.
καὶ ἐδείχη ὁ ZH τοῖς A, B, Γ, Δ, E, ΘK, Λ, Μ καὶ τῆ μονάδι ἴσος. τέλειος δὲ ἀριθμός ἐστιν ὁ τοῖς έαυτοῦ μέρεσιν ἴσος ὤν· τέλειος ἄρα ἐστὶν ὁ ΖΗ· ὅπερ ἔδει δεĩξαι.

measured by no other (numbers) except A, B, C, D, E, HK, L, M, and a unit. For, if possible, let some (number) P measure FG, and let P not be the same as any of A, B, C, D, E, HK, L, M. And as many times as Pmeasures FG, so many units let there be in Q. Thus, Qhas made FG (by) multiplying P. But, in fact, E has also made FG (by) multiplying D. Thus, as E is to Q, so P (is) to D [Prop. 7.19]. And since A, B, C, D are continually proportional, (starting) from a unit, D will thus not be measured by any other numbers except A, B, C [Prop. 9.13]. And P was assumed not (to be) the same as any of A, B, C. Thus, P does not measure D. But, as P (is) to D, so E (is) to Q. Thus, E does not measure Q either [Def. 7.20]. And E is a prime (number). And every prime number [is] prime to every (number) which it does not measure [Prop. 7.29]. Thus, E and Qare prime to one another. And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. And as E is to Q, (so) P (is) to D. Thus, E measures P the same number of times as Q (measures) D. And Dis not measured by any other (numbers) except A, B, C. Thus, Q is the same as one of A, B, C. Let it be the same as B. And as many as is the multitude of B, C, D, let so many (of the set out numbers) have been taken, (starting) from E, (namely) E, HK, L. And E, HK, L are in the same ratio as B, C, D. Thus, via equality, as B (is) to D, (so) E (is) to L [Prop. 7.14]. Thus, the (number created) from (multiplying) B, L is equal to the (number created) from multiplying D, E [Prop. 7.19]. But, the (number created) from (multiplying) D, E is equal to the (number created) from (multiplying) Q, P. Thus, the (number created) from (multiplying) Q, P is equal to the (number created) from (multiplying) B, L. Thus, as Q is to B, (so) L (is) to P [Prop. 7.19]. And Q is the same as B. Thus, L is also the same as P. The very thing (is) impossible. For P was assumed not (to be) the same as any of the (numbers) set out. Thus, FG cannot be measured by any number except A, B, C, D, E, HK, L, M, and a unit. And FG was shown (to be) equal to (the sum of) A, B, C, D, E, HK, L, M, and a unit. And a perfect number is one which is equal to (the sum of) its own parts [Def. 7.22]. Thus, FG is a perfect (number). (Which is) the very thing it was required to show.

[†] This proposition demonstrates that perfect numbers take the form $2^{n-1} (2^n - 1)$ provided that $2^n - 1$ is a prime number. The ancient Greeks knew of four perfect numbers: 6, 28, 496, and 8128, which correspond to n = 2, 3, 5, and 7, respectively.

ELEMENTS BOOK 10

Incommensurable Magnitudes †

^{\dagger} The theory of incommensurable magntidues set out in this book is generally attributed to Theaetetus of Athens. In the footnotes throughout this book, *k*, *k'*, *etc.* stand for distinct ratios of positive integers.

Οροι.

α΄. Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μετρῳ μετρούμενα, ἀσύμμετρα δέ, ῶν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

β'. Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ' αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ' αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον χοινὸν μέτρον γενέσθαι.

γ'. Τούτων ὑποχειμένων δείχνυται, ὅτι τῆ προτεθείση εὐθεία ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήχει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οῦν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήχει καὶ δυνάμει εἴτε δυνάμει μόνον ῥῃταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλογοι καλείσθωσαν.

δ΄. Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ἑητόν, καὶ τὰ τούτῷ σύμμετρα ἑητά, τὰ δὲ τούτῷ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἶη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

Definitions

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.^{\dagger}

2. (Two) straight-lines are commensurable in square[‡] when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them.[§]

3. These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straight-line—those (incommensurable) in length only, and those also (commensurable or incommensurable) in square.[¶] Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.*

4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their square-roots[§] (also be called) irrational—the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).^{||}

[†] In other words, two magnitudes α and β are commensurable if $\alpha : \beta :: 1 : k$, and incommensurable otherwise.

[‡] Literally, "in power".

[§] In other words, two straight-lines of length α and β are commensurable in square if $\alpha : \beta :: 1 : k^{1/2}$, and incommensurable in square otherwise. Likewise, the straight-lines are commensurable in length if $\alpha : \beta :: 1 : k$, and incommensurable in length otherwise.

¶ To be more exact, straight-lines can either be commensurable in square only, incommensurable in length only, or commenusrable/incommensurable in both length and square, with an assigned straight-line.

* Let the length of the assigned straight-line be unity. Then rational straight-lines have lengths expressible as k or $k^{1/2}$, depending on whether the lengths are commensurable in length, or in square only, respectively, with unity. All other straight-lines are irrational.

^{\$} The square-root of an area is the length of the side of an equal area square.

 \parallel The area of the square on the assigned straight-line is unity. Rational areas are expressible as k. All other areas are irrational. Thus, squares whose sides are of rational length have rational areas, and *vice versa*.

α'.

Δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῆ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἕλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους.

Έστω δύο μεγέθη ἄνισα τὰ ΑΒ, Γ, ῶν μεῖζον τὸ ΑΒ·

Proposition 1[†]

If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will λέγω, ὅτι, ἐαν ἀπὸ τοῦ AB ἀφαιρεθῆ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ Γ μεγέθους.



Τὸ Γ γὰρ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ AB μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΔE τοῦ μὲν Γ πολλαπλάσιον, τοῦ δὲ AB μεῖζον, καὶ διηρήσθω τὸ ΔE εἰς τὰ τῷ Γ ἴσα τὰ ΔZ , ZH, HE, καὶ ἀφηρήσθω ἀπὸ μὲν τοῦ AB μεῖζον ἢ τὸ ἤμισυ τὸ BΘ, ἀπὸ δὲ τοῦ AΘ μεῖζον ἢ τὸ ἤμισυ τὸ BΘ, ἀπὸ δὲ τοῦ AΘ μεῖζον ἢ τὸ ἤμισυ τὸ BΘ, ἀπὸ δὲ τοῦ AΘ μεῖζον ἢ τὸ ἤμισυ τὸ GK, καὶ τοῦτο ἀεὶ γιγνέσθω, ἔως ἂν αἱ ἐν τῷ AB διαιρέσεις ἰσοπληθεῖς γένωνται ταῖς ἐν τῷ ΔE διαιρέσειν.

Έστωσαν ούν αί AK, KΘ, ΘB διαιρέσεις ἰσοπληθεῖς οὐσαι ταῖς ΔΖ, ΖΗ, ΗΕ· καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΔΕ τοῦ AB, καὶ ἀφήρηται ἀπὸ μὲν τοῦ ΔΕ ἔλασσον τοῦ ἡμίσεως τὸ EH, ἀπὸ δὲ τοῦ AB μεῖζον ἢ τὸ ἤμισυ τὸ BΘ, λοιπὸν ἄρα τὸ ΗΔ λοιποῦ τοῦ ΘΑ μεῖζόν ἐστιν. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΔ τοῦ ΘΑ, καὶ ἀφήρηται τοῦ μὲν ΗΔ ἤμισυ τὸ HΖ, τοῦ δὲ ΘΑ μεῖζον ἢ τὸ ἤμισυ τὸ ΘΚ, λοιπὸν ἄρα τὸ ΔΖ λοιποῦ τοῦ AK μεῖζόν ἐστιν. ἴσον δὲ τὸ ΔΖ τῷ Γ· καὶ τὸ Γ ἄρα τοῦ AK μεῖζόν ἐστιν. ἕλασσον ἄρα τὸ AK τοῦ Γ.

Καταλείπεται ἄρα ἀπὸ τοῦ ΑΒ μεγέθους τὸ ΑΚ μέγεθος ἕλασσον ὂν τοῦ ἐχχειμένου ἐλάσσονος μεγέθους τοῦ Γ· ὅπερ ἔδει δεῖξαι.— ὑμοίως δὲ δειχθήσεται, κἂν ἡμίση ἢ τὰ ἀφαιρούμενα. be less than the lesser laid out magnitude.

Let AB and C be two unequal magnitudes, of which (let) AB (be) the greater. I say that if (a part) greater than half is subtracted from AB, and (if a part) greater than half (is subtracted) from the remainder, and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the magnitude C.



For C, when multiplied (by some number), will sometimes be greater than AB [Def. 5.4]. Let it have been (so) multiplied. And let DE be (both) a multiple of C, and greater than AB. And let DE have been divided into the (divisions) DF, FG, GE, equal to C. And let BH, (which is) greater than half, have been subtracted from AB. And (let) HK, (which is) greater than half, (have been subtracted) from AH. And let this happen continually, until the divisions in AB become equal in number to the divisions in DE.

Therefore, let the divisions (in AB) be AK, KH, HB, being equal in number to DF, FG, GE. And since DE is greater than AB, and EG, (which is) less than half, has been subtracted from DE, and BH, (which is) greater than half, from AB, the remainder GD is thus greater than the remainder HA. And since GD is greater than HA, and the half GF has been subtracted from GD, and HK, (which is) greater than half, from HA, the remainder DF is thus greater than the remainder AK. And DF (is) equal to C. C is thus also greater than AK. Thus, AK (is) less than C.

Thus, the magnitude AK, which is less than the lesser laid out magnitude C, is left over from the magnitude AB. (Which is) the very thing it was required to show. — (The theorem) can similarly be proved even if the (parts) subtracted are halves.

[†] This theorem is the basis of the so-called *method of exhaustion*, and is generally attributed to Eudoxus of Cnidus.

β΄.

Proposition 2

Έὰν δύο μεγεθῶν [ἐxxειμένων] ἀνίσων ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ καταλειπόμενον μηδέποτε καταμετρῆ τὸ πρὸ ἑαυτοῦ, ἀσύμμετρα ἔσται τὰ μεγέθη.

Δύο γὰρ μεγεθῶν ὄντων ἀνίσων τῶν AB, ΓΔ καὶ ἐλάσσονος τοῦ AB ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ περιλειπόμενον μηδέποτε καταμε-

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.

For, AB and CD being two unequal magnitudes, and AB (being) the lesser, let the remainder never measure

ΓΔ μεγέθη.



Εἰ γάρ ἐστι σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, εἰ δυνατόν, καὶ ἔστω τὸ Ε· καὶ τὸ μὲν AB τὸ Z Δ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΓΖ, τὸ δὲ ΓΖ τὸ ΒΗ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΑΗ, καὶ τοῦτο ἀεὶ γινέσθω, ἕως οῦ λειφθῆ τι μέγεθος, ὄ ἐστιν ἕλασσον τοῦ Ε. γεγονέτω, καὶ λελείφθω τὸ ΑΗ ἔλασσον τοῦ Ε. ἐπεὶ οὖν τὸ Ε τὸ AB μετρεῖ, ἀλλὰ τὸ AB τὸ ΔZ μετρεῖ, καὶ τὸ Ε ẳρα τὸ ΖΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΖ μετρήσει. ἀλλὰ τὸ ΓΖ τὸ ΒΗ μετρεῖ· καὶ τὸ Ε ἄρα τὸ ΒΗ μετρεῖ. μετρεῖ δὲ καὶ ὅλον τὸ ΑΒ· καὶ λοιπὸν ἄρα τὸ ΑΗ μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. ούκ ἄρα τὰ AB, ΓΔ μεγέθη μετρήσει τι μέγεθος ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΓΔ μεγέθη.

Έὰν ἄρα δύο μεγεθῶν ἀνίσων, καὶ τὰ ἑξῆς.

[†] The fact that this will eventually occur is guaranteed by Prop. 10.1.

γ'.

Δύο μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν



Έστω τὰ δοθέντα δύο μεγέθη σύμμετρα τὰ ΑΒ, ΓΔ, ῶν ἕλασσον τὸ ΑΒ· δεῖ δὴ τῶν ΑΒ, ΓΔ τὸ μέγιστον χοινὸν μέτρον εύρεῖν.

Τὸ ΑΒ γὰρ μέγεθος ἤτοι μετρεῖ τὸ ΓΔ ἢ οὔ. εἰ μὲν ούν μετρεῖ, μετρεῖ δὲ καὶ ἑαυτό, τὸ ΑΒ ἄρα τῶν ΑΒ, ΓΔ κοινόν μέτρον ἐστίν· καὶ φανερόν, ὅτι καὶ μέγιστον. μεῖζον γὰρ τοῦ ΑΒ μεγέθους τὸ ΑΒ οὐ μετρήσει.

Μή μετρείτω δή τὸ ΑΒ τὸ ΓΔ. καὶ ἀνθυφαιρουμένου άεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, τὸ περιλειπόμενον μετρήσει ποτὲ τὸ πρὸ ἑαυτοῦ διὰ τὸ μὴ εἶναι ἀσύμμετρα τὰ ΑΒ, ΓΔ· καὶ τὸ μὲν ΑΒ τὸ ΕΔ καταμετροῦν λειπέτω ἑαυτοῦ

τρείτω τὸ πρὸ ἑαυτοῦ· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ AB, the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater. I say that the magnitudes AB and CD are incommensurable.

For if they are commensurable then some magnitude will measure them (both). If possible, let it (so) measure (them), and let it be E. And let AB leave CF less than itself (in) measuring FD, and let CF leave AG less than itself (in) measuring BG, and let this happen continually, until some magnitude which is less than E is left. Let (this) have occurred,^{\dagger} and let AG, (which is) less than E, have been left. Therefore, since E measures AB, but AB measures DF, E will thus also measure FD. And it also measures the whole (of) CD. Thus, it will also measure the remainder CF. But, CF measures BG. Thus, Ealso measures BG. And it also measures the whole (of) AB. Thus, it will also measure the remainder AG, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude cannot measure (both) the magnitudes AB and CD. Thus, the magnitudes AB and *CD* are incommensurable [Def. 10.1].

Thus, if ... of two unequal magnitudes, and so on

Proposition 3

To find the greatest common measure of two given commensurable magnitudes.



Let AB and CD be the two given magnitudes, of which (let) AB (be) the lesser. So, it is required to find the greatest common measure of AB and CD.

For the magnitude AB either measures, or (does) not (measure), CD. Therefore, if it measures (CD), and (since) it also measures itself, AB is thus a common measure of AB and CD. And (it is) clear that (it is) also (the) greatest. For a (magnitude) greater than magnitude ABcannot measure AB.

So let AB not measure CD. And continually subtracting in turn the lesser (magnitude) from the greater, the

284

έλασσον τὸ ΕΓ, τὸ δὲ ΕΓ τὸ ΖΒ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΑΖ, τὸ δὲ ΑΖ τὸ ΓΕ μετρείτω.

Έπει οῦν τὸ ΑΖ τὸ ΓΕ μετρεῖ, ἀλλὰ τὸ ΓΕ τὸ ΖΒ μετρεῖ, καὶ τὸ ΑΖ ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ἑαυτό· καὶ ὅλον ἄρα τὸ ΑΒ μετρήσει τὸ ΑΖ. ἀλλὰ τὸ ΑΒ τὸ ΔΕ μετρεῖ· καὶ τὸ ΑΖ ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ τὸ ΓΕ· καί ὅλον ἄρα τὸ ΓΔ μετρεῖ· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή, ἔσται τι μέγεθος μεῖζον τοῦ ΑΖ, ὃ μετρήσει τὰ ΑΒ, ΓΔ. ἔστω τὸ Η. ἐπεὶ οῦν τὸ Η τὸ ΑΒ μετρεῖ, ἀλλὰ τὸ ΑΒ τὸ ΕΔ μετρεῖ, καὶ τὸ Η ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΕ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΕ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΑΒ, καὶ λοιπὸν τὸ ΑΖ μετρήσει, τὸ μεῖζον τὸ ἕλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μεῖζόν τι μέγεθος τοῦ ΑΖ τὰ ΑΒ, ΓΔ μετρήσει· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον ἐστίν.

Δύο ἄρα μεγεθῶν συμμέτρων δοθέντων τῶν AB, ΓΔ τὸ μέγιστον χοινὸν μέτρον ηὕρηται· ὅπερ ἔδει δεῖξαι.

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι, ἐἀν μέγεθος δύο μεγέθη μετρῆ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

δ'.

Τριῶν μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



Έστω τὰ δοθέντα τρία μεγέθη σύμμετρα τὰ A, B, Γ· δεῖ δὴ τῶν A, B, Γ τὸ μέγιστον χοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν A, B τὸ μέγιστον κοινὸν μέτρον, καὶ ἔστω τὸ Δ · τὸ δὴ Δ τὸ Γ ἤτοι μετρεῖ ἢ οὖ [μετρεῖ]. μετρείτω πρότερον. ἐπεὶ οὖν τὸ Δ τὸ Γ μετρεῖ, μετρεῖ δὲ remaining (magnitude) will (at) some time measure the (magnitude) before it, on account of AB and CD not being incommensurable [Prop. 10.2]. And let AB leave EC less than itself (in) measuring ED, and let EC leave AF less than itself (in) measuring FB, and let AF measure CE.

Therefore, since AF measures CE, but CE measures FB, AF will thus also measure FB. And it also measures itself. Thus, AF will also measure the whole (of) AB. But, AB measures DE. Thus, AF will also measure ED. And it also measures CE. Thus, it also measures the whole of CD. Thus, AF is a common measure of AB and CD. So I say that (it is) also (the) greatest (common measure). For, if not, there will be some magnitude, greater than AF, which will measure (both) ABand CD. Let it be G. Therefore, since G measures AB, but AB measures ED, G will thus also measure ED. And it also measures the whole of CD. Thus, G will also measure the remainder CE. But CE measures FB. Thus, Gwill also measure FB. And it also measures the whole (of) AB. And (so) it will measure the remainder AF, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude greater than AF cannot measure (both) AB and CD. Thus, AF is the greatest common measure of AB and CD.

Thus, the greatest common measure of two given commensurable magnitudes, AB and CD, has been found. (Which is) the very thing it was required to show.

Corollary

So (it is) clear, from this, that if a magnitude measures two magnitudes then it will also measure their greatest common measure.

Proposition 4

To find the greatest common measure of three given commensurable magnitudes.



Let A, B, C be the three given commensurable magnitudes. So it is required to find the greatest common measure of A, B, C.

For let the greatest common measure of the two (magnitudes) *A* and *B* have been taken [Prop. 10.3], and let it καὶ τὰ A, B, τὸ Δ ἄρα τὰ A, B, Γ μετρεῖ· τὸ Δ ἄρα τῶν A, B, Γ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· μεῖζον γὰρ τοῦ Δ μεγέθους τὰ A, B οὐ μετρεῖ.

Μὴ μετρείτω δὴ τὸ
 Δ τὸ Γ. λέγω πρῶτον, ὅτι σύμμετρά έστι τὰ Γ, Δ. ἐπεὶ γὰρ σύμμετρά ἐστι τὰ Α, Β, Γ, μετρήσει τι αὐτὰ μέγεθος, ὃ δηλαδή καὶ τὰ Α, Β μετρήσει ὥστε καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον τὸ Δ μετρήσει. μετρεῖ δὲ καὶ τὸ Γ· ὥστε τὸ εἰρημένον μέγεθος μετρήσει τὰ Γ, Δ · σύμμετρα ἄρα ἐστὶ τὰ Γ, Δ . εἰλήφθω οὖν αὐτῶν τὸ μέγιστον χοινόν μέτρον, χαὶ ἔστω τὸ Ε. ἐπεὶ οὖν τὸ Ε τὸ Δ μετρεῖ, ἀλλὰ τὸ Δ τὰ A, B μετρεῖ, καὶ τὸ E ἄρα τὰ A, B μετρήσει. μετρεῖ δὲ καὶ τὸ Γ. τὸ Ε ἄρα τὰ Α, Β, Γ μετρεῖ· τὸ Ε ἄρα τῶν Α, Β, Γ χοινόν ἐστι μέτρον. λέγω δή, ὅτι χαὶ μέγιστον. εί γὰρ δυνατόν, ἔστω τι τοῦ Ε μεῖζον μέγεθος τὸ Ζ, καὶ μετρείτω τὰ Α, Β, Γ. καὶ ἐπεὶ τὸ Ζ τὰ Α, Β, Γ μετρεῖ, καὶ τὰ Α, Β ἄρα μετρήσει καὶ τὸ τῶν Α, Β μέγιστον κοινόν μέτρον μετρήσει. τό δὲ τῶν Α, Β μέγιστον κοινόν μέτρον ἐστὶ τὸ Δ · τὸ Z ἄρα τὸ Δ μετρεῖ. μετρεῖ δὲ καὶ τὸ Γ · τὸ Z ẳρα τὰ Γ , Δ μετρεῖ· καὶ τὸ τῶν Γ , Δ ẳρα μέγιστον χοινόν μέτρον μετρήσει τὸ Ζ. ἔστι δὲ τὸ Ε· τὸ Ζ ἄρα τὸ Ε μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ άρα μεῖζόν τι τοῦ Ε μεγέθους [μέγεθος] τὰ Α, Β, Γ μετρεῖ· τὸ Ε ἄρα τῶν Α, Β, Γ τὸ μέγιστον χοινὸν μέτρον ἐστίν, ἐὰν μή μετρή τὸ Δ τὸ Γ , ἐὰν δὲ μετρή, αὐτὸ τὸ Δ .

Τριῶν ἄρα μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον κοινὸν μέτρον ηὕρηται [ὅπερ ἔδει δεῖξαι].

Πόρισμα.

Έκ δη τούτου φανερόν, ὅτι, ἐὰν μέγεθος τρία μεγέθη μετρῆ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

Ομοίως δὴ καὶ ἐπὶ πλειόνων τὸ μέγιστον κοινὸν μέτρον ληφθήσεται, καὶ τὸ πόρισμα προχωρήσει. ὅπερ ἔδει δεῖξαι. be *D*. So *D* either measures, or [does] not [measure], *C*. Let it, first of all, measure (*C*). Therefore, since *D* measures *C*, and it also measures *A* and *B*, *D* thus measures *A*, *B*, *C*. Thus, *D* is a common measure of *A*, *B*, *C*. And (it is) clear that (it is) also (the) greatest (common measure). For no magnitude larger than *D* measures (both) *A* and *B*.

So let D not measure C. I say, first, that C and D are commensurable. For if A, B, C are commensurable then some magnitude will measure them which will clearly also measure A and B. Hence, it will also measure D, the greatest common measure of A and B [Prop. 10.3 corr.]. And it also measures C. Hence, the aforementioned magnitude will measure (both) C and D. Thus, C and D are commensurable [Def. 10.1]. Therefore, let their greatest common measure have been taken [Prop. 10.3], and let it be E. Therefore, since E measures D, but D measures (both) A and B, E will thus also measure A and B. And it also measures C. Thus, E measures A, B, C. Thus, E is a common measure of A, B, C. So I say that (it is) also (the) greatest (common measure). For, if possible, let Fbe some magnitude greater than E, and let it measure A, B, C. And since F measures A, B, C, it will thus also measure A and B, and will (thus) measure the greatest common measure of A and B [Prop. 10.3 corr.]. And D is the greatest common measure of A and B. Thus, Fmeasures D. And it also measures C. Thus, F measures (both) C and D. Thus, F will also measure the greatest common measure of C and D [Prop. 10.3 corr.]. And it is E. Thus, F will measure E, the greater (measuring) the lesser. The very thing is impossible. Thus, some [magnitude] greater than the magnitude E cannot measure A, B, C. Thus, if D does not measure C then E is the greatest common measure of A, B, C. And if it does measure (C) then D itself (is the greatest common measure).

Thus, the greatest common measure of three given commensurable magnitudes has been found. [(Which is) the very thing it was required to show.]

Corollary

So (it is) clear, from this, that if a magnitude measures three magnitudes then it will also measure their greatest common measure.

So, similarly, the greatest common measure of more (magnitudes) can also be taken, and the (above) corollary will go forward. (Which is) the very thing it was required to show. ε'.

Τὰ σύμμετρα μεγέθη πρὸς ἄλληλα λόγον ἔχει, ὃν άριθμός πρός άριθμόν.



Έστω σύμμετρα μεγέθη τὰ Α, Β· λέγω, ὅτι τὸ Α πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Έπει γαρ σύμμετρά έστι τα Α, Β, μετρήσει τι αὐτα μέγεθος. μετρείτω, καὶ ἔστω τὸ Γ. καὶ ὁσάκις τὸ Γ τὸ Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ, ὑσάχις δὲ τὸ Γ τὸ Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

Ἐπεὶ οὖν τὸ Γ τὸ Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Δ κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάχις ἄρα ἡ μονὰς τὸν Δ μετρεῖ ἀριθμὸν χαὶ τὸ Γ μέγεθος τὸ Α· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ · ἀνάπαλιν ἄρα, ὡς τὸ Α πρὸς τὸ Γ, οὕτως ἑ Δ πρὸς τὴν μονάδα. πάλιν ἐπεὶ τὸ Γ τὸ Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Ε κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάχις ἄρα ἡ μονὰς τὸν Ε μετρεῖ καὶ τὸ Γ τὸ Β· έστιν ἄρα ὡς τὸ Γ πρὸς τὸ Β, οὕτως ἡ μονὰς πρὸς τὸν Ε. έδείχθη δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, ὁ Δ πρὸς τὴν μονάδα· δι ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως ὁ Δ ἀριθμὸς πρὸς τὸν Ε.

Τὰ ἄρα σύμμετρα μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον έχει, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν Ε· ὅπερ ἔδει δεῖξαι.

Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.



Let A and B be commensurable magnitudes. I say that A has to B the ratio which (some) number (has) to (some) number.

For if A and B are commensurable (magnitudes) then some magnitude will measure them. Let it (so) measure (them), and let it be C. And as many times as C measures A, so many units let there be in D. And as many times as C measures B, so many units let there be in E.

Therefore, since C measures A according to the units in D, and a unit also measures D according to the units in it, a unit thus measures the number D as many times as the magnitude C (measures) A. Thus, as C is to A, so a unit (is) to D [Def. 7.20].[†] Thus, inversely, as A (is) to C, so D (is) to a unit [Prop. 5.7 corr.]. Again, since C measures B according to the units in E, and a unit also measures E according to the units in it, a unit thus measures E the same number of times that C (measures) B. Thus, as C is to B, so a unit (is) to E [Def. 7.20]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to B, so the number D (is) to the (number) E [Prop. 5.22].

Thus, the commensurable magnitudes A and B have to one another the ratio which the number D (has) to the number E. (Which is) the very thing it was required to show.

[†] There is a slight logical gap here, since Def. 7.20 applies to four numbers, rather than two number and two magnitudes.

ਵ'.

Έὰν δύο μεγέθη πρὸς ἄλληλα λόγον ἔχῃ, ὃν ἀριθμὸς πρός ἀριθμόν, σύμμετρα ἔσται τὰ μεγέθη.



Δύο γὰρ μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον ἐχέτω, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν Ε· λέγω, ὅτι σύμμετρά ἐστι $\,$ other the ratio which the number D (has) to the number τὰ Α, Β μεγέθη.

Όσαι γάρ είσιν ἐν τῷ Δ μονάδες, εἰς τοσαῦτα ἴσα rable.

Proposition 6

If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.



For let the two magnitudes A and B have to one an-E. I say that the magnitudes A and B are commensuδιηρήσθω τὸ A, καὶ ἑνὶ αὐτῶν ἴσον ἔστω τὸ Γ · ὅσαι δέ εἰσιν ἐν τῷ E μονάδες, ἐκ τοσούτων μεγεθῶν ἴσων τῷ Γ συγκείσθω τὸ Z.

Έπει οὖν, ὅσαι εἰσιν ἐν τῷ Δ μονάδες, τοσαῦτά εἰσι καὶ έν τῷ Α μεγέθη ἴσα τῷ Γ, ὃ ἄρα μέρος ἐστὶν ἡ μονὰς τοῦ Δ , τὸ αὐτὸ μέρος ἐστὶ καὶ τὸ Γ τοῦ Α· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ A, οὕτως ἡ μονὰς πρὸς τὸν Δ . μετρεῖ δὲ ἡ μονὰς τὸν Δ ἀριθμόν· μετρεῖ ἄρα καὶ τὸ Γ τὸ Α. καὶ ἐπεί ἐστιν ώς τὸ Γ πρὸς τὸ A, οὕτως ἡ μονὰς πρὸς τὸν Δ [ἀριθμόν], άνάπαλιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ ἀριθμὸς πρὸς τὴν μονάδα. πάλιν ἐπεί, ὄσαι εἰσὶν ἐν τῷ Ε μονάδες, τοσαῦτά είσι καὶ ἐν τῷ Z ἴσα τῷ Γ , ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Z, οὕτως ἡ μονὰς πρὸς τὸν Ε [ἀριθμόν]. ἐδείχθη δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως
ὁ Δ πρὸς τὴν μονάδα· δι' ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Ζ, οὕτως ὁ Δ πρὸς τὸν Ε. ἀλλ
՝ ὡς ὁ Δ πρὸς τὸν E, οὕτως ἐστὶ τὸ A πρὸς τὸ B· καὶ ὡς ἄρα τὸ A πρός τὸ Β, οὕτως καὶ πρὸς τὸ Ζ. τὸ Α ἄρα πρὸς ἑκάτερον τῶν Β, Ζ τὸν αὐτὸν ἔχει λόγον ἴσον ἄρα ἐστὶ τὸ Β τῷ Ζ. μετρεῖ δὲ τὸ Γ τὸ Ζ· μετρεῖ ἄρα καὶ τὸ Β. ἀλλὰ μὴν καὶ τὸ A· τὸ Γ ἄρα τὰ A, B μετρεĩ. σύμμετρον ἄρα ἐστὶ τὸ A τῷ В.

Ἐὰν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἑξῆς.

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι, ἐὰν ῶσι δύο ἀριθμοί, ὡς οἱ Δ, Ε, καὶ εὐθεῖα, ὡς ἡ Α, δύνατόν ἐστι ποιῆσαι ὡς ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν, οὕτως τὴν εὐθεῖαν πρὸς εὐθεῖαν. ἐὰν δὲ καὶ τῶν Α, Ζ μέση ἀνάλογον ληφθῆ, ὡς ἡ Β, ἔσται ὡς ἡ Α πρὸς τὴν Ζ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, τουτέστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ἀλλ² ὡς ἡ Α πρὸς τὴν Ζ, οὕτως ἐστὶν ὁ Δ ἀριθμος πρὸς τὸν Ε ἀριθμόν. γέγονεν ἄρα καὶ ὡς ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν, οὕτως τὸ ἀπὸ τῆς Α εὐθείας πρὸς τὸ ἀπὸ τῆς Β εὐθείας. ὅπερ ἔδει δεῖξαι.

ζ'.

Τὰ ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Έστω ἀσύμμετρα μεγέθη τὰ A, B· λέγω, ὅτι τὸ A πρὸς τὸ B λόγον οὐχ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

For, as many units as there are in D, let A have been divided into so many equal (divisions). And let C be equal to one of them. And as many units as there are in E, let F be the sum of so many magnitudes equal to C.

Therefore, since as many units as there are in D, so many magnitudes equal to C are also in A, therefore whichever part a unit is of D, C is also the same part of A. Thus, as C is to A, so a unit (is) to D [Def. 7.20]. And a unit measures the number D. Thus, C also measures A. And since as C is to A, so a unit (is) to the [number] D, thus, inversely, as A (is) to C, so the number D (is) to a unit [Prop. 5.7 corr.]. Again, since as many units as there are in E, so many (magnitudes) equal to C are also in F, thus as C is to F, so a unit (is) to the [number] E[Def. 7.20]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to F, so D(is) to E [Prop. 5.22]. But, as D (is) to E, so A is to B. And thus as A (is) to B, so (it) also is to F [Prop. 5.11]. Thus, A has the same ratio to each of B and F. Thus, B is equal to F [Prop. 5.9]. And C measures F. Thus, it also measures B. But, in fact, (it) also (measures) A. Thus, C measures (both) A and B. Thus, A is commensurable with *B* [Def. 10.1].

Thus, if two magnitudes ... to one another, and so on

Corollary

So it is clear, from this, that if there are two numbers, like D and E, and a straight-line, like A, then it is possible to contrive that as the number D (is) to the number E, so the straight-line (is) to (another) straight-line (*i.e.*, F). And if the mean proportion, (say) B, is taken of A and F, then as A is to F, so the (square) on A (will be) to the (square) on B. That is to say, as the first (is) to the third, so the (figure) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. But, as A (is) to F, so the number D is to the number E. Thus, it has also been contrived that as the number D (is) to the number E, so the (figure) on the straight-line A (is) to the (similar figure) on the straight-line B. (Which is) the very thing it was required to show.

Proposition 7

Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number.

Let A and B be incommensurable magnitudes. I say that A does not have to B the ratio which (some) number (has) to (some) number.
Εἰ γὰρ ἔχει τὸ Α πρὸς τὸ Β λόγον, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρον ἔσται τὸ Α τῷ Β. οὐκ ἔστι δέ· οὐκ ἄρα τὸ Α πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Τὰ ἄρα ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, καὶ τὰ ἑξῆς.

η'.

Έὰν δύο μεγέθη πρὸς ἄλληλα λόγον μὴ ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμόν, ἀσύμμετρα ἔσται τὰ μεγέθη.



Δύο γὰρ μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον μὴ ἐχέτω, ὃν ἀριθμὸς πρὸς ἀριθμόν· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ Α, Β μεγέθη.

Εἰ γὰρ ἕσται σύμμετρα, τὸ Α πρὸς τὸ Β λόγον ἕξει, ὃν ἀριθμὸς πρὸς ἀριθμόν. οὐκ ἔχει δέ. ἀσύμμετρα ἄρα ἐστὶ τὰ Α, Β μεγέθη.

Ἐἀν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἑξῆς.

θ'.

Τὰ ἀπὸ τῶν μήκει συμμέτρων εὐθειῶν τετράγωνα πρὸς ἄλληλα λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὰ τετράγωνα τὰ πρὸς ἄλληλα λόγον ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ τὰς πλευρὰς ἕξει μήκει συμμέτρους. τὰ δὲ ἀπὸ τῶν μήκει ἀσυμμέτρων εὐθειῶν τετράγωνα πρὸς ἄλληλα λόγον οὐκ ἔχει, ὅνπερ τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὰ τετράγωνα τὰ πρὸς ἄλληλα λόγον μὴ ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὰς πλευρὰς ἕξει μήκει συμμέτρους.



Έστωσαν γὰρ ai A, B μήχει σύμμετροι λέγω, ὅτι τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B τετράγωνον λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.



For if A has to B the ratio which (some) number (has) to (some) number then A will be commensurable with B [Prop. 10.6]. But it is not. Thus, A does not have to B the ratio which (some) number (has) to (some) number.

Thus, incommensurable numbers do not have to one another, and so on

Proposition 8

If two magnitudes do not have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be incommensurable.



For let the two magnitudes A and B not have to one another the ratio which (some) number (has) to (some) number. I say that the magnitudes A and B are incommensurable.

For if they are commensurable, A will have to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. But it does not have (such a ratio). Thus, the magnitudes A and B are incommensurable.

Thus, if two magnitudes ... to one another, and so on

Proposition 9

Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either.



For let A and B be (straight-lines which are) commensurable in length. I say that the square on A has to the square on B the ratio which (some) square number (has) to (some) square number. Έπεὶ γὰρ σύμμετρός ἐστιν ἡ A τῆ B μήχει, ἡ A ẵρα πρὸς τὴν B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ἱ Γ πρὸς τὸν Δ. ἐπεὶ οῦν ἐστιν ὡς ἡ A πρὸς τὴν B, οὕτως ἱ Γ πρὸς τὸν Δ, ἀλλὰ τοῦ μὲν τῆς A πρὸς τὴν B λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς A τετραγώνου πρὸς τὸ ἀπὸ τῆς B τετράγωνον· τὰ γὰρ ὅμοια σχήματα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· τοῦ δὲ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τοῦ Γ τετραγώνου πρὸς τὸν ἀπὸ τοῦ Δ τετράγωνον· δύο γὰρ τετραγώνων ἀριθμῶν εῖς μέσος ἀνάλογόν ἐστιν ἀριθμός, καί ὁ τετράγωνος πρὸς τὸν τετράγωνον [ἀριθμὸν] διπλασίονα λόγον ἔχει, ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν· ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B τετράγωνον, οῦτως ὁ ἀπὸ τοῦ Γ τετράγωνος [ἀριθμὸς] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμὸν].

Ἀλλὰ δὴ ἔστω ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς B, οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον]· λέγω, ὅτι σύμμετρός ἐστιν ἡ A τῆ B μήχει.

Έπεὶ γάρ ἐστιν ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον], οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον], ἀλλ' ὁ μὲν τοῦ ἀπὸ τῆς Α τετραγώνου πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγος διπλασίων ἐστὶ τοῦ τῆς Α πρὸς τὴν Β λόγου, ὁ δὲ τοῦ ἀπὸ τοῦ Γ [ἀριθμοῦ] τετραγώνου [ἀριθμοῦ] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμοῦ] λόγος διπλασίων ἐστὶ τοῦ τῆν Β, οὕτως ὁ Γ [ἀριθμὸς] πρὸς τὸν Δ [ἀριθμοῦ]. ἡ Α ἄρα πρὸς τὴν Β λόγον ἔχει, ὃν ἀριθμὸς ὁ Γ πρὸς ἀριθμὸν τὸν Δ· σύμμετρος ἄρα ἐστὶν ἡ Α τῆ Β μήχει.

Άλλὰ δὴ ἀσύμμετρος ἔστω ἡ Α τῆ Β μήχει· λέγω, ὅτι τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον οὐχ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Εἰ γὰρ ἔχει τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, σύμμετρος ἔσται ἡ Α τῆ Β. οὐκ ἔστι δέ· οὐκ ἀρα τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸς.

Πάλιν δὴ τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· λέγω, ὅτι ἀσύμμετρός ἐστιν ἡ Α τῆ Β μήχει.

Εἰ γάρ ἐστι σύμμετρος ἡ Α τῆ Β, ἕξει τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. οὐκ ἔχει δέ οὐκ ἄρα σύμμετρός ἐστιν ἡ Α τῆ Β μήκει.

Τὰ ἄρα ἀπὸ τῶν μήκει συμμέτρων, καὶ τὰ ἑξῆς.

For since A is commensurable in length with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (that) which C (has) to D. Therefore, since as A is to B, so C (is) to D. But the (ratio) of the square on A to the square on B is the square of the ratio of A to B. For similar figures are in the squared ratio of (their) corresponding sides [Prop. 6.20 corr.]. And the (ratio) of the square on C to the square on D is the square of the ratio of the [number] C to the [number] D. For there exits one number in mean proportion to two square numbers, and (one) square (number) has to the (other) square [number] a squared ratio with respect to (that) the side (of the former has) to the side (of the latter) [Prop. 8.11]. And, thus, as the square on A is to the square on B, so the square [number] on the (number) C (is) to the square [number] on the [number] D.[†]

And so let the square on A be to the (square) on B as the square (number) on C (is) to the [square] (number) on D. I say that A is commensurable in length with B.

For since as the square on A is to the [square] on B, so the square (number) on C (is) to the [square] (number) on D. But, the ratio of the square on A to the (square) on B is the square of the (ratio) of A to B [Prop. 6.20 corr.]. And the (ratio) of the square [number] on the [number] C to the square [number] on the [number] D is the square of the ratio of the [number] C to the [number] D [Prop. 8.11]. Thus, as A is to B, so the [number] Calso (is) to the [number] D. A, thus, has to B the ratio which the number C has to the number D. Thus, A is commensurable in length with B [Prop. 10.6].[‡]

And so let A be incommensurable in length with B. I say that the square on A does not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

For if the square on A has to the [square] on B the ratio which (some) square number (has) to (some) square number then A will be commensurable (in length) with B. But it is not. Thus, the square on A does not have to the [square] on the B the ratio which (some) square number (has) to (some) square number.

So, again, let the square on A not have to the [square] on B the ratio which (some) square number (has) to (some) square number. I say that A is incommensurable in length with B.

For if A is commensurable (in length) with B then the (square) on A will have to the (square) on B the ratio which (some) square number (has) to (some) square number. But it does not have (such a ratio). Thus, A is not commensurable in length with B.

Thus, (squares) on (straight-lines which are) com-

mensurable in length, and so on

Πόρισμα.

Καὶ φανερὸν ἐκ τῶν δεδειγμένων ἔσται, ὅτι αἱ μήκει σύμμετροι πάντως καὶ δυνάμει, αἱ δὲ δυνάμει οὐ πάντως καὶ μήκει.

[†] There is an unstated assumption here that if $\alpha : \beta :: \gamma : \delta$ then $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$. [‡] There is an unstated assumption here that if $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$ then $\alpha : \beta :: \gamma : \delta$.

ι´.

Τῆ προτεθείση εὐθεία προσευρεῖν δύο εὐθείας ἀσυμμέτρους, τὴν μέν μήχει μόνον, τὴν δὲ χαὶ δυνάμει.



Έστω ή προτεθεῖσα εὐθεῖα ή Α· δεῖ δὴ τῆ Α προσευρεῖν δύο εὐθείας ἀσυμμέτρους, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.

Έχχείσθωσαν γὰρ δύο αριθμοὶ οἱ Β, Γ πρὸς ἀλλήλους λόγον μὴ ἔχοντες, ὅν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, τουτέστι μὴ ὅμοιοι ἐπίπεδοι, χαὶ γεγονέτω ὡς ὁ Β πρὸς τὸν Γ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Δ τετράγωνον ἐμάθομεν γάρ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Δ τετράγωνον ἐμάθομεν γάρ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Δ τατράγωνον ἀριθμὸς πρὸς τὸν Γ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Δ λογον ἐχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Δ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῆ Δ μήχει. εἰλήφθω τῶν Α, Δ μέση ἀνάλογον ἡ Ε· ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Δ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Ε. ἀσύμμετρος δέ ἑστιν ἡ Α τῆ Δ μήχει· ἀσύμμετρον ἄρα ἐστὶν κὰ Λ τῆς Ε τετραγώνω· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῆ Ε δυνάμει.

Τὴ ἄρα προτεθείση εὐθεία τῆ Α προσεύρηνται δύο εὐθεῖαι ἀσύμμετροι αἱ Δ, Ε, μήκει μὲν μόνον ἡ Δ, δυνάμει δὲ καὶ μήκει δηλαδὴ ἡ Ε [ὅπερ ἔδει δεῖξαι].

Corollary

And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straightlines commensurable) in square (are) not always also (commensurable) in length.

Proposition 10[†]

To find two straight-lines incommensurable with a given straight-line, the one (incommensurable) in length only, the other also (incommensurable) in square.



Let A be the given straight-line. So it is required to find two straight-lines incommensurable with A, the one (incommensurable) in length only, the other also (incommensurable) in square.

For let two numbers, B and C, not having to one another the ratio which (some) square number (has) to (some) square number-that is to say, not (being) similar plane (numbers)-have been taken. And let it be contrived that as B (is) to C, so the square on A (is) to the square on D. For we learned (how to do this) [Prop. 10.6 corr.]. Thus, the (square) on A (is) commensurable with the (square) on D [Prop. 10.6]. And since B does not have to C the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on D the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with D [Prop. 10.9]. Let the (straight-line) E (which is) in mean proportion to Aand D have been taken [Prop. 6.13]. Thus, as A is to D, so the square on A (is) to the (square) on E [Def. 5.9]. And A is incommensurable in length with D. Thus, the square on A is also incommensurble with the square on E [Prop. 10.11]. Thus, A is incommensurable in square with E.

Thus, two straight-lines, D and E, (which are) incommensurable with the given straight-line A, have been found, the one, D, (incommensurable) in length only, the other, E, (incommensurable) in square, and, clearly, also in length. [(Which is) the very thing it was required to show.]

 † This whole proposition is regarded by Heiberg as an interpolation into the original text.

ια'.

Έὰν τέσσαρα μεγέθη ἀνάλογον ἢ, τὸ δὲ πρῶτον τῷ δευτέρῳ σύμμετρον ἢ, καὶ τὸ τρίτον τῷ τετάρτῳ σύμμετρον ἔσται· κἂν τὸ πρῶτον τῷ δευτέρῳ ἀσύμμετρον ἢ, καὶ τὸ τρίτον τῷ τετάρτῳ ἀσύμμετρον ἔσται.



Έστωσαν τέσσαρα μεγέθη ἀνάλογον τὰ A, B, Γ , Δ, ὡς τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ, τὸ A δὲ τῷ B σύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ Γ τῷ Δ σύμμετρον ἔσται.

Έπεὶ γὰρ σύμμετρόν ἐστι τὸ A τῷ B, τὸ A ẳρα πρὸς τὸ B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καί ἐστιν ὡς τὸ A πρὸς τὸ B, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ӑρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· σύμμετρον ἄρα ἐστὶ τὸ Γ τῷ Δ.

Άλλὰ δὴ τὸ Α τῷ Β ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ Γ τῷ Δ ἀσύμμετρον ἔσται. ἐπεὶ γὰρ ἀσύμμετρόν ἐστι τὸ Α τῷ Β, τὸ Α ἄρα πρὸς τὸ Β λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· οὐδὲ τὸ Γ ἅρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ἀσύμμετρον ἅρα ἐστὶ τὸ Γ τῷ Δ.

Ἐὰν ἄρα τέσσαρα μεγέθη, καὶ τὰ ἑξῆς.

ıβ'.

Τὰ τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα.

Έκάτερον γὰρ τῶν Α, Β τῷ Γ ἔστω σύμμετρον. λέγω, ὅτι καὶ τὸ Α τῷ Β ἐστι σύμμετρον.

Έπεὶ γὰρ σύμμετρόν ἐστι τὸ A τῷ Γ, τὸ A ἄρα πρὸς τὸ Γ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Δ πρὸς τὸν Ε. πάλιν, ἐπεὶ σύμμετρόν ἐστι τὸ Γ τῷ Β, τὸ Γ ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Ζ πρὸς τὸν Η. καὶ λόγων δοθέντων ὑποσωνοῦν τοῦ τε, ὃν ἔχει ὁ Δ πρὸς τὸν Ε, καὶ ὁ Ζ πρὸς τὸν Η εἰλήφθωσαν ἀριθμοὶ ἑξῆς ἐν τοῖς δοθεῖσι λόγοις οἱ Θ, Κ, Λ· ὥστε είναι

Proposition 11

If four magnitudes are proportional, and the first is commensurable with the second, then the third will also be commensurable with the fourth. And if the first is incommensurable with the second, then the third will also be incommensurable with the fourth.



Let A, B, C, D be four proportional magnitudes, (such that) as A (is) to B, so C (is) to D. And let A be commensurable with B. I say that C will also be commensurable with D.

For since A is commensurable with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. And as A is to B, so C (is) to D. Thus, C also has to D the ratio which (some) number (has) to (some) number. Thus, C is commensurable with D [Prop. 10.6].

And so let A be incommensurable with B. I say that C will also be incommensurable with D. For since A is incommensurable with B, A thus does not have to B the ratio which (some) number (has) to (some) number [Prop. 10.7]. And as A is to B, so C (is) to D. Thus, C does not have to D the ratio which (some) number (has) to (some) number either. Thus, C is incommensurable with D [Prop. 10.8].

Thus, if four magnitudes, and so on

Proposition 12

(Magnitudes) commensurable with the same magnitude are also commensurable with one another.

For let A and B each be commensurable with C. I say that A is also commensurable with B.

For since A is commensurable with C, A thus has to C the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which D (has) to E. Again, since C is commensurable with B, C thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which F (has) to G. And for any multitude whatsoever ώς μὲν τὸν Δ πρὸς τὸν E, οὕτως τὸν Θ πρὸς τὸν K, ὡς δὲ τὸν Z πρὸς τὸν H, οὕτως τὸν K πρὸς τὸν Λ.



Έπεὶ οῦν ἐστιν ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὸν Ε, ἀλλ' ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Κ, ἔστιν ἄρα xαὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεί ἐστιν ὡς τὸ Γ πρὸς τὸ Β, οῦτως ὁ Ζ πρὸς τὸν Η, ἀλλ' ὡς ὁ Ζ πρὸς τὸν Η, [οὕτως] ὁ Κ πρὸς τὸν Λ, xαὶ ὡς ἄρα τὸ Γ πρὸς τὸ Β, οῦτως ὁ Κ πρὸς τὸν Λ. ἔστι δὲ xαὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ· δι' ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οῦτως ὁ Θ πρὸς τὸν Λ. τὸ Α ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς ὁ Θ πρὸς ἀριθμὸν τὸν Λ· σύμμετρον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα ὅπερ ἔδει δεῖξαι.

ιγ'.

Έλν ῆ δύο μεγέθη σύμμετρα, τὸ δὲ ἕτερον αὐτῶν μεγέθει τινὶ ἀσύμμετρον ῆ, καὶ τὸ λοιπὸν τῷ αὐτῷ ἀσύμμετρον ἔσται.



Έστω δύο μεγέθη σύμμετρα τὰ A, B, τὸ δὲ ἕτερον αὐτῶν τὸ A ἄλλῷ τινὶ τῷ Γ ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ λοιπὸν τὸ B τῷ Γ ἀσύμμετρόν ἐστιν.

Εἰ γάρ ἐστι σύμμετρον τὸ Β τῷ Γ, ἀλλὰ καὶ τὸ Α τῷ Β σύμμετρόν ἐστιν, καὶ τὸ Α ἄρα τῷ Γ σύμμετρόν ἐστιν. ἀλλὰ καὶ ἀσύμμετρον· ὅπερ ἀδύνατον. οὐκ ἄρα σύμμετρόν ἐστι τὸ Β τῷ Γ· ἀσύμμετρον ἄρα.

Ἐὰν ἄρα ἤ δύο μεγέθη σύμμετρα, καὶ τὰ ἑξῆς.

Λῆμμα.

Δύο δοθεισῶν εὐθειῶν ἀνίσων εὑρεῖν, τίνι μεῖζον δύναται ἡ μείζων τῆς ἐλάσσονος. of given ratios—(namely,) those which D has to E, and F to G—let the numbers H, K, L (which are) continuously (proportional) in the(se) given ratios have been taken [Prop. 8.4]. Hence, as D is to E, so H (is) to K, and as F (is) to G, so K (is) to L.



Therefore, since as A is to C, so D (is) to E, but as D (is) to E, so H (is) to K, thus also as A is to C, so H (is) to K [Prop. 5.11]. Again, since as C is to B, so F (is) to G, but as F (is) to G, [so] K (is) to L, thus also as C (is) to B, so K (is) to L [Prop. 5.11]. And also as A is to C, so H (is) to K. Thus, via equality, as A is to B, so H (is) to L [Prop. 5.22]. Thus, A has to B the ratio which the number H (has) to the number L. Thus, A is commensurable with B [Prop. 10.6].

Thus, (magnitudes) commensurable with the same magnitude are also commensurable with one another. (Which is) the very thing it was required to show.

Proposition 13

If two magnitudes are commensurable, and one of them is incommensurable with some magnitude, then the remaining (magnitude) will also be incommensurable with it.



Let A and B be two commensurable magnitudes, and let one of them, A, be incommensurable with some other (magnitude), C. I say that the remaining (magnitude), B, is also incommensurable with C.

For if B is commensurable with C, but A is also commensurable with B, A is thus also commensurable with C [Prop. 10.12]. But, (it is) also incommensurable (with C). The very thing (is) impossible. Thus, B is not commensurable with C. Thus, (it is) incommensurable.

Thus, if two magnitudes are commensurable, and so on

Lemma

For two given unequal straight-lines, to find by (the square on) which (straight-line) the square on the greater



Έστωσαν αί δοθεϊσαι δύο άνισοι εὐθεῖαι αί AB, $\Gamma,$ ῶν μείζων ἔστω ή AB· δεῖ δὴ εὑρεῖν, τίνι μεῖζον δύναται ή AB τῆς $\Gamma.$

Γεγράφθω ἐπὶ τῆς ΑΒ ἡμιχύχλιον τὸ ΑΔΒ, καὶ εἰς αὐτὸ ἐνηρμόσθω τῆ Γ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΒ. φανερὸν δή, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΑΔΒ γωνία, καὶ ὅτι ἡ ΑΒ τῆς ΑΔ, τουτέστι τῆς Γ, μεῖζον δύναται τῆ ΔΒ.

Όμοίως δὲ καὶ δύο δοθεισῶν εὐθειῶν ἡ δυναμένη αὐτὰς εὑρίσκεται οὕτως.

Έστωσαν αί δοθεϊσαι δύο εὐθεῖαι αί ΑΔ, ΔΒ, καὶ δέον ἔστω εὑρεῖν τὴν δυναμένην αὐτάς. κείσθωσαν γάρ, ὥστε ὀρθὴν γωνίαν περιέχειν τὴν ὑπὸ ΑΔ, ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· φανερὸν πάλιν, ὅτι ἡ τὰς ΑΔ, ΔΒ δυναμένη ἐστὶν ἡ AB· ὅπερ ἔδει δεῖξαι. (straight-line is) larger than (the square on) the lesser.[†]



Let AB and C be the two given unequal straight-lines, and let AB be the greater of them. So it is required to find by (the square on) which (straight-line) the square on AB (is) greater than (the square on) C.

Let the semi-circle ADB have been described on AB. And let AD, equal to C, have been inserted into it [Prop. 4.1]. And let DB have been joined. So (it is) clear that the angle ADB is a right-angle [Prop. 3.31], and that the square on AB (is) greater than (the square on) AD—that is to say, (the square on) C—by (the square on) DB [Prop. 1.47].

And, similarly, the square-root of (the sum of the squares on) two given straight-lines is also found likeso.

Let AD and DB be the two given straight-lines. And let it be necessary to find the square-root of (the sum of the squares on) them. For let them have been laid down such as to encompass a right-angle—(namely), that (angle encompassed) by AD and DB. And let AB have been joined. (It is) again clear that AB is the square-root of (the sum of the squares on) AD and DB [Prop. 1.47]. (Which is) the very thing it was required to show.

[†] That is, if α and β are the lengths of two given straight-lines, with α being greater than β , to find a straight-line of length γ such that $\alpha^2 = \beta^2 + \gamma^2$. Similarly, we can also find γ such that $\gamma^2 = \alpha^2 + \beta^2$.

ιδ'.

Έὰν τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, δύνηται δὲ ἡ πρώτη τῆς δευτέρας μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῆ [μήχει], καὶ ἡ τρίτη τῆς τετάρτης μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆ [μήχει]. καὶ ἐὰν ἡ πρώτη τῆς δευτέρας μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ [μήχει], καὶ ἡ τρίτη τῆς τετάρτης μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ [μήχει].

Έστωσαν τέσσαρες εὐθεĩαι ἀνάλογον αἱ A, B, Γ, Δ, ὡς ἡ A πρὸς τὴν B, οὕτως ἡ Γ πρὸς τὴν Δ, καὶ ἡ A μὲν τῆς B μεῖζον δυνάσθω τῷ ἀπὸ τῆς E, ἡ δὲ Γ τῆς Δ μεῖζον δυνάσθω τῷ ἀπὸ τῆς Z· λέγω, ὅτι, εἶτε σύμμετρός ἐστιν ἡ A τῆ E, σύμμετρός ἐστι καὶ ἡ Γ τῆ Z, εἶτε ἀσύμμετρός ἐστιν ἡ A τῆ E, ἀσύμμετρός ἐστι καὶ ὁ Γ τῆ Z.

Proposition 14

If four straight-lines are proportional, and the square on the first is greater than (the square on) the second by the (square) on (some straight-line) commensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) commensurable [in length] with the third. And if the square on the first is greater than (the square on) the second by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the third.

Let A, B, C, D be four proportional straight-lines, (such that) as A (is) to B, so C (is) to D. And let the square on A be greater than (the square on) B by the



Έπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ἀπὸ τῆς Δ. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσα ἐστὶ τὰ ἀπὸ τῶν Ε, Β, τῷ δὲ ἀπὸ τῆς Γ ἴσα ἐστὶ τὰ ἀπὸ τῶν Δ, Ζ. ἔστιν ἄρα ὡς τὰ ἀπὸ τῶν Ε, Β πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὰ ἀπὸ τῶν Δ, Ζ. ἔστιν ἄρα ὡς τὰ ἀπὸ τῶν Ε, Β πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὰ ἀπὸ τῶν Δ, Ζ πρὸς τὸ ἀπὸ τῆς Δ. διελόντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς Δ πὸς τῆς Δ, διελόντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς Δ πρὸς τὸ ἀπὸ τῆς Δ, διελόντι ἀπὸ τῆς Ζ πρὸς τὸ ἀπὸ τῆς Δ. ἔστιν ἄρα καὶ ὡς ἡ Ε πρὸς τὴν Β, οὕτως ἡ Ζ πρὸς τὴν Δ. ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ Β πρὸς τὴν Ε, οὕτως ἡ Γ πρὸς τὴν Δ. δι' ἴσου ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Ε, οῦτως ἡ Γ πρὸς τὴν Ζ. ἔστι οῦν σύμμετρός ἐστιν ἡ Α τῆ Ε, ἀσύμμετρός ἐστι καὶ ἡ Γ τῆ Ζ.

Ἐὰν ἄρα, καὶ τὰ ἑξῆς.

ιε΄.

Έαν δύο μεγέθη σύμμετρα συντεθῆ, καὶ τὸ ὅλον ἑκατέρῷ αὐτῶν σύμμετρον ἔσται· κἂν τὸ ὅλον ἑνὶ αὐτῶν σύμμετρον ἦ, καὶ τὰ ἐξ ἀρχῆς μεγέθη σύμμετρα ἔσται.

Συγκείσθω γὰρ δύο μεγέθη σύμμετρα τὰ AB, BΓ· λέγω, ὅτι καὶ ὅλον τὸ AΓ ἑκατέρῷ τῶν AB, BΓ ἐστι σύμμετρον. (square) on E, and let the square on C be greater than (the square on) D by the (square) on F. I say that Ais either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F.



For since as A is to B, so C (is) to D, thus as the (square) on A is to the (square) on B, so the (square) on C (is) to the (square) on D [Prop. 6.22]. But the (sum of the squares) on E and B is equal to the (square) on A, and the (sum of the squares) on D and F is equal to the (square) on C. Thus, as the (sum of the squares) on E and B is to the (square) on B, so the (sum of the squares) on D and F (is) to the (square) on D. Thus, via separation, as the (square) on E is to the (square) on B, so the (square) on F (is) to the (square) on D [Prop. 5.17]. Thus, also, as E is to B, so F (is) to D[Prop. 6.22]. Thus, inversely, as B is to E, so D (is) to F [Prop. 5.7 corr.]. But, as A is to B, so C also (is) to D. Thus, via equality, as A is to E, so C (is) to F[Prop. 5.22]. Therefore, A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F [Prop. 10.11].

Thus, if, and so on

Proposition 15

If two commensurable magnitudes are added together then the whole will also be commensurable with each of them. And if the whole is commensurable with one of them then the original magnitudes will also be commensurable (with one another).

For let the two commensurable magnitudes AB and BC be laid down together. I say that the whole AC is also commensurable with each of AB and BC.



Έπει γὰρ σύμμετρά ἐστι τὰ AB, BΓ, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οῦν τὸ Δ τὰ AB, BΓ μετρεῖ, καὶ ὅλον τὸ ΑΓ μετρήσει. μετρεῖ δὲ καὶ τὰ AB, BΓ. τὸ Δ ἄρα τὰ AB, BΓ, ΑΓ μετρεῖ· σύμμετρον ἄρα ἐστὶ τὸ ΑΓ ἑκατέρῳ τῶν AB, BΓ.

Άλλὰ δὴ τὸ ΑΓ ἔστω σύμμετρον τῷ AB· λέγω δή, ὅτι καὶ τὰ AB, BΓ σύμμετρά ἐστιν.

Έπει γὰρ σύμμετρά ἐστι τὰ ΑΓ, ΑΒ, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οῦν τὸ Δ τὰ ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸ ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἅρα τὰ ΑΒ, ΒΓ μετρήσει· σύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐὰν ἄρα δύο μεγέϑη, καὶ τὰ ἑξῆς.

เร'.

Έὰν δύο μεγέθη ἀσύμμετρα συντεθῆ, καὶ τὸ ὅλον ἑκατέρῷ αὐτῶν ἀσύμμετρον ἔσται· κἂν τὸ ὅλον ἑνὶ αὐτῶν ἀσύμμετρον ἦ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται.



Συγκείσθω γὰρ δύο μεγέθη ἀσύμμετρα τὰ AB, BΓ· λέγω, ὅτι καὶ ὅλον τὸ AΓ ἑκατέρῳ τῶν AB, BΓ ἀσύμμετρόν ἐστιν.

Εἰ γὰρ μή ἐστιν ἀσύμμετρα τὰ ΓΑ, ΑΒ, μετρήσει τι [αὐτὰ] μέγεθος. μετρείτω, εἰ δυνατόν, καὶ ἔστω τὸ Δ. ἐπεὶ οῦν τὸ Δ τὰ ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸ ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΑΒ, ΒΓ μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ· ὑπέκειντο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΑ, ΑΒ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΓΑ, ΑΒ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὰ ΑΓ, ΓΒ ἀσύμμετρά ἐστιν. τὸ ΑΓ ἄρα ἑκατέρῳ τῶν ΑΒ, ΒΓ ἀσύμμετρόν ἐστιν.

Άλλὰ δὴ τὸ ΑΓ ἑνὶ τῶν ΑΒ, ΒΓ ἀσύμμετρον ἔστω. ἔστω δὴ πρότερον τῷ ΑΒ· λέγω, ὅτι καὶ τὰ ΑΒ, ΒΓ ἀσύμμετρά ἐστιν. εἰ γὰρ ἔσται σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οῦν τὸ Δ τὰ ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον ἄρα τὸ ΑΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἅρα τὰ ΓΑ, ΑΒ μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ



For since AB and BC are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) AB and BC, it will also measure the whole AC. And it also measures AB and BC. Thus, D measures AB, BC, and AC. Thus, AC is commensurable with each of AB and BC [Def. 10.1].

And so let AC be commensurable with AB. I say that AB and BC are also commensurable.

For since AC and AB are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) CA and AB, it will thus also measure the remainder BC. And it also measures AB. Thus, D will measure (both) AB and BC. Thus, AB and BC are commensurable [Def. 10.1].

Thus, if two magnitudes, and so on

Proposition 16

If two incommensurable magnitudes are added together then the whole will also be incommensurable with each of them. And if the whole is incommensurable with one of them then the original magnitudes will also be incommensurable (with one another).



For let the two incommensurable magnitudes AB and BC be laid down together. I say that that the whole AC is also incommensurable with each of AB and BC.

For if CA and AB are not incommensurable then some magnitude will measure [them]. If possible, let it (so) measure (them), and let it be D. Therefore, since D measures (both) CA and AB, it will thus also measure the remainder BC. And it also measures AB. Thus, D measures (both) AB and BC. Thus, AB and BC are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) CA and AB. Thus, CA and AB are incommensurable [Def. 10.1]. So, similarly, we can show that AC and CB are also incommensurable. Thus, AC is incommensurable with each of AB and BC.

And so let AC be incommensurable with one of AB and BC. So let it, first of all, be incommensurable with

ΓΑ, ΑΒ· ὑπέχειτο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΑΒ, ΒΓ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐὰν ἄρα δύο μεγέθη, καὶ τὰ ἑξῆς.

Λῆμμα.

Έλν παρά τινα εὐθεῖαν παραβληθῆ παραλληλόγραμμον ἐλλεῖπον εἶδει τετραγώνω, τὸ παραβληθὲν ἴσον ἐστὶ τῷ ὑπὸ τῶν ἐκ τῆς παραβολής γενομένων τμημάτων τῆς εὐθείας.



Παρὰ γὰρ εὐθεῖαν τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον τὸ ΑΔ ἐλλεῖπον εἴδει τετραγώνω τῷ ΔΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΔ τῷ ὑπὸ τῶν ΑΓ, ΓΒ.

Καί ἐστιν αὐτόθεν φανερόν· ἐπεὶ γὰρ τετράγωνόν ἐστι τὸ ΔB , ἴση ἐστὶν ἡ $\Delta \Gamma$ τῆ ΓB , καί ἐστι τὸ $A\Delta$ τὸ ὑπὸ τῶν $A\Gamma$, $\Gamma\Delta$, τουτέστι τὸ ὑπὸ τῶν $A\Gamma$, ΓB .

Ἐὰν ἄρα παρά τινα εὐθεῖαν, καὶ τὰ ἑξῆς.

[†] Note that this lemma only applies to rectangular parallelograms.

ιζ΄.

Έὰν ῶσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετράτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῆ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρῆ μήκει, ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ συμμέτου ἑαυτῆ [μήκει]. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῆ [μήκει], τῷ δὲ τετράρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῆ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ μήκει.

Έστωσαν δύο εὐθεῖαι ἄνισοι αἱ Α, ΒΓ, ῶν μείζων ἡ

AB. I say that AB and BC are also incommensurable. For if they are commensurable then some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) AB and BC, it will thus also measure the whole AC. And it also measures AB. Thus, D measures (both) CA and AB. Thus, CA and AB are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) AB and BC. Thus, AB and BC are incommensurable [Def. 10.1].

Thus, if two... magnitudes, and so on

Lemma

If a parallelogram,[†] falling short by a square figure, is applied to some straight-line then the applied (parallelogram) is equal (in area) to the (rectangle contained) by the pieces of the straight-line created via the application (of the parallelogram).



For let the parallelogram AD, falling short by the square figure DB, have been applied to the straight-line AB. I say that AD is equal to the (rectangle contained) by AC and CB.

And it is immediately obvious. For since DB is a square, DC is equal to CB. And AD is the (rectangle contained) by AC and CD—that is to say, by AC and CB.

Thus, if ... to some straight-line, and so on

Proposition 17[†]

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable in length, then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater. And if the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the BΓ, τῷ δὲ τετράρτῳ μέρει τοῦ ἀπὸ ἐλάσσονος τῆς Α, τουτέστι τῷ ἀπὸ τῆς ἡμισείας τῆς Α, ἴσον παρὰ τὴν BΓ παραβεβλήσθω ἐλλεῖπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ, σύμμετρος δὲ ἔστω ἡ ΒΔ τῆ ΔΓ μήκει· λέγω, ὂτι ἡ BΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ.



Τετμήσθω γὰρ ή ΒΓ δίγα κατὰ τὸ Ε σημεῖον, καὶ κείσθω τ
ῆ ΔE ἴση ἡ EZ. λοιπὴ ἄρα ἡ $\Delta \Gamma$ ἴση ἐστὶ τῆ BZ. καὶ ἐπεὶ εύθεῖα ἡ ΒΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Ε, εἰς δὲ ἄνισα κατὰ τὸ Δ, τὸ ἄρα ὑπὸ ΒΔ, ΔΓ περειχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΓ τετραγώνω· καὶ τὰ τετραπλάσια· τὸ ἄρα τετράκις ὑπὸ τῶν $B\Delta$, $\Delta\Gamma$ μετὰ τοῦ τετραπλασίου τοῦ ἀπὸ τῆς ΔE ἴσον έστι τῷ τετράχις ἀπὸ τῆς ΕΓ τετραγώνω. ἀλλὰ τῷ μὲν τετραπλασίω τοῦ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ${
m A}$ τετράγωνον, τῷ δὲ τετραπλασίω τοῦ ἀπὸ τῆς $\Delta {
m E}$ ἴσον έστὶ τὸ ἀπὸ τῆς ΔZ τετράγωνον· διπλασίων γάρ ἐστιν ἡ ΔZ τῆς ΔΕ. τῷ δὲ τετραπλασίω τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τὸ άπὸ τῆς ΒΓ τετράγωνον· διπλασίων γάρ ἐστι πάλιν ἡ ΒΓ τῆς ΓΕ. τὰ ἄρα ἀπὸ τῶν Α, ΔΖ τετράγωνα ἴσα ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετράγωνω· ὥστε τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς Α με
ιζόν ἐστι τῷ ἀπὸ τῆς ΔZ^{\cdot} ἡ B
Γ ἀρα τῆς Α με
ιζον δύναται τῆ ΔΖ. δεικτέον, ὅτι καὶ σύμμετρός ἐστιν ἡ ΒΓ τῆ ΔΖ. έπεὶ γὰρ σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήχει, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῆ ΓΔ μήκει. ἀλλὰ ἡ ΓΔ ταῖς ΓΔ, ΒΖ έστι σύμμετρος μήκει· ἴση γάρ ἐστιν ἡ Γ Δ τῆ BZ. καὶ ἡ BΓ άρα σύμμετρός ἐστι ταῖς BZ, ΓΔ μήκει· ὥστε καὶ λοιπῆ τῆ ΖΔ σύμμετρός ἐστιν ἡ ΒΓ μήχει· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ.

Άλλὰ δὴ ἡ ΒΓ τῆς Α μεῖζον δυνάσθω τῷ ἀπὸ συμμέτρου ἑαυτῆ, τῷ δὲ τετράτρῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι σύμμετρός ἐστιν ἡ ΒΔ τῆ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὑμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. δύναται δὲ ἡ greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) commensurable in length.

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater. And let a (rectangle) equal to the fourth part of the (square) on the lesser, A—that is, (equal) to the (square) on half of A—falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BD and DC [see previous lemma]. And let BD be commensurable in length with DC. I say that that the square on BC is greater than the (square on) A by (the square on some straight-line) commensurable (in length) with (BC).



For let BC have been cut in half at the point E [Prop. 1.10]. And let EF be made equal to DE [Prop. 1.3]. Thus, the remainder DC is equal to BF. And since the straight-line BC has been cut into equal (pieces) at E, and into unequal (pieces) at D, the rectangle contained by BD and DC, plus the square on ED, is thus equal to the square on EC [Prop. 2.5]. (The same) also (for) the quadruples. Thus, four times the (rectangle contained) by BD and DC, plus the quadruple of the (square) on DE, is equal to four times the square on EC. But, the square on A is equal to the quadruple of the (rectangle contained) by BD and DC, and the square on DF is equal to the quadruple of the (square) on DE. For DFis double DE. And the square on BC is equal to the quadruple of the (square) on EC. For, again, BC is double CE. Thus, the (sum of the) squares on A and DF is equal to the square on BC. Hence, the (square) on BCis greater than the (square) on A by the (square) on DF. Thus, BC is greater in square than A by DF. It must also be shown that BC is commensurable (in length) with DF. For since BD is commensurable in length with DC, BC is thus also commensurable in length with CD [Prop. 10.15]. But, CD is commensurable in length with CD plus BF. For CD is equal to BF [Prop. 10.6]. Thus, BC is also commensurable in length with BF plus CD [Prop. 10.12]. Hence, BC is also commensurable in length with the remainder FD [Prop. 10.15]. Thus, the square on BC is greater than (the square on) A by the (square) on (some straight-line) commensurable (in length) with (BC).

ΒΓ τῆς Α μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῆ. σύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῆ ΖΔ μήκει· ὥστε καὶ λοιπῆ συναμφοτέρῳ τῆ ΒΖ, ΔΓ σύμμετρός ἐστιν ἡ ΒΓ μήκει. ἀλλὰ συναμφότερος ἡ ΒΖ, ΔΓ σύμμετρός ἐστι τῆ ΔΓ [μήκει]. ὥστε καὶ ἡ ΒΓ τῆ ΓΔ σύμμετρός ἑστι μήκει· καὶ διελόντι ἄρα ἡ ΒΔ τῆ ΔΓ ἑστι σύμμετρος μήκει.

Ἐὰν ἄρα ῶσι δύο εὐθεῖαι ἄνισοι, καὶ τὰ ἑξῆς.

And so let the square on BC be greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC). And let a (rectangle) equal to the fourth (part) of the (square) on A, falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BD and DC. It must be shown that BD is commensurable in length with DC.

For, similarly, by the same construction, we can show that the square on BC is greater than the (square on) A by the (square) on FD. And the square on BC is greater than the (square on) A by the (square) on (some straightline) commensurable (in length) with (BC). Thus, BC is commensurable in length with FD. Hence, BC is also commensurable in length with the remaining sum of BF and DC [Prop. 10.15]. But, the sum of BF and DC is commensurable [in length] with DC [Prop. 10.6]. Hence, BC is also commensurable in length with CD [Prop. 10.12]. Thus, via separation, BD is also commensurable in length with DC [Prop. 10.15].

Thus, if there are two unequal straight-lines, and so on

[†] This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, x = DC, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are commensurable when $\alpha - x$ are *x* are commensurable, and *vice versa*.

*ι*η′.

Έὰν ῶσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῆ ἐλλεῖπον εἴδει τετραγώνῳ, καὶ εἰς ἀσυμμετρα αὐτὴν διαιρῆ [μήκει], ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, τῷ δὲ τετράρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῆ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διαιρεῖ [μήκει].

Έστωσαν δύο εὐθεῖαι ἄνισοι αἰ Α, ΒΓ, ῶν μείζων ἡ ΒΓ, τῷ δὲ τετάρτῳ [μέρει] τοῦ ἀπὸ τῆς ἐλάσσονος τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔΓ, ἀσύμμετρος δὲ ἔστω ἡ ΒΔ τῆ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ.

Proposition 18[†]

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable [in length], then the square on the greater will be larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater. And if the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) incommensurable [in length].

Let *A* and *BC* be two unequal straight-lines, of which (let) *BC* (be) the greater. And let a (rectangle) equal to the fourth [part] of the (square) on the lesser, *A*, falling short by a square figure, have been applied to *BC*. And let it be the (rectangle contained) by *BDC*. And let *BD* be incommensurable in length with *DC*. I say that that the square on *BC* is greater than the (square on) *A* by the (square) on (some straight-line) incommensurable (in length) with (*BC*).



Τῶν γὰρ αὐτῶν κατασκευασθέντων τῷ πρότερον ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. δεικτέον [οὖν], ὅτι ἀσύμμετρός ἐστιν ἡ ΒΓ τῆ ΔΖ μήκει. ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΒΔ τῆ ΔΓ μήκει, ἀσύμμετρος ἀρα ἐστὶ καὶ ἡ ΒΓ τῆ ΓΔ μήκει. ἀλλὰ ἡ ΔΓ σύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ· καὶ ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ· ἡ ΒΓ μήκει. καὶ ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς Χα.

Δυνάσθω δὴ πάλιν ἡ BΓ τῆς Α μεῖζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν BΓ παραβεβλήσθω ἑλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν BΔ, ΔΓ. δεικτέον, ὅτι ἀσύμμετρός ἐστιν ἡ BΔ τῆ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. ἀλλὰ ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῆ ΖΔ μήκει· ὥστε καὶ λοιπῆ συναμφοτέρω τῆ ΒΖ, ΔΓ ἀσύμμετρός ἐστιν ἡ ΒΓ. ἀλλὰ συναμφότερος ἡ ΒΖ, ΔΓ τῆ ΔΓ σύμμετρός ἐστι μήκει· καὶ ἡ ΒΓ ἄρα τῆ ΔΓ ἀσύμμετρός ἐστι μήκει. ὥστε καὶ διελόντι ἡ ΒΔ τῆ ΔΓ ἀσύμμετρός ἐστι μήκει.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι, καὶ τὰ ἑξῆς.



For, similarly, by the same construction as before, we can show that the square on BC is greater than the (square on) A by the (square) on FD. [Therefore] it must be shown that BC is incommensurable in length with DF. For since BD is incommensurable in length with DC, BC is thus also incommensurable in length with CD [Prop. 10.16]. But, DC is commensurable (in length) with the sum of BF and DC [Prop. 10.6]. And, thus, BC is incommensurable (in length) with the remainder FD [Prop. 10.16]. And the square on BC is greater than the (square on) A by the (square) on FD. Thus, the square on BC is greater than the (square on BC is greater than the (square on) A by the (square on) A by the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC).

So, again, let the square on BC be greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC). And let a (rectangle) equal to the fourth [part] of the (square) on A, falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BD and DC. It must be shown that BD is incommensurable in length with DC.

For, similarly, by the same construction, we can show that the square on BC is greater than the (square) on A by the (square) on FD. But, the square on BC is greater than the (square) on A by the (square) on (some straight-line) incommensurable (in length) with (BC). Thus, BC is incommensurable in length with FD. Hence, BC is also incommensurable (in length) with the remaining sum of BF and DC [Prop. 10.16]. But, the sum of BF and DC is commensurable in length with DC [Prop. 10.6]. Thus, BC is also incommensurable in length with DC [Prop. 10.13]. Hence, via separation, BD is also incommensurable in length with DC[Prop. 10.16].

Thus, if there are two ... straight-lines, and so on

[†] This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, x = DC, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are incommensurable when $\alpha - x$ are x are incommensurable, and vice versa.

*ι*ϑ′.

Τὸ ὑπὸ ἑητῶν μήκει συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἑητόν ἐστιν.

Υπὸ γὰρ ἑητῶν μήκει συμμέτρων εὐθειῶν τῶν ${
m AB},{
m B}\Gamma$

Proposition 19

The rectangle contained by rational straight-lines (which are) commensurable in length is rational.

For let the rectangle AC have been enclosed by the

АΓ.



Άναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ρήτὸν ἄρα ἐστὶ τὸ AΔ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ AB τῆ ΒΓ μήχει, ἴση δέ ἐστιν ἡ ΑΒ τῆ ΒΔ, σύμμετρος ἄρα ἐστὶν ή $B\Delta$ τῆ $B\Gamma$ μήχει. χαί ἐστιν ὡς ἡ $B\Delta$ πρὸς τὴν $B\Gamma$, οὕτως τὸ ΔA πρὸς τὸ $A\Gamma$. σύμμετρον ἄρα ἐστὶ τὸ ΔA τῷ $A\Gamma$. ρητὸν δὲ τὸ ΔA ·
ρητὸν ẳρα ἐστὶ καὶ τὸ $A\Gamma$.

Τὸ ἄρα ὑπὸ ῥητῶν μήχει συμμέτρων, χαὶ τὰ ἑξῆς.

χ.

Ἐἀν ἑητὸν παρὰ ἑητὴν παραβληϑῆ, πλάτος ποιεῖ ἑητὴν καὶ σύμμετρον τῆ, παρ' ἢν παράκειται, μήκει.



^eΡητόν γὰρ τὸ ΑΓ παρὰ ἑητὴν τὴν ΑΒ παραβεβλήσθω πλάτος ποιοῦν τὴν ΒΓ· λέγω, ὄτι ἑητή ἐστιν ἡ ΒΓ καὶ σύμμετρος τῆ ΒΑ μήκει.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ρητον ἄρα ἐστὶ το ΑΔ. ρητον δὲ καὶ το ΑΓ· σύμμετρον ἄρα

όρθογώνιον περιεχέσθω τὸ $A\Gamma$ · λέγω, ὅτι ἑητόν ἐστι τὸ rational straight-lines AB and BC (which are) commensurable in length. I say that AC is rational.



For let the square AD have been described on AB. AD is thus rational [Def. 10.4]. And since AB is commensurable in length with BC, and AB is equal to BD, BD is thus commensurable in length with BC. And as BD is to BC, so DA (is) to AC [Prop. 6.1]. Thus, DAis commensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is also rational [Def. 10.4]. Thus, the ... by rational straight-lines ... commensurable, and so on

Proposition 20

If a rational (area) is applied to a rational (straightline) then it produces as breadth a (straight-line which is) rational, and commensurable in length with the (straightline) to which it is applied.



For let the rational (area) AC have been applied to the rational (straight-line) AB, producing the (straight-line) BC as breadth. I say that BC is rational, and commensurable in length with BA.

For let the square AD have been described on AB.

ἐστὶ τὸ ΔA τῷ $A\Gamma$. xaí ἐστιν ὡς τὸ ΔA πρὸς τὸ $A\Gamma$, οὕτως ή ΔB πρὸς τὴν $B\Gamma$. σύμμετρος ἄρα ἐστὶ xaὶ ή ΔB τῆ $B\Gamma$ · ἴση δὲ ή ΔB τῆ BA· σύμμετρος ἄρα xaὶ ή AB τῆ $B\Gamma$. ἑητὴ δέ ἐστιν ή AB· ἑητὴ ἄρα ἐστὶ xaὶ ή $B\Gamma$ xaὶ σύμμετρος τῆ AB μήχει.

Έαν ἄρα ἑητὸν παρὰ ἑητὴν παραβληθῆ, καὶ τὰ ἑξῆς.

κα΄.

Τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλείσθω δὲ μέση.



Υπό γὰρ ἑητῶν δυνάμει μόνον συμμέτρων εὐθειῶν τῶν AB, BΓ ὀρθογώνιον περιεχέσθω τὸ AΓ· λέγω, ὅτι ἄλογόν ἐστι τὸ AΓ, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλείσθω δὲ μέση.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ἑητὸν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΒ τῆ ΒΓ μήκει· δυνάμει γὰρ μόνον ὑπόκεινται σύμμετροι· ἴση δὲ ἡ ΑΒ τῆ ΒΔ, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῆ ΒΓ μήκει. καί ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΑΔ πρὸς τὸ ΑΓ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΑ τῷ ΑΓ. ἑητὸν δὲ τὸ ΔΑ· ἄλογον ἄρα ἐστὶ τὸ ΑΓ· ὥστε καὶ ἡ δυναμένη τὸ ΑΓ [τουτέστιν ἡ ἴσον αὐτῷ τετράγωνον δυναμένη] ἄλογός ἐστιν, καλείσθω δε μέση· ὅπερ ἔδει δεῖξαι. AD is thus rational [Def. 10.4]. And AC (is) also rational. DA is thus commensurable with AC. And as DAis to AC, so DB (is) to BC [Prop. 6.1]. Thus, DB is also commensurable (in length) with BC [Prop. 10.11]. And DB (is) equal to BA. Thus, AB (is) also commensurable (in length) with BC. And AB is rational. Thus, BC is also rational, and commensurable in length with AB [Def. 10.3].

Thus, if a rational (area) is applied to a rational (straight-line), and so on

Proposition 21

The rectangle contained by rational straight-lines (which are) commensurable in square only is irrational, and its square-root is irrational—let it be called medial.^{\dagger}



For let the rectangle AC be contained by the rational straight-lines AB and BC (which are) commensurable in square only. I say that AC is irrational, and its square-root is irrational—let it be called medial.

For let the square AD have been described on AB. AD is thus rational [Def. 10.4]. And since AB is incommensurable in length with BC. For they were assumed to be commensurable in square only. And AB (is) equal to BD. DB is thus also incommensurable in length with BC. And as DB is to BC, so AD (is) to AC [Prop. 6.1]. Thus, DA [is] incommensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is irrational [Def. 10.4]. Hence, its square-root [that is to say, the square-root of the square equal to it] is also irrational [Def. 10.4]—let it be called medial. (Which is) the very thing it was required to show.

[†] Thus, a medial straight-line has a length expressible as $k^{1/4}$.

Λῆμμα.

Έαν ῶσι δύο εὐθεῖαι, ἔστιν ὡς ἡ πρώτη πρὸς τὴν δευτέραν, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ὑπὸ τῶν δύο εὐθειῶν.



Έστωσαν δύο εύθεῖαι αἱ ΖΕ, ΕΗ. λέγω, ὅτι ἐστὶν ὡς ἡ ΖΕ πρός την ΕΗ, οὕτως τὸ ἀπὸ τῆς ΖΕ πρὸς τὸ ὑπὸ τῶν ZE, EH.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΖΕ τετράγωνον τὸ ΔΖ, καὶ συμπεπληρώσθω τὸ ΗΔ. ἐπεὶ οὖν ἐστιν ὡς ἡ ΖΕ πρὸς τὴν EH, οὕτως τὸ ZΔ πρὸς τὸ ΔΗ,
 χαί ἐστι τὸ μὲν ZΔ τὸ ἀπὸ τῆς ΖΕ, τὸ δὲ Δ Η τὸ ὑπὸ τῶν Δ Ε, ΕΗ, τουτέστι τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἔστιν ἄρα ὡς ἡ ΖΕ πρὸς τὴν ΕΗ, οὕτως τὸ άπὸ τῆς ΖΕ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ. ὑμοίως δὲ καὶ ὡς τὸ ύπὸ τῶν ΗΕ, ΕΖ πρὸς τὸ ἀπὸ τῆς ΕΖ, τουτέστιν ὡς τὸ Η Δ πρός τὸ ZΔ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

Τὸ ἀπὸ μέσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ

ρητην και ασύμμετρον τῆ, παρ' ήν παράκειται, μήκει.

В

Г

Lemma

If there are two straight-lines then as the first is to the second, so the (square) on the first (is) to the (rectangle contained) by the two straight-lines.



Let FE and EG be two straight-lines. I say that as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG.

For let the square DF have been described on FE. And let GD have been completed. Therefore, since as FE is to EG, so FD (is) to DG [Prop. 6.1], and FD is the (square) on FE, and DG the (rectangle contained) by *DE* and *EG*—that is to say, the (rectangle contained) by FE and EG—thus as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG. And also, similarly, as the (rectangle contained) by GEand EF is to the (square on) EF—that is to say, as GD(is) to FD—so GE (is) to EF. (Which is) the very thing it was required to show.

Proposition 22

The square on a medial (straight-line), being applied to a rational (straight-line), produces as breadth a (straight-line which is) rational, and incommensurable in length with the (straight-line) to which it is applied.



Έστω μέση μὲν ἡ Α, ῥητὴ δὲ ἡ ΓΒ, καὶ τῷ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΒΔ πλάτος ποιοῦν τὴν ΓΔ· λέγω, ὅτι ῥητή ἐστιν ἡ ΓΔ καὶ ἀσύμμετρος τῆ ΓΒ μήχει.

Δ

E

Έπεὶ γὰρ μέση ἐστὶν ἡ Α, δύναται χωρίον περιεχόμενον ύπὸ ἑητῶν δυνάμει μόνον συμμέτρων. δυνάσθω τὸ ΗΖ.

Let A be a medial (straight-line), and CB a rational (straight-line), and let the rectangular area BD, equal to the (square) on A, have been applied to BC, producing CD as breadth. I say that CD is rational, and incommensurable in length with CB.

For since A is medial, the square on it is equal to a

303

Z

δύναται δ
ἐ καὶ τὸ $B\Delta$ · ἴσον ἄρα ἐστὶ τὸ $B\Delta$ τῷ HZ. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον. τῶν δὲ ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων άντιπεπόνθασιν αί πλευραί αί περί τὰς ἴσας γωνίας ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΕΗ, οὕτως ή ΕΖ πρός την ΓΔ. ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ἀπὸ τῆς EH, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς $\Gamma\Delta$. σύμμετρον δέ έστι τὸ ἀπὸ τῆς ΓΒ τῷ ἀπὸ τῆς ΕΗ· ῥητὴ γάρ έστιν έκατέρα αὐτῶν· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς EZ τῷ ἀπὸ τῆς ΓΔ. ἑητὸν δέ ἐστι τὸ ἀπὸ τῆς EZ· ἑητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ· ἑητὴ ἄρα ἐστὶν ἡ ΓΔ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΖ τῆ ΕΗ μήχει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΕΖ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἀσύμμετρον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς ΕΖ τῷ ὑπὸ τῶν ΖΕ, ΕΗ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΖ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΓΔ· ἑηταὶ γάρ εἰσι δυνάμει· τῷ δὲ ὑπὸ τῶν ΖΕ, ΕΗ σύμμετρόν ἐστι τὸ ὑπὸ τῶν ΔΓ, ΓΒ· ἴσα γάρ ἐστι τῷ ἀπὸ τῆς Α· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ τῷ ὑπὸ τῶν ΔΓ, ΓΒ. ὡς δὲ τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ὑπὸ τῶν $\Delta\Gamma$, ΓB , οὕτως ἐστὶν ἡ $\Delta\Gamma$ πρὸς τὴν ΓB · ἀσύμμετρος ἄρα ἐστὶν
ἡ $\Delta \Gamma$ τῆ ΓB μήκει.
ἑητὴ ἄρα ἐστὶν ἡ ΓΔ καὶ ἀσύμμετρος τῆ ΓΒ μήκει· ὅπερ ἔδει δεῖξαι.

[†] Literally, "rational".

χγ'.

[•]Η τῆ μέση σύμμετρος μέση ἐστίν.

Έστω μέση ή A, καὶ τῆ A σύμμετρος ἔστω ἡ B· λέγω, ὅτι καὶ ἡ B μέση ἐστίν.

Έχχείσθω γὰρ ἑητὴ ἡ ΓΔ, καὶ τῷ μὲν ἀπὸ τῆς Α ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΕΔ· ἑητὴ ἄρα ἐστὶν ἡ ΕΔ καὶ ἀσύμμετρος τὴ ΓΔ μήχει. τῷ δὲ ἀπὸ τῆς Β ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΓΖ πλάτος ποιοῦν τὴν ΔΖ. ἐπεὶ οῦν σύμμετρός ἐστιν ἡ Α τῆ Β, σύμμετρόν ἐστι καὶ τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς Β. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσον ἐστὶ τὸ ΕΓ, τῷ δὲ ἀπὸ τῆς Β ἴσον ἐστὶ τὸ ΓΖ· σύμμετρον ἄρα ἐστὶ τὸ ΕΓ τῷ ΓΖ. καί ἐστιν ὡς τὸ ΕΓ πρὸς τὸ ΓΖ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ·

(rectangular) area contained by rational (straight-lines which are) commensurable in square only [Prop. 10.21]. Let the square on (A) be equal to GF. And the square on (A) is also equal to BD. Thus, BD is equal to GF. And (BD) is also equiangular with (GF). And for equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, proportionally, as BC is to EG, so EF (is) to CD. And, also, as the (square) on BC is to the (square) on EG, so the (square) on EF (is) to the (square) on CD [Prop. 6.22]. And the (square) on CB is commensurable with the (square) on EG. For they are each rational. Thus, the (square) on *EF* is also commensurable with the (square) on CD [Prop. 10.11]. And the (square) on EF is rational. Thus, the (square) on CD is also rational [Def. 10.4]. Thus, CD is rational. And since EF is incommensurable in length with EG. For they are commensurable in square only. And as EF (is) to EG, so the (square) on EF (is) to the (rectangle contained) by FE and EG [see previous lemma]. The (square) on EF [is] thus incommensurable with the (rectangle contained) by FE and EG[Prop. 10.11]. But, the (square) on CD is commensurable with the (square) on EF. For they are rational in square. And the (rectangle contained) by DC and CBis commensurable with the (rectangle contained) by FEand EG. For they are (both) equal to the (square) on A. Thus, the (square) on CD is also incommensurable with the (rectangle contained) by DC and CB [Prop. 10.13]. And as the (square) on CD (is) to the (rectangle contained) by DC and CB, so DC is to CB [see previous lemma]. Thus, DC is incommensurable in length with CB [Prop. 10.11]. Thus, CD is rational, and incommensurable in length with CB. (Which is) the very thing it was required to show.

Proposition 23

A (straight-line) commensurable with a medial (straight-line) is medial.

Let A be a medial (straight-line), and let B be commensurable with A. I say that B is also a medial (staight-line).

Let the rational (straight-line) CD be set out, and let the rectangular area CE, equal to the (square) on A, have been applied to CD, producing ED as width. EDis thus rational, and incommensurable in length with CD[Prop. 10.22]. And let the rectangular area CF, equal to the (square) on B, have been applied to CD, producing DF as width. Therefore, since A is commensurable with B, the (square) on A is also commensurable with σύμμετρος ἄρα ἐστὶν ἡ ΕΔ τῆ ΔΖ μήχει. ῥητὴ δέ ἐστιν ἡ ΕΔ καὶ ἀσύμμετρος τῆ ΔΓ μήχει· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΔΖ καὶ ἀσύμμετρος τῆ ΔΓ μήχει· αἱ ΓΔ, ΔΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἡ δὲ τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων δυναμένη μέση ἐστίν. ἡ ἄρα τὸ ὑπὸ τῶν ΓΔ, ΔΖ δυναμένη μέση ἐστίν· καὶ δύναται τὸ ὑπὸ τῶν ΓΔ, ΔΖ ἡ Β· μέση ἄρα ἐστὶν ἡ Β.



Πόρισμα.

Έκ δη τούτου φανερόν, ὅτι τὸ τῷ μέσῳ χωρίῳ σύμμετρον μέσον ἐστίν.

the (square) on *B*. But, *EC* is equal to the (square) on *A*, and *CF* is equal to the (square) on *B*. Thus, *EC* is commensurable with *CF*. And as *EC* is to *CF*, so *ED* (is) to *DF* [Prop. 6.1]. Thus, *ED* is commensurable in length with *DF* [Prop. 10.11]. And *ED* is rational, and incommensurable in length with *CD*. *DF* is thus also rational [Def. 10.3], and incommensurable in length with *DC* [Prop. 10.13]. Thus, *CD* and *DF* are rational, and commensurable in square only. And the square-root of a (rectangle contained) by rational (straight-lines which are) commensurable in square only is medial [Prop. 10.21]. Thus, the square-root of the (rectangle contained) by *CD* and *DF*. Thus, *B* is a medial (straight-line).



Corollary

And (it is) clear, from this, that an (area) commensurable with a medial area^{\dagger} is medial.

[†] A medial area is equal to the square on some medial straight-line. Hence, a medial area is expressible as $k^{1/2}$.

χδ'.

Τὸ ὑπὸ μέσων μήχει συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον μέσον ἐστίν.

Υπό γὰρ μέσων μήχει συμμέτρων εὐθειῶν τῶν AB, BΓ περιεχέσθω ὀρθογώνιον τὸ AΓ· λέγω, ὅτι τὸ AΓ μέσον ἐστίν.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· μέσον ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΒ τῆ ΒΓ μήκει, ἴση δὲ ἡ ΑΒ τῆ ΒΔ, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῆ ΒΓ μήκει· ὥστε καὶ τὸ ΔΑ τῷ ΑΓ σύμμετρόν ἐστιν. μέσον δὲ τὸ ΔΑ· μέσον ἄρα καὶ τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

Proposition 24

A rectangle contained by medial straight-lines (which are) commensurable in length is medial.

For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in length. I say that AC is medial.

For let the square AD have been described on AB. AD is thus medial [see previous footnote]. And since AB is commensurable in length with BC, and AB (is) equal to BD, DB is thus also commensurable in length with BC. Hence, DA is also commensurable with AC[Props. 6.1, 10.11]. And DA (is) medial. Thus, AC (is) also medial [Prop. 10.23 corr.]. (Which is) the very thing it was required to show.



χε΄.

Τὸ ὑπὸ μέσων δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἦτοι ῥητὸν ἢ μέσον ἐστίν.



Υπό γὰρ μέσων δυνάμει μόνον συμμέτρων εὐθειῶν τῶν AB, BΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι τὸ ΑΓ ἤτοι ῥητὸν ἢ μέσον ἐστίν.

Άναγεγράφθω γὰρ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα τὰ ΑΔ, ΒΕ· μέσον ἄρα ἐστὶν ἑκάτερον τῶν ΑΔ, ΒΕ. καὶ ἐκκείσθω ρητή ή ZH, και τῷ μέν ΑΔ ἴσον παρά την ZH παραβεβλήσθω όρθογώνιον παραλληλόγραμμον τὸ ΗΘ πλάτος ποιοῦν τὴν ΖΘ, τῷ δὲ ΑΓ ἴσον παρὰ τὴν ΘΜ παραβεβλήσθω όρθογώνιον παραλληλόγραμμον τὸ ΜΚ πλάτος ποιοῦν τὴν ΘΚ, καὶ ἕτι τῷ ΒΕ ἴσον ὁμοίως παρὰ τὴν ΚΝ παραβεβλήσθω τὸ ΝΛ πλάτος ποιοῦν τὴν ΚΛ· ἐπ' εὐθείας ἄρα εἰσὶν αἱ ΖΘ, ΘΚ, ΚΛ. ἐπεὶ οὖν μέσον ἐστὶν ἑχάτερον τῶν $A\Delta$, BE, καί ἐστιν ἴσον τὸ μὲν $A\Delta$ τῷ HΘ, τὸ δὲ BE τῷ ΝΛ, μέσον ἄρα καὶ ἑκάτερον τῶν ΗΘ, ΝΛ. καὶ παρὰ ῥητὴν τὴν ΖΗ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΖΘ, ΚΛ καὶ άσύμμετρος τ
 \tilde{n} ΖΗ μήκει. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ
 $A\Delta$ τῷ ΒΕ, σύμμετρον ἄρα ἐστὶ καὶ τὸ ΗΘ τῷ ΝΛ. καί ἐστιν ὡς τὸ ΗΘ πρὸς τὸ ΝΛ, οὕτως
ἡ ΖΘ πρὸς τὴν ΚΛ· σύμμετρος άρα ἐστιν ἡ ΖΘ τῆ ΚΛ μήκει.
αί ΖΘ, ΚΛ ἄρα ἑηταί εἰσι μήκει σύμμετροι· ἑητὸν ἄρα ἐστὶ τὸ ὑπὸ τῶν ΖΘ, ΚΛ. καὶ



Proposition 25

The rectangle contained by medial straight-lines (which are) commensurable in square only is either rational or medial.



For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in square only. I say that AC is either rational or medial.

For let the squares AD and BE have been described on (the straight-lines) AB and BC (respectively). AD and BE are thus each medial. And let the rational (straight-line) FG be laid out. And let the rectangular parallelogram GH, equal to AD, have been applied to FG, producing FH as breadth. And let the rectangular parallelogram MK, equal to AC, have been applied to HM, producing HK as breadth. And, finally, let NL, equal to BE, have similarly been applied to KN, producing KL as breadth. Thus, FH, HK, and KL are in a straight-line. Therefore, since AD and BE are each medial, and AD is equal to GH, and BE to NL, GHand NL (are) thus each also medial. And they are applied to the rational (straight-line) FG. FH and KL are thus each rational, and incommensurable in length with FG [Prop. 10.22]. And since AD is commensurable with BE, GH is thus also commensurable with NL. And as

306

ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῆ ΒΑ, ἡ δὲ ΞΒ τῆ ΒΓ, ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΞ. ἀλλ' ὡς μὲν ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΔΑ πρὸς τὸ ΑΓ· ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΞ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΞ· ἔστιν ἄρα ὡς τὸ ΔΑ πρὸς τὸ ΑΓ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΞ. ἴσον δέ ἐστι τὸ μὲν ΑΔ τῷ ΗΘ, τὸ δὲ ΑΓ τῷ ΜΚ, τὸ δὲ ΓΞ τῷ ΝΛ· ἔστιν ἄρα ὡς τὸ ΗΘ πρὸς τὸ ΜΚ, οὕτως τὸ ΜΚ πρὸς τὸ ΝΛ· ἔστιν ἄρα ὡς ἡ ΖΘ πρὸς τὴν ΘΚ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΛ· τὸ ἄρα ὑπὸ τῶν ΖΘ, ΚΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘΚ. ἑητὸν δὲ τὸ ὑπὸ τῶν ΖΘ, ΚΛ· ἑρτὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ· ἑητὴ ἄρα ἐστὶν ἡ ΘΚ. καὶ εἰ μὲν σύμμετρός ἐστι τῆ ΖΗ μήχει, ἑητόν ἐστι τὸ ΘΝ· εἰ δὲ ἀσύμμετρός ἐστι ΖΗ μήχει, αἱ ΚΘ, ΘΜ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΘΝ. τὸ ΘΝ ἄρα ἤτοι ἑητὸν ἢ μέσον ἐστίν.

Τὸ ἄρα ὑπὸ μέσων δυνάμει μόνον συμμέτρων, καὶ τὰ εξῆς.



Εἰ γὰρ δυνατόν, μέσον τὸ AB μέσου τοῦ AΓ ὑπερεχέτω ἑητῷ τῷ ΔB, καὶ ἐκκείσθω ἑητὴ ἡ EZ, καὶ τῷ AB ἴσον παρὰ τὴν EZ παραβεβλήσθω παραλληλόγραμμον ὀρθογώνιον τὸ ZΘ πλάτος ποιοῦν τὴν EΘ, τῷ δὲ AΓ ἴσον ἀφηρήσθω τὸ ZH· λοιπὸν ἄρα τὸ BΔ λοιπῷ τῷ KΘ ἐστιν ἴσον. ἑητὸν δέ ἐστι τὸ ΔB· ἑητὸν ἄρα ἐστὶ καὶ τὸ KΘ. ἐπεὶ οῦν μέσον ἐστὶν ἑκάτερον τῶν AB, AΓ, καί ἐστι τὸ μὲν AB τῷ ZΘ ἴσον, τὸ δὲ AΓ τῷ ZH, μέσον ἄρα καὶ ἑκάτερον τῶν ZΘ, ZH. καὶ παρὰ ἑητὴν τὴν EZ παράκειται· ἑητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΘE, EH καὶ ἀσύμμετρος τῷ EZ μήκει. καὶ ἐπεὶ ἑῃτόν ἐστι

GH is to NL, so FH (is) to KL [Prop. 6.1]. Thus, FH is commensurable in length with KL [Prop. 10.11]. Thus, FH and KL are rational (straight-lines which are) commensurable in length. Thus, the (rectangle contained) by FH and KL is rational [Prop. 10.19]. And since DB is equal to BA, and OB to BC, thus as DB is to BC, so AB (is) to BO. But, as DB (is) to BC, so DA (is) to AC [Props. 6.1]. And as AB (is) to BO, so AC (is) to CO [Prop. 6.1]. Thus, as DA is to AC, so AC (is) to CO. And AD is equal to GH, and AC to MK, and COto NL. Thus, as GH is to MK, so MK (is) to NL. Thus, also, as FH is to HK, so HK (is) to KL [Props. 6.1, 5.11]. Thus, the (rectangle contained) by FH and KLis equal to the (square) on HK [Prop. 6.17]. And the (rectangle contained) by FH and KL (is) rational. Thus, the (square) on HK is also rational. Thus, HK is rational. And if it is commensurable in length with FG then HN is rational [Prop. 10.19]. And if it is incommensurable in length with FG then KH and HM are rational (straight-lines which are) commensurable in square only: thus, HN is medial [Prop. 10.21]. Thus, HN is either rational or medial. And HN (is) equal to AC. Thus, AC is either rational or medial.

Thus, the ... by medial straight-lines (which are) commensurable in square only, and so on

Proposition 26

A medial (area) does not exceed a medial (area) by a rational (area).^{\dagger}



For, if possible, let the medial (area) AB exceed the medial (area) AC by the rational (area) DB. And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram FH, equal to AB, have been applied to to EF, producing EH as breadth. And let FG, equal to AC, have been cut off (from FH). Thus, the remainder BD is equal to the remainder KH. And DB is rational. Thus, KH is also rational. Therefore, since AB and AC are each medial, and AB is equal to FH, and AC to FG, FH and FG are thus each also medial.

τὸ ΔB καί ἐστιν ἴσον τῷ $K\Theta$, ἑητὸν ἄρα ἐστὶ καὶ τὸ $K\Theta$. καὶ παρὰ ἑητὴν τὴν ΕΖ παράκειται· ἑητὴ ἄρα ἐστὶν ἡ ΗΘ καὶ σύμμετρος τῆ ΕΖ μήκει. ἀλλά καὶ ἡ ΕΗ ῥητή ἐστι καὶ άσύμμετρος τῆ ΕΖ μήχει ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΗ τῆ ΗΘ μήχει. χαί ἐστιν ὡς ἡ ΕΗ πρὸς τὴν ΗΘ, οὕτως τὸ ἀπὸ τῆς ΕΗ πρὸς τὸ ὑπὸ τῶν ΕΗ, ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΗ τῷ ὑπὸ τῶν ΕΗ, ΗΘ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΕΗ, ΗΘ τετράγωνα· ῥητὰ γὰρ άμφότερα· τῷ δὲ ὑπὸ τῶν ΕΗ, ΗΘ σύμμετρόν ἐστι τὸ δὶς ύπὸ τῶν ΕΗ, ΗΘ· διπλάσιον γάρ ἐστιν αὐτοῦ· ἀσύμμετρα άρα ἐστὶ τὰ ἀπὸ τῶν ΕΗ, ΗΘ τῷ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· καὶ συναμφότερα άρα τά τε ἀπὸ τῶν ΕΗ, ΗΘ καὶ τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΕΘ, ἀσύμμετρόν ἐστι τοῖς ἀπὸ τῶν ΕΗ, ΗΘ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΕΗ, ΗΘ· ἄλογον άρα τὸ ἀπὸ τῆς ΕΘ. ἄλογος ἄρα ἐστὶν ἡ ΕΘ. ἀλλὰ καὶ ῥηρή· ὅπερ ἐστὶν ἀδύνατον.

. Μέσον ἄρα μέσου οὐχ ὑπερέχει ῥητῷ· ὅπερ ἔδει δεῖξαι.

[†] In other words, $\sqrt{k} - \sqrt{k'} \neq k''$.

Μέσας εύρεῖν δυνάμει μόνον συμμέτρους ἑητὸν περιεχούσας.



Έκκείσθωσαν δύο ἑηταὶ δυνάμει μόνον σύμμετροι αἰ Α, Β, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Γ, καὶ γεγονέτω ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ.

And they are applied to the rational (straight-line) *EF*. Thus, HE and EG are each rational, and incommensurable in length with EF [Prop. 10.22]. And since DBis rational, and is equal to KH, KH is thus also rational. And (KH) is applied to the rational (straight-line) EF. GH is thus rational, and commensurable in length with EF [Prop. 10.20]. But, EG is also rational, and incommensurable in length with EF. Thus, EG is incommensurable in length with GH [Prop. 10.13]. And as EG is to GH, so the (square) on EG (is) to the (rectangle contained) by EG and GH [Prop. 10.13 lem.]. Thus, the (square) on EG is incommensurable with the (rectangle contained) by EG and GH [Prop. 10.11]. But, the (sum of the) squares on EG and GH is commensurable with the (square) on EG. For (EG and GH are) both rational. And twice the (rectangle contained) by EG and *GH* is commensurable with the (rectangle contained) by EG and GH [Prop. 10.6]. For (the former) is double the latter. Thus, the (sum of the squares) on EG and GH is incommensurable with twice the (rectangle contained) by EG and GH [Prop. 10.13]. And thus the sum of the (squares) on EG and GH plus twice the (rectangle contained) by EG and GH, that is the (square) on EH [Prop. 2.4], is incommensurable with the (sum of the squares) on EG and GH [Prop. 10.16]. And the (sum of the squares) on EG and GH (is) rational. Thus, the (square) on EH is irrational [Def. 10.4]. Thus, EH is irrational [Def. 10.4]. But, (it is) also rational. The very thing is impossible.

Thus, a medial (area) does not exceed a medial (area) by a rational (area). (Which is) the very thing it was required to show.

Proposition 27

To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.



Let the two rational (straight-lines) A and B, (which are) commensurable in square only, be laid down. And let C—the mean proportional (straight-line) to A and B—

Καὶ ἐπεὶ αἱ Α, Β ἑηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Γ, μέσον ἐστίν. μέση ἄρα ἡ Γ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, [οὕτως] ἡ Γ πρὸς τὴν Δ, αἱ δὲ Α, Β δυνάμει μόνον [εἰσὶ] σύμμετροι, καὶ αἱ Γ, Δ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. αἱ Γ, Δ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω, ὅτι καὶ ἑητὸν περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Γ, ἡ Β πρὸς τὴν Δ. ἀλλỉ ὡς ἡ Α πρὸς τὴν Γ, ἡ Γ πρὸς τὴν Β· καὶ ὡς ἄρα ἡ Γ πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Δ· τὸ ἄρα ὑπὸ τῶν Γ, Δ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β. ἑητὸν δὲ τὸ ἀπὸ τῆς Β· ἑητὸν ἄρα [ἐστὶ] καὶ τὸ ὑπὸ τῶν Γ, Δ.

Εὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι ἡητὸν περιέχουσαι ὅπερ ἔδει δεῖξαι.

have been taken [Prop. 6.13]. And let it be contrived that as A (is) to B, so C (is) to D [Prop. 6.12].

And since the rational (straight-lines) A and Bare commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on C[Prop. 6.17]—is thus medial [Prop 10.21]. Thus, C is medial [Prop. 10.21]. And since as A is to B, [so] C (is) to D, and A and B [are] commensurable in square only, C and D are thus also commensurable in square only [Prop. 10.11]. And C is medial. Thus, D is also medial [Prop. 10.23]. Thus, C and D are medial (straight-lines which are) commensurable in square only. I say that they also contain a rational (area). For since as A is to B, so C (is) to D, thus, alternately, as A is to C, so B (is) to D [Prop. 5.16]. But, as A (is) to C, (so) C (is) to B. And thus as C (is) to B, so B (is) to D [Prop. 5.11]. Thus, the (rectangle contained) by C and D is equal to the (square) on B [Prop. 6.17]. And the (square) on B(is) rational. Thus, the (rectangle contained) by C and D [is] also rational.

Thus, (two) medial (straight-lines, C and D), containing a rational (area), (which are) commensurable in square only, have been found.[†] (Which is) the very thing it was required to show.

[†] C and D have lengths $k^{1/4}$ and $k^{3/4}$ times that of A, respectively, where the length of B is $k^{1/2}$ times that of A.

×η'.

Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους μέσον πειριεχούσας.



Έκκείσθωσαν [τρεῖς] ἡηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Δ, καὶ γεγονέτω ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε.

Έπεὶ ai A, B ἑŋτaí εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν A, B, τουτέστι τὸ ἀπὸ τῆς Δ, μέσον ἐστὶν. μέση ἄρα ἡ Δ. καὶ ἐπεὶ αἱ B, Γ δυνάμει μόνον εἰσὶ σύμμετροι, καί ἐστιν ὡς ἡ B πρὸς τὴν Γ, ἡ Δ πρὸς τὴν E, καὶ αἱ Δ, E ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. μέση δὲ ἡ Δ· μέση ἄρα καὶ ἡ E· αἱ Δ, E ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ μέσον περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ B πρὸς τὴν Γ, ἡ Δ πρὸς τὴν E, ἐναλλὰξ ἄρα ὡς ἡ B πρὸς τὴν Δ, ἡ Γ πρὸς τὴν E. ὡς δὲ ἡ B πρὸς τὴν Δ, ἡ Δ πρὸς τὴν Α· καὶ ὡς ἄρα ἡ Δ πρὸς τὴν Α, ἡ Γ πρὸς τὴν E· τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Δ, E. μέσον δὲ τὸ ὑπὸ τῶν Α, Γ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, E.

Έὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι μέσον

Proposition 28

To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.



Let the [three] rational (straight-lines) A, B, and C, (which are) commensurable in square only, be laid down. And let, D, the mean proportional (straight-line) to A and B, have been taken [Prop. 6.13]. And let it be contrived that as B (is) to C, (so) D (is) to E [Prop. 6.12].

Since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on D [Prop. 6.17] is medial [Prop. 10.21]. Thus, D (is) medial [Prop. 10.21]. And since B and C are commensurable in square only, and as B is to C, (so) D (is) to E, D and E are thus commensurable in square only [Prop. 10.11]. And D (is) medial. E (is) thus also medial [Prop. 10.23]. Thus, D and E are medial (straight-lines which are) commensurable in square only. So, I say that they also enclose a medial (area). For since as B is to C, (so) D (is) to E, thus, περιέχουσαι. ὅπερ ἔδει δεῖξαι.

alternately, as B (is) to D, (so) C (is) to E [Prop. 5.16]. And as B (is) to D, (so) D (is) to A. And thus as D (is) to A, (so) C (is) to E. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by D and E [Prop. 6.16]. And the (rectangle contained) by A and C is medial [Prop. 10.21]. Thus, the (rectangle contained) by D and E (is) also medial.

Thus, (two) medial (straight-lines, D and E), containing a medial (area), (which are) commensurable in square only, have been found. (Which is) the very thing it was required to show.

[†] D and E have lengths $k^{1/4}$ and $k'^{1/2}/k^{1/4}$ times that of A, respectively, where the lengths of B and C are $k^{1/2}$ and $k'^{1/2}$ times that of A, respectively.

Λῆμμα α΄.

Εύρειν δύο τετραγώνους ἀριθμούς, ὥστε καὶ τὸν συγκείμενον ἐξ αὐτῶν εἶναι τετράγωνον.



Έχχείσθωσαν δύο ἀριθμοὶ οἱ AB, BΓ, ἔστωσαν δὲ ἤτοι ἄρτιοι ἢ περιττοί. καὶ ἐπεὶ, ἐάν τε ἀπὸ ἀρτίου ἄρτιος ἀφαιρεθῆ, ἐάν τε ἀπὸ περισσοῦ περισσός, ὁ λοιπὸς ἄρτιός ἐστιν, ὁ λοιπὸς ἄρα ὁ ΑΓ ἄρτιός ἐστιν. τετμήσθω ὁ ΑΓ δίχα κατὰ τὸ Δ. ἔστωσαν δὲ καὶ οἱ AB, BΓ ἤτοι ὄμοιοι ἐπίπεδοι ἢ τετράγωνοι, οἶ καὶ αὐτοὶ ὅμοιοί εἰσιν ἐπίπεδοι· ὁ ἄρα ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ τοῦ ΒΔ τετραγώνῳ. καί ἐστι τετράγωνος ὁ ἐκ τῶν AB, BΓ, ἐπειδήπερ ἐδείχθη, ὅτι, ἐὰν δύο ὅμοιοι ἐπίπεδοι πολλαπλασιάσαντες ἀλλήλους ποιῶσι τινα, ὁ γενόμενος τετράγωνός ἐστιν. εὕρηνται ἄρα δύο τετράγωνοι ἀριθμοὶ ὅ τε ἐκ τῶν AB, BΓ καὶ ὁ ἀπὸ τοῦ ΓΔ, οἶ συντεθέντες ποιοῦσι τὸν ἀπὸ τοῦ ΒΔ τετράγωνον.

Καὶ φανερόν, ὅτι εὕρηνται πάλιν δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΓΔ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ὑπὸ ΑΒ, ΒΓ εἶναι τετράγωνον, ὅταν οἱ ΑΒ, ΒΓ ὅμοιοι ὥσιν ἐπίπεδοι. ὅταν δὲ μὴ ὥσιν ὅμοιοι ἐπίπεδοι, εὕρηνται δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΔΓ, ῶν ἡ ὑπεροχὴ ὁ ὑπὸ τῶν ΑΒ, ΒΓ οὐκ ἔστι τετράγωνος· ὅπερ ἔδει δεῖξαι.

Lemma I

To find two square numbers such that the sum of them is also square.



Let the two numbers AB and BC be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even. Let AC have been cut in half at D. And let AB and BC also be either similar plane (numbers), or square (numbers)-which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [Prop. 2.6]. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC, and the (square) on CD—which, (when) added (together), make the square on *BD*.

And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on BD, and the (square) on CD—such that their difference—(namely,) the (rectangle) contained by AB and BC—is square whenever AB and BC are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) have been found—(namely,) the (square) on BD, and the (square) on DC—between which the difference—(namely,) the (rectangle) contained by ABand BC—is not square. (Which is) the very thing it was required to show.

Λῆμμα β'.

Εύρεῖν δύο τετραγώνους ἀριθμούς, ὥστε τὸν ἐξ αὐτῶν συγχείμενον μὴ εἶναι τετράγωνον.

В

А

$$\begin{array}{cccc} A & H & \Delta & Z & \Gamma \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & \Theta & E \end{array}$$

Έστω γὰρ ὁ ἐκ τῶν AB, BΓ, ὡς ἔφαμεν, τετράγωνος, καὶ ἄρτιος ὁ ΓΑ, καὶ τετμήσθω ὁ ΓΑ δίχα τῷ Δ. φανερὸν δή, ὅτι ὁ ἐκ τῶν AB, BΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] BΔ τετραγώνῳ. ἀφηρήσθω μονὰς ἡ ΔΕ[.] ὁ ἄρα ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἐλάσσων ἐστὶ τοῦ ἀπὸ [τοῦ] BΔ τετραγώνου. λέγω οῦν, ὅτι ὁ ἐκ τῶν AB, BΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ οὐκ ἔσται τετράγωνος.

Εἰ γὰρ ἔσται τετράγωνος, ἤτοι ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΕ η ἐλάσσων τοῦ ἀπὸ [τοῦ] ΒΕ, οὐκέτι δὲ καὶ μείζων, ἴνα μή τμηθή ή μονάς. έστω, εί δυνατόν, πρότερον ό έχ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΕ, καὶ ἔστω τῆς ΔΕ μονάδος διπλασίων ὁ ΗΑ. ἐπεὶ οὖν ὅλος ὁ ΑΓ ὅλου τοῦ ΓΔ ἐστι διπλασίων, ῶν ὁ ΑΗ τοῦ ΔΕ ἐστι διπλασίων, καὶ λοιπὸς ἄρα ὁ ΗΓ λοιποῦ τοῦ ΕΓ ἐστι διπλασίων δίγα ἄρα τέτμηται ὁ ΗΓ τῷ Ε. ὁ ἄρα ἐχ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ τετραγώνω. ἀλλὰ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ὑπόχειται τῷ ἀπὸ [τοῦ] ΒΕ τετραγώνω· ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ίσος ἐστὶ τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ. καὶ κοινοῦ άφαιρεθέντος τοῦ ἀπὸ ΓΕ συνάγεται ὁ ΑΒ ἴσος τῷ ΗΒ δπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ. λέγω δή, ὅτι οὐδὲ ἐλάσσων τοῦ ἀπὸ ΒΕ. εἰ γὰρ δυνατόν, ἔστω τῷ ἀπὸ ΒΖ ἴσος, καὶ τοῦ ΔΖ διπλασίων ὁ ΘΑ. καὶ συναχθήσεται πάλιν διπλασίων ὁ ΘΓ τοῦ ΓΖ· ὥστε καὶ τὸν ΓΘ δίγα τετμῆσθαι κατὰ τὸ Ζ, καὶ διὰ τοῦτο τὸν ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΖΓ ἴσον γίνεσθαι τῷ ἀπὸ ΒΖ. ὑπόκειται δὲ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΖ. ὥστε καὶ ὁ ἐκ τῶν ΘΒ, ΒΓ μετά τοῦ ἀπὸ ΓΖ ἴσος ἔσται τῷ ἐκ τῶν ΑΒ, ΒΓ μετά τοῦ ἀπὸ ΓΕ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ [τῷ] ἐλάσσονι τοῦ ἀπὸ ΒΕ. ἐδείχθη δέ, ὅτι οὐδὲ [αὐτῷ] τῷ ἀπὸ ΒΕ. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ τετράγωνός ἐστιν. ὅπερ ἔδει δεῖξαι.

Lemma II

To find two square numbers such that the sum of them is not square.

For let the (number created) from (multiplying) AB and BC, as we said, be square. And (let) CA (be) even. And let CA have been cut in half at D. So it is clear that the square (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [see previous lemma]. Let the unit DE have been subtracted (from BD). Thus, the (number created) from (multiplying) AB and BC, plus the square on BD. I say, therefore, that the square (number created) from (multiplying) AB and BC, plus the (square) on CE, is less than the square on BD. I say, therefore, that the square (number created) from (multiplying) AB and BC, plus the (square) on CE, is not square.

For if it is square, it is either equal to the (square) on BE, or less than the (square) on BE, but cannot any more be greater (than the square on BE), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) AB and BC, plus the (square) on CE, be equal to the (square) on BE. And let GA be double the unit DE. Therefore, since the whole of AC is double the whole of CD, of which AG is double DE, the remainder GC is thus double the remainder EC. Thus, GC has been cut in half at E. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the square on BE [Prop. 2.6]. But, the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the square on BE. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE. And subtracting the (square) on CE from both, AB is inferred (to be) equal to GB. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to the (square) on BE. So I say that (it is) not less than the (square) on BE either. For, if possible, let it be equal to the (square) on BF. And (let) HA (be) double DF. And it can again be inferred that HC (is) double CF. Hence, CH has also been cut in half at F. And, on account of this, the (number created) from (multiplying) HB and BC, plus the (square) on FC, becomes equal to the (square) on BF [Prop. 2.6]. And the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the (square) on *BF*. Hence, the (number created) from (multiplying) *HB* and *BC*, plus the (square) on *CF*, will also be equal to the (number created) from (multiplying) AB and BC,

plus the (square) on CE. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to less than the (square) on BE. And it was shown that (is it) not equal to the (square) on BE either. Thus, the (number created) from (multiplying) AB and BC, plus the square on CE, is not square. (Which is) the very thing it was required to show.

Proposition 29

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line which is) commensurable in length with the greater.





Εύρεῖν δύο ἑητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἑλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῆ μήχει.



Έκκείσθω γάρ τις ἡητὴ ἡ AB καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΔ, ΔΕ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ΓΕ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AZB, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς BA τετράγωνον πρὸς τὸ ἀπὸ τῆς AZ τετράγωνον, καὶ ἐπεζεύχθω ἡ ZB.

Ἐπεὶ [οὖν] ἐστιν ὡς τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ούτως
ό $\Delta\Gamma$ πρὸς τὸν ΓΕ, τὸ ἀπὸ τῆς BA ẳρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὄν ἀριθμὸς ὁ ΔΓ πρὸς ἀριθμὸν τὸν ΓΕ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΖ. ἑητὸν δὲ τὸ ἀπὸ τῆς ΑΒ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΑΖ· ῥητὴ ἄρα καὶ ἡ ΑΖ. καὶ ἐπεὶ ὁ ΔΓ πρὸς τὸν ΓΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν τετράγωνος άριθμός πρός τετράγωνον άριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ AB τῆ AZ μήχει· αἱ BA, AZ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί [ἐστιν] ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ό ΓΔ πρός τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον ἔχει, ὃν τετράγωνος άριθμός πρός τετράγωνον άριθμόν και τὸ ἀπὸ τῆς ΑΒ ἄρα πρός τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρός τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῆ ΒΖ μήκει. καί ἐστι τὸ ἀπὸ τῆς ΑΒ ἴσον τοῖς ἀπὸ τῶν ΑΖ, ΖΒ· ή ΑΒ ἄρα τῆς ΑΖ μεῖζον δύναται τῆ ΒΖ συμμέτρω

For let some rational (straight-line) AB be laid down, and two square numbers, CD and DE, such that the difference between them, CE, is not square [Prop. 10.28 lem. I]. And let the semi-circle AFB have been drawn on AB. And let it be contrived that as DC (is) to CE, so the square on BA (is) to the square on AF [Prop. 10.6 corr.]. And let FB have been joined.

[Therefore,] since as the (square) on BA is to the (square) on AF, so DC (is) to CE, the (square) on BA thus has to the (square) on AF the ratio which the number DC (has) to the number CE. Thus, the (square) on *BA* is commensurable with the (square) on AF [Prop. 10.6]. And the (square) on AB (is) rational [Def. 10.4]. Thus, the (square) on AF (is) also rational. Thus, AF (is) also rational. And since DC does not have to CE the ratio which (some) square number (has) to (some) square number, the (square) on BA thus does not have to the (square) on AF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with AF [Prop. 10.9]. Thus, the rational (straight-lines) BA and AF are commensurable in square only. And since as DC [is] to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on

ἑαυτῆ.

Εὕρηνται ἄρα δύο ἑηταὶ δυνάμει μόνον σύμμετροι αἱ BA, AZ, ὥστε τὴν μεῖζονα τὴν AB τῆς ἐλάσσονος τῆς AZ μεῖζον δύνασθαι τῷ ἀπὸ τῆς BZ συμμέτρου ἑαυτῆ μήχει· ὅπερ ἔδει δεῖζαι. BF [Props. 5.19 corr., 3.31, 1.47]. And CD has to DE the ratio which (some) square number (has) to (some) square number. Thus, the (square) on AB also has to the (square) on BF the ratio which (some) square number has to (some) square number. AB is thus commensurable in length with BF [Prop. 10.9]. And the (square) on AB is equal to the (sum of the squares) on AF and FB [Prop. 1.47]. Thus, the square on AB is greater than (the square on) AF by (the square on) BF, (which is) commensurable (in length) with (AB).

Thus, two rational (straight-lines), BA and AF, commensurable in square only, have been found such that the square on the greater, AB, is larger than (the square on) the lesser, AF, by the (square) on BF, (which is) commensurable in length with (AB).[†] (Which is) the very thing it was required to show.

[†] BA and AF have lengths 1 and $\sqrt{1-k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CD}$.

λ'.

Εύρεῖν δύο ἑητὰς δυνάμει μόνον συμμέτρους, ὤστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήχει.



Έκκείσθω ἑητὴ ἡ AB καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΕ, ΕΔ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΓΔ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AZB, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς BA πρὸς τὸ ἀπὸ τῆς AZ, καὶ ἐπεζεύχθω ἡ ZB.

Όμοίως δὴ δείξομεν τῷ πρὸ τούτου, ὅτι αἱ BA, AZ ἑηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί ἐστιν ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς BA πρὸς τὸ ἀπὸ τῆς AZ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ AB τῆ BZ μήκει. καὶ δύναται ἡ AB τῆς AZ μεῖζον τῷ ἀπὸ τῆς ZB ἀσυμμέτρου ἑαυτῆ.

Proposition 30

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.



Let the rational (straight-line) AB be laid out, and the two square numbers, CE and ED, such that the sum of them, CD, is not square [Prop. 10.28 lem. II]. And let the semi-circle AFB have been drawn on AB. And let it be contrived that as DC (is) to CE, so the (square) on BA (is) to the (square) on AF [Prop. 10.6 corr]. And let FB have been joined.

So, similarly to the (proposition) before this, we can show that BA and AF are rational (straight-lines which are) commensurable in square only. And since as DC is to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, 1.47]. And CD does not have to DE the ratio which (some) square number (has) to (some) square number. Ai AB, AZ ἄρα ἡηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ AB τῆς AZ μεῖζον δύναται τῷ ἀπὸ τῆς ZB ἀσυμμέτρου ἑαυτῇ μήχει· ὅπερ ἔδει δεῖζαι. Thus, the (square) on AB does not have to the (square) on BF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with BF [Prop. 10.9]. And the square on AB is greater than the (square on) AF by the (square) on FB [Prop. 1.47], (which is) incommensurable (in length) with (AB).

Thus, AB and AF are rational (straight-lines which are) commensurable in square only, and the square on AB is greater than (the square on) AF by the (square) on FB, (which is) incommensurable (in length) with (AB).[†] (Which is) the very thing it was required to show.

[†] AB and AF have lengths 1 and $1/\sqrt{1+k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CE}$.

λα΄.

Εύρεῖν δύο μέσας δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει.



Έκκείσθωσαν δύο ἑηταὶ δυνάμει μόνον σύμμετροι αἱ Α, B, ὥστε τὴν Α μείζονα οὕσαν τῆς ἐλάσσονος τῆς Β μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει. καὶ τῷ ὑπὸ τῶν A, B ἴσον ἔστω τὸ ἀπὸ τῆς Γ. μέσον δὲ τὸ ὑπὸ τῶν A, B· μέσον ἄρα καὶ τὸ ἀπὸ τῆς Γ· μέση ἄρα καὶ ἡ Γ. τῷ δὲ ἀπὸ τῆς B ἴσον ἔστω τὸ ὑπὸ τῶν Γ , Δ · ἑητὸν δὲ τὸ ἀπὸ τῆς B· ρητον ἄρα και το ὑπο τῶν Γ, Δ. και ἐπεί ἐστιν ὡς ἡ Α προς τὴν Β, οὕτως τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ἀπὸ τῆς Β, ἀλλὰ τῷ μέν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Γ, τῷ δὲ ἀπὸ τῆς Β ἴσον τὸ ὑπὸ τῶν Γ, Δ, ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ. ὡς δὲ τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ, οὕτως ἡ Γ
 πρὸς τὴν Δ· καὶ ὡς ẳρα ή A πρὸς τὴν B, ο
ὕτως ἡ Γ πρὸς τὴν Δ. σύμμετρος δὲ ἡ A τῆ Β δυνάμει μόνον· σύμμετρος ἄρα καὶ ἡ Γ τῆ Δ δυνάμει μόνον.
 καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. καὶ ἐπεί ἐστιν ώς
ἡ A πρὸς τὴν B, ἡ Γ πρὸς τὴν Δ, ἡ δὲ A τῆς B με
ῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ ἡ Γ ἄρα τῆς Δ με
ῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αί Γ,

Proposition 31

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.



Let two rational (straight-lines), A and B, commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser Bby the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29]. And let the (square) on C be equal to the (rectangle contained) by A and B. And the (rectangle contained by) A and B (is) medial [Prop. 10.21]. Thus, the (square) on C (is) also medial. Thus, C (is) also medial [Prop. 10.21]. And let the (rectangle contained) by C and D be equal to the (square) on B. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D (is) also rational. And since as A is to B, so the (rectangle contained) by A and B (is) to the (square) on B [Prop. 10.21 lem.], but the (square) on C is equal to the (rectangle contained) by A and B, and the (rectangle contained) by C and D to the (square) on B, thus as A (is) to B, so the (square) on C (is) to the (rectangle contained) by C and D. And as the (square) on C (is) to the (rectangle contained) by Δ
ρητὸν περιέχουσαι, καὶ
ἡ Γ τῆς Δ μεῖζον δυνάται τῷ ἀπὸ συμμ
έτρου ἑαυτῆ μήκει.

Όμοίως δὴ δειχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Β μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. *C* and *D*, so *C* (is) to *D* [Prop. 10.21 lem.]. And thus as *A* (is) to *B*, so *C* (is) to *D*. And *A* is commensurable in square only with *B*. Thus, *C* (is) also commensurable in square only with *D* [Prop. 10.11]. And *C* is medial. Thus, *D* (is) also medial [Prop. 10.23]. And since as *A* is to *B*, (so) *C* (is) to *D*, and the square on *A* is greater than (the square on) *B* by the (square) on (some straight-line) commensurable (in length) with (*A*), the square on *C* is thus also greater than (the square on) *D* by the (square) on (some straight-line) commensurable (in length) with (*C*) [Prop. 10.14].

Thus, two medial (straight-lines), C and D, commensurable in square only, (and) containing a rational (area), have been found. And the square on C is greater than (the square on) D by the (square) on (some straight-line) commensurable in length with (C).[†]

So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with *C*), provided that the square on *A* is greater than (the square on *B*) by the (square) on (some straight-line) incommensurable (in length) with (*A*) [Prop. 10.30].[‡]

[†] C and D have lengths $(1 - k^2)^{1/4}$ and $(1 - k^2)^{3/4}$ times that of A, respectively, where k is defined in the footnote to Prop. 10.29.

[‡] C and D would have lengths $1/(1+k^2)^{1/4}$ and $1/(1+k^2)^{3/4}$ times that of A, respectively, where k is defined in the footnote to Prop. 10.30.

λβ΄.

Εύρεῖν δύο μέσας δυνάμει μόνον συμμέτρους μέσον περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῆ.



Έκκείσθωσαν τρεῖς ἑηταὶ δυνάμει μόνον σύμμετροι αἰ A, B, Γ, ὥστε τὴν A τῆς Γ μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ τῷ μὲν ὑπὸ τῶν A, B ἴσον ἔστω τὸ ἀπὸ τὴς Δ. μέσον ἄρα τὸ ἀπὸ τῆς Δ· καὶ ἡ Δ ἄρα μέση ἑστίν. τῷ δὲ ὑπὸ τῶν B, Γ ἴσον ἔστω τὸ ὑπὸ τῶν Δ, Ε. καὶ ἐπεί ἐστιν ὡς τὸ ὑπὸ τῶν A, B πρὸς τὸ ὑπὸ τῶν Δ, Ε. καὶ ἐπεί ἐστιν ὡς τὸ ὑπὸ τῶν A, B πρὸς τὸ ὑπὸ τῶν B, Γ, οὕτως ἡ A πρὸς τὴν Γ, ἀλλὰ τῷ μὲν ὑπὸ τῶν A, B ἴσον ἐστὶ τὸ ἀπὸ τῆς Δ, τῷ δὲ ὑπὸ τῶν B, Γ ἴσον τὸ ὑπὸ τῶν Δ, Ε, ἔστιν ἄρα ὡς ἡ A πρὸς τὴν Γ, οὕτως τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε. ὡς δὲ τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε, οῦτως ἡ Δ πρὸς τὴν Ε· καὶ ὡς ἄρα ἡ A πρὸς τὴν Γ, οὕτως ἡ Δ πρὸς τὴν Ε. σύμμετρος δὲ ἡ A τῆ Γ δυνάμει [μόνον]. σύμμετρος ἄρα καὶ ἡ Δ τῆ Ε δυνάμει μόνον. μέση

Proposition 32

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.



Let three rational (straight-lines), A, B and C, commensurable in square only, be laid out such that the square on A is greater than (the square on C) by the (square) on (some straight-line) commensurable (in length) with (A) [Prop. 10.29]. And let the (square) on D be equal to the (rectangle contained) by A and B. Thus, the (square) on D (is) medial. Thus, D is also medial [Prop. 10.21]. And let the (rectangle contained) by D and E be equal to the (rectangle contained) by B and C. And since as the (rectangle contained) by A and B is to the (rectangle contained) by B and C, so A (is) to C [Prop. 10.21 lem.], but the (square) on D is equal to the (rectangle contained) by A and B, and the (rectangle contained) by A and B.

δὲ ἡ Δ · μέση ἄρα καὶ ἡ E. καὶ ἐπεί ἐστιν ὡς ἡ A πρὸς τὴν Γ, ἡ Δ πρὸς τὴν E, ἡ δὲ A τῆς Γ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ ἡ Δ ἄρα τῆς E μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆ. λέγω δή, ὅτι καὶ μέσον ἐστὶ τὸ ὑπὸ τῶν Δ , E. ἐπεὶ γὰρ ἴσον ἐστὶ τὸ ὑπὸ τῶν B, Γ τῷ ὑπὸ τῶν Δ , E, μέσον δὲ τὸ ὑπὸ τῶν B, Γ [αἱ γὰρ B, Γ ἑῃταί εἰσι δυνάμει μόνον σύμμετροι], μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ , E.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Δ, Ε μέσον περιέχουσαι, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῆ.

Όμοίως δη πάλιν διεχθήσεται και τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Γ μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτη.

contained) by D and E to the (rectangle contained) by B and C, thus as A is to C, so the (square) on D (is) to the (rectangle contained) by D and E. And as the (square) on D (is) to the (rectangle contained) by D and E, so D (is) to E [Prop. 10.21 lem.]. And thus as A (is) to C, so D (is) to E. And A (is) commensurable in square [only] with C. Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial. Thus, E (is) also medial [Prop. 10.23]. And since as A is to C, (so) D (is) to E, and the square on A is greater than (the square on) C by the (square) on (some straight-line) commensurable (in length) with (A), the square on D will thus also be greater than (the square on) E by the (square) on (some straight-line) commensurable (in length) with (D) [Prop. 10.14]. So, I also say that the (rectangle contained) by D and E is medial. For since the (rectangle contained) by B and C is equal to the (rectangle contained) by D and E, and the (rectangle contained) by B and C (is) medial [for B and C are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by D and E (is) thus also medial.

Thus, two medial (straight-lines), D and E, commensurable in square only, (and) containing a medial (area), have been found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.[†].

So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on A is greater than (the square on) C by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].[‡]

[†] *D* and *E* have lengths $k'^{1/4}$ and $k'^{1/4}\sqrt{1-k^2}$ times that of *A*, respectively, where the length of *B* is $k'^{1/2}$ times that of *A*, and *k* is defined in the footnote to Prop. 10.29.

[‡] *D* and *E* would have lengths $k'^{1/4}$ and $k'^{1/4}/\sqrt{1+k^2}$ times that of *A*, respectively, where the length of *B* is $k'^{1/2}$ times that of *A*, and *k* is defined in the footnote to Prop. 10.30.

Λῆμμα.

Έστω τρίγωνον ὀρθογώνιον τὸ ABΓ ὀρθὴν ἔχον τὴν A, καὶ ἦχθω κάθετος ἡ AΔ· λέγω, ὅτι τὸ μὲν ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς BA, τὸ δὲ ὑπὸ τῶν BΓΑ ἴσον τῷ ἀπὸ τῆς ΓΑ, καὶ τὸ ὑπὸ τῶν BΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς AΔ, καὶ ἔτι τὸ ὑπὸ τῶν BΓ, AΔ ἴσον [ἐστὶ] τῶ ὑπὸ τῶν BA, AΓ.

Καὶ πρῶτον, ὅτι τὸ ὑπὸ τῶν ΓΒΔ ἴσον [ἐστὶ] τῷ ἀπὸ τῆς ΒΑ.

Lemma

Let ABC be a right-angled triangle having the (angle) A a right-angle. And let the perpendicular AD have been drawn. I say that the (rectangle contained) by CBDis equal to the (square) on BA, and the (rectangle contained) by BCD (is) equal to the (square) on CA, and the (rectangle contained) by BD and DC (is) equal to the (square) on AD, and, further, the (rectangle contained) by BC and AD [is] equal to the (rectangle contained) by BA and AC.



Έπεὶ γὰρ ἐν ὀρθογωνίω τριγώνω ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΑΔ, τὰ ΑΒΔ, ΑΔΓ ἄρα τρίγωνα ὅμοιά ἐστι τῷ τε ὅλω τῷ ΑΒΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνω, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΒΔ· τὸ ἄρα ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ.

Διὰ τὰ αὐτὰ δὴ
 καὶ τὸ ὑπὸ τῶν
 $B\Gamma\Delta$ ἴσον ἐστὶ τῷ ἀπὸ τῆς
 $A\Gamma.$

Καὶ ἐπεί, ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῆ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν, ἔστιν ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΔΑ, οὕτως ἡ ΑΔ πρὸς τὴν ΔΓ· τὸ ἄρα ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΑ.

Λέγω, ὅτι καὶ τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ. ἐπεὶ γὰρ, ὡς ἔφαμεν, ὅμοιόν ἐστι τὸ ΑΒΓ τῷ ΑΒΔ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΔ. τὸ ἄρα ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ· ὅπερ ἔδει δεῖξαι.

λγ΄.

Εύρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον.

Έκκείσθωσαν δύο ἑηταὶ δυνάμει μόνον σύμμετροι αἰ AB, BΓ, ὥστε τὴν μείζονα τὴν AB τῆς ἐλάσσονος τῆς BΓ μείζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ τετμήσθω ἡ BΓ δίχα κατὰ τὸ Δ, καὶ τῷ ἀφ' ὑποτέρας τῶν BΔ, ΔΓ ἴσον παρὰ τὴν AB παραβεβλήσθω παραλληλόγραμμον ἐλλεῖπον είδει τετραγώνω, καὶ ἔστω τὸ ὑπὸ τῶν AEB, καὶ γεγράφθω ἐπὶ τῆς AB ημικύκλιον τὸ AZB, καὶ ἤχθω τῆ AB πρὸς And, first of all, (let us prove) that the (rectangle contained) by CBD [is] equal to the (square) on BA.



For since AD has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, ABDand ADC are thus triangles (which are) similar to the whole, ABC, and to one another [Prop. 6.8]. And since triangle ABC is similar to triangle ABD, thus as CB is to BA, so BA (is) to BD [Prop. 6.4]. Thus, the (rectangle contained) by CBD is equal to the (square) on AB[Prop. 6.17].

So, for the same (reasons), the (rectangle contained) by BCD is also equal to the (square) on AC.

And since if a (straight-line) is drawn from the rightangle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as BD is to DA, so AD (is) to DC. Thus, the (rectangle contained) by BD and DC is equal to the (square) on DA [Prop. 6.17].

I also say that the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC. For since, as we said, ABC is similar to ABD, thus as BC is to CA, so BA (is) to AD [Prop. 6.4]. Thus, the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC [Prop. 6.16]. (Which is) the very thing it was required to show.

Proposition 33

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Let the two rational (straight-lines) AB and BC, (which are) commensurable in square only, be laid out such that the square on the greater, AB, is larger than (the square on) the lesser, BC, by the (square) on (some straight-line which is) incommensurable (in length) with (AB) [Prop. 10.30]. And let BC have been cut in half at D. And let a parallelogram equal to the (square) on eiόρθὰς ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ.



Καὶ ἐπεὶ [δύο] εὐθεῖαι ἄνισοί εἰσιν αἱ ΑΒ, ΒΓ, καὶ ἡ ΑΒ τῆς ΒΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, τῷ δὲ τετάρτω τοῦ ἀπὸ τῆς ΒΓ, τουτέστι τῷ ἀπὸ τῆς ήμισείας αὐτῆς, ἴσον παρὰ τὴν ΑΒ παραβέβληται παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνω καὶ ποιεῖ τὸ ὑπὸ τῶν ΑΕΒ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΕ τῃ ΕΒ. καί ἐστιν ὡς ή ΑΕ πρός ΕΒ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΕ, ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΕ τῷ ἀπὸ τῆς AZ, τὸ δὲ ὑπὸ τῶν AB, BE τῷ ἑπὸ τῆς BZ ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΖ τῷ ἀπὸ τῆς ΖΒ· αἱ ΑΖ, ΖΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ ΑΒ ἑητή ἐστιν, ἑητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς AB· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΖ, ΖΒ ῥητόν ἐστιν. καὶ ἐπεὶ πάλιν τὸ ὑπὸ τῶν ΑΕ, ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ὑπόκειται δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ καὶ τῷ ἀπὸ τῆς ΒΔ ἴσον, ἴση ἄρα ἐστὶν ἡ ΖΕ τῆ ΒΔ· διπλῆ ἄρα ἡ ΒΓ τὴς ΖΕ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τῷ ὑπὸ τῶν AB, ΕΖ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΕΖ. ἴσον δὲ τὸ ὑπὸ τῶν AB, EZ τῷ ὑπὸ τῶν AZ, ZB· μέσον ἄρα καὶ τὸ ύπὸ τῶν ΑΖ, ΖΒ. ἐδείχθη δὲ καὶ ἑητὸν τὸ συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΖ, ΖΒ ποιοῦσαι τὸ μὲν συγχείμενον ἐχ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ' αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι. ther of BD or DC, (and) falling short by a square figure, have been applied to AB [Prop. 6.28], and let it be the (rectangle contained) by AEB. And let the semi-circle AFB have been drawn on AB. And let EF have been drawn at right-angles to AB. And let AF and FB have been joined.



And since AB and BC are [two] unequal straightlines, and the square on AB is greater than (the square on) BC by the (square) on (some straight-line which is) incommensurable (in length) with (AB). And a parallelogram, equal to one quarter of the (square) on BC that is to say, (equal) to the (square) on half of it-(and) falling short by a square figure, has been applied to AB, and makes the (rectangle contained) by AEB. AE is thus incommensurable (in length) with EB [Prop. 10.18]. And as AE is to EB, so the (rectangle contained) by BAand AE (is) to the (rectangle contained) by AB and BE. And the (rectangle contained) by BA and AE (is) equal to the (square) on AF, and the (rectangle contained) by AB and BE to the (square) on BF [Prop. 10.32 lem.]. The (square) on AF is thus incommensurable with the (square) on FB [Prop. 10.11]. Thus, AF and FB are incommensurable in square. And since AB is rational, the (square) on AB is also rational. Hence, the sum of the (squares) on AF and FB is also rational [Prop. 1.47]. And, again, since the (rectangle contained) by AE and EB is equal to the (square) on EF, and the (rectangle contained) by AE and EB was assumed (to be) equal to the (square) on BD, FE is thus equal to BD. Thus, BCis double *FE*. And hence the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and EF [Prop. 10.6]. And the (rectangle contained) by AB and BC (is) medial [Prop. 10.21]. Thus, the (rectangle contained) by AB and EF (is) also medial [Prop. 10.23 corr.]. And the (rectangle contained) by AB and EF (is) equal to the (rectangle contained) by AF and FB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AF and FB (is) also medial. And the sum of the squares on them was also shown (to be) rational.

Thus, the two straight-lines, AF and FB, (which are) incommensurable in square, have been found, making the sum of the squares on them rational, and the (rectangle contained) by them medial. (Which is) the very thing it was required to show.

[†] AF and FB have lengths $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2}$ and $\sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$ times that of AB, respectively, where k is defined in the footnote to Prop. 10.30.

λδ'.

Εύρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν.



Έκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἰ AB, BΓ ῥητὸν περιέχουσαι τὸ ὑπ' αὐτῶν, ὥστε τὴν AB τῆς BΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ γεγράφθω ἐπὶ τῆς AB τὸ AΔB ἡμικύκλιον, καὶ τετμήσθω ἡ BΓ δίχα κατὰ τὸ Ε, καὶ παραβεβλήσθω παρὰ τὴν AB τῷ ἀπὸ τῆς BE ἴσον παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνω τὸ ὑπὸ τῶν AZB· ἀσύμμετρος ἄρα [ἐστὶν] ἡ AZ τῆ ZB μήκει. καὶ ἦχθω ἀπὸ τοῦ Z τῆ AB πρὸς ὀρθὰς ἡ ZΔ, καὶ ἐπεζεύχθωσαν αἱ AΔ, ΔB.

Έπεὶ ἀσύμμετρός ἐστιν ἡ AZ τῆ ZB, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν BA, AZ τῷ ὑπὸ τῶν AB, BZ. ἴσον δὲ τὸ μὲν ὑπὸ τῶν BA, AZ τῷ ἀπὸ τῆς AΔ, τὸ δὲ ὑπὸ τῶν AB, BZ τῷ ἀπὸ τῆς ΔB· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς AΔ τῷ ἀπὸ τῆς ΔB. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς AB, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AΔ, ΔB. καὶ ἐπεὶ διπλῆ ἐστιν ἡ BΓ τῆς ΔΖ, διπλάσιον ἄρα καὶ τὸ ὑπὸ τῶν AB, BΓ τοῦ ὑπὸ τῶν AB, ZΔ. ἑητὸν δὲ τὸ ὑπὸ τῶν AB, ZΔ ἴσον τῷ ὑπὸ τῶν AΔ, ΔB· ὥστε καὶ τὸ ὑπὸ τῶν AΔ, ΔB ἑŋτόν ἐστιν.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΔ, ΔΒ ποιοῦσαι τὸ [μὲν] συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν· ὅπερ ἔδει δεῖξαι.

Proposition 34

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.



Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.31]. And let the semi-circle ADB have been drawn on AB. And let BC have been cut in half at E. And let a (rectangular) parallelogram equal to the (square) on BE, (and) falling short by a square figure, have been applied to AB, (and let it be) the (rectangle contained by) AFB[Prop. 6.28]. Thus, AF [is] incommensurable in length with FB [Prop. 10.18]. And let FD have been drawn from F at right-angles to AB. And let AD and DB have been joined.

Since AF is incommensurable (in length) with FB, the (rectangle contained) by BA and AF is thus also incommensurable with the (rectangle contained) by ABand BF [Prop. 10.11]. And the (rectangle contained) by BA and AF (is) equal to the (square) on AD, and the (rectangle contained) by AB and BF to the (square) on DB [Prop. 10.32 lem.]. Thus, the (square) on AD is also incommensurable with the (square) on DB. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since BC is double DF [see previous proposition], the (rectangle contained) by AB and BC (is) thus also double the (rectangle contained) by AB and FD. And the (rectangle contained) by AB and BC (is) rational. Thus, the (rectangle contained) by AB and FD (is) also rational [Prop. 10.6, Def. 10.4]. And the (rectangle contained) by AB and FD (is) equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. And hence the (rectangle contained) by AD and DB is rational.

Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the squares on them medial, and the (rectangle

contained) by them rational.^{\dagger} (Which is) the very thing it was required to show.

[†] AD and DB have lengths $\sqrt{[(1+k^2)^{1/2}+k]/[2(1+k^2)]}$ and $\sqrt{[(1+k^2)^{1/2}-k]/[2(1+k^2)]}$ times that of AB, respectively, where k is defined in the footnote to Prop. 10.29.

λε΄.

Εύρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ' αὐτῶν τετραγώνῳ.



Έκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αί AB, BΓ μέσον περιέχουσαι, ώστε τὴν AB τῆς BΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AΔB, καὶ τὰ λοιπὰ γεγονέτω τοῖς ἐπάνω ὁμοίως.

Καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ μήκει, ἀσύμμετρός ἐστι καὶ ἡ ΑΔ τῆ ΔΒ δυνάμει. καὶ ἐπεὶ μέσον ἐστὶ τὸ άπὸ τῆς AB, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ ἴσον ἐστὶ τῷ ἀφ' έκατέρας τῶν BE, ΔΖ, ἴση ἄρα ἐστὶν ἡ BE τ
ỹ ΔΖ· διπλῆ ἄρα ή $m B\Gamma$ τῆς $m Z\Delta^{\cdot}$ ὥστε καὶ τὸ ὑπὸ τῶν m AB, $m B\Gamma$ διπλάσιόν έστι τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. καί ἐστιν ἴσον τῷ ὑπὸ τῶν $A\Delta$, ΔB^{\cdot} μέσον ἄρα καὶ τὸ ὑπὸ τῶν $A\Delta$, ΔB . καὶ ἐπεὶ άσύμμετρός ἐστιν ἡ ΑΒ τῆ ΒΓ μήκει, σύμμετρος δὲ ἡ ΓΒ τῆ ΒΕ, ἀσύμμετρος ἄρα καὶ ἡ ΑΒ τῆ ΒΕ μήκει· ὥστε καὶ τὸ άπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΕ ἀσύμμετρόν ἐστιν. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, τῷ δὲ ύπὸ τῶν ΑΒ, ΒΕ ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ, τουτέστι τὸ ὑπὸ τῶν $A\Delta$, ΔB · ἀσύμμετρον ἄρα ἐστὶ τὸ συγχείμενον έκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ τῷ ὑπὸ τῶν ΑΔ, ΔΒ.

Εύρηνται ἄρα δύο εὐθεῖαι αἱ ΑΔ, ΔΒ δυνάμει ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῷ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων. ὅπερ ἔδει δεῖξαι.

Proposition 35

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.



Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on ABis greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.32]. And let the semi-circle ADB have been drawn on AB. And let the remainder (of the figure) be generated similarly to the above (proposition).

And since AF is incommensurable in length with FB[Prop. 10.18], AD is also incommensurable in square with DB [Prop. 10.11]. And since the (square) on ABis medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since the (rectangle contained) by AF and FB is equal to the (square) on each of BE and DF, BE is thus equal to DF. Thus, BC (is) double FD. And hence the (rectangle contained) by AB and BC is double the (rectangle) contained by AB and FD. And the (rectangle contained) by AB and BC (is) medial. Thus, the (rectangle contained) by ABand FD (is) also medial. And it is equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AD and DB (is) also medial. And since AB is incommensurable in length with BC, and CB (is) commensurable (in length) with BE, AB (is) thus also incommensurable in length with BE[Prop. 10.13]. And hence the (square) on AB is also incommensurable with the (rectangle contained) by ABand BE [Prop. 10.11]. But the (sum of the squares) on AD and DB is equal to the (square) on AB [Prop. 1.47]. And the (rectangle contained) by AB and FD—that is to say, the (rectangle contained) by AD and DB—is equal to the (rectangle contained) by AB and BE. Thus, the

sum of the (squares) on AD and DB is incommensurable with the (rectangle contained) by AD and DB.

Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.[†] (Which is) the very thing it was required to show.

[†] AD and DB have lengths $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2}$ and $k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2}$ times that of AB, respectively, where k and k' are defined in the footnote to Prop. 10.32.

אק'.

Έὰν δύο ἑηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν, χαλείσθω δὲ ἐχ δύο ὀνομάτων.



Συγκείσθωσαν γὰρ δύο ἑηταὶ δυνάμει μόνον σύμμετροι αἱ AB, BΓ· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Έπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ AB τῆ BΓ μήχει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ AB πρὸς τὴν BΓ, οὕτως τὸ ὑπὸ τῶν ABΓ πρὸς τὸ ἀπὸ τῆς BΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν AB, BΓ τῷ ἀπὸ τῆς BΓ. ἀλλὰ τῷ μὲν ὑπὸ τῶν AB, BΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν AB, BΓ, τῷ δὲ ἀπὸ τῆς BΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ· αἱ γὰρ AB, BΓ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν AB, BΓ τοῖς ἀπὸ τῶν AB, BΓ. καὶ συνθέντι τὸ δὶς ὑπὸ τῶν AB, BΓ μετὰ τῶν ἀπὸ τῶν AB, BΓ, τουτέστι τὸ ἀπὸ τῆς AΓ, ἀσύμμετρόν ἐστι τῷ συγχειμένῳ ἐχ τῶν ἀπὸ τῶν AB, BΓ· ἑητὸν δὲ τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν AB, BΓ· ἑητὸν δὲ τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν AB, BΓ· ἑητὸν δὲ τὸ συγχείμενον ἔς τῶν ἀπὸ τῶν AB, BΓ· ἑητὸν δὲ τὸ συγχείμενον ἔς τῶν τῶν AB, BΓ· ἔλογον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς AΓ· ὥστε xaὶ ἡ AΓ ἄλογός ἐστιν, χαλείσθω δὲ ἐχ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

Proposition 36

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is irrational—let it be called a binomial (straight-line).[†]



For let the two rational (straight-lines), AB and BC. (which are) commensurable in square only, be laid down together. I say that the whole (straight-line), AC, is irrational. For since AB is incommensurable in length with BC—for they are commensurable in square only—and as AB (is) to BC, so the (rectangle contained) by ABC (is) to the (square) on BC, the (rectangle contained) by ABand BC is thus incommensurable with the (square) on BC [Prop. 10.11]. But, twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And (the sum of) the (squares) on AB and BC is commensurable with the (square) on BC—for the rational (straight-lines) ABand BC are commensurable in square only [Prop. 10.15]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with (the sum of) the (squares) on ABand BC [Prop. 10.13]. And, via composition, twice the (rectangle contained) by AB and BC, plus (the sum of) the (squares) on AB and BC—that is to say, the (square) on AC [Prop. 2.4]—is incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16]. And the sum of the (squares) on AB and BC (is) rational. Thus, the (square) on AC [is] irrational [Def. 10.4]. Hence, AC is also irrational [Def. 10.4]-let it be called a binomial (straight-line).[‡] (Which is) the very thing it was required to show.

[†] Literally, "from two names".

[‡] Thus, a binomial straight-line has a length expressible as $1 + k^{1/2}$ [or, more generally, $\rho(1 + k^{1/2})$, where ρ is rational—the same proviso applies to the definitions in the following propositions]. The binomial and the corresponding apotome, whose length is expressible as $1 - k^{1/2}$

(see Prop. 10.73), are the positive roots of the quartic $x^4 - 2(1+k)x^2 + (1-k)^2 = 0$.

λζ'.

Έὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ρητον περιέχουσαι, ή όλη άλογός έστιν, καλείσθω δὲ ἐκ δύο μέσων πρώτη.



Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αί ΑΒ, ΒΓ ἑητὸν περιέχουσαι· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῆ ΒΓ μήκει, καὶ τὰ άπὸ τῶν ΑΒ, ΒΓ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ συνθέντι τὰ ἀπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἑητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ὑπόκεινται γὰρ αί ΑΒ, ΒΓ ἑητὸν περιέχουσαι· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· άλογος ἄρα ή ΑΓ, καλείσθω δὲ ἐκ δύο μέσων πρώτη· ὅπερ έδει δεῖξαι.

Proposition 37

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational-let it be called a first bimedial (straight-line).[†]

	¢ 0	· .
A	В	C

For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a rational (area), be laid down together. I say that the whole (straight-line), AC, is irrational.

For since AB is incommensurable in length with BC, (the sum of) the (squares) on AB and BC is thus also incommensurable with twice the (rectangle contained) by AB and BC [see previous proposition]. And, via composition, (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC that is, the (square) on AC [Prop. 2.4]—is incommensurable with the (rectangle contained) by AB and BC[Prop. 10.16]. And the (rectangle contained) by AB and BC (is) rational—for AB and BC were assumed to enclose a rational (area). Thus, the (square) on AC (is) irrational. Thus, AC (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).[‡] (Which is) the very thing it was required to show.

[†] Literally, "first from two medials".

[‡] Thus, a first bimedial straight-line has a length expressible as $k^{1/4} + k^{3/4}$. The first bimedial and the corresponding first apotome of a medial, whose length is expressible as $k^{1/4} - k^{3/4}$ (see Prop. 10.74), are the positive roots of the quartic $x^4 - 2\sqrt{k}(1+k)x^2 + k(1-k)^2 = 0$.

λη'.

Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι If two medial (straight-lines), commensurable in square μέσον περιέχουσαι, ή ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ only, which contain a medial (area), are added together δύο μέσων δυετέρα.



Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αί AB, BΓ μέσον περιέχουσαι· λέγω, ὄτι ἄλογός ἐστιν ή commensurable in square only, (and) containing a medial

Proposition 38

then the whole (straight-line) is irrational-let it be called a second bimedial (straight-line).



For let the two medial (straight-lines), AB and BC,

АΓ.

Ἐκκείσθω γὰρ ἑητὴ ἡ $\Delta ext{E}$, καὶ τῷ ἀπὸ τῆς $ext{A} ext{\Gamma}$ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ. καὶ έπεὶ τὸ ἀπὸ τῆς ΑΓ ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΒ, ΒΓ καὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, παραβεβλήσθω δὴ τοῖς ἀπὸ τῶν ΑΒ, $\rm B\Gamma$ παρὰ τὴν $\rm \Delta E$ ἴσον τὸ $\rm E\Theta\cdot$ λοιπὸν ἄρα τὸ $\rm \Theta Z$ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπεὶ μέση ἐστὶν ἑκατέρα τῶν ΑΒ, ΒΓ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. μέσον δὲ ύπόχειται καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΖΘ· μέσον ἄρα ἑχάτερον τῶν ΕΘ, ΘΖ. χαὶ παρὰ ῥητὴν τὴν ΔE παράκειται· ἑητὴ ἄρα ἐστὶν ἑκατέρα τῶν $\Delta \Theta$, ΘH καὶ ἀσύμμετρος τῆ ΔΕ μήκει. ἐπεὶ οὖν ἀσύμμετρός ἐστιν ἡ ΑΒ τῆ ΒΓ μήχει, χαί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ άπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρόν έστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων, τῶ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀσύμμετρον ἄρα ἐστὶ τὸ συγχείμενον έκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΘΖ. ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΘ τῷ ΘZ^{\cdot} ὥστε καὶ ἡ $\Delta \Theta$ τῆ ΘH ἐστιν ἀσύμμετρος μήκει. αἱ ΔΘ, ΘΗ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. ὥστε ἡ ΔH ἄλογός ἐστιν.
 ἑητὴ δὲ ἡ ΔE · τὸ δὲ ὑπὸ ἀλόγου καὶ ρητῆς περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν. ἄλογον ἄρα έστι το ΔΖ χωρίον, και ή δυναμένη [αὐτο] ἄλογός ἐστιν. δύναται δὲ τὸ ΔΖ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ ἐχ δύο μέσων δευτέρα. ὅπερ ἔδει δεῖξαι.

(area), be laid down together [Prop. 10.28]. I say that AC is irrational.

For let the rational (straight-line) DE be laid down, and let (the rectangle) DF, equal to the (square) on AC, have been applied to DE, making DG as breadth [Prop. 1.44]. And since the (square) on AC is equal to (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC [Prop. 2.4], so let (the rectangle) EH, equal to (the sum of) the squares on AB and BC, have been applied to DE. The remainder *HF* is thus equal to twice the (rectangle contained) by AB and BC. And since AB and BC are each medial, (the sum of) the squares on AB and BC is thus also medial.[‡] And twice the (rectangle contained) by AB and BC was also assumed (to be) medial. And EH is equal to (the sum of) the squares on AB and BC, and FH (is) equal to twice the (rectangle contained) by AB and BC. Thus, EH and HF (are) each medial. And they were applied to the rational (straight-line) DE. Thus, DH and HG are each rational, and incommensurable in length with DE [Prop. 10.22]. Therefore, since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the sum of the squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, the sum of the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.13]. But, EH is equal to (the sum of) the squares on AB and BC, and HF is equal to twice the (rectangle) contained by AB and BC. Thus, EH is incommensurable with *HF*. Hence, *DH* is also incommensurable in length with HG [Props. 6.1, 10.11]. Thus, DH and HG are rational (straight-lines which are) commensurable in square only. Hence, DG is irrational [Prop. 10.36]. And DE (is) rational. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20]. The area DF is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4]. And AC is the square-root of DF. AC is thus irrational-let it be called a second bimedial (straight-line).§ (Which is) the very thing it was required to show.

[†] Literally, "second from two medials".

 $^{^\}ddagger$ Since, by hypothesis, the squares on AB and BC are commensurable—see Props. 10.15, 10.23.

[§] Thus, a second bimedial straight-line has a length expressible as $k^{1/4} + k'^{1/2}/k^{1/4}$. The second bimedial and the corresponding second apotome of a medial, whose length is expressible as $k^{1/4} - k'^{1/2}/k^{1/4}$ (see Prop. 10.75), are the positive roots of the quartic $x^4 - 2[(k + k')/\sqrt{k}]x^2 + k^{1/2}/k^{1/4}$.

 $[(k - k')^2/k] = 0.$

λθ'.

Έὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦςαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ μείζων.



Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ AB, BΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ AΓ.

Έπεὶ γὰρ τὸ ὑπὸ τῶν AB, BΓ μέσον ἐστίν, καὶ τὸ δἰς [ἄρα] ὑπὸ τῶν AB, BΓ μέσον ἐστίν. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB, BΓ ἑῆτόν· ἀσύμμετρον ἄρα ἐστὶ τὸ δἰς ὑπὸ τῶν AB, BΓ τῷ συγκειμένῷ ἐκ τῶν ἀπὸ τῶν AB, BΓ· ὅστε καὶ τὰ ἀπὸ τῶν AB, BΓ μετὰ τοῦ δἰς ὑπὸ τῶν AB, BΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς AΓ, ἀσύμμετρόν ἐστι τῷ συγκειμένῷ ἐκ τῶν ἀπὸ τῶν AB, BΓ [ἑῆτὸν δὲ τὸ συγμείμενον ἐκ τῶν ἀπὸ τῶν AB, BΓ]· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς AΓ. ὥστε καὶ ἡ AΓ ἅλογός ἐστιν, καλείσθω δὲ μείζων. ὅπερ ἕδει δεῖξαι.

Proposition 39

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together then the whole straight-line is irrational—let it be called a major (straight-line).

L		
Á	В	Ċ

For let the two straight-lines, AB and BC, incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33]. I say that AC is irrational.

For since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BCis [thus] also medial [Props. 10.6, 10.23 corr.]. And the sum of the (squares) on AB and BC (is) rational. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the sum of the (squares) on AB and BC [Def. 10.4]. Hence, (the sum of) the squares on ABand BC, plus twice the (rectangle contained) by AB and BC—that is, the (square) on AC [Prop. 2.4]—is also incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16] [and the sum of the (squares) on ABand BC (is) rational]. Thus, the (square) on AC is irrational. Hence, AC is also irrational [Def. 10.4]—let it be called a major (straight-line).[†] (Which is) the very thing it was required to show.

[†] Thus, a major straight-line has a length expressible as $\sqrt{[1+k/(1+k^2)^{1/2}]/2} + \sqrt{[1-k/(1+k^2)^{1/2}]/2}$. The major and the corresponding minor, whose length is expressible as $\sqrt{[1+k/(1+k^2)^{1/2}]/2} - \sqrt{[1-k/(1+k^2)^{1/2}]/2}$ (see Prop. 10.76), are the positive roots of the quartic $x^4 - 2x^2 + k^2/(1+k^2) = 0$.

μ΄.

Έλν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦςαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη.



Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ AB, BΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ AΓ.

Έπει γὰρ τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν AB, BΓ μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν AB, BΓ ῥητόν, ἀσύμμετρον ἀρα ἐστὶ τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν AB, BΓ τῷ δἱς

Proposition 40

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational, are added together then the whole straight-line is irrational—let it be called the square-root of a rational plus a medial (area).



For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34]. I say that AC is irrational.

For since the sum of the (squares) on AB and BC is medial, and twice the (rectangle contained) by AB and
ύπὸ τῶν AB, BΓ· ὥστε καὶ τὸ ἁπὸ τῆς AΓ ἀσύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν AB, BΓ. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν AB, BΓ· ἄλογον ἄρα τὸ ἀπὸ τῆς AΓ. ἄλογος ἄρα ἡ AΓ, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη. ὅπερ ἔδει δεῖξαι. *BC* (is) rational, the sum of the (squares) on *AB* and *BC* is thus incommensurable with twice the (rectangle contained) by *AB* and *BC*. Hence, the (square) on *AC* is also incommensurable with twice the (rectangle contained) by *AB* and *BC* [Prop. 10.16]. And twice the (rectangle contained) by *AB* and *BC* (is) rational. The (square) on *AC* (is) thus irrational. Thus, *AC* (is) irrational [Def. 10.4]—let it be called the square-root of a rational plus a medial (area).[†] (Which is) the very thing it was required to show.

[†] Thus, the square-root of a rational plus a medial (area) has a length expressible as $\sqrt{[(1+k^2)^{1/2}+k]/[2(1+k^2)]} + \sqrt{[(1+k^2)^{1/2}-k]/[2(1+k^2)]}$. This and the corresponding irrational with a minus sign, whose length is expressible as $\sqrt{[(1+k^2)^{1/2}+k]/[2(1+k^2)]} - \sqrt{[(1+k^2)^{1/2}-k]/[2(1+k^2)]}$ (see Prop. 10.77), are the positive roots of the quartic $x^4 - (2/\sqrt{1+k^2})x^2 + k^2/(1+k^2)^2 = 0$.

μα΄.

Έλν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦςαι τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ δύο μέσα δυναμένη.



Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ AB, BΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἡ ΑΓ ἄλογός ἐστιν.

Έχχείσθω ἑητὴ ἡ ΔΕ, καὶ παραβεβλήσθω παρὰ τὴν ΔΕ τοῖς μὲν ἀπὸ τῶν AB, BΓ ἴσον τὸ ΔΖ, τῷ δὲ δἰς ὑπὸ τῶν AB, BΓ ἴσον τὸ HΘ· ὅλον ἄρα τὸ ΔΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς AΓ τετραγώνω. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν AB, BΓ, χαί ἐστιν ἴσον τῷ ΔΖ, μέσον ἄρα ἐστὶ καὶ τὸ ΔΖ. καὶ παρὰ ἑητὴν τὴν ΔΕ παράχειται· ἑητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῆ ΔΕ μήχει. διὰ τὰ αὐτὰ δὴ καὶ ἡ HK ἑητή ἐστι καὶ ἀσύμμετρος τῆ HZ, τουτέστι τῆ ΔΕ, μήχει. καὶ ἐπεὶ ἀσύμμετρό ἐστι τὰ ἀπὸ τῶν AB, BΓ τῷ δἰς ὑπὸ τῶν AB, BΓ, ἀσύμμετρόν ἐστι τὸ ΔΖ τῷ HΘ·

Proposition 41

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational—let it be called the square-root of (the sum of) two medial (areas).



For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35]. I say that AC is irrational.

Let the rational (straight-line) DE be laid out, and let (the rectangle) DF, equal to (the sum of) the (squares) on AB and BC, and (the rectangle) GH, equal to twice the (rectangle contained) by AB and BC, have been applied to DE. Thus, the whole of DH is equal to the square on AC [Prop. 2.4]. And since the sum of the (squares) on AB and BC is medial, and is equal to DF, DF is thus also medial. And it is applied to the rational (straight-line) DE. Thus, DG is rational, and incommenώστε καὶ ἡ Δ H τῆ HK ἀσύμμετρός ἐστιν. καὶ εἰσι ἑηταί· αἰ Δ H, HK ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἄλογος ἄρα ἐστὶν ἡ Δ K ἡ καλουμένη ἐκ δύο ὀνομάτων. ἑητὴ δὲ ἡ Δ E· ἄλογον ἄρα ἐστὶ τὸ Δ Θ καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. δύναται δὲ τὸ Θ Δ ἡ AΓ· ἄλογος ἄρα ἐστὶν ἡ AΓ, καλείσθω δὲ δύο μέσα δυναμένη. ὅπερ ἔδει δεῖξαι.

surable in length with DE [Prop. 10.22]. So, for the same (reasons), GK is also rational, and incommensurable in length with GF—that is to say, DE. And since (the sum of) the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DF is incommensurable with GH. Hence, DG is also incommensurable (in length) with GK [Props. 6.1, 10.11]. And they are rational. Thus, DG and GK are rational (straight-lines which are) commensurable in square only. Thus, DK is irrational, and that (straight-line which is) called binomial [Prop. 10.36]. And DE (is) rational. Thus, DH is irrational, and its square-root is irrational [Def. 10.4]. And AC (is) the square-root of HD. Thus, AC is irrational—let it be called the square-root of (the sum of) two medial (areas).^{\dagger} (Which is) the very thing it was required to show.

[†] Thus, the square-root of (the sum of) two medial (areas) has a length expressible as $k'^{1/4} \left(\sqrt{[1+k/(1+k^2)^{1/2}]/2} + \sqrt{[1-k/(1+k^2)^{1/2}]/2} \right)$. This and the corresponding irrational with a minus sign, whose length is expressible as $k'^{1/4} \left(\sqrt{[1+k/(1+k^2)^{1/2}]/2} - \sqrt{[1-k/(1+k^2)^{1/2}]/2} \right)$ (see Prop. 10.78), are the positive roots of the quartic $x^4 - 2k'^{1/2}x^2 + k'k^2/(1+k^2) = 0$.

Λῆμμα.

Ότι δὲ αἱ εἰρημέναι ἄλογοι μοναχῶς διαιροῦνται εἰς τὰς εὐθείας, ἐξ ῶν σύγκεινται ποιουσῶν τὰ προκείμενα εἴδη, δείζομεν ἤδη προεκθέμενοι λημμάτιον τοιοῦτον·



Έκκείσθω εὐθεῖα ἡ AB καὶ τετμήσθω ἡ ὅλη εἰς ἄνισα καθ' ἑκάτερον τῶν Γ, Δ, ὑποκείσθω δὲ μείζων ἡ ΑΓ τῆς ΔΒ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τῶν ἀπὸ τῶν ΑΔ, ΔΒ.

Τετμήσθω γὰρ ἡ AB δίχα κατὰ τὸ E. καὶ ἐπεὶ μείζων ἐστὶν ἡ AΓ τῆς ΔB, κοινὴ ἀφηρήσθω ἡ ΔΓ· λοιπὴ ἄρα ἡ AΔ λοιπῆς τῆς ΓB μείζων ἐστίν. ἴση δὲ ἡ AE τῆ EB· ἐλάττων ἄρα ἡ ΔΕ τῆς EΓ· τὰ Γ, Δ ἄρα σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. καὶ ἐπεὶ τὸ ὑπὸ τῶν AΓ, ΓB μετὰ τοῦ ἀπὸ τῆς EΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς EB, ἀλλὰ μὴν καὶ τὸ ὑπὸ τῶν AΔ, ΔB μετὰ τοῦ ἀπὸ ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς EB, τὸ ἀρα ὑπὸ τῶν AΓ, ΓB μετὰ τοῦ ἀπὸ τῆς EΓ ἴσον ἐστὶ τῷ ὑπὸ τῶν AΔ, ΔB μετὰ τοῦ ἀπὸ τῆς ΔΕ· ῶν τὸ ἀπὸ τῆς ΔΕ ἕλασσόν ἐστι τοῦ ἀπὸ τῆς EΓ· καὶ λοιπὸν ἄρα τὸ ὑπὸ τῶν AΓ, ΓB ἕλασσόν ἐστι τοῦ ὑπὸ τῶν AΔ, ΔB. ὥστε καὶ τὸ δὶς ὑπὸ τῶν AΓ, ΓΒ ἕλασσόν ἐστι τοῦ δὶς ὑπὸ τῶν AΔ, ΔB. καὶ λοιπὸν ἄρα τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AΔ, ΔΒ. άοιπὸν ἄρα τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AΔ, ὅπερ ἔδει δεῖξαι.

Lemma

We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.



Let the straight-line AB be laid out, and let the whole (straight-line) have been cut into unequal parts at each of the (points) C and D. And let AC be assumed (to be) greater than DB. I say that (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB.

For let AB have been cut in half at E. And since AC is greater than DB, let DC have been subtracted from both. Thus, the remainder AD is greater than the remainder CB. And AE (is) equal to EB. Thus, DE (is) less than EC. Thus, points C and D are not equally far from the point of bisection. And since the (rectangle contained) by AC and CB, plus the (square) on EC, is equal to the (square) on EB [Prop. 2.5], but, moreover, the (rectangle contained) by AD and DB, plus the (square) on DE, is also equal to the (square) on EB [Prop. 2.5], the (rectangle contained) by AC and CB, plus the (square) on EC, is thus equal to the (rectangle contained) by AD and DB, plus the (square) on DE. And, of these, the (square) on DE is less than the (square) on EC. And, thus, the

remaining (rectangle contained) by AC and CB is less than the (rectangle contained) by AD and DB. And, hence, twice the (rectangle contained) by AC and CBis less than twice the (rectangle contained) by AD and DB. And thus the remaining sum of the (squares) on ACand CB is greater than the sum of the (squares) on ADand DB.[†] (Which is) the very thing it was required to show.

[†] Since, $AC^2 + CB^2 + 2ACCB = AD^2 + DB^2 + 2ADDB = AB^2$.

μβ΄.

Ή ἐκ δύο ὀνομάτων κατὰ ἕν μόνον σημεῖον διαιρεῖται εἰς τὰ ὀνόματα.



Έστω ἐκ δύο ἀνομάτων ἡ AB διηρημένη εἰς τὰ ἀνόματα κατὰ τὸ Γ· αἱ AΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται εἰς δύο ῥητὰς δυνάμει μόνον συμμέτρους.

Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ , ὥστε καὶ τὰς ΑΔ, ΔΒ ἑητὰς εἶναι δυνάμει μόνον συμμέτρους. φανερὸν δή, ὅτι ἡ ΑΓ τῆ ΔΒ οὐκ ἔστιν ἡ αὐτή. εἰ γὰρ δυνατόν, ἔστω. ἔσται δὴ καὶ ἡ A
Δ τῆ ΓB ἡ αὐτή· καὶ ἔσται ὡς ἡ A
Γ πρὸς τὴν ΓΒ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ, καὶ ἔσται ἡ ΑΒ κατὰ τὸ αὐτὸ τῇ κατὰ τὸ Γ διαιρέσει διαιρεθεῖσα καὶ κατὰ τὸ Δ· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ἡ ΑΓ τῆ ΔΒ ἐστιν ἡ αὐτή. διὰ δὴ τοῦτο καὶ τὰ Γ, Δ σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. ῷ ἄρα διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν άπὸ τῶν ΑΔ, ΔΒ, τούτω διαφέρει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ διὰ τὸ καὶ τὰ ἀπὸ τῶν ΑΓ, ΓB μετὰ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓB καὶ τὰ ἀπὸ τῶν ΑΔ, ΔB μετὰ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἴσα εἶναι τῷ ἀπὸ τῆς ΑΒ. άλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ διαφέρει ρητῷ·
 ρητὰ γὰρ ἀμφότερα· καὶ τὸ δὶς ẳρα ὑπὸ τῶν
 $A\Delta,\,\Delta B$ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ διαφέρει ῥητῷ μέσα ὄντα ὅπερ άτοπον· μέσον γὰρ μέσου οὐχ ὑπερέχει ῥητῷ.

Οὐχ ἄρα ἡ ἐκ δύο ὀνομάτων κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καϑ' ἕν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

A binomial (straight-line) can be divided into its (component) terms at one point only,^{\dagger}

Proposition 42

А	D	С	В
		1	

Let AB be a binomial (straight-line) which has been divided into its (component) terms at C. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. I say that AB cannot be divided at another point into two rational (straight-lines which are) commensurable in square only.

For, if possible, let it also have been divided at D, such that AD and DB are also rational (straight-lines which are) commensurable in square only. So, (it is) clear that AC is not the same as DB. For, if possible, let it be (the same). So, AD will also be the same as CB. And as AC will be to CB, so BD (will be) to DA. And AB will (thus) also be divided at D in the same (manner) as the division at C. The very opposite was assumed. Thus, ACis not the same as DB. So, on account of this, points C and D are not equally far from the point of bisection. Thus, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount)—on account of both (the sum of) the (squares) on AC and CB, plus twice the (rectangle contained) by AC and CB, and (the sum of) the (squares) on AD and DB, plus twice the (rectangle contained) by AD and DB, being equal to the (square) on AB [Prop. 2.4]. But, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on ADand DB by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by a rational (area, despite both) being medial (areas) [Prop. 10.21]. The very thing is absurd. For a medial (area) cannot exceed a medial (area) by a rational (area) [Prop. 10.26].

Thus, a binomial (straight-line) cannot be divided (into its component terms) at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.

[†] In other words, $k + k'^{1/2} = k'' + k'''^{1/2}$ has only one solution: *i.e.*, k'' = k and k''' = k'. Likewise, $k^{1/2} + k'^{1/2} = k''^{1/2} + k'''^{1/2}$ has only one solution: *i.e.*, k'' = k and k''' = k' (or, equivalently, k'' = k' and k''' = k).

Ή ἐκ δύο μέσων πρώτη καθ' ἕν μόνον σημεῖον διαιρεῖται.



Έστω ἐκ δύο μέσων πρώτη ἡ AB διηρημένη κατὰ τὸ Γ, ὥστε τὰς AΓ, ΓΒ μέσας εἶναι δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας· λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν διηρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ μέσας εἶναι δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας. ἐπεὶ οῦν, ῷ διαφέρει τὸ δἰς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ, τούτῷ διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, ἑητῷ δὲ διαφέρει τὸ δἰς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ· ἑητὰ γὰρ ἀμφότερα· ἑητῷ ἄρα διαφέρει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ μέσα ὄντα· ὅπερ ἄτοπον.

Οὐκ ἄρα ἡ ἐκ δύο μέσων πρώτη κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται εἰς τὰ ὀνόματα· καθ' ἒν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

Proposition 43

A first bimedial (straight-line) can be divided (into its component terms) at one point only.^{\dagger}



Let AB be a first bimedial (straight-line) which has been divided at C, such that AC and CB are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37]. I say that ABcannot be (so) divided at another point.

For, if possible, let it also have been divided at D, such that AD and DB are also medial (straight-lines), commensurable in square only, (and) containing a rational (area). Since, therefore, by whatever (amount) twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by this (same amount) [Prop. 10.41 lem.]. And twice the (rectangle contained) by AC and CB iffers from (they are) both rational (areas). (The sum of) the (squares) on AC and CB by a rational (area). For (they are) both rational (areas). (The sum of) the (squares) on AC and DB by a rational (areas) the (squares) on AD and DB by a rational (areas). The very thing is absurd [Prop. 10.26].

Thus, a first bimedial (straight-line) cannot be divided into its (component) terms at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.

[†] In other words, $k^{1/4} + k^{3/4} = k'^{1/4} + k'^{3/4}$ has only one solution: *i.e.*, k' = k.

μδ'.

Ή ἐκ δύο μέσων δευτέρα καθ' ἕν μόνον σημεῖον διαιρεῖται.

Έστω ἐχ δύο μέσων δευτέρα ἡ AB διηρημένη χατὰ τὸ Γ, ὥστε τὰς AΓ, ΓΒ μέσας εἶναι δυνάμει μόνον συμμέτρους μέσον περιεχούσας· φανερὸν δή, ὅτι τὸ Γ οὐχ ἔστι χατὰ τῆς διχοτομίας, ὅτι οὐχ εἰσὶ μήχει σύμμετροι. λέγω, ὅτι ἡ AB χατ' ἄλλο σημεῖον οὐ διαιρεῖται.

Proposition 44

A second bimedial (straight-line) can be divided (into its component terms) at one point only.^{\dagger}

Let AB be a second bimedial (straight-line) which has been divided at C, so that AC and BC are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38]. So, (it is) clear that C is not (located) at the point of bisection, since (ACand BC) are not commensurable in length. I say that ABcannot be (so) divided at another point.



 ${
m E}$ ί γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ , ὥστε τὴν ΑΓ τῆ ΔΒ μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ' ὑπόθεσιν τὴν ΑΓ· δῆλον δή, ὅτι καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, ὡς ἐπάνω έδείξαμεν, έλάσσονα τῶν ἀπὸ τῶν ΑΓ, ΓΒ \cdot καὶ τὰς Α $\Delta, \Delta B$ μέσας είναι δυνάμει μόνον συμμέτρους μέσον περιεχούσας. καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσον παρὰ την ΕΖ παραλληλόγραμμον όρθογώνιον παραβεβλήσθω το ΕΚ, τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΒ ἴσον ἀφηρήσθω τὸ ΕΗ· λοιπὸν άρα τὸ ΘΚ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. πάλιν δὴ τοῖς άπὸ τῶν ΑΔ, ΔΒ, ἄπερ ἐλάσσονα ἐδείχθη τῶν ἀπὸ τῶν ΑΓ, ΓΒ, ἴσον ἀφηρήσθω τὸ ΕΛ· καὶ λοιπὸν ἄρα τὸ ΜΚ ίσον τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ μέσα ἐστὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα [καὶ] τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράχειται· ἑητὴ ἄρα ἐστὶν ἡ ΕΘ χαὶ ἀσύμμετρος τῆ ΕΖ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΝ ἑητή ἐστι καὶ ἀσύμμετρος τῆ ΕΖ μήχει. χαὶ ἐπεὶ αἱ ΑΓ, ΓΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΓ τῆ ΓΒ μήχει. ὡς δὲ ή ΑΓ πρός την ΓΒ, οὕτως τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ· δυνάμει γάρ εἰσι σύμμετροι αἱ ΑΓ, ΓΒ. τῷ δὲ ὑπὸ τῶν ΑΓ, ΓΒ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον ἐστὶ τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΘΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΗ τῷ ΘΚ· ὥστε καὶ ἡ ΕΘ τῆ ΘΝ ἀσύμμετρός ἐστι μήκει. καί εἰσι ῥηταί· αἱ ΕΘ, ΘΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ δύο ἑηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων ή ΕΝ ἄρα ἐχ δύο ὀνομάτων ἐστὶ διηρημένη χατὰ τὸ Θ. χατὰ τὰ αὐτὰ δὴ δειχθήσονται καὶ αἱ ΕΜ, ΜΝ ῥηταὶ δυνάμει μόνον σύμμετροι· καὶ ἔσται ἡ ΕΝ ἐκ δύο ὀνομάτων κατ ἄλλο καὶ ἄλλο διῃρημένη τό τε Θ καὶ τὸ $\mathrm{M},$ καὶ οὐκ ἔστιν ή ΕΘ τῆ ΜΝ ή αὐτή, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τῶν ἀπὸ τῶν ΑΔ, ΔΒ. ἀλλὰ τὰ ἀπὸ τῶν ΑΔ, ΔΒ μείζονά έστι τοῦ δὶς ὑπὸ ΑΔ, ΔΒ· πολλῷ ἄρα καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΕΗ, μεῖζόν ἐστι τοῦ δἰς ὑπὸ τῶν Α Δ , Δ B, τουτέστι τοῦ ΜΚ· ὥστε καὶ ἡ ΕΘ τῆς ΜΝ μείζων ἐστίν. ἡ άρα ΕΘ τῆ MN οὐκ ἔστιν ἡ αὐτή· ὅπερ ἔδει δεῖξαι.



For, if possible, let it also have been (so) divided at D, so that AC is not the same as DB, but AC (is), by hypothesis, greater. So, (it is) clear that (the sum of) the (squares) on AD and DB is also less than (the sum of) the (squares) on AC and CB, as we showed above [Prop. 10.41 lem.]. And AD and DB are medial (straight-lines), commensurable in square only, (and) containing a medial (area). And let the rational (straightline) EF be laid down. And let the rectangular parallelogram EK, equal to the (square) on AB, have been applied to EF. And let EG, equal to (the sum of) the (squares) on AC and CB, have been cut off (from EK). Thus, the remainder, HK, is equal to twice the (rectangle contained) by AC and CB [Prop. 2.4]. So, again, let EL, equal to (the sum of) the (squares) on AD and DB—which was shown (to be) less than (the sum of) the (squares) on AC and CB—have been cut off (from EK). And, thus, the remainder, MK, (is) equal to twice the (rectangle contained) by AD and DB. And since (the sum of) the (squares) on AC and CB is medial, EG(is) thus [also] medial. And it is applied to the rational (straight-line) EF. Thus, EH is rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF. And since AC and CB are medial (straight-lines which are) commensurable in square only, AC is thus incommensurable in length with CB. And as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by AC and CB [Prop. 10.21 lem.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, (the sum of) the (squares) on AC and CB is commensurable with the (square) on AC. For, AC and CB are commensurable in square [Prop. 10.15]. And twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. And thus (the sum of) the squares on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. But, EG is equal to (the sum of) the (squares) on AC and CB, and HK equal to twice the (rectangle contained) by AC and CB. Thus, EG is incommensurable with HK. Hence, EH is also incom-

mensurable in length with HN [Props. 6.1, 10.11]. And (they are) rational (straight-lines). Thus, EH and HNare rational (straight-lines which are) commensurable in square only. And if two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is that irrational called binomial [Prop. 10.36]. Thus, EN is a binomial (straightline) which has been divided (into its component terms) at H. So, according to the same (reasoning), EM and MN can be shown (to be) rational (straight-lines which are) commensurable in square only. And EN will (thus) be a binomial (straight-line) which has been divided (into its component terms) at the different (points) H and M(which is absurd [Prop. 10.42]). And EH is not the same as MN, since (the sum of) the (squares) on AC and CBis greater than (the sum of) the (squares) on AD and DB. But, (the sum of) the (squares) on AD and DB is greater than twice the (rectangle contained) by AD and DB [Prop. 10.59 lem.]. Thus, (the sum of) the (squares) on AC and CB—that is to say, EG—is also much greater than twice the (rectangle contained) by AD and DB that is to say, MK. Hence, EH is also greater than MN[Prop. 6.1]. Thus, EH is not the same as MN. (Which is) the very thing it was required to show.

[†] In other words, $k^{1/4} + k'^{1/2}/k^{1/4} = k''^{1/4} + k'''^{1/2}/k''^{1/4}$ has only one solution: *i.e.*, k'' = k and k''' = k'.

με΄.

Η μείζων κατά τὸ αὐτὸ μόνον σημεῖον διαιρεῖται.



Έστω μείζων ή AB διηρημένη κατὰ τὸ Γ, ὥστε τὰς AΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AΓ, ΓΒ τετραγώνων ῥητόν, τὸ δ' ὑπὸ τῶν AΓ, ΓΒ μέσον· λέγω, ὅτι ή AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ ἑητόν, τὸ δ' ὑπ' αὐτῶν μέσον. καὶ ἐπεί, ῷ διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, τούτῷ διαφέρει καὶ τὸ δἰς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ, ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ ὑπερέχει ἑητῷ· ἑητὰ γὰρ ἀμφότερα· καὶ τὸ δἰς ὑπὸ τῶν ΑΔ, ΔΒ ἄρα τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ἑητῷ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ μείζων κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· κατὰ τὸ αὐτὸ ἄρα μόνον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

Proposition 45

A major (straight-line) can only be divided (into its component terms) at the same point.^{\dagger}



Let AB be a major (straight-line) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the squares on AC and CB rational, and the (rectangle contained) by AC and CD medial [Prop. 10.39]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D, such that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB rational, and the (rectangle contained) by them medial. And since, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount). But, (the sum of) the (squares) on AC and CB and DB by this (same amount). But, (the sum of) the (squares) on AD and DB by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle

contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, a major (straight-line) cannot be divided (into its component terms) at different points. Thus, it can only be (so) divided at the same (point). (Which is) the very thing it was required to show.

[†] In other words, $\sqrt{[1+k/(1+k^2)^{1/2}]/2} + \sqrt{[1-k/(1+k^2)^{1/2}]/2} = \sqrt{[1+k'/(1+k'^2)^{1/2}]/2} + \sqrt{[1-k'/(1+k'^2)^{1/2}]/2}$ has only one solution: *i.e.*, k' = k.

Ή ἑητὸν καὶ μέσον δυναμένη καθ' ἕν μόνον σημεῖον διαιρεῖται.



Έστω ἡητὸν καὶ μέσον δυναμένη ἡ AB διηρημένη κατὰ τὸ Γ, ὥστε τὰς AΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AΓ, ΓΒ μέσον, τὸ δὲ δὶς ὑπὸ τῶν AΓ, ΓΒ ῥητόν· λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ μέσον, τὸ δὲ δἰς ὑπὸ τῶν ΑΔ, ΔΒ ῥητόν. ἐπεὶ οῦν, ῷ διαφέρει τὸ δἰς ὑπὸ τῶν ΑΓ, ΓΒ τοῦ δἰς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τὸ δὲ δἰς ὑπὸ τῶν ΑΓ, ΓΒ τοῦ δἰς ὑπὸ τῶν ΑΔ, ΔΒ ὑπερέχει ῥητῷ, καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ἑητῷ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἑητὸν καὶ μέσον δυναμένη κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται. κατὰ ἐν ἄρα σημεῖον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

Proposition 46

The square-root of a rational plus a medial (area) can be divided (into its component terms) at one point only.^{\dagger}

Let AB be the square-root of a rational plus a medial (area) which has been divided at C, so that AC and CBare incommensurable in square, making the sum of the (squares) on AC and CB medial, and twice the (rectangle contained) by AC and CB rational [Prop. 10.40]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D, so that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB medial, and twice the (rectangle contained) by AD and DB rational. Therefore, since by whatever (amount) twice the (rectangle contained) by AC and CB differs from twice the (rectangle contained) by AD and DB, (the sum of) the (squares) on AD and DB also differs from (the sum of) the (squares) on AC and CB by this (same amount). And twice the (rectangle contained) by AC and CB exceeds twice the (rectangle contained) by AD and DB by a rational (area). (The sum of) the (squares) on AD and DB thus also exceeds (the sum of) the (squares) on ACand CB by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, the square-root of a rational plus a medial (area) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one point (only). (Which is) the very thing it was required to show.

[†] In other words,
$$\sqrt{[(1+k^2)^{1/2}+k]/[2(1+k^2)]} + \sqrt{[(1+k^2)^{1/2}-k]/[2(1+k^2)]} = \sqrt{[(1+k'^2)^{1/2}+k']/[2(1+k'^2)]} + \sqrt{[(1+k'^2)^{1/2}-k']/[2(1+k'^2)]}$$
 has only one solution: *i.e.*, $k' = k$.

μζ΄.

Proposition 47

Η δύο μέσα δυναμένη καθ' ἕν μόνον σημεῖον διαιρεῖται.

The square-root of (the sum of) two medial (areas) can be divided (into its component terms) at one point only.^{\dagger}



Έστω [δύο μέσα δυναμένη] ή AB διηρημένη κατὰ τὸ Γ, ὥστε τὰς AΓ, ΓB δυνάμει ἀσυμμέτρους εἶναι ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπὸ τῶν AΓ, ΓB μέσον καὶ τὸ ὑπὸ τῶν AΓ, ΓB μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν. λέγω, ὅτι ἡ AB κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται ποιοῦσα τὰ προκείμενα.

Εἰ γὰρ δυνατόν, διηρήσθω κατὰ τὸ Δ, ὥστε πάλιν δηλονότι τὴν ΑΓ τῆ ΔΒ μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ' ὑπόθεσιν τὴν ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ παραβεβλήσθω παρά την ΕΖ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΘΚ· ὅλον ἄρα τὸ ΕΚ ίσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνω. πάλιν δὴ παραβεβλήσθω παρά την EZ τοῖς ἀπὸ τῶν AΔ, ΔB ἴσον τὸ EΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ λοιπῷ τῷ ΜΚ ἴσον έστίν. καὶ ἐπεὶ μέσον ὑπόκειται τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ẳρα ἐστὶ καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ἑητὴ ἄρα ἐστὶν ἡ ΘΕ καὶ ἀσύμμετρος τῆ ΕΖ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΝ ἑητή ἐστι καὶ ἀσύμμετρος τῆ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον έκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, καὶ τὸ ΕΗ ἄρα τῷ ΗΝ ἀσύμμετρόν ἐστιν· ὥστε καὶ ἡ ΕΘ τῇ ΘΝ ἀσύμμετρός ἐστιν. καί εἰσι ῥηταί· αἱ ΕΘ, ΘΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ή ΕΝ ἄρα ἐκ δύο ὀνομάτων ἐστὶ διηρημένη κατά τὸ Θ. ὁμοίως δὴ δείξομεν, ὅτι καὶ κατὰ τὸ Μ διήρηται. καὶ οὐκ ἔστιν ἡ ΕΘ τῆ ΜΝ ἡ αὐτή· ἡ ἄρα ἐκ δύο όνομάτων κατ' άλλο και άλλο σημεῖον διήρηται. ὅπερ ἐστίν άτοπον. ούκ ἄρα ἡ δύο μέσα δυναμένη κατ' ἀλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ' ἕν ἄρα μόνον [σημεῖον] διαιρεῖται.



Let AB be [the square-root of (the sum of) two medial (areas)] which has been divided at C, such that ACand CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and the (rectangle contained) by AC and CB medial, and, moreover, incommensurable with the sum of the (squares) on (ACand CB) [Prop. 10.41]. I say that AB cannot be divided at another point fulfilling the prescribed (conditions).

For, if possible, let it have been divided at D, such that AC is again manifestly not the same as DB, but AC (is), by hypothesis, greater. And let the rational (straight-line) EF be laid down. And let EG, equal to (the sum of) the (squares) on AC and CB, and HK, equal to twice the (rectangle contained) by AC and CB, have been applied to EF. Thus, the whole of EK is equal to the square on AB [Prop. 2.4]. So, again, let EL, equal to (the sum of) the (squares) on AD and DB, have been applied to EF. Thus, the remainder—twice the (rectangle contained) by AD and DB—is equal to the remainder, MK. And since the sum of the (squares) on AC and CB was assumed (to be) medial, EG is also medial. And it is applied to the rational (straight-line) EF. HE is thus rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is thus also incommensurable with GN. Hence, EH is also incommensurable with HN [Props. 6.1, 10.11]. And they are (both) rational (straight-lines). Thus, EH and HN are rational (straight-lines which are) commensurable in square only. Thus, EN is a binomial (straightline) which has been divided (into its component terms) at H [Prop. 10.36]. So, similarly, we can show that it has also been (so) divided at M. And EH is not the same as MN. Thus, a binomial (straight-line) has been divided (into its component terms) at different points. The very thing is absurd [Prop. 10.42]. Thus, the square-root of (the sum of) two medial (areas) cannot be divided (into

its component terms) at different points. Thus, it can be (so) divided at one [point] only.

[†] In other words, $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2} + k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2} = k'''^{1/4}\sqrt{[1+k''/(1+k''^2)^{1/2}]/2} + k'''^{1/4}\sqrt{[1-k''/(1+k''^2)^{1/2}]/2}$ has only one solution: *i.e.*, k'' = k and k''' = k'.

Οροι δεύτεροι.

ε'. Υποχειμένης ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων διηρημένης εἰς τὰ ὀνόματα, ῆς τὸ μεῖζον ὄνομα τοῦ ἐλάσσονος μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει, ἐὰν μὲν τὸ μεῖζον ὄνομα σύμμετρον ῆ μήκει τῆ ἐκκειμένη ἑητῆ, καλείσθω [ἡ ὅλη] ἐκ δύο ὀνομάτων πρώτη.

τ΄. Ἐὰν δὲ τὸ ἐλάσσον ὄνομα σύμμετρον ἢ μήκει τῆ ἐκκειμένῃ ῥητῆ, καλείσθω ἐκ δύο ὀνομάτων δευτέρα.

ζ΄. Ἐὰν δὲ μηδέτερον τῶν ὀνομάτων σύμμετρον ἢ μήκει τῆ ἐκκειμένῃ ἑῃτῆ, καλείσθω ἐκ δύο ὀνομάτων τρίτῃ.

η'. Πάλιν δὴ ἐἀν τὸ μεῖζον ὄνομα [τοῦ ἐλάσσονος] μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήχει, ἐἀν μὲν τὸ μεῖζον ὄνομα σύμμετρον ἦ μήχει τῆ ἐχχειμένῃ ῥητῆ, χαλείσθω ἐχ δύο ὀνομάτων τετάρτη.

θ'. Ἐὰν δὲ τὸ ἔλασσον, πέμπτη.

ι΄. Ἐὰν δὲ μηδέτερον, ἕκτη.

μη'.

Εύρεῖν τὴν ἐκ δύο ὀνομάτων πρώτην.

Έχχείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγχείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΓΑ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐχχείσθω τις ῥητὴ ἡ Δ, καὶ τῆ Δ σύμμετρος ἔστω μήχει ἡ ΕΖ. ἑητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΑΒ πρὸς τὸν ΑΓ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΕΖ ἅρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ὥστε σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς

Definitions II

5. Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line).

6. And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line).

7. And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line).

8. So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line).

9. And if the lesser (term is commensurable), a fifth (binomial straight-line).

10. And if neither (term is commensurable), a sixth (binomial straight-line).

Proposition 48

To find a first binomial (straight-line).

Let two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to CA the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let some rational (straight-line) D be laid down. And let EFbe commensurable in length with D. EF is thus also rational [Def. 10.3]. And let it have been contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And AB has to AC the ratio which (some) number (has) to (some) numZH. καὶ ἐστι ἑητὴ ἡ EZ· ἑητὴ ἄρα καὶ ἡ ZH. καὶ ἐπεὶ ὁ BA πρὸς τὸν AΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς EZ ἄρα πρὸς τὸ ἀπὸ τῆς ZH λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ EZ τῆ ZH μήκει. αἱ EZ, ZH ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἐχ δύο ἄρα ὀνομάτων ἐστὶν ἡ EH. λέγω, ὅτι καὶ πρώτη.



Έπει γάρ ἑστιν ὡς ὁ BA ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, μείζων δὲ ὁ BA τοῦ ΑΓ, μείζων ἄρα καὶ τὸ ἀπὸ τῆς ΕΖ τοῦ ἀπὸ τῆς ΖΗ. ἔστω οὖν τῷ ἀπὸ τῆς ΕΖ ἴσα τὰ ἀπὸ τῶν ΖΗ, Θ. καὶ ἐπεί ἐστιν ὡς ὁ BA πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, ἀναστρέψαντι ἄρα ἑστὶν ὡς ὁ AB πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸν ΔΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸν ΔΓ, οῦτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸν ἀπὸ τῆς ΕΖ πρὸς τὸν ἀπὸ τῆς ΕΖ κρὸς τὸν ἀπὸ τῆς ΕΖ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ AB πρὸς τὸν BΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ τὸ ἀπὸ τῆς ΕΖ ἅρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῆ Θ μήκει· ἡ ΕΖ ἄρα τῆς ΖΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καί εἰσι ἑρταὶ αἱ ΕΖ, ΖΗ, καὶ σύμμετρος ἡ ΕΖ τῆ Δ μήκει.

Ή ΕΗ ἄρα ἐχ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

ber. Thus, the (square) on EF also has to the (square) on FG the ratio which (some) number (has) to (some) number. Hence, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. And EF is rational. Thus, FG (is) also rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, thus the (square) on EFdoes not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG[Prop 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straight-line) [Prop. 10.36]. I say that (it is) also a first (binomial straight-line).



For since as the number BA is to AC, so the (square) on EF (is) to the (square) on FG, and BA (is) greater than AC, the (square) on EF (is) thus also greater than the (square) on FG [Prop. 5.14]. Therefore, let (the sum of) the (squares) on FG and H be equal to the (square) on EF. And since as BA is to AC, so the (square) on EF (is) to the (square) on FG, thus, via conversion, as AB is to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And AB has to BCthe ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, EF is commensurable in length with H [Prop. 10.9]. Thus, the square on EF is greater than (the square on) FG by the (square) on (some straight-line) commensurable (in length) with (EF). And EF and FG are rational (straight-lines). And EF (is) commensurable in length with D.

Thus, *EG* is a first binomial (straight-line) [Def. 10.5].[†] (Which is) the very thing it was required to show.

[†]If the rational straight-line has unit length then the length of a first binomial straight-line is $k + k\sqrt{1 - k'^2}$. This, and the first apotome, whose length is $k - k\sqrt{1 - k'^2}$ [Prop. 10.85], are the roots of $x^2 - 2kx + k^2k'^2 = 0$.

μθ'.

Εύρεῖν τὴν ἐκ δύο ὀνομάτων δευτέραν.

Proposition 49

To find a second binomial (straight-line).



Έχχείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγχείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐχχείσθω ῥητὴ ἡ Δ, καὶ τῆ Δ σύμμετρος ἔστω ἡ ΕΖ μήχει· ῥητὴ ἄρα ἐστὶν ἡ ΕΖ. γεγονέτω δὴ καὶ ὡς ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ. ἑητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ λὸγον οὐχ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμον, οὐδὲ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῆ ΖΗ μήχει· αἱ ΕΖ, ΖΗ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἐχ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ. δειχτέον δή, ὅτι καὶ δευτέρα.

Έπει γὰρ ἀνάπαλίν ἐστιν ὡς ὁ BA ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς ΖΕ, μείζων δὲ ὁ BA τοῦ ΑΓ, μεῖζον ἄρα [καὶ] τὸ ἀπὸ τῆς ΖΕ, μείζων δὲ ὁ BA τοῦ ΑΓ, μεῖζον ἄρα [καὶ] τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ' ἀναστρέψαντι ἄρα ἐστιν ὡς ὁ AB πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Θ. ἀλλ' ὁ AB πρὸς τὸν BΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΖΗ τῆς ΔΗ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμος. σύμμετρος ἄρα ἐστιν ἡ ΖΗ τῆ Θ μήκει· ὥστε ἡ ΖΗ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καί εἰσι ἑηταὶ αἱ ΖΗ, ΖΕ δυνάμει μόνον σύμμετροι, καὶ τὸ ΕΖ ἕλασσον ὄνομα τῆ ἐκκειμένη ἑητῆ σύμμετρόν ἐστι τῆ Δ μήκει.

Ή ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα ὅπερ ἔδει δεῖξαι.



Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D. EF is thus a rational (straight-line). So, let it also have been contrived that as the number CA (is) to AB, so the (square) on EF(is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straightline). And since the number CA does not have to ABthe ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straightline) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

For since, inversely, as the number BA is to AC, so the (square) on GF (is) to the (square) on FE [Prop. 5.7 corr.], and BA (is) greater than AC, the (square) on GF (is) thus [also] greater than the (square) on FE[Prop. 5.14]. Let (the sum of) the (squares) on EF and H be equal to the (square) on GF. Thus, via conversion, as AB is to BC, so the (square) on FG (is) to the (square) on H [Prop. 5.19 corr.]. But, AB has to BCthe ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, FG is commensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) commensurable in length with (FG). And *FG* and *FE* are rational (straight-lines which are) commensurable in square only. And the lesser term EFis commensurable in length with the rational (straightline) D (previously) laid down.

Thus, EG is a second binomial (straight-line) [Def. 10.6].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a second binomial straight-line is $k/\sqrt{1-k'^2} + k$. This, and the second apotome,



whose length is $k/\sqrt{1-k'^2} - k$ [Prop. 10.86], are the roots of $x^2 - (2k/\sqrt{1-k'^2})x + k^2[k'^2/(1-k'^2)] = 0$.

Έχχείσθωσαν δύο άριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγχείμενον έξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρός δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἐχχείσθω δέ τις χαὶ ἄλλος μὴ τετράγωνος ἀριθμὸς ὁ Δ, καὶ πρὸς ἑκάτερον τῶν ΒΑ, ΑΓ λόγον μή ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον άριθμόν και έκκείσθω τις δητή εύθεῖα ή Ε, και γεγονέτω ώς ὁ Δ πρὸς τὸν AB, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ε τῷ ἀπὸ τῆς ΖΗ. καί ἐστι ῥητὴ
ἡ Ε· ῥητὴ ἄρα ἐστὶ καὶ ἡ ZH. καὶ ἐπεὶ
ὑ Δ πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῆ ΖΗ μήχει. γεγονέτω δη πάλιν ώς ή BA ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρός τὸ ἀπὸ τῆς ΗΘ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ἑητὴ δὲ ἡ ΖΗ· ἑητὴ ἄρα καὶ ἡ ΗΘ. καὶ έπει ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος άριθμός πρός τετράγωνον άριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρός τὸ ἀπὸ τῆς ΘΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῆ ΗΘ μήχει. αἱ ΖΗ, ΗΘ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. ή ΖΘ ἄρα ἐχ δύο ὀνομάτων ἐστίν. λέγω δή, ὅτι καὶ τρίτη.

Έπει γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν AB, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ZH, ὡς δὲ ὁ BA πρὸς τὸν AΓ, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς HΘ, δι' ἴσου ἄρα ἑστιν ὡς ὁ Δ πρὸς τὸν AΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς HΘ. ὁ δὲ Δ πρὸς τὸν AΓ λόγον οὐx ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς HΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἑστιν ἡ Ε τῆ HΘ μήχει. καὶ ἐπεί ἐστιν ὡς ὁ BA πρὸς τὸν AΓ, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς HΘ, μεῖζον ἄρα τὸ ἀπὸ τῆς ZH τοῦ ἀπὸ τῆς HΘ. ἔστω οῦν τῷ ἀπὸ τῆς ZH ἴσα τὰ ἀπὸ τῶν HΘ, Κ· ἀναστρέψαντι ἄρα [ἐστιν] ὡς ὁ AB πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς X. ὁ δὲ



Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number. And let some other nonsquare number D also be laid down, and let it not have to each of BA and AC the ratio which (some) square number (has) to (some) square number. And let some rational straight-line E be laid down, and let it have been contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E is commensurable with the (square) on FG [Prop. 10.6]. And E is a rational (straight-line). Thus, FG is also a rational (straight-line). And since D does not have to AB the ratio which (some) square number has to (some) square number, the (square) on E does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. E is thus incommensurable in length with FG[Prop. 10.9]. So, again, let it have been contrived that as the number BA (is) to AC, so the (square) on FG(is) to the (square) on GH [Prop. 10.6 corr.]. Thus, the (square) on FG is commensurable with the (square) on *GH* [Prop. 10.6]. And *FG* (is) a rational (straight-line). Thus, GH (is) also a rational (straight-line). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on HG the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH[Prop. 10.9]. FG and GH are thus rational (straightlines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a third (binomial straight-line).

For since as D is to AB, so the (square) on E (is) to the (square) on FG, and as BA (is) to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D (is) to AC, so the (square) on E (is) to the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not

τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ZH ἄρα πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα [ἐστὶν] ἡ ZH τῆ Κ μήκει. ἡ ZH ἄρα τῆς HΘ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καί εἰσιν aὶ ZH, HΘ ἑηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῆ Ε μήκει.

Ή ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on Edoes not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, E is incommensurable in length with GH[Prop. 10.9]. And since as *BA* is to *AC*, so the (square) on FG (is) to the (square) on GH, the (square) on FG(is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG. Thus, via conversion, as AB [is] to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number. Thus, FG [is] commensurable in length with K [Prop. 10.9]. Thus, the square on FG is greater than (the square on) GHby the (square) on (some straight-line) commensurable (in length) with (FG). And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with E.

Thus, FH is a third binomial (straight-line) [Def. 10.7].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a third binomial straight-line is $k^{1/2} (1 + \sqrt{1 - k'^2})$. This, and the third apotome, whose length is $k^{1/2} (1 - \sqrt{1 - k'^2})$ [Prop. 10.87], are the roots of $x^2 - 2k^{1/2}x + kk'^2 = 0$.



Έκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς τὸν ΒΓ λόγον μὴ ἔχειν μήτε μὴν πρὸς τὸν ΑΓ, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ ἐκκείσθω ῥητὴ ἡ Δ, καὶ τῆ Δ σύμμετρος ἔστω μήκει ἡ ΕΖ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῆ ΖΗ μήκει. αἱ ΕΖ, ΖΗ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ὥστε ἡ ΕΗ ἐκ δύο ὀνομάτων ἐστίν. λέγω δή,

Proposition 51



Let the two numbers AC and CB be laid down such that AB does not have to BC, or to AC either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D. Thus, EF is also a rational (straight-line). And let it have been contrived that as the number BA (is) to AC, so the (square) on EF(is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straightline). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, ὄτι καὶ τετάρτη.

Έπεὶ γάρ ἐστιν ὡς ὁ BA πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς ZH [μείζων δὲ ὁ BA τοῦ AΓ], μεῖζον ἄρα τὸ ἀπὸ τῆς EZ τοῦ ἀπὸ τῆς ZH. ἔστω οῦν τῷ ἀπὸ τῆς EZ ἴσα τὰ ἀπὸ τῶν ZH, Θ· ἀναστρέψαντι ἄρα ὡς ὁ AB ἀριθμὸς πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς EZ πρὸς τὸν BΓ, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς EZ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς Θλόγον ἐχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς EZ τῆ Θ μήχει· ἡ EZ ἅρα τῆς HZ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί εἰσιν αἱ EZ, ZH ἑηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ EZ τῆ Δ σύμμετρός ἐστι μήχει.

Ή ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. Thus, EF and FG are rational (straight-lines which are) commensurable in square only. Hence, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fourth (binomial straight-line).

For since as BA is to AC, so the (square) on EF (is) to the (square) on FG [and BA (is) greater than AC], the (square) on EF (is) thus greater than the (square) on FG[Prop. 5.14]. Therefore, let (the sum of) the squares on FG and H be equal to the (square) on EF. Thus, via conversion, as the number AB (is) to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And ABdoes not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with H[Prop. 10.9]. Thus, the square on EF is greater than (the square on) GF by the (square) on (some straight-line) incommensurable (in length) with (EF). And EF and FGare rational (straight-lines which are) commensurable in square only. And EF is commensurable in length with D. Thus, EG is a fourth binomial (straight-line) [Def. 10.8].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a fourth binomial straight-line is $k(1 + 1/\sqrt{1 + k'})$. This, and the fourth apotome, whose length is $k(1 - 1/\sqrt{1 + k'})$ [Prop. 10.88], are the roots of $x^2 - 2kx + k^2k'/(1 + k') = 0$.



Έχχείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐχκείσθω ῥητή τις εὐθεῖα ἡ Δ, καὶ τῆ Δ σύμμετρος ἔστω [μήχει] ἡ ΕΖ· ἑητὴ ἄρα ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΓΑ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΓΑ πρὸς τὸν ΑΒ λόγον οὐχ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὑδὲ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. αἱ



Let the two numbers AC and CB be laid down such that AB does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.]. And let some rational straight-line D be laid down. And let EF be commensurable [in length] with D. Thus, EF (is) a rational (straight-line). And let it have been contrived that as CA (is) to AB, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And CA does not have to AB the ra-

EZ, ZH ἄρα ἡηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ EH. λέγω δή, ὅτι καὶ πέμπτη.

Έπεὶ γάρ ἐστιν ὡς ὁ ΓΑ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, ἀνάπαλιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΖΕ· μεῖζον ἀρα τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω οῦν τῷ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω οῦν τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ· ἀναστρέψαντι ἄρα ἐστὶν ὡς ἱ ΑΒ ἀριθμὸς πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς Θ. ἱ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐx ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς ΖΗ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΕΛ τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Ε μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί εἰσιν αί ΗΖ, ΖΕ ἑρταὶ δυνάμει μόνον σύμμετροι, καὶ τὸ ΕΖ ἔλαττον ὄνομα σύμμετρόν ἑστι τῆ ἑχκειμένῃ ῥητῆ Τῃ Δ μήχει.

Ή ΕΗ ἄρα ἐχ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι. tio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF and FG are rational (straight-lines which are) commensurable in square only [Prop. 10.9]. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For since as CA is to AB, so the (square) on EF(is) to the (square) on FG, inversely, as BA (is) to AC, so the (square) on FG (is) to the (square) on FE[Prop. 5.7 corr.]. Thus, the (square) on GF (is) greater than the (square) on FE [Prop. 5.14]. Therefore, let (the sum of) the (squares) on EF and H be equal to the (square) on GF. Thus, via conversion, as the number AB is to BC, so the (square) on GF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BCthe ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FEby the (square) on (some straight-line) incommensurable (in length) with (FG). And GF and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line previously) laid down, D.

Thus, EG is a fifth binomial (straight-line).[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a fifth binomial straight-line is $k(\sqrt{1+k'}+1)$. This, and the fifth apotome, whose length is $k(\sqrt{1+k'}-1)$ [Prop. 10.89], are the roots of $x^2 - 2k\sqrt{1+k'}x + k^2k' = 0$.



Έχχείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς ἑχάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔστω δὲ καὶ ἕτερος ἀριθμὸς ὁ Δ μὴ τετράγωνος ῶν μηδὲ πρὸς ἑχάτερον τῶν ΒΑ, ΑΓ λόγον ἔχων, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ ἐχχείσθω τις ῥητὴ εὐθεῖα ἡ Ε, καὶ γεγονέτω ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Ε τῷ ἀπὸ



Let the two numbers AC and CB be laid down such that AB does not have to each of them the ratio which (some) square number (has) to (some) square number. And let D also be another number, which is not square, and does not have to each of BA and AC the ratio which (some) square number (has) to (some) square number either [Prop. 10.28 lem. I]. And let some rational straightline E be laid down. And let it have been contrived that τῆς ΖΗ. καί ἐστι ἑητὴ ἡ Ε· ἑητὴ ἄρα καὶ ἡ ΖΗ. καὶ ἐπεὶ οὐκ ἔχει ὁ Δ πρὸς τὸν AB λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἡ Ε τῆ ΖΗ μήχει. γεγονέτω δὴ πάλιν ὡς ὁ BA πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ. σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΘΗ. ἑητὸν ἄρα τὸ ἀπὸ τῆς ΘΗ· ἑητὴ ἄρα ἡ ΘΗ. καὶ ἐπεὶ ὁ BA πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῆ ΗΘ μήχει. αἱ ΖΗ, ΗΘ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἐχ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΖΘ. δειχτέον δή, ὅτι χαὶ ἕχτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ἔστι δὲ καὶ ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ έχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ούδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον έχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. άσύμμετρος ἄρα ἐστὶν ἡ Ε τῆ ΗΘ μήχει. ἐδείχθη δὲ χαὶ τῆ ΖΗ ἀσύμμετρος· ἑκατέρα ἄρα τῶν ΖΗ, ΗΘ ἀσύμμετρός έστι τῆ Ε μήκει. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μεῖζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῶ ἀπὸ [τῆς] ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα ὡς ὁ ΑΒ πρός ΒΓ, οὕτως τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ὥστε οὐδὲ τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον άριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῆ Κ μήκει· ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί εἰσιν αί ΖΗ, ΗΘ ἑηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι μήχει τῆ ἐχχειμένῃ ῥητῃ τῆ Ε.

'Η ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἕκτη· ὅπερ ἔδει δεῖξαι.

as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E (is) commensurable with the (square) on FG [Prop. 10.6]. And E is rational. Thus, FG (is) also rational. And since D does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on E thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, E (is) incommensurable in length with FG [Prop. 10.9]. So, again, let it have be contrived that as BA (is) to AC, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.]. The (square) on FG (is) thus commensurable with the (square) on HG[Prop. 10.6]. The (square) on HG (is) thus rational. Thus, HG (is) rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a sixth (binomial straight-line).

For since as D is to AB, so the (square) on E (is) to the (square) on FG, and also as BA is to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D is to AC, so the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on Edoes not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. E is thus incommensurable in length with GH[Prop. 10.9]. And (E) was also shown (to be) incommensurable (in length) with FG. Thus, FG and GHare each incommensurable in length with E. And since as BA is to AC, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG. Thus, via conversion, as AB (is) to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Hence, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with K [Prop. 10.9]. The square on FG is thus greater than (the square on) GHby the (square) on (some straight-line which is) incom-

mensurable (in length) with (FG). And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with the rational (straight-line) E (previously) laid down.

Thus, *FH* is a sixth binomial (straight-line) [Def. 10.10].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a sixth binomial straight-line is $\sqrt{k} + \sqrt{k'}$. This, and the sixth apotome, whose length is $\sqrt{k} - \sqrt{k'}$ [Prop. 10.90], are the roots of $x^2 - 2\sqrt{k}x + (k - k') = 0$.

Λῆμμα.

Έστω δύο τετράγωνα τὰ AB, BΓ καὶ κείσθωσαν ὥστε ἐπ' εὐθείας εἶναι τὴν ΔB τῆ BE· ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ ZB τῆ BH. καὶ συμπεπληρώσθω τὸ AΓ παραλληλόγραμμον· λέγω, ὅτι τετράγωνόν ἐστι τὸ AΓ, καὶ ὅτι τῶν AB, BΓ μέσον ἀνάλογόν ἐστι τὸ ΔΗ, καὶ ἕτι τῶν AΓ, ΓB μέσον ἀνάλογόν ἐστι τὸ ΔΓ.



Έπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΔΒ τῆ ΒΖ, ἡ δὲ ΒΕ τῆ ΒΗ, ὅλη ἄρα ἡ ΔΕ ὅλῃ τῆ ΖΗ ἐστιν ἴση. ἀλλ' ἡ μὲν ΔΕ ἑκατέρα τῶν ΑΘ, ΚΓ ἐστιν ἴσῃ, ἡ δὲ ΖΗ ἑκατέρα τῶν ΑΚ, ΘΓ ἐστιν ἴσῃ· καὶ ἑκατέρα ἄρα τῶν ΑΘ, ΚΓ ἑκατέρα τῶν ΑΚ, ΘΓ ἐστιν ἴσŋ. ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΓ παραλληλόγραμμον. ἔστι δὲ καὶ ὀρθογώνιον· τετράγωνον ἄρα ἐστὶ τὸ ΑΓ.

Καὶ ἐπεὶ ἐστιν ὡς ἡ ZB πρὸς τὴν BH, οὕτως ἡ ΔB πρὸς τὴν BE, ἀλλ' ὡς μὲν ἡ ZB πρὸς τὴν BH, οὕτως τὸ AB πρὸς τὸ ΔH, ὡς δὲ ἡ ΔB πρὸς τὴν BE, οὕτως τὸ ΔH πρὸς τὸ BΓ, καὶ ὡς ἄρα τὸ AB πρὸς τὸ ΔH, οὕτως τὸ ΔH πρὸς τὸ BΓ. τῶν AB, BΓ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΔH.

Λέγω δή, ὅτι καὶ τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν [ἑστι] τὸ
 $\Delta \Gamma.$

Έπεὶ γάρ ἐστιν ὡς ἡ ΑΔ πρὸς τὴν ΔΚ, οὕτως ἡ ΚΗ πρὸς τὴν ΗΓ· ἴση γάρ [ἐστιν] ἑϫατέρα ἑϫατέρα· καὶ συνθέντι ὡς ἡ ΑΚ πρὸς ΚΔ, οὕτως ἡ ΚΓ πρὸς ΓΗ, ἀλλ' ὡς μὲν ἡ ΑΚ πρὸς ΚΔ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΔ, ὡς δὲ ἡ ΚΓ πρὸς ΓΗ, οὕτως τὸ ΔΓ πρὸς ΓΒ, καὶ ὡς ἄρα τὸ ΑΓ πρὸς ΔΓ, οὕτως τὸ ΔΓ πρὸς τὸ ΒΓ. τῶν ΑΓ, ΓΒ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΔΓ· ἂ προέκειτο δεῖξαι.

Lemma

Let AB and BC be two squares, and let them be laid down such that DB is straight-on to BE. FB is, thus, also straight-on to BG. And let the parallelogram AChave been completed. I say that AC is a square, and that DG is the mean proportional to AB and BC, and, moreover, DC is the mean proportional to AC and CB.



For since DB is equal to BF, and BE to BG, the whole of DE is thus equal to the whole of FG. But DE is equal to each of AH and KC, and FG is equal to each of AK and HC [Prop. 1.34]. Thus, AH and KC are also equal to AK and HC, respectively. Thus, the parallelogram AC is equilateral. And (it is) also right-angled. Thus, AC is a square.

And since as FB is to BG, so DB (is) to BE, but as FB (is) to BG, so AB (is) to DG, and as DB (is) to BE, so DG (is) to BC [Prop. 6.1], thus also as AB (is) to DG, so DG (is) to BC [Prop. 5.11]. Thus, DG is the mean proportional to AB and BC.

So I also say that DC [is] the mean proportional to AC and CB.

For since as AD is to DK, so KG (is) to GC. For [they are] respectively equal. And, via composition, as AK (is) to KD, so KC (is) to CG [Prop. 5.18]. But as AK (is) to KD, so AC (is) to CD, and as KC (is) to CG, so DC(is) to CB [Prop. 6.1]. Thus, also, as AC (is) to DC, so DC (is) to BC [Prop. 5.11]. Thus, DC is the mean proportional to AC and CB. Which (is the very thing) it

was prescribed to show.

νδ΄.

Έὰν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.



Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ἑητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων πρώτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.

Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶ πρώτη ἡ $\mathrm{A}\Delta,$ διηρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω τὸ μεῖζον ὄνομα τὸ ΑΕ. φανερὸν δή, ὅτι αἱ ΑΕ, ΕΔ ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτη, καὶ ἡ ΑΕ σύμμετρός ἐστι τῆ ἐκκειμένη ρητῆ τῆ AB μήκει. τετμήσθω δὴ ἡ EΔ δίχα κατὰ τὸ Z σημεῖον. καὶ ἐπεὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, ἐὰν ἄρα τῷ τετάρτω μέρει τοῦ ἀπὸ τῆς έλάσσονος, τουτέστι τῷ ἀπὸ τῆς ΕΖ, ἴσον παρὰ τὴν μείζονα τὴν ΑΕ παραβληθῆ ἐλλεῖπον εἴδει τετραγώνω, εἰς σύμμετρα αὐτὴν διαιρεῖ. παραβεβλήσθω οὖν παρὰ τὴν ΑΕ τῷ ἀπὸ τῆς ΕΖ ἴσον τὸ ὑπὸ ΑΗ, ΗΕ· σύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῆ ΕΗ μήχει. χαὶ ἤχθωσαν ἀπὸ τῶν Η, Ε, Ζ ὑποτέρα τῶν ΑΒ, ΓΔ παράλληλοι αί ΗΘ, ΕΚ, ΖΛ· και τῷ μέν ΑΘ παραλληλογράμμω ίσον τετράγωνον συνεστάτω το ΣΝ, τῶ δὲ ΗΚ ίσον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ' εὐθείας εἶναι τὴν ΜΝ τῆ ΝΞ· ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ ΡΝ τῆ ΝΟ. καὶ συμπεπληρώσθω τὸ ΣΠ παραλληλόγραμμον. τετράγωνον ἄρα έστι τὸ ΣΠ. και ἐπει τὸ ὑπὸ τῶν ΑΗ, ΗΕ ἴσον ἐστι τῷ ἀπὸ τῆς ΕΖ, ἔστιν ἄρα ὡς ἡ ΑΗ πρὸς ΕΖ, οὕτως ἡ ΖΕ πρὸς ΕΗ· καὶ ὡς ἄρα τὸ ΑΘ πρὸς ΕΛ, τὸ ΕΛ πρὸς ΚΗ· τῶν ΑΘ, ΗΚ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἀλλὰ τὸ μὲν ΑΘ ίσον ἐστὶ τῷ ΣΝ, τὸ δὲ ΗΚ ἴσον τῷ ΝΠ· τῶν ΣΝ, ΝΠ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἔστι δὲ τῶν αὐτῶν τῶν ΣΝ, ΝΠ μέσον ἀνάλογον καὶ τὸ ΜΡ· ἴσον ἄρα ἐστὶ τὸ ΕΛ τῷ MP· ώστε καὶ τῷ ΟΞ ἴσον ἐστίν. ἔστι δὲ καὶ τὰ AΘ, HK τοῖς Σ N, NΠ ἴσα· ὅλον ἄρα τὸ AΓ ἴσον ἐστὶν ὅλω τ $\tilde{\omega}$ Σ Π, τουτέστι τῷ ἀπὸ τῆς ΜΞ τετραγώνω· τὸ ΑΓ ἄρα δύναται ἡ ΜΞ. λέγω, ὅτι ἡ ΜΞ ἐκ δύο ὀνομάτων ἐστίν.

Ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΑΗ τῆ ΗΕ, σύμμετρός ἐστι καὶ ἡ ΑΕ ἑκατέρα τῶν ΑΗ, ΗΕ. ὑπόκειται δὲ καὶ ἡ ΑΕ τῆ

Proposition 54

If an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called binomial.^{\dagger}



For let the area AC be contained by the rational (straight-line) AB and by the first binomial (straight-line) AD. I say that square-root of area AC is the irrational (straight-line which is) called binomial.

For since AD is a first binomial (straight-line), let it have been divided into its (component) terms at E, and let AE be the greater term. So, (it is) clear that AE and ED are rational (straight-lines which are) commensurable in square only, and that the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and that AE is commensurable (in length) with the rational (straight-line) AB (first) laid out [Def. 10.5]. So, let ED have been cut in half at point F. And since the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), thus if a (rectangle) equal to the fourth part of the (square) on the lesser (term)—that is to say, the (square) on *EF*—falling short by a square figure, is applied to the greater (term) AE, then it divides it into (terms which are) commensurable (in length) [Prop 10.17]. Therefore, let the (rectangle contained) by AG and GE, equal to the (square) on EF, have been applied to AE. AG is thus commensurable in length with EG. And let GH, EK, and FL have been drawn from (points) G, E, and F (respectively), parallel to either of AB or CD. And let the square SN, equal to the parallelogram AH, have been constructed, and (the square) NQ, equal to (the parallelogram) GK [Prop. 2.14]. And let MN be laid down so as to be straight-on to NO. RN is thus also straight-on to NP. And let the parallelogram SQ have been completed. SQ is thus a square [Prop. 10.53 lem.]. And since the (rectangle contained) by AG and GE is equal to the (square) on EF, thus as AG is to EF, so FE (is) to EG[Prop. 6.17]. And thus as AH (is) to EL, (so) EL (is)

ΑΒ σύμμετρος και αι ΑΗ, ΗΕ άρα τη ΑΒ σύμμετροί είσιν. καί ἐστι ῥητὴ ἡ ΑΒ· ῥητὴ ἄρα ἐστὶ καὶ ἑκατέρα τῶν ΑΗ, ΗΕ· δητόν ἄρα ἐστιν ἑκάτερον τῶν ΑΘ, ΗΚ, καί ἐστι σύμμετρον τὸ ΑΘ τῷ ΗΚ. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἴσον ἐστίν, τὸ δὲ ΗΚ τῷ ΝΠ· καὶ τὰ ΣΝ, ΝΠ ἄρα, τουτέστι τὰ ἀπὸ τῶν ΜΝ, ΝΞ, ἑητά ἐστι καὶ σύμμετρα. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῆ ΕΔ μήχει, ἀλλ' ἡ μὲν ΑΕ τῆ ΑΗ ἐστι σύμμετρος, ή δὲ ΔΕ τῆ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα καὶ ἡ ΑΗ τῆ EZ· ώστε καὶ τὸ $A\Theta$ τῶ $E\Lambda$ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν AΘ τῷ ΣΝ ἐστιν ἴσον, τὸ δὲ ΕΛ τῷ MP· καὶ τὸ ΣΝ ἄρα τῶ MP ἀσύμμετρόν ἐστιν. ἀλλ' ὡς τὸ ΣΝ πρὸς MP, ἡ ΟΝ πρὸς τὴν NP· ἀσύμμετρος ἄρα ἐστὶν ἡ ΟΝ τῆ NP. ἴση δὲ ἡ μέν ΟΝ τῆ ΜΝ, ἡ δὲ ΝΡ τῆ ΝΞ ἀσύμμετρος ἄρα ἐστὶν ἡ ΜΝ τῆ ΝΞ. καί ἐστι τὸ ἀπὸ τῆς ΜΝ σύμμετρον τῷ ἀπὸ τῆς ΝΞ, καὶ ῥητὸν ἑκάτερον· αἱ ΜΝ, ΝΞ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι.

Ή ΜΞ ἄρα ἐχ δύο ὀνομάτων ἐστὶ χαὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

to KG [Prop. 6.1]. Thus, EL is the mean proportional to AH and GK. But, AH is equal to SN, and GK (is) equal to NQ. EL is thus the mean proportional to SN and NQ. And MR is also the mean proportional to the same—(namely), SN and NQ [Prop. 10.53 lem.]. EL is thus equal to MR. Hence, it is also equal to PO [Prop. 1.43]. And AH plus GK is equal to SN plus NQ. Thus, the whole of AC is equal to the whole of SQ—that is to say, to the square on MO. Thus, MO (is) the square-root of (area) AC. I say that MO is a binomial (straight-line).

For since AG is commensurable (in length) with GE, AE is also commensurable (in length) with each of AGand GE [Prop. 10.15]. And AE was also assumed (to be) commensurable (in length) with AB. Thus, AGand GE are also commensurable (in length) with AB[Prop. 10.12]. And AB is rational. AG and GE are thus each also rational. Thus, AH and GK are each rational (areas), and AH is commensurable with GK[Prop. 10.19]. But, AH is equal to SN, and GK to NQ. SN and NQ—that is to say, the (squares) on MN and NO (respectively)-are thus also rational and commensurable. And since AE is incommensurable in length with ED, but AE is commensurable (in length) with AG, and DE (is) commensurable (in length) with EF, AG (is) thus also incommensurable (in length) with EF[Prop. 10.13]. Hence, AH is also incommensurable with EL [Props. 6.1, 10.11]. But, AH is equal to SN, and EL to MR. Thus, SN is also incommensurable with MR. But, as SN (is) to MR, (so) PN (is) to NR[Prop. 6.1]. *PN* is thus incommensurable (in length) with NR [Prop. 10.11]. And PN (is) equal to MN, and NR to NO. Thus, MN is incommensurable (in length) with NO. And the (square) on MN is commensurable with the (square) on NO, and each (is) rational. MNand NO are thus rational (straight-lines which are) commensurable in square only.

Thus, MO is (both) a binomial (straight-line) [Prop. 10.36], and the square-root of AC. (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then this proposition states that the square-root of a first binomial straight-line is a binomial straight-line is a binomial straight-line is a length $k + k\sqrt{1 - k'^2}$ whose square-root can be written $\rho (1 + \sqrt{k''})$, where $\rho = \sqrt{k(1 + k')/2}$ and k'' = (1 - k')/(1 + k'). This is the length of a binomial straight-line (see Prop. 10.36), since ρ is rational.

νε΄.

Έλν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὑνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων πρώτη. If an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called first bimedial.^{\dagger}

Proposition 55



Περιεχέσθω γὰρ χωρίον τὸ ΑΒΓΔ ὑπὸ ἑητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων δυετέρας τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐκ δύο μέσων πρώτη ἐστίν.

Έπεὶ γὰρ ἐκ δύο ὀνομάτων δευτέρα ἐστὶν ἡ ΑΔ, διηρήσθω είς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον όνομα είναι τὸ ΑΕ· αί ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ τὸ ἔλαττον ὄνομα ἡ ΕΔ σύμμετρόν έστι τῆ ΑΒ μήκει. τετμήσθω ἡ ΕΔ δίχα κατὰ τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνω τὸ ὑπὸ τῶν ΑΗΕ· σύμμετρος ἄρα ἡ ΑΗ τῆ ΗΕ μήχει. χαὶ διὰ τῶν Η, Ε, Ζ παράλληλοι ἤχθωσαν ταῖς ΑΒ, ΓΔ αἱ ΗΘ, ΕΚ, ΖΛ, καὶ τῷ μὲν ΑΘ παραλληλογράμμω ίσον τετράγωνον συνεστάτω το ΣΝ, τῷ δὲ ΗΚ ίσον τετράγωνον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ' εὐθείας εἶναι τὴν MN τῆ NΞ· ἐπ' εὐθείας ἄρα [ἐστὶ] καὶ ἡ PN τῆ NO. καὶ συμπεπληρώσθω τὸ ΣΠ τετράγωνον φανερὸν δὴ ἐκ τοῦ προδεδειγμένου, ὅτι τὸ ΜΡ μέσον ἀνάλογόν ἐστι τῶν ΣΝ, ΝΠ, καὶ ἴσον τῷ ΕΛ, καὶ ὅτι τὸ ΑΓ χωρίον δύναται ἡ ΜΞ. δειχτέον δή, ὄτι ἡ ΜΞ ἐχ δύο μέσων ἐστὶ πρώτη.

Έπει ἀσύμμετρός ἐστιν ἡ ΑΕ τῆ ΕΔ μήχει, σύμμετρος δὲ ἡ ΕΔ τῆ ΑΒ, ἀσύμμετρος ἄρα ἡ ΑΕ τῆ ΑΒ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΗ τῆ ΕΗ, σύμμετρός ἐστι καὶ ἡ ΑΕ έκατέρα τῶν AH, HE. ἀλλὰ ἡ AE ἀσύμμετρος τῆ AB μήκει· καὶ αἱ AH, HE ẳρα ἀσύμμετροί εἰσι τῆ AB. αἱ BA, AH, ΗΕ άρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. ὥστε μέσον έστιν ἑκάτερον τῶν ΑΘ, ΗΚ. ὥστε και ἑκάτερον τῶν ΣΝ, ΝΠ μέσον ἐστίν. καὶ αἱ ΜΝ, ΝΞ ἄρα μέσαι εἰσίν. καὶ έπει σύμμετρος ή ΑΗ τῆ ΗΕ μήχει, σύμμετρόν ἐστι χαι τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΠ, τουτέστι τὸ ἀπὸ τῆς ΜΝ τῷ ἀπὸ τῆς ΝΞ [ὥστε δυνάμει εἰσὶ σύμμετροι αἱ MN, NΞ]. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῆ ΕΔ μήκει, άλλ' ή μέν ΑΕ σύμμετρός έστι τῆ ΑΗ, ή δὲ ΕΔ τῆ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα ἡ ΑΗ τῆ ΕΖ. ὥστε καὶ τὸ ΑΘ τῷ ΕΛ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΣΝ τῷ ΜΡ, τουτέστιν ὁ ΟΝ τῆ ΝΡ, τουτέστιν ἡ ΜΝ τῆ ΝΞ ἀσύμμετρός έστι μήχει. έδείχθησαν δὲ αἱ MN, NΞ χαὶ μέσαι οὔσαι χαὶ δυνάμει σύμμετροι· αί ΜΝ, ΝΞ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὄτι καὶ ἑητὸν περιέχουσιν. ἐπεὶ γὰρ ἡ ΔΕ ὑπόκειται ἑκατέρα τῶν ΑΒ, ΕΖ σύμμετρος, σύμμετρος ἄρα καὶ ἡ ΕΖ τῆ ΕΚ. καὶ ῥητὴ ἑκατέρα αὐτῶν· ῥητὸν ἄρα τὸ ΕΛ, τουτέστι τὸ MP· τὸ δὲ MP ἐστι τὸ ὑπὸ τῶν MNΞ. έὰν δὲ δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν



For let the area ABCD be contained by the rational (straight-line) AB and by the second binomial (straight-line) AD. I say that the square-root of area AC is a first bimedial (straight-line).

For since *AD* is a second binomial (straight-line), let it have been divided into its (component) terms at E, such that AE is the greater term. Thus, AE and ED are rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and the lesser term EDis commensurable in length with AB [Def. 10.6]. Let ED have been cut in half at F. And let the (rectangle contained) by AGE, equal to the (square) on EF, have been applied to AE, falling short by a square figure. AG (is) thus commensurable in length with GE[Prop. 10.17]. And let GH, EK, and FL have been drawn through (points) G, E, and F (respectively), parallel to AB and CD. And let the square SN, equal to the parallelogram AH, have been constructed, and the square NQ, equal to GK. And let MN be laid down so as to be straight-on to NO. Thus, RN [is] also straight-on to NP. And let the square SQ have been completed. So, (it is) clear from what has been previously demonstrated [Prop. 10.53 lem.] that MR is the mean proportional to SN and NQ, and (is) equal to EL, and that MO is the square-root of the area AC. So, we must show that MOis a first bimedial (straight-line).

Since AE is incommensurable in length with ED, and ED (is) commensurable (in length) with AB, AE (is) thus incommensurable (in length) with AB[Prop. 10.13]. And since AG is commensurable (in length) with EG, AE is also commensurable (in length) with each of AG and GE [Prop. 10.15]. But, AE is incommensurable in length with AB. Thus, AG and GEare also (both) incommensurable (in length) with AB[Prop. 10.13]. Thus, BA, AG, and (BA, and) GE are (pairs of) rational (straight-lines which are) commensurable in square only. And, hence, each of AH and GKis a medial (area) [Prop. 10.21]. Hence, each of SNand NQ is also a medial (area). Thus, MN and NOare medial (straight-lines). And since AG (is) commensurable in length with GE, AH is also commensurable περιέχουσαι, ή ὄλη ἄλογός ἐστιν, καλεῖται δὲ ἐκ δύο μέσων πρώτη.

Η άρα ΜΞ ἐκ δύο μέσων ἐστὶ πρώτη ὅπερ ἔδει δεῖξαι.

with GK—that is to say, SN with NQ—that is to say, the (square) on MN with the (square) on NO [hence, *MN* and *NO* are commensurable in square] [Props. 6.1, 10.11]. And since AE is incommensurable in length with ED, but AE is commensurable (in length) with AG, and ED commensurable (in length) with EF, AG (is) thus incommensurable (in length) with EF [Prop. 10.13]. Hence, AH is also incommensurable with EL—that is to say, SN with MR—that is to say, PN with NR—that is to say, MN is incommensurable in length with NO[Props. 6.1, 10.11]. But MN and NO have also been shown to be medial (straight-lines) which are commensurable in square. Thus, MN and NO are medial (straightlines which are) commensurable in square only. So, I say that they also contain a rational (area). For since DE was assumed (to be) commensurable (in length) with each of AB and EF, EF (is) thus also commensurable with EK[Prop. 10.12]. And they (are) each rational. Thus, ELthat is to say, MR—(is) rational [Prop. 10.19]. And MR is the (rectangle contained) by MNO. And if two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together, then the whole is (that) irrational (straight-line which is) called first bimedial [Prop. 10.37].

Thus, *MO* is a first bimedial (straight-line). (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then this proposition states that the square-root of a second binomial straight-line is a first bimedial straight-line: *i.e.*, a second binomial straight-line has a length $k/\sqrt{1-k'^2} + k$ whose square-root can be written $\rho(k''^{1/4} + k''^{3/4})$, where $\rho = \sqrt{(k/2)(1+k')/(1-k')}$ and k'' = (1-k')/(1+k'). This is the length of a first bimedial straight-line (see Prop. 10.37), since ρ is rational.

Έαν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.



Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ἑητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τρίτης τῆς ΑΔ διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ῶν μεῖζόν ἐστι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

Κατεσκευάσθω γάρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ which is) called second bimedial.

Proposition 56

If an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called second bimedial.[†]



For let the area ABCD be contained by the rational (straight-line) AB and by the third binomial (straight-line) AD, which has been divided into its (component) terms at E, of which AE is the greater. I say that the square-root of area AC is the irrational (straight-line which is) called second bimedial.

ἐκ δύο ὀνομάτων ἐστὶ τρίτη ἡ ΑΔ, αἰ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ οὐδετέρα τῶν ΑΕ, ΕΔ σύμμετρός [ἐστι] τῆ ΑΒ μήκει. ὁμοίως δὴ τοῖς προδεδειγμένοις δείξομεν, ὅτι ἡ ΜΞ ἐστιν ἡ τὸ ΑΓ χωρίον δυναμένη, καὶ αἱ MN, ΝΞ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. ὥστε ἡ ΜΞ ἐκ δύο μέσων ἐστίν. δεικτέον δή, ὅτι καὶ δευτέρα.

[Kaì] ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΔΕ τῆ ΑΒ μήχει, τουτέστι τῆ ΕΚ, σύμμετρος δὲ ἡ ΔΕ τῆ ΕΖ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῆ ΕΚ μήχει. χαί εἰσι ῥηταί· αἰ ΖΕ, ΕΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. μέσον ἄρα [ἐστὶ] τὸ ΕΛ, τουτέστι τὸ MP· χαὶ περιέχεται ὑπὸ τῶν MNΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν MNΞ.

Ή ΜΞ ἄρα ἐκ δύο μέσων ἐστὶ δευτέρα· ὅπερ ἔδει δεῖξαι.

For let the same construction be made as previously. And since AD is a third binomial (straight-line), AE and ED are thus rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and neither of AE and ED [is] commensurable in length with AB [Def. 10.7]. So, similarly to that which has been previously demonstrated, we can show that MO is the square-root of area AC, and MN and NO are medial (straight-lines which are) commensurable in square only. Hence, MO is bimedial. So, we must show that (it is) also second (bimedial).

[And] since DE is incommensurable in length with AB—that is to say, with EK—and DE (is) commensurable (in length) with EF, EF is thus incommensurable in length with EK [Prop. 10.13]. And they are (both) rational (straight-lines). Thus, FE and EK are rational (straight-lines which are) commensurable in square only. EL—that is to say, MR—[is] thus medial [Prop. 10.21]. And it is contained by MNO. Thus, the (rectangle contained) by MNO is medial.

Thus, *MO* is a second bimedial (straight-line) [Prop. 10.38]. (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then this proposition states that the square-root of a third binomial straight-line is a second bimedial straight-line: *i.e.*, a third binomial straight-line has a length $k^{1/2} (1 + \sqrt{1 - k'^2})$ whose square-root can be written $\rho (k^{1/4} + k''^{1/2}/k^{1/4})$, where $\rho = \sqrt{(1 + k')/2}$ and k'' = k (1 - k')/(1 + k'). This is the length of a second bimedial straight-line (see Prop. 10.38), since ρ is rational.

Έὰν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.



Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ἑητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης τῆς ΑΔ διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ῶν μεῖζον ἔστω τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.

Έπεὶ γὰρ ἡ ΑΔ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη, αἱ ΑΕ, ΕΔ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ ἡ ΑΕ τῆ ΑΒ σύμμετρός [ἐστι] μήκει. τετμήσθω ἡ ΔΕ δίχα κατὰ

Proposition 57

If an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called major.^{\dagger}



For let the area AC be contained by the rational (straight-line) AB and the fourth binomial (straight-line) AD, which has been divided into its (component) terms at E, of which let AE be the greater. I say that the square-root of AC is the irrational (straight-line which is) called major.

For since AD is a fourth binomial (straight-line), AE and ED are thus rational (straight-lines which are) com-

τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω παραλληλόγραμμον τὸ ὑπὸ ΑΗ, ΗΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῆ ΗΕ μήχει. ἤχθωσαν παράλληλοι τῆ ΑΒ αἰ ΗΘ, ΕΚ, ΖΛ, καὶ τὰ λοιπὰ τὰ αὐτὰ τοῖς πρὸ τούτου γεγονέτω· φανερὸν δή, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δειχτέον δή, ὅτι ἡ ΜΞ ἄλογός ἐστιν ἡ χαλουμένη μείζων.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΗ τῆ ΕΗ μήχει, ἀσύμμετρόν έστι καὶ τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΠ αἱ ΜΝ, ΝΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ σύμμετρός ἐστιν ή ΑΕ τῆ ΑΒ μήκει, ἑητόν ἐστι τὸ ΑΚ· καί ἐστιν ἴσον τοῖς ἀπὸ τῶν MN, NΞ· ῥητὸν ἄρα [ἐστὶ] καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. καὶ ἐπεὶ ἀσύμμετρός [ἐστιν] ἡ ΔΕ τῆ ΑΒ μήκει, τουτέστι τῆ ΕΚ, ἀλλὰ ἡ ΔΕ σύμμετρός ἐστι τῆ ΕΖ, ἀσύμμετρος ἄρα ἡ ΕΖ τῆ ΕΚ μήχει. αἱ ΕΚ, ΕΖ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΛΕ, τουτέστι τὸ MP. καὶ περιέχεται ὑπὸ τῶν MN, NΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν MN, ΝΞ. καὶ ῥητὸν τὸ [συγκείμενον] έκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, καί εἰσιν ἀσύμμετροι αἱ ΜΝ, ΝΞ δυνάμει. ἐὰν δὲ δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγχείμενον ἐχ τῶν ἀπ' αὐτῶν τετραγώνων ρητόν, τὸ δ' ὑπ' αὐτῶν μέσον, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ μείζων.

Ή ΜΞ ἄρα ἄλογός ἐστιν ἡ καλουμένη μείζων, καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ ἔδει δεῖξαι.

mensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) incommensurable (in length) with (AE), and AE [is] commensurable in length with AB[Def. 10.8]. Let DE have been cut in half at F, and let the parallelogram (contained by) AG and GE, equal to the (square) on EF, (and falling short by a square figure) have been applied to AE. AG is thus incommensurable in length with GE [Prop. 10.18]. Let GH, EK, and FLhave been drawn parallel to AB, and let the rest (of the construction) have been made the same as the (proposition) before this. So, it is clear that MO is the square-root of area AC. So, we must show that MO is the irrational (straight-line which is) called major.

Since AG is incommensurable in length with EG, AHis also incommensurable with GK—that is to say, SNwith NQ [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AE is commensurable in length with AB, AK is rational [Prop. 10.19]. And it is equal to the (sum of the squares) on MNand NO. Thus, the sum of the (squares) on MN and NO [is] also rational. And since DE [is] incommensurable in length with AB [Prop. 10.13]—that is to say, with EK—but DE is commensurable (in length) with EF, EF (is) thus incommensurable in length with EK[Prop. 10.13]. Thus, EK and EF are rational (straightlines which are) commensurable in square only. LEthat is to say, MR—(is) thus medial [Prop. 10.21]. And it is contained by MN and NO. The (rectangle contained) by MN and NO is thus medial. And the [sum] of the (squares) on MN and NO (is) rational, and MN and NO are incommensurable in square. And if two straightlines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together, then the whole is the irrational (straight-line which is) called major [Prop. 10.39].

Thus, MO is the irrational (straight-line which is) called major. And (it is) the square-root of area AC. (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then this proposition states that the square-root of a fourth binomial straight-line is a major straight-line i.e., a fourth binomial straight-line has a length $k (1 + 1/\sqrt{1 + k'})$ whose square-root can be written $\rho \sqrt{[1 + k''/(1 + k''^2)^{1/2}]/2} + \rho \sqrt{[1 - k''/(1 + k''^2)^{1/2}]/2}$, where $\rho = \sqrt{k}$ and $k''^2 = k'$. This is the length of a major straight-line (see Prop. 10.39), since ρ is rational.

νη'.

Έὰν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἑητὸν καὶ μέσον δυναμένη.

Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ἑητῆς τῆς ΑΒ καὶ

Proposition 58

If an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).[†]

τῆς ἐχ δύο ὀνομάτων πέμπτης τῆς ΑΔ διηρημένης εἰς τὰ ἀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· λέγω [δή], ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἑητὸν καὶ μέσον δυναμένη.



Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον δεδειγμένοις· φανερὸν δή, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δεικτέον δή, ὅτι ἡ ΜΞ ἐστιν ἡ ἑητὸν καὶ μέσον δυναμένη.

Έπει γὰρ ἀσύμμετρός ἐστιν ἡ AH τῆ HE, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ AΘ τῷ ΘΕ, τουτέστι τὸ ἀπὸ τῆς MN τῷ ἀπὸ τῆς NΞ· αἱ MN, NΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ AΔ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη, καί [ἐστιν] ἔλασσον αὐτῆς τμῆμα τὸ ΕΔ, σύμμετρος ἄρα ἡ ΕΔ τῆ AB μήκει. ἀλλὰ ἡ AE τῆ ΕΔ ἐστιν ἀσύμμετρος· καὶ ἡ AB ἄρα τῆ AE ἐστιν ἀσύμμετρος μήκει [αἱ BA, AE ῥηταί εἰσι δυνάμει μόνον σύμμετροι]· μέσον ἄρα ἐστὶ τὸ AK, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν MN, NΞ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῆ AB μήκει, τουτέστι τῆ ΕΚ, ἀλλὰ ἡ ΔΕ τῆ EZ σύμμετρός ἐστιν, καὶ ἡ ΕΖ ἄρα τῆ ΕΚ σύμμετρός ἐστιν. καὶ ῥητὴ ἡ ΕΚ· ἑητὸν ἄρα καὶ τὸ ΕΛ, τουτέστι τὸ MP, τουτέστι τὸ ὑπὸ MNΞ· αἱ MN, NΞ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ² αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπʾ αὐτῶν ἑητόν.

Ή ΜΞ ἄρα ἑητὸν καὶ μέσον δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ ἔδει δεῖξαι.

For let the area AC be contained by the rational (straight-line) AB and the fifth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term. [So] I say that the square-root of area AC is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).



For let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of area AC. So, we must show that MO is the square-root of a rational plus a medial (area).

For since AG is incommensurable (in length) with GE [Prop. 10.18], AH is thus also incommensurable with HE—that is to say, the (square) on MN with the (square) on NO [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AD is a fifth binomial (straight-line), and ED [is] its lesser segment, ED (is) thus commensurable in length with AB[Def. 10.9]. But, AE is incommensurable (in length) with ED. Thus, AB is also incommensurable in length with AE [BA and AE are rational (straight-lines which are) commensurable in square only] [Prop. 10.13]. Thus, AK—that is to say, the sum of the (squares) on MNand NO-is medial [Prop. 10.21]. And since DE is commensurable in length with AB—that is to say, with EK—but, DE is commensurable (in length) with EF, EF is thus also commensurable (in length) with EK[Prop. 10.12]. And EK (is) rational. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—(is) also rational [Prop. 10.19]. MN and NO are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

Thus, MO is the square-root of a rational plus a medial (area) [Prop. 10.40]. And (it is) the square-root of area AC. (Which is) the very thing it was required to show.

 $\rho \sqrt{[(1+k^{\prime\prime\,2})^{1/2}+k^{\prime\prime}]/[2\,(1+k^{\prime\prime\,2})]} + \rho \sqrt{[(1+k^{\prime\prime\,2})^{1/2}-k^{\prime\prime}]/[2\,(1+k^{\prime\prime\,2})]}, \text{ where } \rho = \sqrt{k\,(1+k^{\prime\prime\,2})} \text{ and } k^{\prime\prime\,2} = k^{\prime}. \text{ This is the length of } k^{\prime\prime\,2} = k^{\prime}.$

[†] If the rational straight-line has unit length then this proposition states that the square-root of a fifth binomial straight-line is the square root of a rational plus a medial area: *i.e.*, a fifth binomial straight-line has a length $k(\sqrt{1+k'}+1)$ whose square-root can be written

the square root of a rational plus a medial area (see Prop. 10.40), since ρ is rational.

νθ'.

Έὰν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη δύο μέσα δυναμένη.



Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης τῆς ΑΔ διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

Κατεσκευάσθω [γάρ] τὰ αὐτὰ τοῖς προδεδειγμένοις. φανερὸν δή, ὅτι [ή] τὸ ΑΓ δυναμένη ἐστὶν ἡ ΜΞ, καὶ ότι ἀσύμμετρός ἐστιν ἡ ΜΝ τῆ ΝΞ δυνάμει. καὶ ἐπεὶ άσύμμετρός ἐστιν ἡ ΕΑ τῆ ΑΒ μήχει, αἱ ΕΑ, ΑΒ ἄρα ρηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ AK, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. πάλιν, έπει ἀσύμμετρός ἐστιν ἡ ΕΔ τῆ ΑΒ μήχει, ἀσύμμετρος ἄρα έστι και ή ΖΕ τῆ ΕΚ· αί ΖΕ, ΕΚ ἄρα þηταί είσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΕΛ, τουτέστι τὸ MP, τουτέστι τὸ ὑπὸ τῶν ΜΝΞ. καὶ ἐπεὶ ἀσύμμετρος ἡ ΑΕ τῆ ΕΖ, καὶ τὸ ΑΚ τῷ ΕΛ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν ΑΚ ἐστι τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, τὸ δὲ ΕΛ ἐστι τὸ ὑπὸ τῶν ΜΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝΞ τῷ ὑπὸ τῶν ΜΝΞ. καί έστι μέσον έκάτερον αὐτῶν, καὶ αἱ ΜΝ, ΝΞ δυνάμει εἰσὶν ἀσύμμετροι.

Ή ΜΞ ἄρα δύο μέσα δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

Proposition 59

If an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of (the sum of) two medial (areas).[†]



For let the area ABCD be contained by the rational (straight-line) AB and the sixth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term. So, I say that the square-root of AC is the square-root of (the sum of) two medial (areas).

[For] let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of AC, and that MN is incommensurable in square with NO. And since EA is incommensurable in length with AB [Def. 10.10], EA and AB are thus rational (straightlines which are) commensurable in square only. Thus, AK—that is to say, the sum of the (squares) on MNand NO-is medial [Prop. 10.21]. Again, since ED is incommensurable in length with AB [Def. 10.10], FE is thus also incommensurable (in length) with EK[Prop. 10.13]. Thus, FE and EK are rational (straightlines which are) commensurable in square only. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—is medial [Prop. 10.21]. And since AE is incommensurable (in length) with EF, AK is also incommensurable with EL [Props. 6.1, 10.11]. But, AKis the sum of the (squares) on MN and NO, and EL is the (rectangle contained) by MNO. Thus, the sum of the (squares) on MNO is incommensurable with the (rectangle contained) by MNO. And each of them is medial. And *MN* and *NO* are incommensurable in square.

Thus, MO is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. And (it is) the square-root of AC. (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then this proposition states that the square-root of a sixth binomial straight-line is the square root of the sum of two medial areas: *i.e.*, a sixth binomial straight-line has a length $\sqrt{k} + \sqrt{k'}$ whose square-root can be written

 $k^{1/4}\left(\sqrt{[1+k''/(1+k''^2)^{1/2}]/2} + \sqrt{[1-k''/(1+k''^2)^{1/2}]/2}\right), \text{ where } k''^2 = (k-k')/k'. \text{ This is the length of the square-root of the sum of } k''^2 = (k-k')/k'.$

two medial areas (see Prop. 10.41).

Λῆμμα.

Έαν εύθεῖα γραμμὴ τμηθῆ εἰς ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τετράγωνα μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ἀνίσων περιεχομένου ὀρθογωνίου.



Έστω εὐθεῖα ἡ AB καὶ τετμήσθω εἰς ἄνισα κατὰ τὸ Γ, καὶ ἔστω μείζων ἡ AΓ· λέγω, ὅτι τὰ ἀπὸ τῶν AΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν AΓ, ΓΒ.

Τετμήσθω γὰρ ή AB δίχα κατὰ τὸ Δ. ἐπεὶ οὖν εὐθεĩα γραμμὴ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Δ, εἰς δὲ ἄνισα κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν AΓ, ΓΒ μετὰ τοῦ ἀπὸ ΓΔ ἴσον ἐστὶ τῷ ἀπὸ AΔ· ὥστε τὸ ὑπὸ τῶν AΓ, ΓΒ ἔλαττον ἐστι τοῦ ἀπὸ AΔ· τὸ ἄρα δἰς ὑπὸ τῶν AΓ, ΓΒ ἔλαττον ἢ διπλάσιόν ἐστι τοῦ ἀπὸ AΔ. ἀλλὰ τὰ ἀπὸ τῶν AΓ, ΓΒ διπλάσιά [ἐστι] τῶν ἀπὸ τῶν AΔ, ΔΓ· τὰ ἄρα ἀπὸ τῶν AΓ, ΓΒ μείζονά ἐστι τοῦ δἰς ὑπὸ τῶν AΓ, ΓΒ· ὅπερ ἔδει δεῖξαι.

Lemma

If a straight-line is cut unequally then (the sum of) the squares on the unequal (parts) is greater than twice the rectangle contained by the unequal (parts).

	1		
Å	$\dot{\mathrm{D}}$	Ċ	B
4 5 1			

Let AB be a straight-line, and let it have been cut unequally at C, and let AC be greater (than CB). I say that (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB.

For let AB have been cut in half at D. Therefore, since a straight-line has been cut into equal (parts) at D, and into unequal (parts) at C, the (rectangle contained) by AC and CB, plus the (square) on CD, is thus equal to the (square) on AD [Prop. 2.5]. Hence, the (rectangle contained) by AC and CB is less than the (square) on AD. Thus, twice the (rectangle contained) by AC and CB is less than the (square) on AD. Thus, twice the (rectangle contained) by AC and CB is less than double the (square) on AD. But, (the sum of) the (squares) on AC and CB [is] double (the sum of) the (squares) on AD and DC [Prop. 2.9]. Thus, (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB. (Which is) the very thing it was required to show.

Proposition 60

The square on a binomial (straight-line) applied to a rational (straight-line) produces as breadth a first binomial (straight-line).^{\dagger}



Let AB be a binomial (straight-line), having been divided into its (component) terms at C, such that AC is the greater term. And let the rational (straight-line) DE be laid down. And let the (rectangle) DEFG, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a first binomial (straight-line).

For let DH, equal to the (square) on AC, and KL, equal to the (square) on BC, have been applied to DE.

ξ'.

Τὸ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην.



Έστω ἐκ δύο ὀνομάτων ἡ AB διηρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ AΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΕΖΗ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ πρώτη.

Παραβεβλήσθω γὰρ παρὰ τὴν ΔΕ τῷ μὲν ἀπὸ τῆς ΑΓ ἴσον τὸ ΔΘ, τῷ δὲ ἀπὸ τῆς ΒΓ ἴσον τὸ ΚΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον ἐστὶ τῷ ΜΖ. τετμήσθω ἡ ΜΗ δίχα κατὰ τὸ Ν, καὶ παράλληλος ἤχθω ἡ ΝΞ [ἑκατέρα

τῶν MA, HZ]. ἑκάτερον ἄρα τῶν ΜΞ, NZ ἴσον ἐστὶ τῷ άπαξ ὑπὸ τῶν ΑΓΒ. καὶ ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ διηρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΒ ἑητά έστι καὶ σύμμετρα ἀλλήλοις. ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ. καί ἐστιν ἴσον τῷ $\Delta \Lambda$ · ἑητὸν ἄρα έστι τὸ ΔΛ. και παρὰ ῥητὴν τὴν ΔΕ παράχειται· ῥητὴ ἄρα ἐστὶν ἡ ΔM καὶ σύμμετρος τῆ ΔE μήκει. πάλιν, ἐπεὶ αἱ $A\Gamma$, ΓΒ φηταί εἰσι δυνάμει μόνον σύμμετροι, μέσον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΜΖ. καὶ παρὰ ῥητὴν τὴν ΜΛ παράκειται· ῥητὴ ἄρα καὶ ἡ ΜΗ καὶ ἀσύμμετρος τῆ ΜΛ, τουτέστι τῆ ΔΕ, μήχει. ἔστι δὲ χαὶ ἡ ΜΔ ῥητὴ καὶ τ
ῆ ΔE μήκει σύμμετρος· ἀσύμμετρος ἄρα ἐστὶν
ἡ ΔM τῆ MH μήχει.
 χαί εἰσι ῥηταί αὶ $\Delta M,~MH$ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστίν ἡ ΔΗ. δεικτέον δή, ὄτι καὶ πρώτη.

Έπεὶ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν ἐστι τὸ ύπὸ τῶν ΑΓΒ, καὶ τῶν ΔΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΜΞ. ἔστιν ἄρα ὡς τὸ $\Delta\Theta$ πρὸς τὸ ΜΞ, οὕτως τὸ ΜΞ πρὸς τὸ $K\Lambda$, τουτέστιν ὡς ἡ ΔK πρὸς τὴν MN, ἡ MN πρὸς τὴν ΜΚ· τὸ ἄρα ὑπὸ τῶν ΔΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, σύμμετρόν ἐστι καὶ τὸ $\Delta \Theta$ τῷ ΚΛ· ὥστε καὶ ἡ ΔK τῆ ΚΜ σύμμετρός ἐστιν. καὶ ἐπεὶ μείζονά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, μεῖζον ἄρα καὶ τὸ ΔΛ τοῦ ΜΖ· ὥστε καὶ ἡ ΔΜ τῆς ΜΗ μείζων ἐστίν. καί ἐστιν ἴσον τὸ ὑπὸ τῶν ΔΚ, ΚΜ τῷ ἀπὸ τῆς MN, τουτέστι τῷ τετάρτω τοῦ ἀπὸ τῆς ΜΗ, καὶ σύμμετρος ἡ ΔΚ τῆ ΚΜ. έὰν δὲ ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτω μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῃ ἐλλεῖπον είδει τετραγώνω καὶ εἰς σύμμετρα αὐτὴν διαιρῆ, ἡ μείζων τῆς ἐλάσσονος μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καί εἰσι ῥηταὶ αἱ ΔΜ, MH, καὶ ἡ ΔΜ μεῖζον ὄνομα οὖσα σύμμετρός έστι τῆ ἐκκειμένῃ ῥητῃ τῆ ΔΕ μήκει.

Ή ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

Thus, the remaining twice the (rectangle contained) by AC and CB is equal to MF [Prop. 2.4]. Let MG have been cut in half at N, and let NO have been drawn parallel [to each of ML and GF]. MO and NF are thus each equal to once the (rectangle contained) by ACB. And since AB is a binomial (straight-line), having been divided into its (component) terms at C, AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Thus, the (squares) on ACand CB are rational, and commensurable with one another. And hence the sum of the (squares) on AC and CB (is rational) [Prop. 10.15], and is equal to DL. Thus, DL is rational. And it is applied to the rational (straightline) DE. DM is thus rational, and commensurable in length with DE [Prop. 10.20]. Again, since AC and CBare rational (straight-lines which are) commensurable in square only, twice the (rectangle contained) by AC and CB—that is to say, MF—is thus medial [Prop. 10.21]. And it is applied to the rational (straight-line) ML. MG is thus also rational, and incommensurable in length with ML—that is to say, with DE [Prop. 10.22]. And *MD* is also rational, and commensurable in length with DE. Thus, DM is incommensurable in length with MG[Prop. 10.13]. And they are rational. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a first (binomial straight-line).

Since the (rectangle contained) by ACB is the mean proportional to the squares on AC and CB[Prop. 10.53 lem.], MO is thus also the mean proportional to DH and KL. Thus, as DH is to MO, so MO (is) to KL—that is to say, as DK (is) to MN, (so) MN (is) to MK [Prop. 6.1]. Thus, the (rectangle contained) by DK and KM is equal to the (square) on MN [Prop. 6.17]. And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL. Hence, DK is also commensurable with KM [Props. 6.1, 10.11]. And since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59 lem.], DL (is) thus also greater than MF. Hence, DM is also greater than MG [Props. 6.1, 5.14]. And the (rectangle contained) by DK and KM is equal to the (square) on MN—that is to say, to one quarter the (square) on MG. And DK (is) commensurable (in length) with KM. And if there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable (in length), then the square on the greater is larger

than (the square on) the lesser by the (square) on (some straight-line) commensurable (in length) with the greater [Prop. 10.17]. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM). And DM and MG are rational. And DM, which is the greater term, is commensurable in length with the (previously) laid down rational (straight-line) DE.

Thus, DG is a first binomial (straight-line) [Def. 10.5]. (Which is) the very thing it was required to show.

 † In other words, the square of a binomial is a first binomial. See Prop. 10.54.

ξα΄.

Τὸ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν.



Έστω ἐκ δύο μέσων πρώτη ἡ AB διηρημένη εἰς τὰς μέσας κατὰ τὸ Γ, ῶν μείζων ἡ AΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ παραβεβλήσθω παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς AB ἴσον παραλληλόγραμμον τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

Κατεσχευάσθω γὰρ τὰ αὐτὰ τοῖς πρὸ τούτου. καὶ ἐπεὶ ἡ AB ἐκ δύο μέσων ἐστὶ πρώτη διῃρημένη κατὰ τὸ Γ, αἱ AΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ἑητὸν περιέχουσαι· ὥστε καὶ τὰ ἀπὸ τῶν AΓ, ΓΒ μέσα ἐστίν. μέσον ἄρα ἐστὶ τὸ ΔΛ. καὶ παρὰ ἑητὴν τὴν ΔΕ παραβέβληται· ἑητὴ ἄρα ἐστίν ἡ ΜΔ καὶ ἀσύμμετρος τῆ ΔΕ μήκει. πάλιν, ἐπεὶ ἑητόν ἐστι τὸ δἰς ὑπὸ τῶν AΓ, ΓΒ, ἑητόν ἐστι καὶ τὸ MZ. καὶ παρὰ ἑητὴν τὴν ΜΛ παράκειται· ἑητὴ ἄρα [ἐστὶ] καὶ ἡ MH καὶ μήκει σύμμετρος τῆ MΛ, τουτέστι τῆ ΔΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΜ τῆ MH μήκει. καί εἰσι ἑηταί· αἱ ΔΜ, MH ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον δή, ὅτι καὶ δευτέρα.

Έπεὶ γὰρ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ, μεῖζον ἄρα καὶ τὸ ΔΛ τοῦ ΜΖ· ὥστε καὶ ἡ ΔΜ τῆς ΜΗ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, σύμμετρόν ἐστι καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε καὶ ἡ ΔΚ τῆ ΚΜ σύμμετρός ἐστιν. καί ἐστι τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ

Proposition 61

The square on a first bimedial (straight-line) applied to a rational (straight-line) produces as breadth a second binomial (straight-line).[†]



Let AB be a first bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, of which AC (is) the greater. And let the rational (straight-line) DE be laid down. And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a second binomial (straight-line).

For let the same construction have been made as in the (proposition) before this. And since AB is a first bimedial (straight-line), having been divided at C, AC and CB are thus medial (straight-lines) commensurable in square only, and containing a rational (area) [Prop. 10.37]. Hence, the (squares) on AC and CBare also medial [Prop. 10.21]. Thus, DL is medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) DE. MD is thus rational, and incommensurable in length with *DE* [Prop. 10.22]. Again, since twice the (rectangle contained) by AC and CB is rational, MF is also rational. And it is applied to the rational (straight-line) ML. Thus, MG [is] also rational, and commensurable in length with ML—that is to say, with DE [Prop. 10.20]. DM is thus incommensurable in length with MG [Prop. 10.13]. And they are rational. DM and MG are thus rational, and commensuάπὸ συμμέτρου ἑαυτῆ. καί ἐστιν ἡ ΜΗ σύμμετρος τῆ ΔΕ μήκει.

Η ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

[†]In other words, the square of a first bimedial is a second binomial. See Prop. 10.55.

ξβ΄.

Τὸ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην.



Έστω ἐχ δύο μέσων δευτέρα ή AB διηρημένη εἰς τὰς μέσας χατὰ τὸ Γ, ὥστε τὸ μεῖζον τμῆμα εἶναι τὸ AΓ, ῥητὴ δέ τις ἔστω ή ΔE , χαὶ παρὰ τὴν ΔE τῷ ἀπὸ τῆς AB ἴσον παραλληλόγραμμον παραβεβλήσθω τὸ ΔZ πλάτος ποιοῦν τὴν ΔH · λέγω, ὅτι ή ΔH ἐχ δύο ὀνομάτων ἐστὶ τρίτη.

Κατεσχευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ ἐκ δύο μέσων δευτέρα ἐστὶν ἡ AB διηρημένη κατὰ τὸ Γ, αἱ AΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AΓ, ΓΒ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΔΛ· μέσον ἄρα καὶ τὸ ΔΛ. καὶ παράκειται παρὰ ῥητὴν τὴν ΔΕ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΜΔ καὶ ἀσύμμετρος τῆ ΔΕ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ MH ἑητή ἐστι καὶ ἀσύμμετρος τῆ MΛ, τουτέστι τῆ ΔΕ, μήκει· ἑητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΜ, MH καὶ ἀσύμμετρος τῆ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΓ τῆ ΓΒ μήκει, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ

Proposition 62

Thus, DG is a second binomial (straight-line) [Def.

rable in square only. DG is thus a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second

For since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59], DL (is) thus also greater than MF. Hence, DM (is) also (greater) than MG [Prop. 6.1]. And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL. Hence, DK is also commensurable (in length) with KM [Props. 6.1, 10.11]. And the (rectangle contained) by DKM is equal to the (square) on MN. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17]. And MG is commen-

(binomial straight-line).

surable in length with DE.

10.6].

The square on a second bimedial (straight-line) applied to a rational (straight-line) produces as breadth a third binomial (straight-line).^{\dagger}



Let AB be a second bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, such that AC is the greater segment. And let DE be some rational (straight-line). And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a third binomial (straight-line).

Let the same construction be made as that shown previously. And since AB is a second bimedial (straightline), having been divided at C, AC and CB are thus medial (straight-lines) commensurable in square only, and containing a medial (area) [Prop. 10.38]. Hence, the sum of the (squares) on AC and CB is also medial [Props. 10.15, 10.23 corr.]. And it is equal to DL. Thus, DL (is) also medial. And it is applied to the rational (straight-line) DE. MD is thus also rational, and inύπὸ τῶν ΑΓΒ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓΒ. ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓΒ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΔΛ τῷ ΜΖ· ὥστε καὶ ἡ ΔΜ τῷ ΜΗ ἀσύμμετρός ἐστιν. καί εἰσι ἑηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον [δή], ὅτι καὶ τρίτη.

Όμοίως δὴ τοῖς προτέροις ἐπιλογιούμεθα, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ, καὶ σύμμετρος ἡ ΔΚ τῆ ΚΜ. καί ἐστι τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῆ ΔΕ μήχει.

Ή ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

commensurable in length with DE [Prop. 10.22]. So, for the same (reasons), MG is also rational, and incommensurable in length with ML—that is to say, with DE. Thus, DM and MG are each rational, and incommensurable in length with DE. And since AC is incommensurable in length with CB, and as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by ACB [Prop. 10.21 lem.], the (square) on AC (is) also incommensurable with the (rectangle contained) by ACB[Prop. 10.11]. And hence the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by ACB—that is to say, DL with MF[Props. 10.12, 10.13]. Hence, DM is also incommensurable (in length) with MG [Props. 6.1, 10.11]. And they are rational. DG is thus a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a third (binomial straight-line).

So, similarly to the previous (propositions), we can conclude that DM is greater than MG, and DK (is) commensurable (in length) with KM. And the (rectangle contained) by DKM is equal to the (square) on MN. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17]. And neither of DM and MG is commensurable in length with DE.

Thus, DG is a third binomial (straight-line) [Def. 10.7]. (Which is) the very thing it was required to show.

 † In other words, the square of a second bimedial is a third binomial. See Prop. 10.56.



Τὸ ἀπὸ τῆς μείζονος παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐχ δύο ὀνομάτων τετάρτην.



Έστω μείζων ή AB διηρημένη κατὰ τὸ Γ, ὥστε μείζονα είναι τὴν AΓ τῆς ΓB, ἑητὴ δὲ ή ΔE , καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΔE παραβεβλήσθω τὸ ΔZ παραλληλόγραμμον πλάτος ποιοῦν τὴν ΔH · λέγω, ὅτι ή ΔH ἐκ δύο ὀνομάτων ἐστὶ τετάρτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ μείζων ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ δυνάμει

Proposition 63

The square on a major (straight-line) applied to a rational (straight-line) produces as breadth a fourth binomial (straight-line).[†]



Let AB be a major (straight-line) having been divided at C, such that AC is greater than CB, and (let) DE(be) a rational (straight-line). And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a fourth binomial (straight-line).

Let the same construction be made as that shown pre-

εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγχείμενον ἐχ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ' αὐτῶν μέσον. ἐπεὶ οῦν ῥητόν ἐστι τὸ συγχείμενον ἐχ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, ῥητὸν ἄρα ἐστὶ τὸ ΔΛ· ῥητὴ ἄρα χαὶ ἡ ΔΜ καὶ σύμμετρος τῆ ΔΕ μήχει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ MZ, καὶ παρὰ ῥητήν ἐστι τὴν MΛ, ῥητὴ ἄρα ἐστὶ χαὶ ἡ MH καὶ ἀσύμμετρος τῆ ΔΕ μήχει· ἀσύμμετρος ἄρα ἐστὶ χαὶ ἡ ΔΜ τῆ MH μήχει. αἱ ΔΜ, MH ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐχ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δειχτέον [δή], ὅτι χαὶ τετάρτη.

Όμοίως δὴ δείξομεν τοῖς πρότερον, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ, καὶ ὅτι τὸ ὑπὸ ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. ἐπεὶ οὕν ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΔΘ τῷ ΚΛ· ὅστε ἀσύμμετρος καὶ ἡ ΔΚ τῆ ΚΜ ἐστιν. ἐὰν δὲ ὅσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παραλληλόγραμμον παρὰ τὴν μείζονα παραβληθῆ ἐλλεῖπον είδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρῆ, ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ ἀσύμμέτρου ἑαυτῆ μήχει· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί εἰσιν αἱ ΔΜ, ΜΗ ἑηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΜ σύμμετρός ἐστι τῆ ἐχχειμένη ἑητῆ τῆ ΔΕ.

Ή ΔH ἄρα ἐ
κ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἕδει δε
ῖζαι.

[†] In other words, the square of a major is a fourth binomial. See Prop. 10.57.

ξδ΄.

Τὸ ἀπὸ τῆς ἑητὸν καὶ μέσον δυναμένης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην.

Έστω ἡητὸν καὶ μέσον δυναμένη ἡ AB διηρημένη εἰς τὰς εὐθείας κατὰ τὸ Γ, ὥστε μείζονα εἶναι τὴν AΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη. viously. And since AB is a major (straight-line), having been divided at C, AC and CB are incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. Therefore, since the sum of the (squares) on AC and CB is rational, DL is thus rational. Thus, DM (is) also rational, and commensurable in length with DE [Prop. 10.20]. Again, since twice the (rectangle contained) by AC and CB—that is to say, MF—is medial, and is (applied to) the rational (straight-line) ML, MG is thus also rational, and incommensurable in length with DE [Prop. 10.22]. DM is thus also incommensurable in length with MG [Prop. 10.13]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a fourth (binomial straight-line).

So, similarly to the previous (propositions), we can show that DM is greater than MG, and that the (rectangle contained) by DKM is equal to the (square) on MN. Therefore, since the (square) on AC is incommensurable with the (square) on CB, DH is also incommensurable with KL. Hence, DK is also incommensurable with *KM* [Props. 6.1, 10.11]. And if there are two unequal straight-lines, and a parallelogram equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable (in length), then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) incommensurable in length with the greater [Prop. 10.18]. Thus, the square on DM is greater than (the square on) MGby the (square) on (some straight-line) incommensurable (in length) with (DM). And DM and MG are rational (straight-lines which are) commensurable in square only. And DM is commensurable (in length) with the (previously) laid down rational (straight-line) DE.

Thus, DG is a fourth binomial (straight-line) [Def. 10.8]. (Which is) the very thing it was required to show.

Proposition 64

The square on the square-root of a rational plus a medial (area) applied to a rational (straight-line) produces as breadth a fifth binomial (straight-line).[†]

Let AB be the square-root of a rational plus a medial (area) having been divided into its (component) straightlines at C, such that AC is greater. And let the rational (straight-line) DE be laid down. And let the (parallelogram) DF, equal to the (square) on AB, have been ap-



Κατεσχευάσθω τὰ αὐτα τοῖς πρὸ τούτου. ἐπεὶ οῦν ἑητὸν καὶ μέσον δυναμένη ἐστὶν ἡ ΑΒ διηρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ἑητόν. ἐπεὶ οῦν μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα ἐστὶ τὸ ΔΛ· ὥστε ἑητή ἐστιν ἡ ΔΜ καὶ μήκει ἀσύμμετρος τῆ ΔΕ. πάλιν, ἐπεὶ ἑητόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓΒ, τουτέστι τὸ MZ, ἑητὴ ἄρα ἡ MH καὶ σύμμετρος τῆ ΔΕ. ἀσύμμετρος ἄρα ἡ ΔΜ τῆ MH· αἱ ΔΜ, MH ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. λέγω δή, ὅτι καὶ πέμπτη.

Ομοίως γὰρ διεχθήσεται, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς MN, καὶ ἀσύμμετρος ἡ ΔΚ τῆ KM μήκει· ἡ ΔΜ ἄρα τῆς MH μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί εἰσιν αἱ ΔM, MH [ἑηταὶ] δυνάμει μόνον σύμμετροι, καὶ ἡ ἐλάσσων ἡ MH σύμμετρος τῆ ΔΕ μήκει.

Ή ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

plied to DE, producing DG as breadth. I say that DG is a fifth binomial straight-line.



Let the same construction be made as in the (propositions) before this. Therefore, since AB is the square-root of a rational plus a medial (area), having been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. Therefore, since the sum of the (squares) on AC and CB is medial, DL is thus medial. Hence, DM is rational and incommensurable in length with DE [Prop. 10.22]. Again, since twice the (rectangle contained) by ACB that is to say, MF—is rational, MG (is) thus rational and commensurable (in length) with DE [Prop. 10.20]. DM (is) thus incommensurable (in length) with MG[Prop. 10.13]. Thus, DM and MG are rational (straightlines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For, similarly (to the previous propositions), it can be shown that the (rectangle contained) by DKM is equal to the (square) on MN, and DK (is) incommensurable in length with KM. Thus, the square on DMis greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM) [Prop. 10.18]. And DM and MG are [rational] (straight-lines which are) commensurable in square only, and the lesser MG is commensurable in length with DE.

Thus, DG is a fifth binomial (straight-line) [Def. 10.9]. (Which is) the very thing it was required to show.

[†] In other words, the square of the square-root of a rational plus medial is a fifth binomial. See Prop. 10.58.

ξε΄.

Τὸ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην.

Έστω δύο μέσα δυναμένη ή AB διηρημένη κατὰ τὸ Γ, ἑητὴ δὲ ἔστω ή ΔΕ, καὶ παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς AB ἴσον παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη.

Proposition 65

The square on the square-root of (the sum of) two medial (areas) applied to a rational (straight-line) produces as breadth a sixth binomial (straight-line).[†]

Let AB be the square-root of (the sum of) two medial (areas), having been divided at C. And let DE be a rational (straight-line). And let the (parallelogram) DF, equal to the (square) on AB, have been applied to DE,



Κατεσχευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἡ AB δύο μέσα δυναμένη ἐστὶ διῃρημένη κατὰ τὸ Γ, αἱ AΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων συγκείμενον τῷ ὑπ᾽ αὐτῶν· ὥστε κατὰ τὰ προδεδειγμένα μέσον ἐστὶν ἑκάτερον τῶν ΔΛ, MZ. καὶ παρὰ ἑῃτὴν τὴν ΔΕ παράκειται· ἑῃτὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΜ, MH καὶ ἀσύμμετρος τῷ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρον ἄρα ἐστὶ τὸ ΔΛ τῷ MZ. ἀσύμμετρος ἄρα καὶ ἡ ΔΜ τῷ MH· αἱ ΔΜ, MH ἄρα ἑῃτὶν ἑωτἰρι Δύνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἑστὶν ἡ ΔΗ. λέγω δή, ὅτι καὶ ἔκτη.

Όμοίως δὴ πάλιν δεῖξομεν, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς MN, καὶ ὅτι ἡ ΔΚ τῆ ΚΜ μήκει ἐστὶν ἀσύμμετρος· καὶ διὰ τὰ αὐτὰ δὴ ἡ ΔΜ τῆς MH μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήκει. καὶ οὐδετέρα τῶν ΔΜ, MH σύμμετρός ἐστι τῆ ἐκκειμένῃ ἑητῆ τῆ ΔΕ μήκει.

Ή ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἕκτη· ὅπερ ἔδει δεῖξαι.

producing DG as breadth. I say that DG is a sixth binomial (straight-line).



For let the same construction be made as in the previous (propositions). And since AB is the square-root of (the sum of) two medial (areas), having been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.41]. Hence, according to what has been previously demonstrated, DL and MF are each medial. And they are applied to the rational (straight-line) DE. Thus, DM and MG are each rational, and incommensurable in length with DE[Prop. 10.22]. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, DL is thus incommensurable with MF. Thus, DM (is) also incommensurable (in length) with MG [Props. 6.1, 10.11]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a sixth (binomial straight-line).

So, similarly (to the previous propositions), we can again show that the (rectangle contained) by DKM is equal to the (square) on MN, and that DK is incommensurable in length with KM. And so, for the same (reasons), the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable in length with (DM) [Prop. 10.18]. And neither of DM and MG is commensurable in length with the (previously) laid down rational (straight-line) DE.

Thus, DG is a sixth binomial (straight-line) [Def. 10.10]. (Which is) the very thing it was required to show.

 † In other words, the square of the square-root of two medials is a sixth binomial. See Prop. 10.59.

ξς΄.

Ή τῆ ἐκ δύο ὀνομάτων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῆ τάζει ἡ αὐτή.

Έστω ἐκ δύο ὀνομάτων ἡ AB, καὶ τῇ AB μήκει in order.

Proposition 66

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order. σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῆ τάξει ἡ αὐτὴ τῆ ΑΒ.



Έπει γὰρ ἐx δύο ὀνομάτων ἐστιν ἡ AB, διηρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω μεῖζον ὄνομα τὸ AE· αἱ AE, EB ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. γεγονέτω ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ AE πρὸς τὴν ΓΖ· καὶ λοιπὴ ἄρα ἡ EB πρὸς λοιπὴν τὴν ΖΔ ἐστιν, ὡς ἡ AB πρὸς τὴν ΓΔ. σύμμετρος δὲ ἡ AB τῆ ΓΔ μήκει· σύμμετρος ἄρα ἐστὶ καὶ ἡ μὲν AE τῆ ΓΖ, ἡ δὲ EB τῆ ΖΔ. καί εἰσι ἑηταὶ αἱ AE, EB· ἑηταὶ ἄρα εἰσὶ καὶ αἱ ΓΖ, ΖΔ. καὶ ἐστιν ὡς ἡ AE πρὸς ΓΖ, ἡ EB πρὸς ΖΔ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ AE πρὸς EB, ἡ ΓΖ πρὸς ΖΔ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ AE πρὸς EB, ἡ ΓΖ πρὸς ΖΔ. αἱ δὲ AE, EB δυνάμει μόνον [εἰσι] σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καὶ εἰσι ἑηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΓΔ. λέγω δή, ὅτι τῆ τάξει ἑστὶν ἡ αὐτὴ τῆ AB.

Ή γὰρ ΑΕ τῆς ΕΒ μεῖζον δύναται ἤτοι τῷ ἀπὸ συμμέτρου έαυτῆ ἢ τῷ ἀπὸ ἀσυμμέτρου. εἰ μὲν οὖν ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ ή ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῆ ἐκκειμένῃ ῥητῆ, καὶ ή ΓΖ σύμμετρος αὐτῆ ἔσται, καὶ διὰ τοῦτο ἑκατέρα τῶν ΑΒ, ΓΔ ἐκ δύο ὀνομάτων ἐστὶ πρώτη, τουτέστι τῆ τάξει ή αὐτή. εἰ δὲ ή ΕΒ σύμμετρός ἐστι τῆ ἐκκειμένῃ ῥητῆ, καὶ ἡ ΖΔ σύμμετρός ἐστιν αὐτῆ, καὶ διὰ τοῦτο πάλιν τῆ τάξει ή αὐτὴ ἔσται τῆ ΑΒ· ἑκατέρα γὰρ αὐτῶν ἔσται ἐκ δύο όνομάτων δευτέρα. εί δὲ οὐδετέρα τῶν ΑΕ, ΕΒ σύμμετρός έστι τῆ ἐχχειμένῃ ῥητῆ, οὐδετέρα τῶν ΓΖ, ΖΔ σύμμετρος αὐτῆ ἔσται, καί ἐστιν ἑκατέρα τρίτη. εἰ δὲ ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ ἡ ΓZ τὴς $Z\Delta$ μεϊζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καὶ εἰ μὲν ἡ ΑΕ σύμμετρός ἐστι τῆ ἐκκειμένῃ ῥητῃ, καὶ ἡ ΓΖ σύμμετρός έστιν αὐτῆ, καὶ ἐστιν ἑκατέρα τετάρτη. εἰ δὲ ἡ ΕΒ, καὶ ή ΖΔ, καὶ ἔσται ἑκατέρα πέμπτη. εἰ δὲ οὐδετέρα τῶν ΑΕ, EB, καὶ τῶν ΓΖ, ΖΔ οὐδετέρα σύμμετρός ἐστι τ
 ἢ ἐκκειμένῃ ἡητῆ, καὶ ἔσται ἑκατέρα ἕκτη.

Ωστε ή τῆ ἐκ δύο ὀνομάτων μήκει σύμμετρος ἐκ δύο ὀνομάτων ἐστὶ καὶ τῆ τάξει ἡ αὐτή ὅπερ ἔδει δεῖξαι. Let AB be a binomial (straight-line), and let CD be commensurable in length with AB. I say that CD is a binomial (straight-line), and (is) the same in order as AB.



For since AB is a binomial (straight-line), let it have been divided into its (component) terms at E, and let AE be the greater term. AE and EB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Let it have been contrived that as AB (is) to CD, so AE (is) to CF [Prop. 6.12]. Thus, the remainder EB is also to the remainder FD, as AB (is) to CD[Props. 6.16, 5.19 corr.]. And AB (is) commensurable in length with CD. Thus, AE is also commensurable (in length) with CF, and EB with FD [Prop. 10.11]. And AE and EB are rational. Thus, CF and FD are also rational. And as AE is to CF, (so) EB (is) to FD[Prop. 5.11]. Thus, alternately, as AE is to EB, (so) CF (is) to FD [Prop. 5.16]. And AE and EB [are] commensurable in square only. Thus, CF and FD are also commensurable in square only [Prop. 10.11]. And they are rational. *CD* is thus a binomial (straight-line) [Prop. 10.36]. So, I say that it is the same in order as AB.

For the square on AE is greater than (the square on) EB by the (square) on (some straight-line) either commensurable or incommensurable (in length) with (AE). Therefore, if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with (some previously) laid down rational (straight-line) then CF will also be commensurable (in length) with it [Prop. 10.12]. And, on account of this, AB and CD are each first binomial (straightlines) [Def. 10.5]—that is to say, the same in order. And if *EB* is commensurable (in length) with the (previously) laid down rational (straight-line) then FD is also commensurable (in length) with it [Prop. 10.12], and, again, on account of this, (CD) will be the same in order as AB. For each of them will be second binomial (straightlines) [Def. 10.6]. And if neither of AE and EB is commensurable (in length) with the (previously) laid down rational (straight-line) then neither of CF and FD will be commensurable (in length) with it [Prop. 10.13], and each (of AB and CD) is a third (binomial straight-line)

[Def. 10.7]. And if the square on AE is greater than (the square on) EB by the (square) on (some straightline) incommensurable (in length) with (AE) then the square on CF is also greater than (the square on) FDby the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with the (previously) laid down rational (straight-line) then CF is also commensurable (in length) with it [Prop. 10.12], and each (of AB and CD) is a fourth (binomial straight-line) [Def. 10.8]. And if EB (is commensurable in length with the previously laid down rational straight-line) then FD (is) also (commensurable in length with it), and each (of AB and CD) will be a fifth (binomial straight-line) [Def. 10.9]. And if neither of AE and EB (is commensurable in length with the previously laid down rational straight-line) then also neither of CF and FD is commensurable (in length) with the laid down rational (straight-line), and each (of AB and CD) will be a sixth (binomial straight-line) [Def. 10.10].

Hence, a (straight-line) commensurable in length with a binomial (straight-line) is a binomial (straightline), and the same in order. (Which is) the very thing it was required to show.

Proposition 67

A (straight-line) commensurable in length with a bimedial (straight-line) is itself also bimedial, and the same in order.



Let AB be a bimedial (straight-line), and let CD be commensurable in length with AB. I say that CD is bimedial, and the same in order as AB.

For since AB is a bimedial (straight-line), let it have been divided into its (component) medial (straight-lines) at E. Thus, AE and EB are medial (straight-lines) which are) commensurable in square only [Props. 10.37, 10.38]. And let it have been contrived that as AB (is) to CD, (so) AE (is) to CF [Prop. 6.12]. And thus as the remainder EB is to the remainder FD, so AB (is) to CD[Props. 5.19 corr., 6.16]. And AB (is) commensurable in length with CD. Thus, AE and EB are also commensurable (in length) with CF and FD, respectively [Prop. 10.11]. And AE and EB (are) medial. Thus, CFand FD (are) also medial [Prop. 10.23]. And since as AE is to EB, (so) CF (is) to FD, and AE and EB are commensurable in square only, CF and FD are [thus]

ξζ΄.

Ή τῆ ἐκ δύο μέσων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο μέσων ἐστὶ καὶ τῆ τάξει ἡ αὐτή.



Έστω ἐκ δύο μέσων ἡ AB, καὶ τῆ AB σύμμετρος ἔστω μήκει ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ ἐκ δύο μέσων ἐστὶ καὶ τῆ τάξει ἡ αὐτὴ τῆ AB.

Έπει γὰρ ἐx δύο μέσων ἐστιν ἡ AB, διηρήσθω εἰς τὰς μέσας xατὰ τὸ E· αἱ AE, EB ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. xαὶ γεγονέτω ὡς ἡ AB πρὸς ΓΔ, ἡ AE πρὸς ΓΖ· xαὶ λοιπὴ ἄρα ἡ EB πρὸς λοιπὴν τὴν ΖΔ ἐστιν, ὡς ἡ AB πρὸς ΓΔ. σύμμετρος δὲ ἡ AB τῆ ΓΔ μήχει· σύμμετρος ἄρα xαὶ ἑxατέρα τῶν AE, EB ἑxατέρα τῶν ΓΖ, ΖΔ. μέσαι δὲ αἱ AE, EB· μέσαι ἄρα xαὶ αἱ ΓΖ, ΖΔ. xαὶ ἐπεί ἐστιν ὡς ἡ AE πρὸς EB, ἡ ΓΖ πρὸς ΖΔ, αἱ δὲ AE, EB δυνάμει μόνον σύμμετροί εἰσιν, xαὶ αἱ ΓΖ, ΖΔ [ӑρα] δυνάμει μόνον σύμμετροί εἰσιν, ἑδείχθησαν δὲ xαὶ μέσαι· ἡ ΓΔ ἄρα ἐx δύο μέσων ἐστίν. λέγω δή, ὅτι xαὶ τῆ τάξει ἡ αὐτή ἐστι τῆ AB.

Ἐπεὶ γάρ ἐστιν ὡς ἡ AE πρὸς EB, ἡ ΓΖ πρὸς ΖΔ, καὶ ὡς ἄρα τὸ ἀπὸ τῆς AE πρὸς τὸ ὑπὸ τῶν AEB, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖΔ· ἐναλλὰξ ὡς τὸ ἀπὸ τῆς

ΑΕ πρὸς τὸ ἀπὸ τῆς ΓΖ, οὕτως τὸ ὑπὸ τῶν ΑΕΒ πρὸς τὸ ὑπὸ τῶν ΓΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΖ΄ σύμμετρον ἄρα καὶ τὸ ὑπὸ τῶν ΑΕΒ τῷ ὑπὸ τῶν ΓΖΔ. εἴτε οῦν ἑητόν ἐστι τὸ ὑπὸ τῶν ΑΕΒ, καὶ τὸ ὑπὸ τῶν ΓΖΔ ἑητόν ἐστιν [καὶ διὰ τοῦτό ἐστιν ἐκ δύο μέσων πρώτη]. εἴτε μέσον, μέσον, καί ἐστιν ἑκατέρα δευτέρα.

Καὶ διὰ τοῦτο ἔσται ἡ ΓΔ τῆ ΑΒ τῆ τάξει ἡ αὐτή· ὅπερ ἔδει δεῖξαι.





Έστω μείζων ή AB, καὶ τῆ AB σύμμετρος ἔστω ή ΓΔ λέγω, ὅτι ἡ ΓΔ μείζων ἐστίν.

Διηρήσθω ή AB κατὰ τὸ Ε· αί AE, EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγχείμενον ἐχ τῶν ἀπ' αὐτῶν τετραγώνων ἑητόν, τὸ δ' ὑπ' αὐτῶν μέσον· καὶ γεγονέτω τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν $\Gamma\Delta$, οὕτως ἤ τε ΑΕ πρὸς τὴν ΓΖ καὶ ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ὡς ἄρα ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ. σύμμετρος δὲ ἡ ΑΒ τῆ ΓΔ· σύμμετρος ἄρα καὶ ἑκατέρα τῶν ΑΕ, ΕΒ ἑκατέρα τῶν ΓΖ, ΖΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ἐναλλὰξ ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ, καὶ συνθέντι ἄρα ἑστὶν ὡς ή AB πρός την BE, οὕτως ή ΓΔ πρός την ΔΖ· καὶ ὡς ἄρα τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BE, οὕτως τὸ ἀπὸ τῆς Γ Δ πρὸς τὸ ἀπὸ τῆς ΔZ . ὑμοίως δὴ δείξομεν, ὅτι καὶ ὡς τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς AE, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΓΖ. καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὰ ἀπὸ τῶν AE, EB, οὕτως τὸ ἀπὸ τῆς Γ Δ πρὸς τὰ ἀπὸ τῶν ΓZ, Z Δ ·

also commensurable in square only [Prop. 10.11]. And they were also shown (to be) medial. Thus, CD is a bimedial (straight-line). So, I say that it is also the same in order as AB.

For since as AE is to EB, (so) CF (is) to FD, thus also as the (square) on AE (is) to the (rectangle contained) by AEB, so the (square) on CF (is) to the (rectangle contained) by CFD [Prop. 10.21 lem.]. Alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AEB (is) to the (rectangle contained) by CFD [Prop. 5.16]. And the (square) on AE (is) commensurable with the (square) on CF. Thus, the (rectangle contained) by AEB (is) also commensurable with the (rectangle contained) by CFD [Prop. 10.11]. Therefore, either the (rectangle contained) by AEB is rational, and the (rectangle contained) by CFD is rational [and, on account of this, (AE and CD) are first bimedial (straight-lines)], or (the rectangle contained by AEB is) medial, and (the rectangle contained by CFD is) medial, and (AB and CD) are each second (bimedial straight-lines) [Props. 10.23, 10.37, 10.38].

And, on account of this, CD will be the same in order as AB. (Which is) the very thing it was required to show.

Proposition 68

A (straight-line) commensurable (in length) with a major (straight-line) is itself also major.



Let AB be a major (straight-line), and let CD be commensurable (in length) with AB. I say that CD is a major (straight-line).

Let AB have been divided (into its component terms) at E. AE and EB are thus incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. And let (the) same (things) have been contrived as in the previous (propositions). And since as AB is to CD, so AE (is) to CF and EB to FD, thus also as AE (is) to CF, so EB (is) to FD [Prop. 5.11]. And AB (is) commensurable (in length) with CD. Thus, AE and EB (are) also commensurable (in length) with CT and FD, respectively [Prop. 10.11]. And since as AE is to CF, so EB (is) to FD [Prop. 5.16], and thus, via composition, as AB is to BE, so CD (is) to DF [Prop. 5.18]. And thus as the (square) on AB (is) to the (square) on BE, so the
καὶ ἐναλλὰξ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς ΓΔ, οὕτως τὰ ἀπὸ τῶν AE, EB πρὸς τὰ ἀπὸ τῶν ΓΖ, ΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς AB τῷ ἀπὸ τῆς ΓΔ· σύμμετρα ἄρα καὶ τὰ ἀπὸ τῶν AE, EB τοῖς ἀπὸ τῶν ΓΖ, ΖΔ. καί ἐστι τὰ ἀπὸ τῶν AE, EB ἅμα ῥητόν, καὶ τὰ ἀπὸ τῶν ΓΖ, ΖΔ ἅμα ῥητόν ἐστιν. ὑμοίως δὲ καὶ τὸ δὶς ὑπὸ τῶν ΑΕ, EB σύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν ΓΖ, ΖΔ. καί ἐστι μέσον τὸ δὶς ὑπὸ τῶν AE, EB· μέσον ἄρα καὶ τὸ δὶς ὑπὸ τῶν ΓΖ, ΖΔ. αἱ ΓΖ, ΖΔ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπʾ αὐτῶν τετραγώνων ἅμα ῥητόν, τὸ δὲ δὶς ὑπʾ αὐτῶν μέσον· ὅλη ἄρα ἡ ΓΔ ἅλογός ἐστιν ἡ καλουμένη μείζων.

Ή ἄρα τῆ μείζονι σύμμετρος μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

ξθ'.

Ή τῆ ἑητὸν καὶ μέσον δυναμένη σύμμετρος [καὶ αὐτὴ] ἑητὸν καὶ μέσον δυναμένη ἐστίν.



Έστω ἡητὸν καὶ μέσον δυναμένη ἡ AB, καὶ τῆ AB σύμμετρος ἔστω ἡ ΓΔ· δεικτέον, ὅτι καὶ ἡ ΓΔ ἡητὸν καὶ μέσον δυναμένη ἐστίν.

Διηρήσθω ή AB εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ AE, EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ἑητόν· καὶ τὰ αὐτὰ κατεσκευάσθω τοῖς πρότερον. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι, καὶ σύμμετρον τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AE, EB τῷ συγκειμένῷ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ AE, EB τῷ ὑπὸ ΓΖ, ΖΔ· ὥστε καὶ τὸ [μὲν] συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων ἐστὶ μέσον, τὸ δ' ὑπὸ τῶν ΓΖ,

(square) on CD (is) to the (square) on DF [Prop. 6.20]. So, similarly, we can also show that as the (square) on AB (is) to the (square) on AE, so the (square) on CD(is) to the (square) on CF. And thus as the (square) on AB (is) to (the sum of) the (squares) on AE and EB, so the (square) on CD (is) to (the sum of) the (squares) on CF and FD. And thus, alternately, as the (square) on ABis to the (square) on CD, so (the sum of) the (squares) on AE and EB (is) to (the sum of) the (squares) on CF and FD [Prop. 5.16]. And the (square) on AB (is) commensurable with the (square) on CD. Thus, (the sum of) the (squares) on AE and EB (is) also commensurable with (the sum of) the (squares) on CF and FD [Prop. 10.11]. And the (squares) on AE and EB (added) together are rational. The (squares) on CF and FD (added) together (are) thus also rational. So, similarly, twice the (rectangle contained) by AE and EB is also commensurable with twice the (rectangle contained) by CF and FD. And twice the (rectangle contained) by AE and EB is medial. Therefore, twice the (rectangle contained) by CFand FD (is) also medial [Prop. 10.23 corr.]. CF and FD are thus (straight-lines which are) incommensurable in square [Prop 10.13], simultaneously making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. The whole, CD, is thus that irrational (straight-line) called major [Prop. 10.39].

Thus, a (straight-line) commensurable (in length) with a major (straight-line) is major. (Which is) the very thing it was required to show.

Proposition 69

A (straight-line) commensurable (in length) with the square-root of a rational plus a medial (area) is [itself also] the square-root of a rational plus a medial (area).



Let AB be the square-root of a rational plus a medial (area), and let CD be commensurable (in length) with AB. We must show that CD is also the square-root of a rational plus a medial (area).

Let AB have been divided into its (component) straight-lines at E. AE and EB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and that the sum of the (squares) on AE and

Υρτόν ἄρα και μέσον δυναμένη ἐστιν ἡ ΓΔ· ὅπερ ἔδει δεῖξαι.

o′.

Ή τῆ δύο μέσα δυναμένη σύμμετρος δύο μέσα δυναμένη ἐστίν.



Έστω δύο μέσα δυναμένη ή AB, καὶ τῆ AB σύμμετρος ή ΓΔ· δεικτέον, ὅτι καὶ ή ΓΔ δύο μέσα δυναμένη ἐστίν.

Έπεὶ γὰρ δύο μέσα δυναμένη ἑστὶν ἡ AB, διηρήσθω εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ AE, EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπʾ αὐτῶν [τετραγώνων] μέσον καὶ τὸ ὑπʾ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AE, EB τετραγώνων τῷ ὑπὸ τῶν AE, EB· καὶ κατεσκευάσθω τὰ αὐτὰ τοῖς πρότερον. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι καὶ σύμμετρον τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AE, EB τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον ἐστὶ καὶ τὸ ὑπὸ τῶν ΛΖ, ΖΔ μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων τῷ ὑπὸ τῶν ΓΖ, ΖΔ.

Η ἄρα ΓΔ δύο μέσα δυναμένη ἐστίν· ὅπερ ἔδει δεῖξαι.

οα'.

⁶Ρητοῦ καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἤτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ἑητὸν καὶ μέσον δυναμένη. *EB* (is) commensurable with the sum of the (squares) on *CF* and *FD*, and the (rectangle contained) by *AE* and *EB* with the (rectangle contained) by *CF* and *FD*. And hence the sum of the squares on *CF* and *FD* is medial, and the (rectangle contained) by *CF* and *FD* (is) rational.

Thus, CD is the square-root of a rational plus a medial (area) [Prop. 10.40]. (Which is) the very thing it was required to show.

Proposition 70

A (straight-line) commensurable (in length) with the square-root of (the sum of) two medial (areas) is (itself also) the square-root of (the sum of) two medial (areas).



Let AB be the square-root of (the sum of) two medial (areas), and (let) CD (be) commensurable (in length) with AB. We must show that CD is also the square-root of (the sum of) two medial (areas).

For since AB is the square-root of (the sum of) two medial (areas), let it have been divided into its (component) straight-lines at E. Thus, AE and EB are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on AEand EB incommensurable with the (rectangle) contained by AE and EB [Prop. 10.41]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and (that) the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Hence, the sum of the squares on CFand FD is also medial, and the (rectangle contained) by CF and FD (is) medial, and, moreover, the sum of the squares on CF and FD (is) incommensurable with the (rectangle contained) by CF and FD.

Thus, CD is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. (Which is) the very thing it was required to show.

Proposition 71

When a rational and a medial (area) are added together, four irrational (straight-lines) arise (as the squareroots of the total area)—either a binomial, or a first biΈστω ἡητὸν μὲν τὸ AB, μέσον δὲ τὸ ΓΔ· λέγω, ὅτι ἡ τὸ AΔ χωρίον δυναμένη ἤτοι ἐκ δύο ὀνομάτων ἐστὶν ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.



Τὸ γὰρ ΑΒ τοῦ ΓΔ ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. έστω πρότερον μεῖζον· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ παραβεβλήσθω παρὰ τὴν ΕΖ τῷ ΑΒ ἴσον τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΘ· τῷ δὲ ΔΓ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΘΙ πλάτος ποιοῦν τὴν ΘΚ. καὶ ἐπεὶ ῥητόν ἐστι τὸ ΑΒ καί έστιν ίσον τῷ ΕΗ, ῥητὸν ἄρα καὶ τὸ ΕΗ. καὶ παρὰ [ῥητὴν] τὴν ΕΖ παραβέβληται πλάτος ποιοῦν τὴν ΕΘ· ἡ ΕΘ ἄρα ρητή ἐστι καὶ σύμμετρος τῆ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον έστι τὸ ΓΔ καί ἐστιν ἴσον τῷ ΘΙ, μέσον ẳρα ἐστι και τὸ ΘΙ. καὶ παρὰ ἑητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΚ· ἑητὴ ἄρα ἐστὶν ἡ ΘΚ καὶ ἀσύμμετρος τῆ ΕΖ μήκει. καὶ έπεὶ μέσον ἐστὶ τὸ ΓΔ, ῥητὸν δὲ τὸ ΑΒ, ἀσύμμετρον ἄρα έστι τὸ ΑΒ τῷ ΓΔ· ὥστε και τὸ ΕΗ ἀσύμμετρόν ἐστι τῷ ΘΙ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΙ, οὕτως ἐστὶν ἡ ΕΘ πρὸς τὴν ΘΚ· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΕΘ τῆ ΘΚ μήκει. καί εἰσιν άμφότεραι ἑηταί· αἱ ΕΘ, ΘΚ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστίν ἡ ΕΚ διηρημένη κατὰ τὸ Θ. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΑΒ τοῦ ΓΔ, ἴσον δὲ τὸ μέν ΑΒ τῷ ΕΗ, τὸ δὲ ΓΔ τῷ ΘΙ, μεῖζον ἄρα καὶ τὸ ΕΗ τοῦ ΘΙ· καὶ ἡ ΕΘ ἄρα μείζων ἐστὶ τῆς ΘΚ. ἤτοι οὖν ἡ ΕΘ τῆς ΘΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῃ μήκει ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου έαυτῆ· καί ἐστιν ἡ μείζων ἡ ΘΕ σύμμετρος τῆ ἐκκειμένῃ ρητη τῆ ΕΖ· ἡ ἄρα ΕΚ ἐχ δύο ὀνομάτων ἐστὶ πρώτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο όνομάτων πρώτης, ή τὸ χωρίον δυναμένη ἐκ δύο ὀνομάτων έστίν. ἡ ἄρα τὸ ΕΙ δυναμένη ἐχ δύο ὀνομάτων ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη ἐκ δύο ὀνομάτων ἐστίν. ἀλλὰ δὴ δυνάσθω ή ΕΘ τῆς ΘΚ μεῖζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ· καί ἐστιν ἡ μείζων ἡ ΕΘ σύμμετρος τῆ ἐκκειμένῃ ἑητῆ τῆ ΕΖ μήκει· ή ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη. ἑητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο

medial, or a major, or the square-root of a rational plus a medial (area).

Let AB be a rational (area), and CD a medial (area). I say that the square-root of area AD is either binomial, or first bimedial, or major, or the square-root of a rational plus a medial (area).



For AB is either greater or less than CD. Let it, first of all, be greater. And let the rational (straight-line) EF be laid down. And let (the rectangle) EG, equal to AB, have been applied to EF, producing EH as breadth. And let (the recatangle) HI, equal to DC, have been applied to EF, producing HK as breadth. And since ABis rational, and is equal to EG, EG is thus also rational. And it has been applied to the [rational] (straight-line) EF, producing EH as breadth. EH is thus rational, and commensurable in length with EF [Prop. 10.20]. Again, since CD is medial, and is equal to HI, HI is thus also medial. And it is applied to the rational (straight-line) EF, producing HK as breadth. HK is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since CD is medial, and AB rational, AB is thus incommensurable with CD. Hence, EG is also incommensurable with HI. And as EG (is) to HI, so EH is to HK [Prop. 6.1]. Thus, EH is also incommensurable in length with HK [Prop. 10.11]. And they are both rational. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line), having been divided (into its component terms) at H [Prop. 10.36]. And since ABis greater than CD, and AB (is) equal to EG, and CDto HI, EG (is) thus also greater than HI. Thus, EH is also greater than HK [Prop. 5.14]. Therefore, the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable in length with (EH), or by the (square) on (some straightline) incommensurable (in length with EH). Let it, first of all, be greater by the (square) on (some straight-line) commensurable (in length with EH). And the greater όνομάτων τετάρτης, ή τὸ χωρίον δυναμένη ἄλογός ἐστιν ή καλουμένη μείζων. ή ἄρα τὸ ΕΙ χωρίον δυναμένη μείζων ἐστίν· ὥστε καὶ ή τὸ ΑΔ δυναμένη μείζων ἐστίν.

Άλλὰ δὴ ἔστω ἔλασσον τὸ ΑΒ τοῦ ΓΔ· καὶ τὸ ΕΗ άρα ἕλασσόν ἐστι τοῦ ΘΙ· ὥστε καὶ ἡ ΕΘ ἐλάσσων ἐστὶ τῆς ΘΚ. ἤτοι δὲ ἡ ΘΚ τῆς ΕΘ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῃ μήκει· καί ἐστιν ἡ ἐλάσσων ἡ ΕΘ σύμμετρος τῆ ἐκκειμένῃ ῥητῆ τῆ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ δευτέρα. ἑητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέγηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη έκ δύο μέσων έστι πρώτη. ὥστε και ή τὸ $A\Delta$ δυναμένη ἐ
κ δύο μέσων ἐστὶ πρώτη. ἀλλὰ δὴ ἡ ΘΚ τῆς ΘΕ μεῖζον δυνάσθω τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί ἐστιν ή ἐλάσσων ἡ ΕΘ σύμμετρος τῆ ἐχχειμένῃ ῥητῆ τῆ ΕΖ· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ή τὸ χωρίον δυναμένη ἑητὸν καὶ μέσον δυναμένη έστίν. ή ἄρα τὸ ΕΙ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν ὥστε καὶ ἡ τὸ ΑΔ χωρίον δυναμένη ἑητὸν καὶ μέσον δυναμένη ἐστίν.

Υρτοῦ ἄρα καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἤτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη· ὅπερ ἔδει δεῖξαι.

(of the two components of EK) HE is commensurable (in length) with the (previously) laid down (straightline) *EF*. *EK* is thus a first binomial (straight-line) [Def. 10.5]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is a binomial (straight-line) [Prop. 10.54]. Thus, the square-root of EI is a binomial (straight-line). Hence the squareroot of AD is also a binomial (straight-line). And, so, let the square on EH be greater than (the square on) HKby the (square) on (some straight-line) incommensurable (in length) with (EH). And the greater (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a fourth binomial (straight-line) [Def. 10.8]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line) called major [Prop. 10.57]. Thus, the square-root of area EI is a major (straight-line). Hence, the square-root of AD is also major.

And so, let AB be less than CD. Thus, EG is also less than HI. Hence, EH is also less than HK [Props. 6.1. 5.14]. And the square on HK is greater than (the square on) EH either by the (square) on (some straightline) commensurable (in length) with (HK), or by the (square) on (some straight-line) incommensurable (in length) with (HK). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (HK). And the lesser (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EKis a second binomial (straight-line) [Def. 10.6]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is a first bimedial (straightline) [Prop. 10.55]. Thus, the square-root of area EI is a first bimedial (straight-line). Hence, the square-root of AD is also a first bimedial (straight-line). And so, let the square on HK be greater than (the square on) HEby the (square) on (some straight-line) incommensurable (in length) with (HK). And the lesser (of the two components of EK) EH is commensurable (in length) with the (previously) laid down rational (straight-line) EF. Thus, EK is a fifth binomial (straight-line) [Def. 10.9]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the square-root of a rational plus a medial (area) [Prop. 10.58]. Thus, the square-root of area EI is the square-root of a rational plus a medial (area). Hence, the square-root of area AD is also the

ξβ΄.

Δύο μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἰ λοιπαὶ δύο ἄλογοι γίγνονται ἤτοι ἐχ δύο μέσων δευτέρα ἢ [ή] δύο μέσα δυναμένη.



Συγκείσθω γὰρ δύο μέσα ἀσύμμετρα ἀλλήλοις τὰ AB, ΓΔ· λέγω, ὅτι ἡ τὸ ΑΔ χωρίον δυναμένη ἤτοι ἐκ δύο μέσων ἐστὶ δευτέρα ἢ δύο μέσα δυναμένη.

Τὸ γὰρ ΑΒ τοῦ ΓΔ ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. ἔστω, εἰ τύχον, πρότερον μεῖζον τὸ ΑΒ τοῦ ΓΔ· καὶ ἐκκείσθω ρητή ή EZ, και τῷ μέν AB ἴσον παρὰ τὴν EZ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΘ, τῷ δὲ ΓΔ ἴσον τὸ ΘΙ πλάτος ποιοῦν τὴν ΘΚ. καὶ ἐπεὶ μέσον ἐστὶν ἑκάτερον τῶν AB, ΓΔ, μέσον ἄρα καὶ ἐκάτερον τῶν ΕΗ, ΘΙ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράχειται πλάτος ποιοῦν τὰς ΕΘ, ΘΚ· ἑχατέρα ἄρα τῶν ΕΘ, ΘΚ ἑητή ἐστι καὶ ἀσύμμετρος τῆ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΑΒ τῷ ΓΔ, καί ἐστιν ἴσον τὸ μὲν ΑΒ τῷ EH, τὸ δὲ Γ Δ τῷ ΘΙ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ EH τῷ ΘΙ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΙ, οὕτως ἐστὶν ἡ ΕΘ πρὸς ΘΚ· άσύμμετρος ἄρα ἐστὶν ἡ ΕΘ τῆ ΘΚ μήχει. αἱ ΕΘ, ΘΚ ἄρα ρηταί είσι δυνάμει μόνον σύμμετροι. ἐκ δύο ἄρα ὀνομάτων έστιν ή ΕΚ. ήτοι δε ή ΕΘ τῆς ΘΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· καὶ οὐδετέρα τῶν ΕΘ, ΘΚ σύμμετρός ἐστι τῆ ἐκκειμένῃ ῥητῃ τῆ ΕΖ μήκει· ἡ ΕΚ ἄρα έκ δύο ὀνομάτων ἐστὶ τρίτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἐχ δύο μέσων ἐστὶ δευτέρα. ἡ ἄρα τὸ ΕΙ, τουτέστι τὸ ΑΔ, δυναμένη ἐχ δύο μέσων ἐστὶ δευτέρα.

square-root of a rational plus a medial (area).

Thus, when a rational and a medial area are added together, four irrational (straight-lines) arise (as the squareroots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area). (Which is) the very thing it was required to show.

Proposition 72

When two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedial, or the squareroot of (the sum of) two medial (areas).



For let the two medial (areas) AB and CD, (which are) incommensurable with one another, have been added together. I say that the square-root of area AD is either a second bimedial, or the square-root of (the sum of) two medial (areas).

For AB is either greater than or less than CD. By chance, let AB, first of all, be greater than CD. And let the rational (straight-line) EF be laid down. And let EG, equal to AB, have been applied to EF, producing EH as breadth, and HI, equal to CD, producing HKas breadth. And since AB and CD are each medial, EGand HI (are) thus also each medial. And they are applied to the rational straight-line FE, producing EH and HK (respectively) as breadth. Thus, EH and HK are each rational (straight-lines which are) incommensurable in length with EF [Prop. 10.22]. And since AB is incommensurable with CD, and AB is equal to EG, and CDto HI, EG is thus also incommensurable with HI. And as EG (is) to HI, so EH is to HK [Prop. 6.1]. EH is thus incommensurable in length with HK [Prop. 10.11]. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line) [Prop. 10.36]. And the square on EH is greater than (the square on) HK either by the (square)

άλλα δὴ ἡ ΕΘ τῆς ΘΚ μεῖζον δυνάσθω τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήκει· καὶ ἀσύμμετρός ἐστιν ἑκατέρα τῶν ΕΘ, ΘΚ τῆ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη. ἐὰν δὲ χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

[Όμοίως δη δείξομεν, ὅτι κἂν ἔλαττον ἢ τὸ ΑΒ τοῦ ΓΔ, ἡ τὸ ΑΔ χωρίον δυναμένη ἢ ἐκ δύο μέσων δευτέρα ἐστὶν ἤτοι δύο μέσα δυναμένη].

Δύο ἄρα μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλογοι γίγνονται ἤτοι ἐχ δύο μέσων δευτέρα ἢ δύο μέσα δυναμένη.

Ή ἐκ δύο ὀνομάτων καὶ αἱ μετ' αὐτὴν ἄλογοι οὔτε τῆ μέση οὔτε ἀλλήλαις εἰσὶν αἱ αὐταί. τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἑητὴν καὶ άσύμμετρον τῆ παρ' ἢν παράχειται μήχει. τὸ δὲ ἀπὸ τῆς έκ δύο ὀνομάτων παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐχ δύο ὀνομάτων πρώτην. τὸ δὲ ἀπὸ τῆς ἐχ δύο μέσων πρώτης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρά ἑητὴν παραβαλλόμενον πλάτος ποιεῖ την έχ δύο όνομάτων τρίτην. το δε άπο της μείζονος παρά ρητήν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην. τὸ δὲ ἀπὸ τῆς ῥητὸν καὶ μέσον δυναμένης παρὰ ρητήν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην. τὸ δὲ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην. τὰ δ' εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ἑητή ἐστιν, ἀλλήλων δέ, ὅτι τῆ τάξει ούχ εἰσὶν αἱ αὐταί· ὥστε χαὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων.

on (some straight-line) commensurable (in length) with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (EH). And neither of EH or HK is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a third binomial (straight-line) [Def. 10.7]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is a second bimedial (straight-line) [Prop. 10.56]. Thus, the square-root of EI-that is to say, of ADis a second bimedial. And so, let the square on EHbe greater than (the square) on HK by the (square) on (some straight-line) incommensurable in length with (EH). And EH and HK are each incommensurable in length with EF. Thus, EK is a sixth binomial (straightline) [Def. 10.10]. And if an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the square-root of (the sum of) two medial (areas) [Prop. 10.59]. Hence, the square-root of area AD is also the square-root of (the sum of) two medial (areas).

[So, similarly, we can show that, even if AB is less than CD, the square-root of area AD is either a second bimedial or the square-root of (the sum of) two medial (areas).]

Thus, when two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the squareroots of the total area)—either a second bimedial, or the square-root of (the sum of) two medial (areas).

A binomial (straight-line), and the (other) irrational (straight-lines) after it, are neither the same as a medial (straight-line) nor (the same) as one another. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straightline which is) also incommensurable in length with (the straight-line) to which it is applied [Prop. 10.22]. And the (square) on a binomial (straight-line), applied to a rational (straight-line), produces as breadth a first binomial [Prop. 10.60]. And the (square) on a first bimedial (straight-line), applied to a rational (straight-line), produces as breadth a second binomial [Prop. 10.61]. And the (square) on a second bimedial (straight-line), applied to a rational (straight-line), produces as breadth a third binomial [Prop. 10.62]. And the (square) on a major (straight-line), applied to a rational (straight-line), produces as breadth a fourth binomial [Prop. 10.63]. And the (square) on the square-root of a rational plus a medial

ογ'.

Έὰν ἀπὸ ἑητῆς ἑητὴ ἀφαιρεθῆ δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἀποτομή.



Άπὸ γὰρ ἑητῆς τῆς ΑΒ ἑητὴ ἀφηρήσθω ἡ ΒΓ δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλη· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Έπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ AB τῆ BΓ μήχει, καί ἐστιν ὡς ἡ AB πρὸς τὴν BΓ, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ὑπὸ τῶν AB, BΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τῷ ὑπὸ τῶν AB, BΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς AB σύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ τετράγωνα, τῷ δὲ ὑπὸ τῶν AB, BΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν AB, BΓ. καὶ ἐπειδήπερ τὰ ἀπὸ τῶν AB, BΓ ἴσα ἐστὶ τῷ δὶς ὑπὸ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΑ, καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς AΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν AB, BΓ· ἄλογος ἄρα ἐστὶν ἡ AΓ· καλείσθω δὲ ἀποτομή. ὅπερ ἕδει δεῖξαι.

[†] See footnote to Prop. 10.36.

οδ'.

Έλν ἀπὸ μέσης μέση ἀφαιρεθῆ δυνάμει μόνον σύμμετρος οὖσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη. (area), applied to a rational (straight-line), produces as breadth a fifth binomial [Prop. 10.64]. And the (square) on the square-root of (the sum of) two medial (areas), applied to a rational (straight-line), produces as breadth a sixth binomial [Prop. 10.65]. And the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational—and from one another, because they are not the same in order. Hence, the (previously mentioned) irrational (straightlines) themselves also differ from one another.

Proposition 73

If a rational (straight-line), which is commensurable in square only with the whole, is subtracted from a(nother) rational (straight-line) then the remainder is an irrational (straight-line). Let it be called an apotome.



For let the rational (straight-line) BC, which commensurable in square only with the whole, have been subtracted from the rational (straight-line) AB. I say that the remainder AC is that irrational (straight-line) called an apotome.

For since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the (sum of the) squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And, inasmuch as the (sum of the squares) on AB and BC is equal to twice the (rectangle contained) by AB and BC plus the (square) on CA[Prop. 2.7], the (sum of the squares) on AB and BC is thus also incommensurable with the remaining (square) on AC [Props. 10.13, 10.16]. And the (sum of the squares) on AB and BC is rational. AC is thus an irrational (straight-line) [Def. 10.4]. And let it be called an apotome.[†] (Which is) the very thing it was required to show.

Proposition 74

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a rational (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational

B



Άπὸ γὰρ μέσης τῆς AB μέση ἀφηρήσθω ἡ BΓ δυνάμει μόνον σύμμετρος οὕσα τῆ AB, μετὰ δὲ τῆς AB ἑητὸν ποιοῦσα τὸ ὑπὸ τῶν AB, BΓ· λέγω, ὅτι ἡ λοιπὴ ἡ AΓ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

Έπει γὰρ ai AB, BΓ μέσαι εἰσίν, μέσα ἐστὶ καὶ τὰ ἀπὸ τῶν AB, BΓ. ἑητὸν δὲ τὸ δὶς ὑπὸ τῶν AB, BΓ· ἀσύμμετρα ἄρα τὰ ἀπὸ τῶν AB, BΓ τῷ δἰς ὑπὸ τῶν AB, BΓ· καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς AΓ ἀσύμμετρόν ἐστι τὸ δἰς ὑπὸ τῶν AB, BΓ, ἐπεὶ κἂν τὸ ὅλον ἑνὶ αὐτῶν ἀσύμμετρον ἢ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται. ἑητὸν δὲ τὸ δἰς ὑπὸ τῶν AB, BΓ· ἅλογον ἄρα τὸ ἀπὸ τῆς AΓ· ἅλογος ἄρα ἐστὶν ἡ AΓ· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

[†] See footnote to Prop. 10.37.

οε'.

Έὰν ἀπὸ μέσης μέση ἀφαιρεθῆ δυνάμει μόνον σύμμετρος οὕσα τῆ ὅλη, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα, ἡ λοιπὴ ἀλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

Άπὸ γὰρ μέσης τῆς ΑΒ μέση ἀφηρήσθω ἡ ΓΒ δυνάμει μόνον σύμμετρος οὕσα τῆ ὅλη τῆ ΑΒ, μετὰ δὲ τῆς ὅλης τῆς ΑΒ μέσον περιέχουσα τὸ ὑπὸ τῶν ΑΒ, ΒΓ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα. (straight-line). Let it be called a first apotome of a medial (straight-line).

С

А

For let the medial (straight-line) BC, which is commensurable in square only with AB, and which makes with AB the rational (rectangle contained) by AB and BC, have been subtracted from the medial (straight-line) AB [Prop. 10.27]. I say that the remainder AC is an irrational (straight-line). Let it be called the first apotome of a medial (straight-line).

For since AB and BC are medial (straight-lines), the (sum of the squares) on AB and BC is also medial. And twice the (rectangle contained) by AB and BC (is) rational. The (sum of the squares) on AB and BC (is) thus incommensurable with twice the (rectangle contained) by AB and BC. Thus, twice the (rectangle contained) by AB and BC. Thus, twice the (rectangle contained) by AB and BC is also incommensurable with the remaining (square) on AC [Prop. 2.7], since if the whole is incommensurable with one of the (constituent magnitudes) then the original magnitudes will also be incommensurable (with one another) [Prop. 10.16]. And twice the (rectangle contained) by AB and BC (is) rational. Thus, the (square) on AC is irrational. Thus, AC is an irrational (straight-line) [Def. 10.4]. Let it be called a first apotome of a medial (straight-line).[†]

Proposition 75

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a medial (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).

For let the medial (straight-line) CB, which is commensurable in square only with the whole, AB, and which contains with the whole, AB, the medial (rectangle contained) by AB and BC, have been subtracted from the medial (straight-line) AB [Prop. 10.28]. I say that the remainder AC is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).



Έκκείσθω γὰρ ῥητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΘ πλάτος ποιοῦν τὴν ΔΖ· λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καὶ ἐπεὶ μέσα καὶ σύμμετρά έστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, μέσον ἄρα καὶ τὸ ΔΕ. καὶ παρὰ ρητήν την ΔΙ παράχειται πλάτος ποιοῦν την ΔΗ· ρητή ἄρα έστιν ή ΔΗ και ἀσύμμετρος τῆ ΔΙ μήκει. πάλιν, ἐπεὶ μέσον έστι τὸ ὑπὸ τῶν ΑΒ, ΒΓ, και τὸ δὶς ἄρα ὑπὸ τῶν ΑΒ,
ΒΓ μέσον ἐστίν. καί ἐστιν ἴσον τῷ $\Delta \Theta$ · καὶ τὸ $\Delta \Theta$ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔΙ παραβέβληται πλάτος ποιοῦν τὴν ΔZ · ἑητὴ ẳρα ἐστὶν ἡ ΔZ καὶ ἀσύμμετρος τῃ ΔI μήχει. χαὶ ἐπεὶ αἱ ΑΒ, ΒΓ δυνάμει μόνον σύμμετροί εἰσιν, άσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῆ ΒΓ μήχει· ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τετράγωνον τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν AB, BΓ. ἴσον δὲ τοῖς μὲν ἀπὸ τῶν AB, BΓ τὸ ΔE , τῷ δὲ δὶς ὑπὸ τῶν AB, BΓ τὸ $\Delta\Theta$ · ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔE τῷ $\Delta \Theta$. ὡς δὲ τὸ ΔE πρὸς τὸ $\Delta \Theta$, οὕτως ἡ H Δ πρὸς τὴν ΔZ^{\cdot} ἀσύμμετρος ἄρα ἐστὶν ἡ $H\Delta$ τῃ ΔZ . καί εἰσιν άμφότεραι ἑηταί· αἱ ἄρα ΗΔ, ΔΖ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ή ΖΗ ἄρα ἀποτομή ἐστιν. ἡητὴ δὲ ή ΔΙ· τὸ δὲ ύπὸ ἑητῆς καὶ ἀλόγου περιεχόμενον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. καὶ δύναται τὸ ΖΕ ἡ ΑΓ· ἡ ΑΓ ἄρα ἄλογός ἐστιν· χαλείσθω δὲ μέσης ἀποτομὴ δευτέρα. ὅπερ ἔδει δεῖξαι.



For let the rational (straight-line) DI be laid down. And let DE, equal to the (sum of the squares) on ABand BC, have been applied to DI, producing DG as breadth. And let DH, equal to twice the (rectangle contained) by AB and BC, have been applied to DI, producing DF as breadth. The remainder FE is thus equal to the (square) on AC [Prop. 2.7]. And since the (squares) on AB and BC are medial and commensurable (with one another). DE (is) thus also medial [Props. 10.15. 10.23 corr.]. And it is applied to the rational (straightline) DI, producing DG as breadth. Thus, DG is rational, and incommensurable in length with DI [Prop. 10.22]. Again, since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BCis thus also medial [Prop. 10.23 corr.]. And it is equal to DH. Thus, DH is also medial. And it has been applied to the rational (straight-line) DI, producing DF as breadth. DF is thus rational, and incommensurable in length with DI [Prop. 10.22]. And since AB and BC are commensurable in square only, AB is thus incommensurable in length with BC. Thus, the square on AB (is) also incommensurable with the (rectangle contained) by AB and BC [Props. 10.21 lem., 10.11]. But, the (sum of the squares) on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the (sum of the squares) on AB and BC [Prop. 10.13]. And DE is equal to the (sum of the squares) on AB and BC, and DH to twice the (rectangle contained) by AB and BC. Thus, DE [is] incommensurable with DH. And as DE (is) to DH, so GD (is) to DF [Prop. 6.1]. Thus, GD is incommensurable with DF[Prop. 10.11]. And they are both rational (straight-lines). Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And DI (is) rational. And the (area) contained by a rational and an irrational (straight-line) is irrational [Prop. 10.20], and its square-root is irrational.

And AC is the square-root of FE. Thus, AC is an irrational (straight-line) [Def. 10.4]. And let it be called the second apotome of a medial (straight-line).[†] (Which is) the very thing it was required to show.

Proposition 76

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the (squares) on them (added) together rational, and the (rectangle contained) by them medial, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called a minor (straight-line).

A C BFor let the straight-line *BC*, which is incommensu-

For let the straight-line BC, which is incommensurable in square with the whole, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.33]. I say that the remainder AC is that irrational (straight-line) called minor.

For since the sum of the squares on AB and BC is rational, and twice the (rectangle contained) by AB and BC (is) medial, the (sum of the squares) on AB and BCis thus incommensurable with twice the (rectangle contained) by AB and BC. And, via conversion, the (sum of the squares) on AB and BC is incommensurable with the remaining (square) on AC [Props. 2.7, 10.16]. And the (sum of the squares) on AB and BC (is) rational. The (square) on AC (is) thus irrational. Thus, AC (is) an irrational (straight-line) [Def. 10.4]. Let it be called a minor (straight-line).[†] (Which is) the very thing it was required to show.

Proposition 77

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called that which makes with a rational (area) a medial whole.

For let the straight-line BC, which is incommensurable in square with AB, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.34]. I say that the remainder AC is the

[†] See footnote to Prop. 10.38.

oτ'.

Έλν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθὴ δυνάμει ἀσύμμετρος οὕσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ μὲν ἀπ' αὐτῶν ἅμα ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἐλάσσων.



Άπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφηρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οῦσα τῆ ὅλῃ ποιοῦσα τὰ προχείμενα. λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Έπει γὰρ τὸ μὲν συγχείμενον ἐχ τῶν ἀπὸ τῶν AB, BΓ τετραγώνων ῥητόν ἐστιν, τὸ δὲ δἰς ὑπὸ τῶν AB, BΓ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν AB, BΓ τῷ δἰς ὑπὸ τῶν AB, BΓ· χαὶ ἀναστρέψαντι λοιπῷ τῷ ἀπὸ τῆς AΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν AB, BΓ· ἅλογον ἄρα τὸ ἀπὸ τῆς AΓ· ἅλογος ἄρα ἡ AΓ· χαλείσθω δὲ ἐλάσσων. ὅπερ ἕδει δεῖξαι.

[†] See footnote to Prop. 10.39.

οζ΄.

Έλν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῆ δυνάμει ἀσύμμετρος οῦσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δὲ δἰς ὑπ᾽ αὐτῶν ἑητόν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα.



Άπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφηρήσθω ἡ ΒΓ δυνάμει ἀσύμμετος οὕσα τῆ ΑΒ ποιοῦσα τὰ προχείμενα λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ προειρημένη.

Έπει γάρ τὸ μέν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ

τετραγώνων μέσον ἐστίν, τὸ δὲ δἰς ὑπὸ τῶν AB, BΓ ἑητόν, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν AB, BΓ τῷ δἰς ὑπὸ τῶν AB, BΓ· καὶ λοιπὸν ἄρα τὸ ἀπὸ τῆς AΓ ἀσύμμετρόν ἐστι τῷ δἰς ὑπὸ τῶν AB, BΓ. καί ἐστι τὸ δἰς ὑπὸ τῶν AB, BΓ ἑητόν· τὸ ἄρα ἀπὸ τῆς AΓ ἄλογόν ἐστιν· ἄλογος ἄρα ἐστὶν ἡ AΓ· καλείσθω δὲ ἡ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.40.

oη'.

Έλν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῆ δυνάμει ἀσύμμετρος οῦσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλῃς ποιοῦσα τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον τό τε δὶς ὑπ᾽ αὐτῶν μέσον καὶ ἔτι τὰ ἀπ᾽ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δἰς ὑπ᾽ αὐτῶν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.



Άπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφηρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οῦσα τῆ ΑΒ ποιοῦσα τὰ προχείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Έκκείσθω γὰρ ἑητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν AB, BΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δὶς ὑπὸ τῶν AB, BΓ ἴσον ἀφηρήσθω τὸ ΔΘ [πλάτος ποιοῦν τὴν ΔΖ]. λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ· ὥστε ἡ ΑΓ δύναται τὸ ΖΕ. καὶ ἐπεὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB, BΓ τετραγώνων μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΔΕ, μέσον ἄρα [ἐστὶ] τὸ ΔΕ. καὶ παρὰ ἑητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΗ· ἑητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῆ ΔΙ μήκει. πάλιν, ἐπεὶ τὸ δἰς ὑπὸ τῶν AB, BΓ μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΔΘ, τὸ ἄρα aforementioned irrational (straight-line).

For since the sum of the squares on AB and BC is medial, and twice the (rectangle contained) by AB and BC rational, the (sum of the squares) on AB and BCis thus incommensurable with twice the (rectangle contained) by AB and BC. Thus, the remaining (square) on AC is also incommensurable with twice the (rectangle contained) by AB and BC [Props. 2.7, 10.16]. And twice the (rectangle contained) by AB and BC is rational. Thus, the (square) on AC is irrational. Thus, ACis an irrational (straight-line) [Def. 10.4]. And let it be called that which makes with a rational (area) a medial whole.[†] (Which is) the very thing it was required to show.

Proposition 78

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called that which makes with a medial (area) a medial whole.



For let the straight-line BC, which is incommensurable in square AB, and fulfils the (other) prescribed (conditions), have been subtracted from the (straightline) AB [Prop. 10.35]. I say that the remainder AC is the irrational (straight-line) called that which makes with a medial (area) a medial whole.

For let the rational (straight-line) DI be laid down. And let DE, equal to the (sum of the squares) on AB and BC, have been applied to DI, producing DG as breadth. And let DH, equal to twice the (rectangle contained) by AB and BC, have been subtracted (from DE) [producing DF as breadth]. Thus, the remainder FE is equal to the (square) on AC [Prop. 2.7]. Hence, AC is the square-root of FE. And since the sum of the squares on ΔΘ μέσον ἐστίν. καὶ παρὰ ἑητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΖ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΔΖ καὶ ἀσύμμετρος τῆ ΔΙ μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ τῷ δὶς ὑπὸ τῶν AB, BΓ, ἀσύμμετρον ἄρα καὶ τὸ ΔΕ τῷ ΔΘ. ὡς δὲ τὸ ΔΕ πρὸς τὸ ΔΘ, οῦτως ἐστὶ καὶ ἡ ΔΗ πρὸς τὴν ΔΖ· ἀσύμμετρος ἄρα ἡ ΔΗ τῆ ΔΖ. καί εἰσιν ἀμφότεραι ἑηταί· αἱ HΔ, ΔΖ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. ἀποτομὴ ἄρα ἐστίν ἡ ΖΗ· ἑητὴ δὲ ἡ ΖΘ. τὸ δὲ ὑπὸ ἑητῆς καὶ ἀποτομῆς περιεχόμενον [ὀρθογώνιον] ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν· καὶ δύναται τὸ ΖΕ ἡ ΑΓ· ἡ ΑΓ ἅρα ἄλογός ἑστιν· καλείσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.41.

oθ'.

Τῆ ἀποτομῆ μία [μόνον] προσαρμόζει εὐθεῖα ἑητὴ δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλη.



Έστω ἀποτομὴ ἡ AB, προσαρμόζουσα δὲ αὐτῆ ἡ BΓ· αἰ AΓ, ΓΒ ἀρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· λέγω, ὅτι τῆ AB ἑτέρα οὐ προσαρμόζει ἑητὴ δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλῆ.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί, ῷ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· τῷ γὰρ αὐτῷ τῷ ἀπὸ τῆς ΑΒ ἀμφότερα ὑπερέχει· ἐναλλὰξ ἄρα, ῷ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει [καὶ] τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. τὰ δὲ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ἑρτῷ· ἑρτὰ γὰρ ἀμφότερα. καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ἑρτῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γὰρ ἀμφότερα, μέσον δὲ μέσου οὐχ ὑπερέχει ἑρτῷ. τῆ ἄρα ΑΒ ἑτέρα οὐ προσαρμόζει ἑρτὴ δυνάμει μόνον σύμμετρος οὕσα τῆ ὅλη.

Μία ἄρα μόνη τῆ ἀποτομῆ προσαρμόζει ἑητὴ δυνάμει

AB and BC is medial, and is equal to DE, DE [is] thus medial. And it is applied to the rational (straight-line) DI, producing DG as breadth. Thus, DG is rational, and incommensurable in length with DI [Prop 10.22]. Again, since twice the (rectangle contained) by AB and BC is medial, and is equal to DH, DH is thus medial. And it is applied to the rational (straight-line) DI, producing DFas breadth. Thus, DF is also rational, and incommensurable in length with DI [Prop. 10.22]. And since the (sum of the squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DE(is) also incommensurable with DH. And as DE (is) to DH, so DG also is to DF [Prop. 6.1]. Thus, DG (is) incommensurable (in length) with DF [Prop. 10.11]. And they are both rational. Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And FH(is) rational. And the [rectangle] contained by a rational (straight-line) and an apotome is irrational [Prop. 10.20], and its square-root is irrational. And AC is the squareroot of FE. Thus, AC is irrational. Let it be called that which makes with a medial (area) a medial whole.[†] (Which is) the very thing it was required to show.

Proposition 79

[Only] one rational straight-line, which is commensurable in square only with the whole, can be attached to an apotome.[†]



Let AB be an apotome, with BC (so) attached to it. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.73]. I say that another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And since by whatever (area) the (sum of the squares) on ADand DB exceeds twice the (rectangle contained) by ADand DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For both exceed by the same (area)— (namely), the (square) on AB [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on ACand CB, twice the (rectangle contained) by AD and DB[also] exceeds twice the (rectangle contained) by AC and μόνον σύμμετρος ούσα τῆ ὅλη. ὅπερ ἔδει δεῖξαι.

CB by this (same area). And the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB by a rational (area). For both (are) rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.21], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26]. Thus, another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

Thus, only one rational (straight-line), which is commensurable in square only with the whole, can be attached to an apotome. (Which is) the very thing it was required to show.

[†] This proposition is equivalent to Prop. 10.42, with minus signs instead of plus signs.

π'.

Τῆ μέσης ἀποτομῆ πρώτῃ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

А	В	$\Gamma \Delta$

Έστω γὰρ μέσης ἀποτομὴ πρώτη ἡ AB, καὶ τῆ AB προσαρμοζέτω ἡ BΓ· αἱ AΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν AΓ, ΓΒ· λέγω, ὅτι τῆ AB ἑτέρα οὐ προσαρμόζει μέση δυνάμει μόνον σύμμετρος οὕσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω καὶ ἡ ΔΒ· αἱ ἄρα ΑΔ, ΔΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεί, ῷ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· τῷ γὰρ αὐτῷ [πάλιν] ὑπερέχουσι τῷ ἀπὸ τῆς ΑΒ· ἐναλλὰξ ἄρα, ῷ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα. καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΛ, ΓΒ [τετραγώνων] ὑπερέχει ῥητῷ· ὅπερ ἐστιν ἀδύνατον· μέσα γάρ ἑστιν ἀμφότερα, μέσον δὲ μέσου οὐχ ὑπερέχει ἑητῷ.

Τῆ ἄρα μέσης ἀποτομῆ πρώτη μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ἑητὸν περιέχουσα[.] ὅπερ ἔδει δεῖξαι.

Proposition 80

Only one medial straight-line, which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line).[†]

For let AB be a first apotome of a medial (straightline), and let BC be (so) attached to AB. Thus, ACand CB are medial (straight-lines which are) commensurable in square only, containing a rational (area)— (namely, that contained) by AC and CB [Prop. 10.74]. I say that a(nother) medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, cannot be attached to AB.

For, if possible, let DB also be (so) attached to AB. Thus, AD and DB are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that) contained by AD and DB[Prop. 10.74]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For [again] both exceed by the same (area)—(namely), the (square) on AB[Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice

the (rectangle contained) by AC and CB by a rational (area). For both (are) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the) [squares] on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Props. 10.15, 10.23 corr.], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

Thus, only one medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line). (Which is) the very thing it was required to show.

[†] This proposition is equivalent to Prop. 10.43, with minus signs instead of plus signs.

πα΄.

Τῆ μέσης ἀποτομῆ δευτέρα μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος τῆ ὅλη, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.



Έστω μέσης ἀποτομὴ δευτέρα ἡ AB καὶ τῆ AB προσαρμόζουσα ἡ BΓ· αἱ ἄρα AΓ, ΓΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν AΓ, ΓΒ· λέγω, ὅτι τῆ AB ἑτέρα οὐ προσαρμόσει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὕσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.

Eἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ ἀ ΑΔ, ΔΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΜ· τῷ δὲ δἰς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον ἀφηρήσθω τὸ ΘΗ πλάτος ποιοῦν τὴν ΘΜ· λοιπὸν ἄρα τὸ ΕΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ· ὥστε ἡ ΑΒ δύναται τὸ ΕΛ. πάλιν δὴ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΙ πλάτος ποιοῦν τὴν ΕΝ· ἔστι δὲ καὶ τὸ ΕΛ ἴσον τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ· λοιπὸν ἄρα τὸ ΘΙ ἴσον ἐστὶ τῷ δἰς ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ μέσαι εἰσιν αἱ ΑΓ, ΓΒ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ. καί ἐστιν ἴσα τῷ ΕΗ· μέσον ἄρα καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν

Proposition 81

Only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line).[†]



Let AB be a second apotome of a medial (straightline), with BC (so) attached to AB. Thus, AC and CBare medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AC and CB [Prop. 10.75]. I say that a(nother) medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, cannot be attached to AB.

For, if possible, let BD be (so) attached. Thus, AD and DB are also medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AD and DB [Prop. 10.75]. And let the rational (straight-line) EF be laid down. And let EG, equal to the (sum of the squares) on AC and CB, have been applied to EF, producing EM as breadth. And let HG, equal to twice the (rectangle contained) by AC and CB, have been subtracted (from EG), producing HM as breadth. The remainder EL is thus equal to the (square) on AB [Prop. 2.7]. Hence, AB is the

τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῆ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, καὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΘΗ καὶ τὸ ΘΗ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΜ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΘΜ καὶ ἀσύμμετρος τῆ ΕΖ μήκει. καὶ ἐπεὶ αἱ ΑΓ, ΓΒ δυνάμει μόνον σύμμετροί είσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΓ τῆ ΓΒ μήχει. ὡς δὲ ἡ ΑΓ πρός την ΓΒ, οὕτως ἐστὶ τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ, τῷ δὲ ὑπὸ τῶν ΑΓ, ΓΒ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΗ τῷ ΘΗ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΗ, οὕτως ἐστὶν ἡ ΕΜ πρὸς τὴν ΘΜ· ἀσύμμετρος ἄρα ἐστὶν ή ΕΜ τῆ ΜΘ μήχει. χαί εἰσιν ἀμφότεραι ῥηταί· αἱ ΕΜ, ΜΘ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ή ΕΘ, προσαρμόζουσα δὲ αὐτῆ ἡ ΘΜ. ὁμοίως δὴ δείξομεν, ότι καὶ ἡ ΘΝ αὐτῇ προσαρμόζει· τῇ ἄρα ἀποτομῇ ἄλλῃ καὶ άλλη προσαρμόζει εύθεῖα δυνάμει μόνον σύμμετρος οὕσα τῆ ὅλη. ὅπερ ἐστὶν ἀδύνατον.

Τῆ ἄρα μέσης ἀποτομῆ δευτέρα μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οῦσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα[.] ὅπερ ἔδει δεῖξαι.

square-root of EL. So, again, let EI, equal to the (sum of the squares) on AD and DB have been applied to EF, producing EN as breadth. And EL is also equal to the square on AB. Thus, the remainder HI is equal to twice the (rectangle contained) by AD and DB [Prop. 2.7]. And since AC and CB are (both) medial (straight-lines), the (sum of the squares) on AC and CB is also medial. And it is equal to EG. Thus, EG is also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) EF, producing EM as breadth. Thus, EM is rational, and incommensurable in length with EF[Prop. 10.22]. Again, since the (rectangle contained) by AC and CB is medial, twice the (rectangle contained) by AC and CB is also medial [Prop. 10.23 corr.]. And it is equal to HG. Thus, HG is also medial. And it is applied to the rational (straight-line) EF, producing HMas breadth. Thus, HM is also rational, and incommensurable in length with EF [Prop. 10.22]. And since ACand CB are commensurable in square only, AC is thus incommensurable in length with CB. And as AC (is) to CB, so the (square) on AC is to the (rectangle contained) by AC and CB [Prop. 10.21 corr.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, the (sum of the squares) on AC and CB is commensurable with the (square) on AC, and twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. Thus, the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. And EG is equal to the (sum of the squares) on AC and CB. And GH is equal to twice the (rectangle contained) by AC and CB. Thus, EG is incommensurable with HG. And as EG (is) to HG, so EMis to HM [Prop. 6.1]. Thus, EM is incommensurable in length with MH [Prop. 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], and HM(is) attached to it. So, similarly, we can show that HN(is) also (commensurable in square only with EN and is) attached to (EH). Thus, different straight-lines, which are commensurable in square only with the whole, are attached to an apotome. The very thing is impossible [Prop. 10.79].

Thus, only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line). (Which is) the very thing it was required to show. [†] This proposition is equivalent to Prop. 10.44, with minus signs instead of plus signs.

πβ'.

Τῆ ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῆ ὅλη ποιοῦσα μετὰ τῆς ὅλης τὸ μὲν ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ἑητόν, τὸ δὲ δἰς ὑπ' αὐτῶν μέσον.

A B
$$\Gamma \Delta$$

Έστω ή ἐλάσσων ή AB, καὶ τῆ AB προσαρμόζουσα ἔστω ή BΓ· αἱ ἄρα AΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ' αὐτῶν μέσον· λέγω, ὅτι τῆ AB ἑτέρα εὐθεῖα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. καὶ ἐπεί, ῷ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τὰ δὲ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα τῶν ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνων ὑπερέχει ῥητῷ· ῥητὰ γάρ ἐστιν ἀμφότερα· καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἄρα τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα.

Τῆ ἄρα ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῆ ὅλῃ καὶ ποιοῦσα τὰ μὲν ἀπ' αὐτῶν τετράγωνα ἅμα ῥητόν, τὸ δὲ δὶς ὑπ' αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.

Proposition 82

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the (sum of the) squares on them rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).

Let AB be a minor (straight-line), and let BC be (so) attached to AB. Thus, AC and CB are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial [Prop. 10.76]. I say that another another straight-line fulfilling the same (conditions) cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76]. And since by whatever (area) the (sum of the squares) on AD and DBexceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area) [Prop. 2.7]. And the (sum of the) squares on AD and DB exceeds the (sum of the) squares on ACand CB by a rational (area). For both are rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by ACand CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, only one straight-line, which is incommensurable in square with the whole, and (with the whole) makes the squares on them (added) together rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line). (Which is) the very thing it was required to show.

[†] This proposition is equivalent to Prop. 10.45, with minus signs instead of plus signs.

πγ΄.

Τῆ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιούση μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῆ ὅλη, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ' αὐτῶν ἑητόν.



Proposition 83

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, can be attached to that (straight-line) which with a rational (area) makes a medial whole.[†]



Έστω ή μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ AB, καὶ τῆ AB προσαρμοζέτω ἡ BΓ· αἱ ἄρα AΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι τῆ AB ἑτέρα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα. ἐπεὶ οὐν, ῷ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῷ ὑπερέχει καὶ τὸ δἰς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ ἀκολούθως τοῖς πρὸ αὐτοῦ, τὸ δὲ δἰς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δἰς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ἑητῷ· ἑητὰ γάρ ἐστιν ἀμφότερα· καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ἑητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα.

Οὐκ ἄρα τῆ AB ἑτέρα προσαρμόσει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῆ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ προειρημένα μία ἄρα μόνον προσαρμόσει· ὅπερ ἔδει δεῖξαι. Let AB be a (straight-line) which with a rational (area) makes a medial whole, and let BC be (so) attached to AB. Thus, AC and CB are (straight-lines which are) incommensurable in square, fulfilling the (other) proscribed (conditions) [Prop. 10.77]. I say that another (straight-line) fulfilling the same (conditions) cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also straight-lines (which are) incommensurable in square, fulfilling the (other) prescribed (conditions) [Prop. 10.77]. Therefore, analogously to the (propositions) before this, since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area). For they are (both) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the squares) on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, another straight-line cannot be attached to AB, which is incommensurable in square with the whole, and fulfills the (other) aforementioned (conditions) with the whole. Thus, only one (such straight-line) can be (so) attached. (Which is) the very thing it was required to show.

[†] This proposition is equivalent to Prop. 10.46, with minus signs instead of plus signs.

πδ'.

Τῆ μετὰ μέσου μέσον τὸ ὅλον ποιούση μία μόνη προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῆ ὅλη, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγχείμενον ἐχ τῶν ἀπ' αὐτῶν τετραγώνων μέσον τό τε δὶς ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγχειμένῷ ἐχ τῶν ἀπ' αὐτῶν.

Έστω ή μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ AB, προσαρμόζουσα δὲ αὐτῆ ἡ BΓ· αί ἄρα AΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. λέγω, ὅτι τῆ AB ἑτέρα οὐ προσαρμόσει ποιοῦσα προειρημένα.

Proposition 84

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the (squares) on them, can be attached to that (straight-line) which with a medial (area) makes a medial whole.[†]

Let AB be a (straight-line) which with a medial (area) makes a medial whole, BC being (so) attached to it. Thus, AC and CB are incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78]. I say that a(nother) (straight-line) fulfilling the aforementioned (conditions) cannot be attached to AB.



Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τά τε ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα ἄμα μέσον καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ μέσον καὶ ἔτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἀσύμμετρα τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μέν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΜ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ίσον παρὰ τὴν EZ παραβεβλήσθω τὸ ΘΗ πλάτος ποιοῦν τὴν ΘM^{\cdot} λοιπὸν ắρα τὸ ἀπὸ τῆς AB ἴσον ἐστὶ τῷ $E\Lambda^{\cdot}$ ἡ άρα ΑΒ δύναται τὸ ΕΛ. πάλιν τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΙ πλάτος ποιοῦν τὴν ΕΝ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς AB ἴσον τῷ ΕΛ· λοιπὸν ἄρα τὸ δὶς ύπὸ τῶν ΑΔ, ΔΒ ἴσον [ἐστὶ] τῷ ΘΙ. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ καί ἐστιν ἴσον τῷ ΕΗ, μέσον ἄρα ἐστὶ χαὶ τὸ ΕΗ. χαὶ παρὰ ῥητὴν τὴν ΕΖ παράχειται πλάτος ποιοῦν τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῆ ΕΖ μήχει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ καί ἐστιν ἴσον τῷ ΘΗ, μέσον ἄρα καὶ τὸ ΘΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΜ· ρητή ἄρα ἐστίν ή ΘΜ καὶ ἀσύμμετρος τῆ ΕΖ μήκει. καὶ ἐπεὶ άσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρόν ἐστι καὶ τὸ ΕΗ τῷ ΘΗ· ἀσύμμετρος ἄρα έστι και ή ΕΜ τῆ ΜΘ μήκει. και είσιν ἀμφότεραι ῥηται· αί ἄρα EM, MΘ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ άρα ἐστὶν ἡ ΕΘ, προσαρμόζουσα δὲ αὐτῆ ἡ ΘΜ. ὑμοίως δὴ δείξομεν, ὅτι ἡ ΕΘ πάλιν ἀποτομή ἐστιν, προσαρμόζουσα δὲ αὐτῆ ἡ ΘΝ. τῆ ἄρα ἀποτομῆ ἄλλη καὶ ἄλλη προσαρμόζει ρητή δυνάμει μόνον σύμμετρος ούσα τῆ ὅλῃ. ὅπερ ἐδείχϑη άδύνατον. οὐκ ἄρα τῆ ΑΒ ἑτέρα προσαρμόσει εὐθεῖα.

Τῆ ἄρα AB μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οῦσα τῆ ὅλη, μετὰ δὲ τῆς ὅλης ποιοῦσα τά τε ἀπ' αὐτῶν τετράγωνα ἅμα μέσον καὶ τὸ δὶς ὑπ' αὐτῶν μέσον καὶ ἔτι τὰ ἀπ' αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ' αὐτῶν· ὅπερ ἔδει δεῖξαι.



For, if possible, let BD be (so) attached. Hence, AD and DB are also (straight-lines which are) incommensurable in square, making the squares on AD and DB (added) together medial, and twice the (rectangle contained) by AD and DB medial, and, moreover, the (sum of the squares) on AD and DB incommensurable with twice the (rectangle contained) by AD and DB[Prop. 10.78]. And let the rational (straight-line) EF be laid down. And let *EG*, equal to the (sum of the squares) on AC and CB, have been applied to EF, producing EMas breadth. And let HG, equal to twice the (rectangle contained) by AC and CB, have been applied to EF, producing HM as breadth. Thus, the remaining (square) on AB is equal to EL [Prop. 2.7]. Thus, AB is the squareroot of EL. Again, let EI, equal to the (sum of the squares) on AD and DB, have been applied to EF, producing EN as breadth. And the (square) on AB is also equal to EL. Thus, the remaining twice the (rectangle contained) by AD and DB [is] equal to HI [Prop. 2.7]. And since the sum of the (squares) on AC and CB is medial, and is equal to EG, EG is thus also medial. And it is applied to the rational (straight-line) EF, producing EM as breadth. EM is thus rational, and incommensurable in length with EF [Prop. 10.22]. Again, since twice the (rectangle contained) by AC and CB is medial, and is equal to HG, HG is thus also medial. And it is applied to the rational (straight-line) EF, producing HM as breadth. HM is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is also incommensurable with HG. Thus, EMis also incommensurable in length with MH [Props. 6.1. 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], with HM attached to it. So, similarly, we can show that EH is again an apotome, with HNattached to it. Thus, different rational (straight-lines), which are commensurable in square only with the whole, are attached to an apotome. The very thing was shown

(to be) impossible [Prop. 10.79]. Thus, another straightline cannot be (so) attached to AB.

Thus, only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the squares on them (added) together medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with the (rectangle contained) by them, can be attached to AB. (Which is) the very thing it was required to show.

[†] This proposition is equivalent to Prop. 10.47, with minus signs instead of plus signs.

Οροι τρίτοι.

ια'. Υποχειμένης ἑητῆς καὶ ἀποτομῆς, ἐἀν μὲν ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήχει, καὶ ἡ ὅλη σύμμετρος ἦ τῆ ἐχχειμένῃ ἑητῆ μήχει, καλείσθω ἀποτομὴ πρώτη.

ιβ΄. Ἐἀν δὲ ἡ προσαρμόζουσα σύμμετρος ἢ τῆ ἐκκειμένῃ ἑητῆ μήκει, καὶ ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καλείσθω ἀποτομὴ δευτέρα.

ιγ΄. Ἐὰν δὲ μηδετέρα σύμμετρος ἢ τῆ ἐxxειμένῃ ἑητῆ μήχει, ἡ δὲ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καλείσθω ἀποτομὴ τρίτη.

ιδ΄. Πάλιν, ἐὰν ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ [μήκει], ἐὰν μὲν ἡ ὅλη σύμμετρος ἢ τῆ ἐκκειμένῃ ῥητῆ μήκει, καλείσθω ἀποτομὴ τετάρτη.

ιε΄. Ἐὰν δὲ ἡ προσαρμόζουσα, πέμπτη.

ις΄. Ἐὰν δὲ μηδετέρα, ἕκτη.

Definitions III

11. Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome.

12. And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome.

13. And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome.

14. Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome.

15. And if the attached (straight-line is commensurable), a fifth (apotome).

16. And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

Proposition 85

To find a first apotome.

πε΄.

Εύρεῖν τὴν πρώτην ἀποτομήν.



Έκκείσθω ἑητὴ ἡ Α, καὶ τῆ Α μήκει σύμμετρος ἔστω ή ΒΗ· ρητή ἄρα ἐστὶ καὶ ή ΒΗ. καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔΕ, ΕΖ, ῶν ἡ ὑπεροχὴ ὁ ΖΔ μὴ έστω τετράγωνος· οὐδ' ἄρα ὁ ΕΔ πρὸς τὸν ΔΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ώς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ τετράγωνον πρός τὸ ἀπὸ τὴς ΗΓ τετράγωνον· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς BH τῷ ἀπὸ τῆς HΓ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΒΗ· ἑητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΓ· ἑητὴ ἄρα ἐστὶ καὶ ή ΗΓ. καὶ ἐπεὶ ὁ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ άπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος άριθμός πρός τετράγωνον άριθμόν. ἀσύμμετρος ἄρα ἐστίν ή ΒΗ τῆ ΗΓ μήχει. χαί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα BΓ ἀποτομή έστιν. λέγω δή, ὄτι καὶ πρώτη.

[°]Ωι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς HΓ, ἔστω τὸ ἀπὸ τῆς Θ. καὶ ἐπεί ἐστιν ὡς ὁ ΕΔ πρὸς τὸν ΖΔ, οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς HΓ, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς HB πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΔΕ πρὸς τὸν ΕΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἑκάτερος γὰρ τετράγωνος ἐστιν· καὶ τὸ ἀπὸ τῆς HB ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ BH τῆ Θ μήκει. καὶ δύναται ἡ BH τῆς HΓ μεῖζον τῷ ἀπὸ τῆς Θ ἡ BH ἄρα τῆς HΓ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει. καί ἐστιν ἡ ὅλη ἡ BH σύμμετρος τῆ ἑκκειμένη ἑρτῆ μήκει τῆ Α. ἡ BΓ ἄρα ἀποτομή ἐστι πρώτη.

Εὕρηται ἄρα ή πρώτη ἀποτομὴ ή ΒΓ· ὅπερ ἔδει εὑρεῖν.

[†] See footnote to Prop. 10.48.

Εύρεῖν τὴν δευτέραν ἀποτομήν.



Let the rational (straight-line) A be laid down. And let BG be commensurable in length with A. BG is thus also a rational (straight-line). And let two square numbers DE and EF be laid down, and let their difference FD be not square [Prop. 10.28 lem. I]. Thus, ED does not have to DF the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as ED (is) to DF, so the square on BG(is) to the square on GC [Prop. 10.6. corr.]. Thus, the (square) on BG is commensurable with the (square) on GC [Prop. 10.6]. And the (square) on BG (is) rational. Thus, the (square) on GC (is) also rational. Thus, GC is also rational. And since ED does not have to DFthe ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). Thus, BG and GCare rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

Let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC[Prop. 10.13 lem.]. And since as ED is to FD, so the (square) on BG (is) to the (square) on GC, thus, via conversion, as DE is to EF, so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.]. And DE has to EFthe ratio which (some) square-number (has) to (some) square-number. For each is a square (number). Thus, the (square) on GB also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, BG is commensurable in length with H[Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG). And the whole, BG, is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, BC is a first apotome [Def. 10.11].

Thus, the first apotome BC has been found. (Which is) the very thing it was required to find.

Proposition 86

To find a second apotome.

Έχχείσθω ἑητὴ ἡ Α καὶ τῆ Α σύμμετρος μήχει ἡ ΗΓ. ἑητὴ ἄρα ἐστὶν ἡ ΗΓ. καὶ ἐχχείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔΕ, ΕΖ, ῶν ἡ ὑπεροχὴ ὁ ΔΖ μὴ ἔστω τετράγωνος. καὶ πεποιήσθω ὡς ὁ ΖΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΓΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΒ τετράγωνον. σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΓΗ τετράγωνον τῷ ἀπὸ τῆς ΗΒ τετραγώνω. ἑητὸν δὲ τὸ ἀπὸ τῆς ΓΗ. ἑητὸν ἄρα [ἐστὶ] καὶ τὸ ἀπὸ τῆς ΗΒ· ἑητὴ ἄρα ἐστὶν ἡ ΒΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΗΓ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΒ λόγον οὐχ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀσύμμετρός ἐστιν ἡ ΓΗ τῆ ΗΒ μήχει. καί εἰσιν ἀμφότεραι ἑηταί· αἱ ΓΗ, ΗΒ ἅρα ρηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΒΓ ἅρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ δευτέρα.



[°]Ωι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς HΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οῦν ἐστιν ὡς τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς HΓ, οὕτως ὁ ΕΔ ἀριθμὸς πρὸς τὸν ΔΖ ἀριθμόν, ἀναστρέψαντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς Θ, οὕτως ὁ ΔΕ πρὸς τὸν ΕΖ. καί ἐστιν ἑκάτερος τῶν ΔΕ, ΕΖ τετράγωνος· τὸ ἄρα ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ BH τῆ Θ μήκει. καὶ δύναται ἡ BH τῆς HΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ BH ἄρα τῆς HΓ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΓΗ τῆ ἐκκειμένη ἑητῆ σύμμετρος τῆ Α. ἡ BΓ ἅρα ἀποτομή ἐστι δευτέτα.

Εύρηται άρα δευτέρα άποτομή ή ΒΓ· ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.49.

πζ΄.

Εύρεῖν τὴν τρίτην ἀποτομήν.

Let the rational (straight-line) A, and GC (which is) commensurable in length with A, be laid down. Thus, GC is a rational (straight-line). And let the two square numbers DE and EF be laid down, and let their difference DF be not square [Prop. 10.28 lem. I]. And let it have been contrived that as FD (is) to DE, so the square on CG (is) to the square on GB [Prop. 10.6 corr.]. Thus, the square on CG is commensurable with the square on GB [Prop. 10.6]. And the (square) on CG (is) rational. Thus, the (square) on GB [is] also rational. Thus, BG is a rational (straight-line). And since the square on GC does not have to the (square) on GB the ratio which (some) square number (has) to (some) square number, CG is incommensurable in length with GB [Prop. 10.9]. And they are both rational (straight-lines). Thus, CG and GBare rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that it is also a second (apotome).



For let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC[Prop. 10.13 lem.]. Therefore, since as the (square) on BG is to the (square) on GC, so the number ED (is) to the number DF, thus, also, via conversion, as the (square) on BG is to the (square) on H, so DE (is) to EF [Prop. 5.19 corr.]. And DE and EF are each square (numbers). Thus, the (square) on BG has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, BG is commensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG). And the attachment CGis commensurable (in length) with the (prevously) laid down rational (straight-line) A. Thus, BC is a second apotome [Def. 10.12].[†]

Thus, the second apotome BC has been found. (Which is) the very thing it was required to show.

Proposition 87

To find a third apotome.



Έκκείσθω ἑητὴ ἡ Α, καὶ ἐκκείσθωσαν τρεῖς ἀριθμοὶ οί Ε, ΒΓ, ΓΔ λόγον μη έχοντες πρός αλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ ΓΒ πρός τὸν ΒΔ λόγον ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ πεποιήσθω ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ τετράγωνον, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τὴς ΗΘ. ἐπεὶ οὖν ἐστιν ώς ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρός τὸ ἀπὸ τῆς ΖΗ τετράγωνον, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Α τετράγωνον τῷ ἀπὸ τῆς ΖΗ τετραγώνω. ρητὸν δὲ τὸ ἀπὸ τῆς A τετράγωνον. ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΖΗ· ἑητὴ ἄρα ἐστὶν ἡ ΖΗ. καὶ ἐπεὶ ὁ Ε πρὸς τὸν ΒΓ λόγον ούχ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον άριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ [τετράγωνον] λόγον ἔχει, ὄν τετράγωνος ἀριθμὸς πρός τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῆ ΖΗ μήχει. πάλιν, ἐπεί ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ άπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΘ, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΖΗ· ἑητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΘ· ἑητὴ ἄρα ἐστὶν ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΓ πρὸς τὸν ΓΔ λόγον οὐκ ἔγει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρός τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῆ ΗΘ μήκει. καί είσιν ἀμφότεραι ῥηταί· αἱ ΖΗ, ΗΘ ἄρα ῥηταί είσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστίν ή ΖΘ. λέγω δή, ὅτι καὶ τρίτη.

Ἐπεὶ γάρ ἐστιν ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν Γ Δ , οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς Θ H, δι ἴσου άρα ἐστὶν ὡς ὁ Ε πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΘΗ. ὁ δὲ Ε πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ άπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἡ Α τῆ ΗΘ μήχει. οὐδετέρα ἄρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῆ έκκειμένη ἑητῆ τῆ Α μήκει. ῷ οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ, ἔστω τὸ ἀπὸ τῆς Κ. ἐπεὶ οὖν ἐστιν ώς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΒΓ πρὸς τὸν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΒΓ πρός τὸν ΒΔ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ τὸ ἑπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον



Let the rational (straight-line) A be laid down. And let the three numbers, E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. And let CB have to BD the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as E (is) to BC, so the square on A (is) to the square on FG, and as BC (is) to CD, so the square on FG (is) to the (square) on GH [Prop. 10.6 corr.]. Therefore, since as E is to BC, so the square on A (is) to the square on FG, the square on A is thus commensurable with the square on FG [Prop. 10.6]. And the square on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the square on Athus does not have to the [square] on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with FG[Prop. 10.9]. Again, since as BC is to CD, so the square on FG is to the (square) on GH, the square on FG is thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH is a rational (straightline). And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on *GH* the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). FG and GH are thus rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since as E is to BC, so the square on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on HG, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on HG [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number (has) to (some) and H (square) on GH the ratio which (some) square number (has) to (some) square number (has) to (some) square number (has) to (some) and H (square) on GH the ratio which (some) square number (has) to (some) square (has) to

ἀριθμόν. σύμμετρός ἄρα ἐστὶν ἡ ΖΗ τῆ Κ μήχει, καὶ δύναται ἡ ΖΗ τῆς ΗΘ μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῆ. καὶ οὐδετέρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῆ ἐχχειμένῃ ῥητῆ τῆ Α μήχει: ἡ ΖΘ ἄρα ἀποτομή ἐστι τρίτη.

Εὕρηται ἄρα ή τρίτη ἀποτομή ή ΖΘ· ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.50.

πη΄. Εύρεῖν τὴν τετάρτην ἀποτομήν. Α Δ Ζ Ε

Έχχείσθω ἡητὴ ἡ Α καὶ τῆ Α μήχει σύμμετρος ἡ BH· ἡητὴ ἄρα ἐστὶ καὶ ἡ BH. καὶ ἐχχείσθωσαν δύο ἀριθμοὶ οἱ ΔΖ, ΖΕ, ὥστε τὸν ΔΕ ὅλον πρὸς ἑκάτερον τῶν ΔΖ, ΕΖ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς BH τετράγωνον πρὸς τὸ ἀπὸ τῆς HΓ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς BH τῷ ἀπὸ τῆς HΓ. ῥητὸν δὲ τὸ ἀπὸ τῆς BH· ἑητὸν ἄρα καὶ τὸ ἀπὸ τῆς HΓ· ἑητὴ ἄρα ἐστὶν ἡ HΓ. καὶ ἐπεὶ ὁ ΔΕ πρὸς τὸν ΕΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς HΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῆ HΓ μήχει. καί εἰσιν ἀμφότεραι ἑηταί· αί BH, HΓ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ BΓ. [λέγω δή, ὅτι καὶ τετάρτη.]

²Ωι οῦν μεῖζόν ἐστι τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς HΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οῦν ἐστιν ὡς ὁ ΔΕ πρὸς τὸν EZ, οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς HΓ, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς HB πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον

(previously) laid down rational (straight-line) A. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH[Prop. 10.13 lem.]. Therefore, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, thus, via conversion, as BC is to BD, so the square on FG (is) to the square on K [Prop. 5.19 corr.]. And BC has to BDthe ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number. FG is thus commensurable in length with K [Prop. 10.9]. And the square on FG is (thus) greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, FH is a third apotome [Def. 10.13].

Thus, the third apotome FH has been found. (Which is) very thing it was required to show.



Let the rational (straight-line) A, and BG (which is) commensurable in length with A, be laid down. Thus, BG is also a rational (straight-line). And let the two numbers DF and FE be laid down such that the whole, DE, does not have to each of DF and EF the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as DE (is) to EF, so the square on BG (is) to the (square) on GC[Prop. 10.6 corr.]. The (square) on BG is thus commensurable with the (square) on GC [Prop. 10.6]. And the (square) on BG (is) rational. Thus, the (square) on GC (is) also rational. Thus, GC (is) a rational (straightline). And since DE does not have to EF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). Thus, BG and GC are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. [So, I say that (it ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς HB πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὅν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῆ Θ μήκει. καὶ δύναται ἡ BH τῆς HΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ἄρα BH τῆς HΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καὶ ἐστιν ὅλη ἡ BH σύμμετρος τῆ ἐκκειμένῃ ἑητῆ μήκει τῆ Α. ἡ ἄρα BΓ ἀποτομή ἐστι τετάρτῃ.

Εὕρηται ἄρα ή τετάρτη ἀποτομή· ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.51.

(⊢) ⊦

πϑ΄.



Έκκείσθω ἑητὴ ἡ Α, καὶ τῆ Α μήκει σύμμετρος ἔστω ἡ ΓΗ· ἑητὴ ἄρα [ἐστὶν] ἡ ΓΗ. καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔΖ, ΖΕ, ὥστε τὸν ΔΕ πρὸς ἑκάτερον τῶν ΔΖ, ΖΕ λόγον πάλιν μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς ὁ ΖΕ πρὸς τὸν ΕΔ, οὕτως τὸ ἀπὸ τῆς ΓΗ πρὸς τὸ ἀπὸ τῆς ΗΒ. ἑητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΒ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐπεί ἐστιν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, ὁ δὲ ΔΕ πρὸς τὸν ΕΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῆ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ἑηταί· αἱ ΒΗ, ΗΓ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΒΓ ἅρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ πέμπτη.

²Ωι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς HΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οῦν ἐστιν ὡς τὸ ἀπὸ τὴς BH πρὸς τὸ ἀπὸ τῆς HΓ, οὕτως ὁ ΔΕ πρὸς τὸν ΕΖ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς Θ, ὁ δὲ ΕΔ πρὸς τὸν ΔΖ λόγον οὐx ἔχει, ὃν is) also a fourth (apotome).]

Now, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC[Prop. 10.13 lem.]. Therefore, since as DE is to EF, so the (square) on BG (is) to the (square) on GC, thus, also, via conversion, as ED is to DF, so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.]. And EDdoes not have to DF the ratio which (some) square number (has) to (some) square number. Thus, the (square) on GB does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with H[Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on BG is greater than (the square) on GC by the (square) on (some straight-line) incommensurable (in length) with (BG). And the whole, BG, is commensurable in length with the the (previously) laid down rational (straightline) A. Thus, BC is a fourth apotome [Def. 10.14].^{\dagger}

Thus, a fourth apotome has been found. (Which is) the very thing it was required to show.

Proposition 89

To find a fifth apotome.



Let the rational (straight-line) A be laid down, and let CG be commensurable in length with A. Thus, CG [is] a rational (straight-line). And let the two numbers DFand FE be laid down such that DE again does not have to each of DF and FE the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as FE (is) to ED, so the (square) on CG (is) to the (square) on GB. Thus, the (square) on GB(is) also rational [Prop. 10.6]. Thus, BG is also rational. And since as DE is to EF, so the (square) on BG (is) to the (square) on GC. And DE does not have to EF the ratio which (some) square number (has) to (some) square number. The (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). BG and GC are thus rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῆ Θ μήχει. καὶ δύναται ἡ BH τῆς HΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ HB ἄρα τῆς HΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήχει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΓΗ σύμμετρος τῆ ἐχχειμένῃ ἑῃτῆ τῆ Α μήχει· ἡ ἄρα BΓ ἀποτομή ἐστι πέμπτη.

Εὕρηται ἄρα ἡ πέμπτη ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.52.

۲.

Εύρεῖν τὴν ἕχτην ἀποτομήν.

Έκκείσθω ἑητὴ ἡ A καὶ τρεῖς ἀριθμοὶ οἱ E, BΓ, ΓΔ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔτι δὲ καὶ ὁ ΓΒ πρὸς τὸν ΒΔ λόγον μὴ ἐχετώ, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς μὲν ὁ E πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς ZH, ὡς δὲ ὁ BΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς HΘ.



Έπεὶ οὕν ἐστιν ὡς ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, σύμμετρον ἄρα τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς ΖΗ. ἑητὸν δὲ τὸ ἀπὸ τῆς Α· ἑητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΖΗ· ἑητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ Ε πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῆ ΖΗ μήκει. πάλιν, ἐπεί ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ἑητὸν

For, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC[Prop. 10.13 lem.]. Therefore, since as the (square) on BG (is) to the (square) on GC, so DE (is) to EF, thus, via conversion, as ED is to DF, so the (square) on BG(is) to the (square) on H [Prop. 5.19 corr.]. And ED does not have to DF the ratio which (some) square number (has) to (some) square number. Thus, the (square) on BG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on GB is greater than (the square on) GC by the (square) on (some straight-line) incommensurable in length with (GB). And the attachment CG is commensurable in length with the (previously) laid down rational (straightline) A. Thus, BC is a fifth apotome [Def. 10.15].^{\dagger}

Thus, the fifth apotome BC has been found. (Which is) the very thing it was required to show.

Proposition 90

To find a sixth apotome.

Let the rational (straight-line) A, and the three numbers E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. Furthermore, let CB also not have to BD the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as E (is) to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].



Therefore, since as E is to BC, so the (square) on A (is) to the (square) on FG, the (square) on A (is) thus commensurable with the (square) on FG [Prop. 10.6]. And the (square) on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is also a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number (has)

δὲ τὸ ἀπὸ τῆς ZH· ἑητὸν ἄρα καὶ τὸ ἀπὸ τῆς HΘ· ἑητὴ ἄρα καὶ ἡ HΘ. καὶ ἐπεὶ ὁ BΓ πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς HΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ZH τῆ HΘ μήχει. καί εἰσιν ἀμφότεραι ἑηταί· αἱ ZH, HΘ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα ZΘ ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ ἔχτη.

Έπεὶ γάρ ἐστιν ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς ZH, ὡς δὲ ἑ BΓ πρὸς τὸν Γ Δ , οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ. ἑ δὲ Ε πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρός τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῆ ΗΘ μήχει· οὐδετέρα ἄρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῆ Α ῥητῆ μήχει. ῷ οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ, έστω τὸ ἀπὸ τῆς Κ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΓΒ πρὸς τὸν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΓΒ πρὸς τὸν ΒΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. οὐδ' ἄρα τὸ άπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔγει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῆ Κ μήκει. καὶ δύναται ἡ ΖΗ τῆς ΗΘ μεῖζον τῷ ἀπὸ τῆς Κ· ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου έαυτῆ μήκει. καὶ οὐδετέρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῆ έκκειμένη ἑητῆ μήκει τῆ Α. ἡ ἄρα ΖΘ ἀποτομή ἐστιν ἕκτη.

Εὕρηται ἄρα ή ἕχτη ἀποτομὴ ἡ ΖΘ· ὅπερ ἔδει δεῖξαι.

[†] See footnote to Prop. 10.53.

commensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH(is) also rational. Thus, GH (is) also rational. And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square (number) has to (some) square (number) either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straightlines). Thus, FG and GH are rational (straight-lines) which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since as E is to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on *GH* [Prop. 5.22]. And *E* does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) GH the ratio which (some) square number (has) to (some) square number either. A is thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the rational (straight-line) A. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on *GH* [Prop. 10.13 lem.]. Therefore, since as BC is to CD, so the (square) on FG(is) to the (square) on GH, thus, via conversion, as CB is to BD, so the (square) on FG (is) to the (square) on K[Prop. 5.19 corr.]. And CB does not have to BD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. FG is thus incommensurable in length with K [Prop. 10.9]. And the square on FG is greater than (the square on) GH by the (square) on K. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straightline) incommensurable in length with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, FHis a sixth apotome [Def. 10.16].

Thus, the sixth apotome FH has been found. (Which is) the very thing it was required to show.

4α'.

Έὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης, ἡ τὸ χωρίον δυναμένη ἀπορομή ἐστιν.

Περιεχέσθω γὰρ χωρίον τὸ AB ὑπὸ ῥητῆς τῆς AΓ καὶ ἀποτομῆς πρώτης τῆς AΔ· λέγω, ὅτι ἡ τὸ AB χωρίον δυ-ναμένη ἀποτομή ἐστιν.



Έπει γὰρ ἀποτομή ἐστι πρώτη ἡ ΑΔ, ἔστω αὐτῆ προσαρμόζουσα ἡ ΔΗ· αί ΑΗ, ΗΔ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ὅλη ἡ ΑΗ σύμμετρός ἐστι τῆ ἐκκειμένῃ ἑητῆ τῆ ΑΓ, καὶ ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει· ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῆ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. τετμήσθω ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· σύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῆ ΖΗ. καὶ διὰ τῶν Ε, Ζ, Η σημείων τῆ ΑΓ παράλληλοι ἤχθωσαν αἱ ΕΘ, ΖΙ, ΗΚ.

Καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΖ τῆ ΖΗ μήκει, καὶ ἡ ΑΗ ἄρα ἐκατέρα τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ἀλλὰ ἡ ΑΗ σύμμετρός ἐστι τῆ ΑΓ· καὶ ἑκατέρα ἄρα τῶν ΑΖ, ΖΗ σύμμετρός ἐστι τῆ ΑΓ μήκει. καί ἐστι ῥητὴ ἡ ΑΓ· ἑητὴ ἄρα καὶ ἑκατέρα τῶν ΑΖ, ΖΗ· ὥστε καὶ ἑκάτερον τῶν ΑΙ, ΖΚ ἑητόν ἐστιν. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῆ ΕΗ μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρα τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ἑητὴ δὲ ἡ ΔΗ καὶ ἀσύμμετρος τῆ ΑΓ μήκει· ἑητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ ἀσύμμετρος τῆ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν.

Κείσθω δὴ τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ZK ἴσον τετράγωνον ἀφηρήσθω χοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΛΟΜ τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, χαὶ χαταγεγράφθω τὸ σχῆμα. ἐπεὶ οῦν ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΖ, ZH περιεχόμενον ὀρθογώνιον τῷ ἀπὸ τῆς EH τετραγώνῳ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν EH, οὕτως ἡ EH πρὸς τὴν ZH. ἀλλ᾽ ὡς μὲν ἡ ΑΖ πρὸς τὴν EH, οὕτως τὸ AI πρὸς τὸ EK, ὡς δὲ ἡ EH πρὸς τὴν ZH, οὕτως ἐστὶ τὸ EK πρὸς τὸ KZ[·] τῶν ἄρα AI, KZ μέσον ἀνάλογόν ἐστι τὸ EK. ἔστι δὲ xαὶ τῶν ΛΜ, ΝΞ μέσον ἀνάλογον τὸ MN, ὡς ἐν τοῖς ἔμπροσθεν ἑδείχθη, xaí ἐστι τὸ [μὲν] AI τῷ ΛΜ τετραγώνῳ ἴσον, τὸ δὲ KZ τῷ ΝΞ· xαὶ τὸ MN ἄρα τῷ EK ἴσον ἐστίν. ἀλλὰ τὸ μὲν EK τῷ ΔΘ ἐστιν ἴσον, τὸ δὲ MN τῷ ΛΞ· τὸ ἄρα

Proposition 91

If an area is contained by a rational (straight-line) and a first apotome then the square-root of the area is an apotome.

For let the area AB have been contained by the rational (straight-line) AC and the first apotome AD. I say that the square-root of area AB is an apotome.



For since AD is a first apotome, let DG be its attachment. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And the whole, AG, is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.11]. Thus, if (an area) equal to the fourth part of the (square) on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Let DG have been cut in half at E. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. AF is thus commensurable (in length) with FG. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC.

And since AF is commensurable in length with FG, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. But AG is commensurable (in length) with AC. Thus, each of AF and FG is also commensurable in length with AC [Prop. 10.12]. And AC is a rational (straight-line). Thus, AF and FG (are) each also rational (straight-lines). Hence, AI and FK are also each rational (areas) [Prop. 10.19]. And since DE is commensurable in length with EG, DG is thus also commensurable in length with each of DE and EG [Prop. 10.15]. And DG (is) rational, and incommensurable in length with AC. DE and EG (are) thus each rational, and incommensurable in length with AC [Prop. 10.13]. Thus, DH and EK are each medial (areas) [Prop. 10.21].

τὸ δὲ KZ τῷ NΞ· καὶ τὸ MN ἄρα τῷ EK ἴσον ἐστίν. ἀλλὰ So let the square LM, equal to AI, be laid down. τὸ μὲν EK τῷ ΔΘ ἐστιν ἴσον, τὸ δὲ MN τῷ ΛΞ· τὸ ἄρα And let the square NO, equal to FK, have been sub ΔK σον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ AK ἴσον τοῖς ΛΜ, ΝΞ τετραγώνοις· λοιπὸν ἄρα τὸ AB ἴσον ἐστὶ τῷ ΣΤ. τὸ δὲ ΣΤ τὸ ἀπὸ τῆς ΛΝ ἐστι τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΛΝ τετράγωνον ἴσον ἐστὶ τῷ AB· ἡ ΛΝ ἄρα δύναται τὸ AB. λέγω δή, ὅτι ἡ ΛΝ ἀποτομή ἐστιν.

Έπει γὰρ ἑητόν ἐστιν ἑχάτερον τῶν ΑΙ, ΖΚ, χαί ἐστιν ἴσον τοῖς ΛΜ, ΝΞ, και ἑκάτερον ἄρα τῶν ΛΜ, ΝΞ ἑητόν ἐστιν, τουτέστι τὸ ἀπὸ ἑκατέρας τῶν ΛΟ, ΟΝ· και ἑκατέρα ἄρα τῶν ΛΟ, ΟΝ ἑητή ἐστιν. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ΔΘ καί ἐστιν ἴσον τῷ ΛΞ, μέσον ἄρα ἐστὶ καὶ τὸ ΛΞ. ἐπεὶ οῦν τὸ μὲν ΛΞ μέσον ἐστίν, τὸ δὲ ΝΞ ἑητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΛΞ τῷ ΝΞ· ὡς δὲ τὸ ΛΞ πρὸς τὸ ΝΞ, οῦτως ἐστὶν ἡ ΛΟ πρὸς τὴν ΟΝ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΛΟ τῆ ΟΝ μήκει. καί εἰσιν ἀμφότεραι ἑηταί· αἱ ΛΟ, ΟΝ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἑστὶν ἡ ΛΝ. καὶ δύναται τὸ ΑΒ χωρίον· ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη ἀποτομή ἐστιν.

Έαν ἄρα χωρίον περιέχηται ὑπὸ ἑητῆς καὶ τὰ ἑξῆς.

4β'.

Έἀν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς δευτέρας, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη.



tracted (from LM), having with it the common angle LPM. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the rectangle contained by AF and FG is equal to the square EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI (is) to EK, and as EG (is) to FG, so EK is to KF [Prop. 6.1]. Thus, EK is the mean proportional to AI and KF [Prop. 5.11]. And MN is also the mean proportional to LM and NO, as shown before [Prop. 10.53 lem.]. And AI is equal to the square LM, and KF to NO. Thus, MN is also equal to EK. But, EKis equal to DH, and MN to LO [Prop. 1.43]. Thus, DKis equal to the gnomon UVW and NO. And AK is also equal to (the sum of) the squares LM and NO. Thus, the remainder AB is equal to ST. And ST is the square on LN. Thus, the square on LN is equal to AB. Thus, LN is the square-root of AB. So, I say that LN is an apotome.

For since AI and FK are each rational (areas), and are equal to LM and NO (respectively), thus LM and NO—that is to say, the (squares) on each of LP and PN(respectively)—are also each rational (areas). Thus, LP and PN are also each rational (straight-lines). Again, since DH is a medial (area), and is equal to LO, LOis thus also a medial (area). Therefore, since LO is medial, and NO rational, LO is thus incommensurable with NO. And as LO (is) to NO, so LP is to PN [Prop. 6.1]. LP is thus incommensurable in length with *PN* [Prop. 10.11]. And they are both rational (straightlines). Thus, LP and PN are rational (straight-lines which are) commensurable in square only. Thus, LN is an apotome [Prop. 10.73]. And it is the square-root of area AB. Thus, the square-root of area AB is an apotome.

Thus, if an area is contained by a rational (straightline), and so on

Proposition 92

If an area is contained by a rational (straight-line) and a second apotome then the square-root of the area is a first apotome of a medial (straight-line).



Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ἑητῆς τῆς AΓ καὶ ἀποτομῆς δευτέρας τῆς AΔ· λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη.

Έστω γὰρ τῆ Α Δ προσαρμόζουσα ἡ $\Delta \mathrm{H}^{.}$ αἱ ἄρα ΑΗ, ΗΔ ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ή ΔΗ σύμμετρός ἐστι τῆ ἐκκειμένῃ ἑητῆ τῆ ΑΓ, ή δὲ ὅλη ή ΑΗ τῆς προσαρμοζούσης τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήχει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, ἐὰν ἄρα τῷ τετάρτω μέρει τοῦ ἀπὸ τῆς ΗΔ ἴσον παρὰ τὴν ΑΗ παραβληθή έλλειπον είδει τετραγώνω, είς σύμμετρα αὐτὴν διαιρεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε΄ καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνω, και έστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· σύμμετρος ἄρα έστιν ή ΑΖ τῆ ΖΗ μήκει. και ή ΑΗ ἄρα ἑκατέρα τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήχει. ἑητὴ δὲ ἡ ΑΗ χαὶ ἀσύμμετρος τῆ ΑΓ μήκει· καὶ ἑκατέρα ἄρα τῶν ΑΖ, ΖΗ ῥητή ἐστι καὶ ἀσύμμετρος τῆ ΑΓ μήχει· ἑχάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔE τῆ EH, καὶ ἡ ΔH ἄρα ἑχατέρα τῶν ΔΕ, ΕΗ σύμμετρός ἐστιν. ἀλλ' ἡ ΔΗ σύμμετρός ἐστι τῆ ΑΓ μήκει [ἑητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ σύμμετρος τῆ ΑΓ μήκει]. ἑκάτερον ἄρα τῶν $\Delta\Theta$, EK phytón έστιν.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφηρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὂν τῷ ΛΜ τὴν ὑπὸ τῶν ΛΟΜ· περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὰ ΑΙ, ΖΚ μέσα ἐστὶ καί ἐστιν ἴσα τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ [ἄρα] μέσα ἐστίν· καὶ αἱ ΛΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. χαὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ· ἀλλ' ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως τὸ ΑΙ πρός τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως [ἐστὶ] τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. έστι δὲ χαὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ MN· καί ἐστιν ἴσον τὸ μὲν AI τῷ AM, τὸ δὲ ZK τῷ NΞ· καὶ τὸ ΜΝ ἄρα ἴσον ἐστὶ τῷ ΕΚ. ἀλλὰ τῷ μὲν ΕΚ ἴσον [ἐστὶ] τὸ $\Delta\Theta$, τῷ δὲ MN ἴσον τὸ $\Lambda\Xi$ · ὅλον ἄρα τὸ ΔK ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι χαὶ τῷ ΝΞ. ἐπεὶ οὖν ὅλον τὸ ΑΚ ἴσον έστὶ τοῖς ΛM , NΞ, ῶν τὸ ΔK ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ, λοιπὸν ẳρα τὸ AB ἴσον ἐστὶ τῷ ΤΣ. τὸ δ
ὲ ΤΣ έστι τὸ ἀπὸ τῆς ΛN · τὸ ἀπὸ τῆς ΛN ἄρα ἴσον ἐστὶ τῷ ABχωρίω· ή ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω [δή], ὅτι ή ΛΝ μέσης ἀποτομή ἐστι πρώτη.

Έπεὶ γὰρ ἑητόν ἐστι τὸ ΕΚ καί ἐστιν ἴσον τῷ Λ Ξ, ἑητὸν which are) commensurable in square only.[†] And since ἄρα ἐστὶ τὸ Λ Ξ, τουτέστι τὸ ὑπὸ τῶν Λ Ο, ΟΝ. μέσον δὲ ἐδείχϑη τὸ ΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ Λ Ξ τῷ ΝΞ· ὡς the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) δὲ τὸ Λ Ξ πρὸς τὸ ΝΞ, οὕτως ἐστὶν ἡ Λ Ο πρὸς ΟΝ· αἰ Λ Ο, ΟΝ ἄρα ἀσύμμετροί εἰσι μήκει. αἱ ἄρα Λ Ο, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ἑητὸν περιέχουσαι· ἡ Λ Ν ἄρα [Prop. 6.1]. Thus, EK is the mean proportional to AI

For let the area AB have been contained by the rational (straight-line) AC and the second apotome AD. I say that the square-root of area AB is the first apotome of a medial (straight-line).

For let DG be an attachment to AD. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment DG is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, GD, by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.12]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the (square) on GD is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let DG have been cut in half at E. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. Thus, AF is commensurable in length with FG. AG is thus also commensurable in length with each of AF and FG[Prop. 10.15]. And AG (is) a rational (straight-line), and incommensurable in length with AC. AF and FG are thus also each rational (straight-lines), and incommensurable in length with AC [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable (in length) with EG, thus DG is also commensurable (in length) with each of DE and EG[Prop. 10.15]. But, DG is commensurable in length with AC [thus, DE and EG are also each rational, and commensurable in length with AC]. Thus, DH and EK are each rational (areas) [Prop. 10.19].

Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, which is about the same angle LPM as LM, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since AI and FK are medial (areas), and are equal to the (squares) on LP and PN (respectively), [thus] the (squares) on LP and PN are also medial. Thus, LP and PN are also medial (straight-lines which are) commensurable in square only.[†] And since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 10.17]. But, as AF (is) to EG, so AI(is) to EK. And as EG (is) to FG, so EK [is] to FK[Prop. 6.1]. Thus, EK is the mean proportional to AI μέσης ἀποτομή ἐστι πρώτη· καὶ δύναται τὸ ΑΒ χωρίον.

Ή ἄρα τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη ὅπερ ἔδει δεῖξαι.

and FK [Prop. 5.11]. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is equal to LM, and FK to NO. Thus, MN is also equal to EK. But, DH [is] equal to EK, and LO equal to MN [Prop. 1.43]. Thus, the whole (of) DK is equal to the gnomon UVW and NO. Therefore, since the whole (of) AK is equal to LM and NO, of which DK is equal to the gnomon UVW and NO, the remainder AB is thus equal to TS. And TS is the (square) on LN. Thus, the square-root of area AB. [So], I say that LN is the first apotome of a medial (straight-line).

For since EK is a rational (area), and is equal to LO, LO—that is to say, the (rectangle contained) by LP and PN—is thus a rational (area). And NO was shown (to be) a medial (area). Thus, LO is incommensurable with NO. And as LO (is) to NO, so LP is to PN [Prop. 6.1]. Thus, LP and PN are incommensurable in length [Prop. 10.11]. LP and PN are thus medial (straight-lines which are) commensurable in square only, and which contain a rational (area). Thus, LN is the first apotome of a medial (straight-line) [Prop. 10.74]. And it is the square-root of area AB.

Thus, the square root of area AB is the first apotome of a medial (straight-line). (Which is) the very thing it was required to show.

[†] There is an error in the argument here. It should just say that LP and PN are commensurable in square, rather than in square only, since LP and PN are only shown to be incommensurable in length later on.



Έὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα.



Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ἑητῆς τῆς AΓ καὶ ἀποτομῆς τρίτης τῆς AΔ· λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα.

Έστω γὰρ τῆ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ΑΗ, ΗΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα τῶν ΑΗ, ΗΔ σύμμετρός ἐστι μήκει τῆ ἐκκειμένῃ ῥητῆ τῆ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τὴς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. ἐπεὶ οῦν ἡ ΑΗ τῆς ΗΔ μεῖζον

Proposition 93

If an area is contained by a rational (straight-line) and a third apotome then the square-root of the area is a second apotome of a medial (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the third apotome AD. I say that the square-root of area AB is the second apotome of a medial (straight-line).

For let DG be an attachment to AD. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of AG and GD is commensurable in length with the (previ-

δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῃ ἐλλεῖπον εἴδει τετραγώνω, εἰς σύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ή ΔΗ δίχα κατά τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω έλλεῖπον εἴδει τετραγώνω, καὶ ἔστω τὸ ύπὸ τῶν ΑΖ, ΖΗ. χαὶ ἤχθωσαν διὰ τῶν Ε, Ζ, Η σημείων τῆ ΑΓ παράλληλοι αί ΕΘ, ΖΙ, ΗΚ· σύμμετροι ἄρα εἰσὶν αί AZ, ZH σύμμετρον ἄρα καὶ τὸ AI τῷ ZK. καὶ ἐπεὶ αἱ AZ, ΖΗ σύμμετροί εἰσι μήχει, χαὶ ἡ ΑΗ ἄρα ἑχατέρα τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήχει. ἑητὴ δὲ ἡ ΑΗ χαὶ ἀσύμμετρος τῆ ΑΓ μήκει· ὥστε καὶ αἱ ΑΖ, ΖΗ. ἑκάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔE τῆ EH μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρα τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ἑητὴ δὲ ἡ ΗΔ καὶ ἀσύμμετρος τῆ ΑΓ μήκει· ἑητὴ ἄρα καὶ ἑκατέρα τῶν $\Delta \mathrm{E, \, EH}$ καὶ ἀσύμμετρος τῆ ΑΓ μήκει· έκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν. καὶ ἐπεὶ αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶ μήχει ή ΑΗ τῆ ΗΔ. ἀλλ' ή μέν ΑΗ τῆ ΑΖ σύμμετρός ἐστι μήχει
ἡ δὲ ΔH τῆ EH· ἀσύμμετρος ἄρα ἐστὶν ἡ AZ τῆ EH μήχει. ὡς δὲ ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΕΚ.

Συνεστάτω οὐν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῆρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὂν τῷ ΛΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ ούν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα ώς ή ΑΖ πρός την ΕΗ, οὕτως ή ΕΗ πρός την ΖΗ. ἀλλ' ὡς μέν ή ΑΖ πρός την ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρός τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΖΚ· καὶ ώς ἄρα τὸ ΑΙ πρὸς τὸ ΕΚ, οὕτως τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα AI, ZK μέσον ἀνάλογόν ἐστι τὸ EK. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ· καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ἄρα ἴσον ἐστὶ τῷ MN. ἀλλὰ τὸ μὲν MN ἴσον ἐστὶ τῷ $\Lambda \Xi$, τὸ δὲ ΕΚ ἴσον [ἐστὶ] τῷ $\Delta \Theta$ · καὶ ὅλον ἄρα τὸ ΔK ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ ΑΚ ἴσον τοῖς ΛM , NΞ· λοιπὸν ἄρα τὸ AB ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ άπὸ τῆς ΛΝ τετραγώνω· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω, ὄτι ἡ ΛΝ μέσης ἀποτομή ἐστι δευτέρα.

Έπει γὰρ μέσα ἐδείχθη τὰ AI, ZK καί ἐστιν ἴσα τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, μέσον ἄρα και ἑκάτερον τῶν ἀπὸ τῶν ΛΟ, ΟΝ· μέση ἄρα ἑκατέρα τῶν ΛΟ, ΟΝ. και ἐπει σύμμετρόν ἐστι τὸ AI τῷ ZK, σύμμετρον ἄρα και τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ. πάλιν, ἐπει ἀσύμμετρον ἐδείχθη τὸ AI τῷ EK, ἀσύμμετρον ἄρα ἐστι και τὸ ΛΜ τῷ MN, τουτέστι τὸ ἀπὸ τῆς ΛΟ τῷ ὑπὸ τῶν ΛΟ, ΟΝ· ὥστε και ἡ ΛΟ ἀσύμμετρός ἐστι μήκει τῆ ΟΝ· αἱ ΛΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι και μέσον περιέχουσιν.

Έπει γὰρ μέσον ἐδείχθη τὸ ΕΚ καί ἐστιν ἴσον τῷ ὑπὸ τῶν ΛΟ, ΟΝ, μέσον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΛΟ, ΟΝ· ὥστε αἱ ΛΟ, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον

ously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straightline) commensurable (in length) with (AG) [Def. 10.13]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the square on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let *DG* have been cut in half at E. And let (an area) equal to the (square) on EGhave been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC. Thus, AF and FG are commensurable (in length). AI (is) thus also commensurable with FK [Props. 6.1, 10.11]. And since AF and FG are commensurable in length, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. And AG (is) rational, and incommensurable in length with AC. Hence, AF and FG (are) also (rational, and incommensurable in length with AC) [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable in length with EG, DG is also commensurable in length with each of DE and EG [Prop. 10.15]. And GD (is) rational, and incommensurable in length with AC. Thus, DE and EG (are) each also rational, and incommensurable in length with AC [Prop. 10.13]. DH and EK are thus each medial (areas) [Prop. 10.21]. And since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD. But, AG is commensurable in length with AF, and DG with EG. Thus, AF is incommensurable in length with EG [Prop. 10.13]. And as AF (is) to EG, so AI is to EK [Prop. 6.1]. Thus, AI is incommensurable with EK [Prop. 10.11].

Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, which is about the same angle as LM, have been subtracted (from LM). Thus, LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI is to EK [Prop. 6.1]. And as EG (is) to FG, so EK is to FK[Prop. 6.1]. And thus as AI (is) to EK, so EK (is) to FK [Prop. 5.11]. Thus, EK is the mean proportional to AI and FK. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is περιέχουσαι. ή ΛΝ ἄρα μέσης ἀποτομή ἐστι δευτέρα· καὶ δύναται τὸ ΑΒ χωρίον.

Ή ἄρα τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα· ὅπερ ἔδει δεῖξαι.

4δ'.

Ἐἀν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ ἀποτομῆς τετάρτης, ἡ τὸ χωρίον δυναμένη ἐλάσσων ἐστίν.



Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ἑητῆς τῆς AΓ καὶ ἀποτομῆς τετάρτης τῆς AΔ· λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη ἐλάσσων ἐστίν. equal to LM, and FK to NO. Thus, EK is also equal to MN. But, MN is equal to LO, and EK [is] equal to DH [Prop. 1.43]. And thus the whole of DK is equal to the gnomon UVW and NO. And AK (is) also equal to LM and NO. Thus, the remainder AB is equal to ST—that is to say, to the square on LN. Thus, LN is the square-root of area AB. I say that LN is the second apotome of a medial (straight-line).

For since AI and FK were shown (to be) medial (areas), and are equal to the (squares) on LP and PN (respectively), the (squares) on each of LP and PN (are) thus also medial. Thus, LP and PN (are) each medial (straight-lines). And since AI is commensurable with FK [Props. 6.1, 10.11], the (square) on LP (is) thus also commensurable with the (square) on PN. Again, since AI was shown (to be) incommensurable with EK, LM is thus also incommensurable with MN—that is to say, the (square) on LP with the (rectangle contained) by LP and PN. Hence, LP is also incommensurable in length with PN [Props. 6.1, 10.11]. Thus, LP and PNare medial (straight-lines which are) commensurable in square only. So, I say that they also contain a medial (area).

For since EK was shown (to be) a medial (area), and is equal to the (rectangle contained) by LP and PN, the (rectangle contained) by LP and PN is thus also medial. Hence, LP and PN are medial (straight-lines which are) commensurable in square only, and which contain a medial (area). Thus, LN is the second apotome of a medial (straight-line) [Prop. 10.75]. And it is the square-root of area AB.

Thus, the square-root of area AB is the second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

Proposition 94

If an area is contained by a rational (straight-line) and a fourth apotome then the square-root of the area is a minor (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the fourth apotome AD. I say that the square-root of area AB is a minor (straight-

Έστω γὰρ τῆ Α Δ προσαρμόζουσα ἡ $\Delta {
m H}^{.}$ αἱ ἄρα ΑΗ, ΗΔ ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΗ σύμμετρός ἐστι τῆ ἐκκειμένῃ ῥητῃ τῆ ΑΓ μήκει, ἡ δὲ ὅλη ή ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήχει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ μήκει, ἐὰν ἄρα τῷ τετάρτω μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθή έλλεῖπον εἴδει τετραγώνω, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὐν ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνω, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶ μήχει ἡ ΑΖ τῆ ΖΗ. ἤχθωσαν οὖν διὰ τῶν Ε, Ζ, Η παράλληλοι ταῖς ΑΓ, ΒΔ αἱ ΕΘ, ΖΙ, ΗΚ. ἐπεὶ οὖν ῥητή ἐστιν ή ΑΗ καὶ σύμμετρος τῆ ΑΓ μήκει, ἑητὸν ἄρα ἐστὶν ὅλον τὸ ΑΚ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΔΗ τῆ ΑΓ μήχει, καί είσιν ἀμφότεραι ἑηταί, μέσον ἄρα ἐστὶ τὸ ΔΚ. πάλιν, ἐπεὶ άσύμμετρός ἐστὶν ἡ ΑΖ τῆ ΖΗ μήχει, ἀσύμμετρον ἄρα χαὶ τὸ ΑΙ τῷ ΖΚ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφηρήσθω περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ τῶν ΛOM τὸ NΞ. περὶ τὴν αὐτὴν ắρα διάμετόν ἐστι τὰ $\Lambda M,$ NΞ τετράγωνα. ἔστω αὐτῶν διάμετος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ή ΕΗ πρός την ΖΗ. ἀλλ' ὡς μὲν ἡ ΑΖ πρός την ΕΗ, οὕτως έστι τὸ ΑΙ πρὸς τὸ ΕΚ, ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως έστι τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν έστι τὸ ΕΚ. ἕστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον άνάλογον τὸ MN, καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ẳρα ἴσον ἐστὶ τῷ ΜΝ. ἀλλὰ τῷ μὲν EK ἴσον ἐστὶ τὸ $\Delta\Theta$, τῷ δὲ MN ἴσον ἐστὶ τὸ $\Lambda\Xi$ · ὅλον άρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἐπεὶ ούν όλον τὸ ΑΚ ἴσον ἐστὶ τοῖς ΛΜ, ΝΞ τετραγώνοις, ῶν τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ τετραγώνῳ, λοιπόν ἄρα τὸ AB ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ ἀπὸ τῆς ΛΝ τετραγώνω· ή ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω, ότι ή ΛΝ ἄλογός ἐστιν ή καλουμένη ἐλάσσων.

Έπει γὰρ ἑητόν ἐστι τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ τετράγωνοις, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ ἑητόν ἐστιν. πάλιν, ἐπεὶ τὸ ΔΚ μέσον ἐστίν, καί ἐστιν ἴσον τὸ ΔΚ τῷ δἰς ὑπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα δἰς ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τετράγωνον τῷ ἀπὸ τῆς ΟΝ τετραγώνῳ. αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπʾ αὐτῶν τετραγώνων ἑητόν, τὸ δὲ δἰς ὑπʾ αὐτῶν μέσον. ἡ ΛΝ ἄρα ἄλογός ἑστιν ἡ καλουμένη ἐλάσσων· καὶ δύναται τὸ ΑΒ χωρίον.

Ή ἄρα τὸ AB χωρίον δυναμένη ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι. line). For let DG be an attachment to AD. Thus, AGand DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and AG is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the square on (some straight-line) incommensurable in length with (AG) [Def. 10.14]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG), thus if (some area), equal to the fourth part of the (square) on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been cut in half at E, and let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure, and let it be the (rectangle contained) by AF and FG. Thus, AF is incommensurable in length with FG. Therefore, let EH, FI, and GK have been drawn through E, F, and G (respectively), parallel to AC and BD. Therefore, since AGis rational, and commensurable in length with AC, the whole (area) AK is thus rational [Prop. 10.19]. Again, since DG is incommensurable in length with AC, and both are rational (straight-lines), DK is thus a medial (area) [Prop. 10.21]. Again, since AF is incommensurable in length with FG, AI (is) thus also incommensurable with FK [Props. 6.1, 10.11].

Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, (and) about the same angle, LPM, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus, proportionally, as AFis to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI is to EK, and as EG (is) to FG, so EK is to FK [Prop. 6.1]. Thus, EK is the mean proportional to AI and FK [Prop. 5.11]. And MN is also the mean proportional to the squares LM and NO [Prop. 10.13 lem.], and AI is equal to LM, and FK to NO. EK is thus also equal to MN. But, DH is equal to EK, and LO is equal to MN [Prop. 1.43]. Thus, the whole of DK is equal to the gnomon UVW and NO. Therefore, since the whole of AK is equal to the (sum of the) squares LMand NO, of which DK is equal to the gnomon UVWand the square NO, the remainder AB is thus equal to ST—that is to say, to the square on LN. Thus, LN is the square-root of area AB. I say that LN is the irrational (straight-line which is) called minor.

For since AK is rational, and is equal to the (sum of the) squares LP and PN, the sum of the (squares) on LP and PN is thus rational. Again, since DK is medial, and DK is equal to twice the (rectangle contained) by LP and PN, thus twice the (rectangle contained) by LP and PN is medial. And since AI was shown (to be) incommensurable with FK, the square on LP (is) thus also incommensurable with the square on PN. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. LN is thus the irrational (straight-line) called minor [Prop. 10.76]. And it is the square-root of area AB.

Thus, the square-root of area AB is a minor (straightline). (Which is) the very thing it was required to show.

Proposition 95

If an area is contained by a rational (straight-line) and a fifth apotome then the square-root of the area is that (straight-line) which with a rational (area) makes a medial whole.



For let the area AB have been contained by the rational (straight-line) AC and the fifth apotome AD. I say that the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole.

For let DG be an attachment to AD. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment GD is commensurable in length the the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straight-line) incommensurable (in length) with (AG) [Def. 10.15]. Thus, if (some area), equal to the fourth part of the (square) on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been divided in half at point *E*, and let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure, and let it be the (rectangle contained) by AF and FG. Thus, AF is incommensurable in length with FG. And since AG is incommensurable

4ε'.

Ἐἀν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς πέμπτης, ἡ τὸ χωρίον δυναμένη [ή] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.



Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ἑητῆς τῆς AΓ καὶ ἀποτομῆς πέμπτης τῆς AΔ· λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη [ἡ] μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Έστω γὰρ τῆ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα AH, HΔ ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ἡ HΔ σύμμετρός ἐστι μήκει τῆ ἐκκειμένη ἑητῆ τῆ AΓ, ἡ δὲ ὅλη ἡ AH τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν AH παραβληθῆ ἑλλεῖπον είδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οῦν ἡ ΔΗ δίχα κατὰ τὸ Ε σημεῖον, καὶ τῷ ἀπὸ τῆς EH ἴσον παρὰ τὴν AH παραβεβλήσθω ἐλλεῖπον είδει τετραγώνῳ καὶ ἔστω τὸ ὑπὸ τῶν AZ, ZH· ἀσύμμετρος ἄρα ἐστὶν ἡ AZ τῆ ZH μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστὶν ἡ AH τῆ ΓΑ μήκει, καί εἰσιν ἀμφότεραι ἑηταί, μέσον ἄρα ἐστὶ τὸ AK. πάλιν, ἐπεὶ ἑητή ἐστιν ἡ ΔΗ καὶ σύμμετρος τῆ AΓ μήκει, ἑητόν ἐστι τὸ ΔK.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον τετράγωνον ἀφηρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ ΛΟΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΛΝ δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΛΝ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Έπει γὰρ μέσον ἐδείχθη τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπει ῥητόν ἐστι τὸ ΔΚ καί ἐστιν ἴσον τῷ δἰς ὑπὸ τῶν ΛΟ, ΟΝ, καὶ αὑτὸ ῥητόν ἐστιν. καὶ ἐπει ἀσύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δἰς ὑπ' αὐτῶν ῥητόν. ἡ λοιπὴ ἄρα ἡ ΛΝ ἄλογός ἐστιν ἡ καλουμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα· καὶ δύναται τὸ ΑΒ χωρίον.

Ή τὸ AB ẳρα χωρίον δυναμένη μετὰ ἡητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

45'.

Έὰν χωρίον περιέχηται ὑπὸ ἑητῆς καὶ ἀποτομῆς ἕκτης, ἡ τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.



Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ῥητῆς τῆς AΓ καὶ ἀποτομῆς ἕκτης τῆς AΔ· λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Έστω γὰρ τῆ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα in length with CA, and both are rational (straight-lines), AK is thus a medial (area) [Prop. 10.21]. Again, since DG is rational, and commensurable in length with AC, DK is a rational (area) [Prop. 10.19].

Therefore, let the square LM, equal to AI, have been constructed. And let the square NO, equal to FK, (and) about the same angle, LPM, have been subtracted (from NO). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly (to the previous propositions), we can show that LN is the square-root of area AB. I say that LN is that (straight-line) which with a rational (area) makes a medial whole.

For since AK was shown (to be) a medial (area), and is equal to (the sum of) the squares on LP and PN, the sum of the (squares) on LP and PN is thus medial. Again, since DK is rational, and is equal to twice the (rectangle contained) by LP and PN, (the latter) is also rational. And since AI is incommensurable with FK, the (square) on LP is thus also incommensurable with the (square) on PN. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational. Thus, the remainder LN is the irrational (straight-line) called that which with a rational (area) makes a medial whole [Prop. 10.77]. And it is the square-root of area AB.

Thus, the square-root of area AB is that (straightline) which with a rational (area) makes a medial whole. (Which is) the very thing it was required to show.

Proposition 96

If an area is contained by a rational (straight-line) and a sixth apotome then the square-root of the area is that (straight-line) which with a medial (area) makes a medial whole.



For let the area AB have been contained by the rational (straight-line) AC and the sixth apotome AD. I say that the square-root of area AB is that (straight-line) which with a medial (area) makes a medial whole.

For let DG be an attachment to AD. Thus, AG and

αὐτῶν σύμμετρός ἐστι τῆ ἐκκειμένῃ ῥητῃ τῆ ΑΓ μήκει, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῃ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἐαυτῃ μήκει, ἑὰν ἄρα τῷ τετάρτω μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθή έλλεῖπον εἴδει τετραγώνω, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οῦν ἡ ΔΗ δίχα κατὰ τὸ Ε [σημεῖον], καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνω, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ AZ τῆ ZH μήκει. ὡς δὲ ἡ AZ πρὸς τὴν ZH, ούτως έστι τὸ ΑΙ πρὸς τὸ ΖΚ· ἀσύμμετρον ἄρα ἐστι τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΗ, ΑΓ ἑηταί εἰσι δυνάμει μόνον σύμμετροι, μέσον ἐστὶ τὸ ΑΚ. πάλιν, ἐπεὶ αἱ ΑΓ, ΔΗ ῥηταί είσι καὶ ἀσύμμετροι μήκει, μέσον ἐστὶ καὶ τὸ ΔΚ. ἐπεὶ οὖν αί ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα έστιν ή AH τ
ῆ HΔ μήχει.
ὡς δὲ ή AH πρὸς τὴν HΔ, οὕτως έστι τὸ ΑΚ πρὸς τὸ ΚΔ· ἀσύμμετρον ἄρα ἐστι τὸ ΑΚ τῷ KΔ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφηρήσθω περὶ τὴν αὐτὴν γωνίαν τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ OP, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ τοῖς ἐπάνω δείξομεν, ὅτι ἡ ΛΝ δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΛΝ [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Έπει γὰρ μέσον ἐδείχθη τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπεὶ μέσον ἐδείχθη τὸ ΔΚ καί ἐστιν ἴσον τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, καὶ τὸ δὶς ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΚ τῷ ΔΚ, ἀσύμμετρα [ἄρα] ἐστὶ καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ τετράγωνα τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ. καὶ ἐπεὶ ἀσύμμετρον ἐστι τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπʾ αὐτῶν τετραγώνων μέσον καὶ τὸ ἀμὸ τῶν κοι το δὶς ὑπʾ αὐτῶν. ἡ ἄρα ΛΝ ἄλογός ἐστιν ἡ καλουμέμη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα· καὶ δύναται τὸ ΑΒ χωρίον.

Ή ἄρα τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of them is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straight-line) incommensurable in length with (AG) [Def. 10.16]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG), thus if (some area), equal to the fourth part of square on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been cut in half at [point] E. And let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. AF is thus incommensurable in length with FG. And as AF (is) to FG, so AI is to FK [Prop. 6.1]. Thus, AI is incommensurable with FK [Prop. 10.11]. And since AG and ACare rational (straight-lines which are) commensurable in square only, AK is a medial (area) [Prop. 10.21]. Again, since AC and DG are rational (straight-lines which are) incommensurable in length, DK is also a medial (area) [Prop. 10.21]. Therefore, since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD. And as AG (is) to GD, so AK is to *KD* [Prop. 6.1]. Thus, *AK* is incommensurable with *KD* [Prop. 10.11].

Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, (and) about the same angle, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly to the above, we can show that LN is the square-root of area AB. I say that LN is that (straight-line) which with a medial (area) makes a medial whole.

For since AK was shown (to be) a medial (area), and is equal to the (sum of the) squares on LP and PN, the sum of the (squares) on LP and PN is medial. Again, since DK was shown (to be) a medial (area), and is equal to twice the (rectangle contained) by LP and PN, twice the (rectangle contained) by LP and PN is also medial. And since AK was shown (to be) incommensurable with DK, [thus] the (sum of the) squares on LPand PN is also incommensurable with twice the (rectangle contained) by LP and PN. And since AI is incommensurable with FK, the (square) on LP (is) thus also incommensurable with the (square) on PN. Thus, LP and PN are (straight-lines which are) incommensu-
4ζ΄.

Τὸ ἀπὸ ἀποτομῆς παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην.



Έστω ἀποτομὴ ἡ ΑΒ, ἑητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι πρώτη.

Έστω γὰρ τῆ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ ἑηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ άπὸ τῆς ΒΗ τὸ ΚΛ. ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· ὤν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ· λοιπὸν άρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω ή ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ ἤχθω διὰ τοῦ Ν τῆ ΓΔ παράλληλος ή ΝΞ· ἑχάτερον ἄρα τῶν ΖΞ, ΛΝ ἴσον ἐστὶ τῷ ύπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ ῥητά ἐστιν, καί ἐστι τοῖς ἀπὸ τῶν AH, HB ἴσον τὸ ΔΜ, ῥητὸν ἄρα έστι τὸ ΔΜ. και παρὰ ἑητὴν τὴν ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ἑητὴ ẳρα ἐστὶν ἡ ΓΜ καὶ σύμμετρος τῆ Γ Δ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, καὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ, μέσον ἄρα τὸ ΖΛ. καὶ παρὰ ἑητὴν τὴν ΓΔ παράχειται πλάτος ποιοῦν τὴν ΖΜ· ἑητὴ άρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῆ ΓΔ μήκει. καὶ ἐπεὶ τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ ῥητά ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ύπὸ τῶν AH, HB. καὶ τοῖς μὲν ἀπὸ τῶν AH, HB ἴσον ἐστὶ τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν AH, HB τὸ $Z\Lambda$ · ἀσύμμετρον ἄρα ἐστὶ τὸ ΔM τῷ $Z\Lambda$. ὡς δὲ τὸ ΔM πρὸς τὸ $Z\Lambda$, οὕτως ἐστὶν ή ΓΜ πρός την ΖΜ. ἀσύμμετρος ἄρα ἐστιν ή ΓΜ τῆ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι

rable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by medial, and, furthermore, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them. Thus, LN is the irrational (straight-line) called that which with a medial (area) makes a medial whole [Prop. 10.78]. And it is the square-root of area AB.

Thus, the square-root of area (AB) is that (straightline) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

Proposition 97

The (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.



Let AB be an apotome, and CD a rational (straightline). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a first apotome.

For let BG be an attachment to AB. Thus, AG and GB are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let CH, equal to the (square) on AG, and KL, (equal) to the (square) on BG, have been applied to CD. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB. The remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Let FM have been cut in half at point N. And let NO have been drawn through N, parallel to CD. Thus, FO and LN are each equal to the (rectangle contained) by AG and GB. And since the (sum of the squares) on AG and GB is rational, and DM is equal to the (sum of the squares) on AG and GB, DM is thus rational. And it has been applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and commensurable in length with CD [Prop. 10.20]. Again, since twice the (rectangle contained) by AG and GB is medial, and FL (is) equal to twice the (rectangle contained) by AG and GB, FL (is) thus a medial (area). And it is applied to the rational (straight-line) CD, producing FM as breadth. FM is δυνάμει μόνον σύμμετροι· ή ΓΖ ἄρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ πρώτη.

Έπει γαρ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ $\Gamma\Theta$, τῷ δὲ ἀπὸ τῆς BH ἴσον τὸ KΛ, τῷ δὲ ὑπὸ τῶν AH, ΗΒ τὸ ΝΛ, καὶ τῶν ΓΘ, ΚΛ ẳρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ Γ Θ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ' ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν NM· ὡς δὲ τὸ NΛ πρὸς τὸ KΛ, οὕτως ἐστὶν ή ΝΜ πρός την ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῶ ἀπὸ τῆς ΝΜ, τουτέστι τῶ τετάρτω μέρει τοῦ ἀπὸ τῆς ΖΜ. καὶ επεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρόν [ἐστι] καὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ σύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῆ ΚΜ. ἐπεὶ οῦν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται έλλεῖπον εἴδει τετραγώνω τὸ ὑπὸ τῶν ΓΚ, ΚΜ, καί ἐστι σύμμετρος ἡ ΓΚ τῆ ΚΜ, ἡ ἄρα ΓΜ τῆς ΜΖ μεϊζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῃ μήκει. καί ἐστιν ἡ ΓM σύμμετρος τ
ῆ ἐκκειμένῃ ἑητῆ τῆ ΓΔ μήκει· ἡ ἄρα ΓZ ἀποτομή ἐστι πρώτη.

Τὸ ἄρα ἀπὸ ἀποτομῆς παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην ὅπερ ἔδει δεῖξαι.

4η'.

Τὸ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν.

Έστω μέσης ἀποτομὴ πρώτη ἡ AB, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος thus rational, and incommensurable in length with CD [Prop. 10.22]. And since the (sum of the squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB. And GB. And CL is equal to the (sum of the squares) on AG and GB, and FL to twice the (rectangle contained) by AG and GB. And GB, and FL to twice the (rectangle contained) by AG and GB. DM is thus incommensurable with FL. And as DM (is) to FL, so CM is to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

For since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB[Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on BG, and NL to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NMis to *KM* [Prop. 6.1]. Thus, the (rectangle contained) by CK and KM is equal to the (square) on NM that is to say, to the fourth part of the (square) on FM[Prop. 6.17]. And since the (square) on AG is commensurable with the (square) on GB, CH [is] also commensurable with KL. And as CH (is) to KL, so CK (is) to *KM* [Prop. 6.1]. *CK* is thus commensurable (in length) with KM [Prop. 10.11]. Therefore, since CM and MFare two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and CK is commensurable (in length) with KM, the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. And CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a first apotome [Def. 10.15].

Thus, the (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth. (Which is) the very thing it was required to show.

Proposition 98

The (square) on a first apotome of a medial (straightline), applied to a rational (straight-line), produces a second apotome as breadth.

Let AB be a first apotome of a medial (straight-line),

ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι δευτέρα.

Έστω γὰρ τῆ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράχειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ή ΓΜ καὶ ἀσύμμετρος τῆ ΓΔ μήκει. καὶ ἐπεὶ τὸ ΓΛ ἴσον έστι τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ῶν τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστι τῶ ΓΕ, λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τῶ ΖΛ. ἑητὸν δέ [ἐστι] τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ· ἑητὸν ἄρα τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ἡητὴ ἄρα ἐστὶ καὶ ἡ ΖΜ καὶ σύμμετρος τῆ ΓΔ μήκει. έπει οῦν τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΓΛ, μέσον έστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΖΛ, ῥητόν άσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἡ ΓΜ τῆ ΖΜ μήχει. χαί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ή ΓΖ ἄρα ἀποτομή έστιν. λέγω δή, ὅτι καὶ δευτέρα.



Τετμήσθω γὰρ ή ZM δίχα κατὰ τὸ N, καὶ ἤχθω διὰ τοῦ N τῆ ΓΔ παράλληλος ή NΞ· ἑκάτερον ἄρα τῶν ZΞ, NΛ ἴσον ἑστὶ τῷ ὑπὸ τῶν AH, HB. καὶ ἐπεὶ τῶν ἀπὸ τῶν AH, HB τετραγώνων μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν AH, HB, καί ἐστιν ἴσον τὸ μὲν ἀπὸ τῆς AH τῷ ΓΘ, τὸ δὲ ὑπὸ τῶν AH, HB τῷ NΛ, τὸ δὲ ἀπὸ τῆς BH τῷ KΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ NΛ, οὕτως τὸ NΛ πρὸς τὸ ΚΛ. ἀλλ² ὡς μὲν τὸ ΓΘ πρὸς

and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a second apotome.

For let BG be an attachment to AB. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a rational (area) [Prop. 10.74]. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on GB, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. Thus, CL (is) also a medial (area) [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) CD, producing CM as breadth. CMis thus rational, and incommensurable in length with CD[Prop. 10.22]. And since CL is equal to the (sum of the squares) on AG and GB, of which the (square) on ABis equal to CE, the remainder, twice the (rectangle contained) by AG and GB, is thus equal to FL [Prop. 2.7]. And twice the (rectangle contained) by AG and GB [is] rational. Thus, FL (is) rational. And it is applied to the rational (straight-line) FE, producing FM as breadth. FM is thus also rational, and commensurable in length with CD [Prop. 10.20]. Therefore, since the (sum of the squares) on AG and GB—that is to say, CL—is medial, and twice the (rectangle contained) by AG and GB that is to say, FL—(is) rational, CL is thus incommensurable with FL. And as CL (is) to FL, so CM is to FM [Prop. 6.1]. Thus, CM (is) incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a second (apotome).



For let FM have been cut in half at N. And let NO have been drawn through (point) N, parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since the (rectangle contained) by AG and GB is the mean proportional to the squares on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (rectangle contained) by AG and GB to NL, and the (square) on

τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΜΚ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΝΜ, οὕτως ἑστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ [καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς BH, σύμμετρόν ἐστι καὶ τὸ ΓΘ τῷ ΚΛ, τουτέστιν ἡ ΓΚ τῆ KM]. ἐπεὶ οῦν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάτρῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν μείζονα τὴν ΓΜ παραβέβληται ἐλλεῖπον είδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος μήκει τῆ ἐκκειμένῃ ἑῆτῆ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι δευτέρα.

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν· ὅπερ ἔδει δεῖξαι.

BG to KL, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL[Prop. 5.11]. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to MK [Prop. 6.1]. Thus, as CK (is) to NM, so NM is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on NM [Prop. 6.17]—that is to say, to the fourth part of the (square) on FM [and since the (square) on AG is commensurable with the (square) on BG, CH is also commensurable with KL—that is to say, CK with KM]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to the greater CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. The attachment FM is also commensurable in length with the (previously) laid down rational (straight-line) CD. CF is thus a second apotome [Def. 10.16].

Thus, the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth. (Which is) the very thing it was required to show.

Proposition 99

The (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.



Let AB be the second apotome of a medial (straightline), and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a third apotome.

For let BG be an attachment to AB. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a medial (area) [Prop. 10.75]. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth. And let KL,

4ϑ′.

Τὸ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην.



Έστω μέσης ἀποτομὴ δευτέρα ἡ ΑΒ, ἑητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι τρίτη. Έστω γὰρ τῆ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον παρὰ τὴν ΚΘ παραβεβλήσθω τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ [καί ἐστι μέσα τὰ ἀπὸ τῶν ΑΗ, ΗΒ]· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ἑῃτὴν τὴν

ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ἑητὴ ἄρα ἐστὶν ή ΓΜ καὶ ἀσύμμετρος τῆ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ ΓΛ ίσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ῶν τὸ ΓΕ ἴσον ἐστὶ τῷ άπὸ τῆς AB, λοιπὸν ἄρα τὸ ΛΖ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω ούν ή ΖΜ δίχα κατά τὸ Ν σημεῖον, καὶ τῆ ΓΔ παράλληλος ἦχθω ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῶ ὑπὸ τῶν ΑΗ, ΗΒ. μέσον δὲ τὸ ὑπὸ τῶν AH, HB· μέσον ἄρα ἐστὶ καὶ τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράχειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα χαὶ ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει μόνον είσι σύμμετροι, ἀσύμμετρος ἄρα [ἐστί] μήχει ἡ ΑΗ τῆ ΗΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν AH, HB, τῷ δὲ ὑπὸ τῶν AH, HB τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον έστὶ τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν AH, HB ἴσον ἐστὶ τὸ ZA· άσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἐστὶν ή ΓΜ τῆ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ή ΓΖ. λέγω δή, ὄτι καὶ τρίτη.

Έπει γαρ σύμμετρόν έστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς HB, σύμμετρον ἄρα καὶ τὸ $\Gamma\Theta$ τῷ $K\Lambda$ · ὥστε καὶ ἡ ΓK τῆ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν έστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ' ὡς μὲν τὸ $\Gamma\Theta$ πρὸς τὸ ΝΛ, οὕτως έστιν ή ΓΚ πρός την ΝΜ, ώς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως έστιν ή ΝΜ πρός την ΚΜ· ώς ἄρα ή ΓΚ πρός την ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ [ἀπὸ τῆς MN, τουτέστι τῷ] τετάρτω μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτω μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται έλλεῖπον εἴδει τετραγώνω καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ΓΜ ἄρα τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καὶ οὐδετέρα τῶν ΓΜ, ΜΖ σύμμετρός έστι μήκει τῆ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι τρίτη.

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην· ὅπερ ἔδει δεῖξαι.

equal to the (square) on BG, have been applied to KH, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB [and the (sum of the squares) on AG and GB is medial]. CL (is) thus also medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder LF is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at point N. And let NO have been drawn parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) EF, producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since AG and GB are commensurable in square only, AG [is] thus incommensurable in length with GB. Thus, the (square) on AG is also incommensurable with the (rectangle contained) by AG and GB [Props. 6.1, 10.11]. But, the (sum of the squares) on AG and GB is commensurable with the (square) on AG, and twice the (rectangle contained) by AG and GBwith the (rectangle contained) by AG and GB. The (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.13]. But, CL is equal to the (sum of the squares) on AG and GB, and FL is equal to twice the (rectangle contained) by AG and GB. Thus, CL is incommensurable with FL. And as CL (is) to FL, so CMis to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since the (square) on AG is commensurable with the (square) on GB, CH (is) thus also commensurable with KL. Hence, CK (is) also (commensurable in length) with KM [Props. 6.1, 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM (is) to KM [Prop. 6.1].

Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11]. Thus, the (rectangle contained) by CK and KM is equal to the [(square) on MN—that is to say, to the] fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MFby the (square) on (some straight-line) commensurable (in length) with (CM) [Prop. 10.17]. And neither of CM and MF is commensurable in length with the (previously) laid down rational (straight-line) CD. CF is thus a third apotome [Def. 10.13].

Thus, the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth. (Which is) the very thing it was required to show.

Proposition 100

The (square) on a minor (straight-line), applied to a rational (straight-line), produces a fourth apotome as breadth.



Let AB be a minor (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to the rational (straight-line) CD, producing CF as breadth. I say that CF is a fourth apotome.

For let BG be an attachment to AB. Thus, AG and GB are incommensurable in square, making the sum of the squares on AG and GB rational, and twice the (rectangle contained) by AG and GB medial [Prop. 10.76]. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on BG, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB is rational. CL is thus also rational. And it is applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM (is) also rational, and commensurable in length with CD [Prop. 10.20]. And since the

ρ΄.

Τὸ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην.



Έστω ἐλάσσων ἡ AB, ἑητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ ἑητὴν τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι τετάρτη.

Έστω γὰρ τῆ AB προσαρμόζουσα ἡ BH· ai ἄρα AH, HB δυνάμει εἰσιν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AH, HB τετραγώνων ῥητόν, τὸ δὲ δἰς ὑπὸ τῶν AH, HB μέσον. καὶ τῷ μὲν ἀπὸ τῆς AH ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς BH ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν KM· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν AH, HB. καί ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AH, HB ἑητόν· ἑητὸν ἄρα ἐστὶ καὶ τὸ ΓΛ. καὶ παρὰ ἑητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ἑητὴ ἄρα καὶ ἡ ΓΜ καὶ σύμμετρος τῆ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν AH, HB, ῶν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δἰς ὑπὸ τῶν AH, HB. τετμήσθω οῦν ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ ἦχθω δὶα τοῦ Ν ὁποτέρα τῶν ΓΔ, ΜΛ παράλληλος ή ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὸ δἰς ὑπὸ τῶν ΑΗ, ΗΒ μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΖΛ, καὶ τὸ ΖΛ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῆ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστιν, τὸ δὲ δἰς ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα [ἄρα] ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δἰς ὑπὸ τῶν ΑΗ, ΗΒ. ἴσον δέ [ἐστι] τὸ ΓΛ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ δἰς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῆ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λέγω [δή], ὅτι καὶ τετάρτη.

Έπει γαρ αί ΑΗ, ΗΒ δυνάμει είσιν ασύμμετροι, ασύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. καί ἐστι τῷ μέν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ $K\Lambda$ · ἀσύμμετρον ἄρα ἐστὶ τὸ $\Gamma\Theta$ τῷ $K\Lambda$. ὡς δὲ τὸ $\Gamma\Theta$ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ ἀσύμμετρος ἄρα έστιν ή ΓΚ τῆ ΚΜ μήχει. χαι ἐπει τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστιν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΛ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΛ, τῶν ἄρα ΓΘ, ΚΛ μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ' ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστίν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρός τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρός τὴν ΚΜ· τὸ ἄρα ύπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, τουτέστι τῷ τετάρτω μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αί $\Gamma M,~MZ,$ καὶ τῷ τετράρτ
ῳ μέρει τοῦ ἀπὸ τῆς MZίσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνω τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καί έστιν όλη ή ΓM σύμμετρος μήκει τ
 τῆ ἐκκειμένῃ ἑητῆ τῆ $\Gamma \Delta$ · ή ἄρα ΓΖ ἀποτομή ἐστι τετάρτη.

Τὸ ἄρα ἀπὸ ἐλάσσονος καὶ τὰ ἑξῆς.

whole of CL is equal to the (sum of the squares) on AGand GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at point N. And let NO have been drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is medial, and is equal to FL, FL is thus also medial. And it is applied to the rational (straight-line) FE, producing FM as breadth. Thus, FM is rational, and incommensurable in length with CD[Prop. 10.22]. And since the sum of the (squares) on AG and *GB* is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AGand GB is [thus] incommensurable with twice the (rectangle contained) by AG and GB. And CL (is) equal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB. CL[is] thus incommensurable with FL. And as CL (is) to FL, so CM is to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. *CF* is thus an apotome [Prop. 10.73]. [So], I say that (it is) also a fourth (apotome).

For since AG and GB are incommensurable in square, the (square) on AG (is) thus also incommensurable with the (square) on GB. And CH is equal to the (square) on AG, and KL equal to the (square) on GB. Thus, CH is incommensurable with KL. And as CH (is) to KL, so CK is to KM [Prop. 6.1]. CK is thus incommensurable in length with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (square) on GB to KL, and the (rectangle contained) by AG and GB to NL, NL is thus the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to KM [Prop. 6.1]. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on MN—that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MFby the (square) on (some straight-line) incommensurable

(in length) with (CM) [Prop. 10.18]. And the whole of CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a fourth apotome [Def. 10.14].

Thus, the (square) on a minor, and so on ...

Proposition 101

The (square) on that (straight-line) which with a rational (area) makes a medial whole, applied to a rational (straight-line), produces a fifth apotome as breadth.



Let AB be that (straight-line) which with a rational (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a fifth apotome.

Let BG be an attachment to AB. Thus, the straightlines AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational [Prop. 10.77]. And let CH, equal to the (square) on AG, have been applied to CD, and KL, equal to the (square) on GB. The whole of CL is thus equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB together is medial. Thus, CL is medial. And it has been applied to the rational (straight-line) CD, producing CM as breadth. CM is thus rational, and incommensurable (in length) with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at N. And let NO have been drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is rational, and [is] equal to FL, FL is thus rational. And it is applied to the rational (straight-line) EF, producing FM as breadth. Thus, FM is rational, and commensurable in length with CD[Prop. 10.20]. And since *CL* is medial, and *FL* rational,

ρα΄.

Τὸ ἀπὸ τῆς μετὰ ἑητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην.



Έστω ή μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ AB, ἑητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι πέμπτη.

Έστω γὰρ τῆ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ εύθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον έκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ' αὐτῶν ῥητόν, καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ἅμα μέσον έστίν μέσον ἄρα ἐστὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράχειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῆ ΓΔ. ϰαὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ῶν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οῦν ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν ὁποτέρα τῶν ΓΔ, ΜΛ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν AH, HB, ϰαὶ ἐπεὶ τὸ δὶς ὑπὸ τῶν AH, ΗΒ ἑητόν ἐστι καί [ἐστιν] ἴσον τῷ $Z\Lambda$, ἑητὸν ἄρα ἐστὶ τὸ ΖΛ. καὶ παρὰ ἑητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ἑητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ σύμμετρος τῆ ΓΔ μήκει. καὶ έπει τὸ μὲν ΓΛ μέσον ἐστίν, τὸ δὲ ΖΛ ῥητόν, ἀσύμμετρον άρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ή ΓΜ πρός τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ή ΓΜ τῆ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λὲγω δή, ὄτι καὶ πέμπτη.

Όμοίως γὰρ δείξομεν, ὅτι τὸ ὑπὸ τῶν ΓΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς

ΖΜ. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, ἴσον δὲ τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΛ, ἀσύμμετρον ἄρα τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἡ ΓΚ τῆ ΚΜ μήκει. ἐπεὶ οῦν δύο εὐθεῖαι ἄνισοί εἰσιν αἰ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσύμμέτρου ἑαυτῆ. καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος τῆ ἐκκειμένη ῥητῆ Γῶ· ἡ ἄρα ΓΖ ἀποτομή ἐστι πέμπτη· ὅπερ ἕδει δεῖξαι.

ρβ΄.

Τὸ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕχτην.



Έστω ή μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ AB, ἑητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστιν ἕκτη.

Έστω γὰρ τῆ AB προσαρμόζουσα ἡ BH· αἱ ἄρα AH, HB δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ δἰς ὑπὸ τῶν AH, HB μέσον καὶ ἀσύμμετρον τὰ ἀπὸ τῶν AH, HB τῷ CL is thus incommensurable with FL. And as CL (is) to FL, so CM (is) to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

For, similarly (to the previous propositions), we can show that the (rectangle contained) by CKM is equal to the (square) on NM—that is to say, to the fourth part of the (square) on FM. And since the (square) on AGis incommensurable with the (square) on GB, and the (square) on AG (is) equal to CH, and the (square) on GB to KL, CH (is) thus incommensurable with KL. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1]. Thus, CK (is) incommensurable in length with KM[Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM)[Prop. 10.18]. And the attachment FM is commensurable with the (previously) laid down rational (straightline) CD. Thus, CF is a fifth apotome [Def. 10.15]. (Which is) the very thing it was required to show.

Proposition 102

The (square) on that (straight-line) which with a medial (area) makes a medial whole, applied to a rational (straight-line), produces a sixth apotome as breadth.



Let AB be that (straight-line) which with a medial (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a sixth apotome.

For let BG be an attachment to AB. Thus, AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle

δὶς ὑπὸ τῶν ΑΗ, ΗΒ. παραβεβλήσθω οὖν παρὰ τὴν ΓΔ τῷ μέν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα [ἐστί] καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράχειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ή ΓΜ καὶ ἀσύμμετρος τῆ ΓΔ μήκει. ἐπεὶ οὖν τὸ ΓΛ ἴσον έστι τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ῶν τὸ ΓΕ ἴσον τῶ ἀπὸ τῆς ΑΒ, λοιπόν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. καί έστι τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον καὶ τὸ ΖΛ ἄρα μέσον έστίν. και παρά ρητήν τήν ΖΕ παράκειται πλάτος ποιοῦν τήν ΖΜ· ἑητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῆ ΓΔ μήκει. καὶ έπει τὰ ἀπὸ τῶν ΑΗ, ΗΒ ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ, ἀσύμμετρος ἄρα [ἐστὶ] τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ή ΓΜ πρὸς τὴν MZ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῆ MZ μήκει. καί εἰσιν ἀμφότεραι ῥηταί. αἱ ΓΜ, ΜΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λέγω δή, ὄτι καὶ ἕκτη.

Ἐπεὶ γὰρ τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, τετμήσθω δίχα ή ΖΜ κατά τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν τῆ ΓΔ παράλληλος ή ΝΞ· ἑχάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. χαὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει εἰσὶν άσύμμετροι, άσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΗ τῶ ἀπὸ τῆς ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον ἐστὶ τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς HB ἴσον ἐστὶ τὸ Κ $\Lambda\cdot$ ἀσύμμετρον ἄρα ἐστὶ τὸ Γ Θ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῆ ΚΜ. καὶ ἐπεὶ τῶν άπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, HB, καὶ ἐστι τῷ μὲν ἀπὸ τῆς AH ἴσον τὸ $\Gamma\Theta$, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΛ, καὶ τῶν ắρ
α $\Gamma\Theta,\, \mathrm{K}\Lambda$ μέσον ἀνάλογόν ἐστι τὸ $\mathrm{N}\Lambda^.$ ἔστιν ἄρα ώς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ.
 καὶ διὰ τὰ αὐτὰ ἡ ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου έαυτῆ. καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῆ ἐκκειμένῃ ρητῆ τῆ ΓΔ· ἡ ΓΖ ἄρα ἀποτομή ἐστιν ἕκτη· ὅπερ ἕδει δεῖξαι.

contained) by AG and GB medial, and the (sum of the squares) on AG and GB incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.78]. Therefore, let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on BG. Thus, the whole of CLis equal to the (sum of the squares) on AG and GB. CL [is] thus also medial. And it is applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. Therefore, since CL is equal to the (sum of the squares) on AG and GB, of which CE (is) equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB[Prop. 2.7]. And twice the (rectangle contained) by AGand GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) FE, producing FMas breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since the (sum of the squares) on AG and GB is incommensurable with twice the (rectangle contained) by AG and GB, and CLequal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB, CL [is] thus incommensurable with FL. And as CL(is) to FL, so CM is to MF [Prop. 6.1]. Thus, CM is incommensurable in length with MF [Prop. 10.11]. And they are both rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since FL is equal to twice the (rectangle contained) by AG and GB, let FM have been cut in half at N, and let NO have been drawn through N, parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since AG and GBare incommensurable in square, the (square) on AG is thus incommensurable with the (square) on GB. But, CH is equal to the (square) on AG, and KL is equal to the (square) on GB. Thus, CH is incommensurable with KL. And as CH (is) to KL, so CK is to KM[Prop. 6.1]. Thus, CK is incommensurable (in length) with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL(is) to KL. And for the same (reasons as the preceding propositions), the square on CM is greater than (the square on) MF by the (square) on (some straight-line)

incommensurable (in length) with (CM) [Prop. 10.18]. And neither of them is commensurable with the (previously) laid down rational (straight-line) CD. Thus, CFis a sixth apotome [Def. 10.16]. (Which is) the very thing it was required to show.

Proposition 103



ργ΄.

Έστω ἀποτομὴ ἡ AB, καὶ τῆ AB μήκει σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ ἀποτομή ἐστι καὶ τῆ τάξει ἡ αὐτὴ τῆ AB.

Έπεὶ γὰρ ἀποτομή ἑστιν ἡ AB, ἔστω αὐτῆ προσαρμόζουσα ἡ BE· αἱ AE, EB ἄρα ἑρηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ τῆς AB πρὸς τὴν ΓΔ λόγῳ ὁ αὐτὸς γεγονέτω ὁ τῆς BE πρὸς τὴν ΔΖ· καὶ ὡς ἕν ἄρα πρὸς ἕν, πάντα [ἐστὶ] πρὸς πάντα· ἔστιν ἄρα καὶ ὡς ὅλη ἡ AE πρὸς ὅλην τὴν ΓΖ, οὕτως ἡ AB πρὸς τὴν ΓΔ. σύμμετρος δὲ ἡ AB τῆ ΓΔ μήκει· σύμμετρος ἄρα καὶ ἡ AE μὲν τῆ ΓΖ, ἡ δὲ BE τῆ ΔΖ. καὶ αἱ AE, EB ἑηταί εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι [ἀποτομὴ ἄρα ἐστὶν ἡ ΓΔ. λέγω δή, ὅτι καὶ τῆ τάξει ἡ αὐτὴ τῆ AB].

Έπεὶ οὕν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΕ πρὸς τὴν ΔΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ. ἤτοι δὴ ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ ἢ τῷ ἀπὸ ἀσυμμέτρου. εἰ μὲν οὕν ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῆ ἐκκειμένῃ ῥητῆ μήκει, καὶ ἡ ΓΖ, εἰ δὲ ἡ ΒΕ, καὶ ἡ ΔΖ, εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, καὶ οὐδετέρα τῶν ΓΖ, ΖΔ. εἰ δὲ ἡ ΑΕ [τῆς ΕΒ] μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καὶ εἰ μὲν σύμμετρός ἑστιν ἡ ΑΕ τῆ ἐκκειμένῃ ῥητῆ μήκει, καὶ ἡ ΓΖ, εἰ δὲ ἡ ΒΕ, καὶ ἡ ΔΖ, εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, οὐδετέρα τῶν ΓΖ, ΖΔ.

Άποτομή ἄρα ἐστὶν ἡ ΓΔ καὶ τῆ τάξει ἡ αὐτὴ τῆ AB· ὅπερ ἔδει δεῖξαι. A (straight-line) commensurable in length with an apotome is an apotome, and (is) the same in order.



Let AB be an apotome, and let CD be commensurable in length with AB. I say that CD is also an apotome, and (is) the same in order as AB.

For since AB is an apotome, let BE be an attachment to it. Thus, AE and EB are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let it have been contrived that the (ratio) of BE to DFis the same as the ratio of AB to CD [Prop. 6.12]. Thus, also, as one is to one, (so) all [are] to all [Prop. 5.12]. And thus as the whole AE is to the whole CF, so AB(is) to CD. And AB (is) commensurable in length with CD. AE (is) thus also commensurable (in length) with CF, and BE with DF [Prop. 10.11]. And AE and BE are rational (straight-lines which are) commensurable in square only. Thus, CF and FD are also rational (straight-lines which are) commensurable in square only [Prop. 10.13]. [CD is thus an apotome. So, I say that (it is) also the same in order as AB.]

Therefore, since as AE is to CF, so BE (is) to DF, thus, alternately, as AE is to EB, so CF (is) to FD [Prop. 5.16]. So, the square on AE is greater than (the square on) EB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (AE). Therefore, if the (square) on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AEis commensurable in length with a (previously) laid down rational (straight-line) then so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF, and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13]. And if the (square) on AE is greater [than (the square on) EB] by the (square) on (some straight-line) incommensurable (in ρδ΄.

Ή τῆ μέσης ἀποτομῆ σύμμετρος μέσης ἀποτομή ἐστι καὶ τῆ τάξει ἡ αὐτή.



Έστω μέσης ἀποτομὴ ἡ AB, καὶ τῆ AB μήκει σύμμετρος ἕστω ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μέσης ἀποτομή ἐστι καὶ τῆ τάξει ἡ αὐτὴ τῆ AB.

Έπεὶ γὰρ μέσης ἀποτομή ἐστιν ἡ AB, ἔστω αὐτῆ προσαρμόζουσα ἡ EB. αί AE, EB ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέτω ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ BE πρὸς τὴν ΔΖ· σύμμετρος ἄρα [ἐστὶ] καὶ ἡ AE τῆ ΓΖ, ἡ δὲ BE τῆ ΔΖ. αἱ δὲ AE, EB μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· μέσης ἄρα ἀποτομή ἐστιν ἡ ΓΔ. λέγω δή, ὅτι καὶ τῆ τάξει ἐστὶν ἡ αὐτὴ τῆ AB.

Έπεὶ [γάρ] ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ [ἀλλ' ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, ὡς δὲ ἡ ΓΖ πρὸς τὴν ΖΔ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ], ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ [καὶ ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ [καὶ ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ [καὶ ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ [καὶ ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΓΖ[.] σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ. εἴτε οῦν ῥητόν ἐστι τὸ ὑπὸ τῶν ΑΕ, ΕΒ, ῥητὸν ἔσται καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ, εἴτε μέσον [ἐστὶ] τὸ ὑπὸ τῶν ΑΕ, ΕΒ, μέσον [ἐστὶ] καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ.

Μέσης ἄρα ἀποτομή ἐστιν ἡ ΓΔ καὶ τῆ τάξει ἡ αὐτὴ τῆ ΑΒ· ὅπερ ἔδει δεῖξαι. length) with (AE) then the (square) on CF will also be greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable in length with a (previously) laid down rational (straightline), so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF, and if neither of AE or EB(are commensurable), neither (are) either of CF or FD[Prop. 10.13].

Thus, CD is an apotome, and (is) the same in order as AB [Defs. 10.11—10.16]. (Which is) the very thing it was required to show.

Proposition 104

A (straight-line) commensurable (in length) with an apotome of a medial (straight-line) is an apotome of a medial (straight-line), and (is) the same in order.



Let AB be an apotome of a medial (straight-line), and let CD be commensurable in length with AB. I say that CD is also an apotome of a medial (straight-line), and (is) the same in order as AB.

For since AB is an apotome of a medial (straightline), let EB be an attachment to it. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.74, 10.75]. And let it have been contrived that as AB is to CD, so BE (is) to DF[Prop. 6.12]. Thus, AE [is] also commensurable (in length) with CF, and BE with DF [Props. 5.12, 10.11]. And AE and EB are medial (straight-lines which are) commensurable in square only. CF and FD are thus also medial (straight-lines which are) commensurable in square only [Props. 10.23, 10.13]. Thus, CD is an apotome of a medial (straight-line) [Props. 10.74, 10.75]. So, I say that it is also the same in order as AB.

[For] since as AE is to EB, so CF (is) to FD[Props. 5.12, 5.16] [but as AE (is) to EB, so the (square) on AE (is) to the (rectangle contained) by AE and EB, and as CF (is) to FD, so the (square) on CF (is) to the (rectangle contained) by CF and FD], thus as the (square) on AE is to the (rectangle contained) by AEand EB, so the (square) on CF also (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.] [and, alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AE and EB (is) to the (rectangle contained) by CF and FD]. And the (square) on AE (is) commensurable with the (square)



[•]Η τῆ ἐλάσσονι σύμμετρος ἐλάσσων ἐστίν.



Έστω γὰρ ἐλάσσων ἡ ΑΒ καὶ τῆ ΑΒ σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ ἐλάσσων ἐστίν.

Γεγονέτω γὰρ τὰ αὐτά· καὶ ἐπεὶ αἱ ΑΕ, ΕΒ δυνάμει εἰσὶν άσύμμετροι, καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. έπει οῦν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οῦτως ἡ ΓΖ πρὸς τὴν $Z\Delta$, ἔστιν ẳρα καὶ ὡς τὸ ἀπὸ τῆς AE πρὸς τὸ ἀπὸ τῆς EB, οὕτως τὸ ἀπὸ τῆς ΓZ πρὸς τὸ ἀπὸ τῆς $Z\Delta$. συνθέντι ἄρα ἐστὶν ὡς τὰ ἀπὸ τῶν ΑΕ, ΕΒ πρὸς τὸ ἀπὸ τῆς ΕΒ, οὕτως τὰ ἀπὸ τῶν ΓΖ, Ζ Δ πρὸς τὸ ἀπὸ τῆς Ζ Δ [καὶ ἐναλλάξ]· σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς BE τῷ ἀπὸ τῆς $\Delta \operatorname{Z}$ · σύμμετρον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένω ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων. ἑητὸν δέ ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων ἑητὸν ἄρα ἐστὶ καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, Ζ Δ τετραγώνων. πάλιν, ἐπεί ἐστιν ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ, σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τετράγωνον τῷ ἀπὸ τῆς ΓΖ τετραγώνῳ, σύμμετρον ἄρα έστι και τὸ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ· αί ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγχείμενον έχ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ύπ' αὐτῶν μέσον.

Ἐλάσσων ἄρα ἐστὶν ἡ ΓΔ· ὅπερ ἔδει δεῖξαι.

on *CF*. Thus, the (rectangle contained) by *AE* and *EB* is also commensurable with the (rectangle contained) by *CF* and *FD* [Props. 5.16, 10.11]. Therefore, either the (rectangle contained) by *AE* and *EB* is rational, and the (rectangle contained) by *CF* and *FD* will also be rational [Def. 10.4], or the (rectangle contained) by *AE* and *EB* [is] medial, and the (rectangle contained) by *CF* and *FD* will also be *AE* and *FD* [is] also medial [Prop. 10.23 corr.].

Therefore, CD is the apotome of a medial (straightline), and is the same in order as AB [Props. 10.74, 10.75]. (Which is) the very thing it was required to show.

Proposition 105

A (straight-line) commensurable (in length) with a minor (straight-line) is a minor (straight-line).



For let AB be a minor (straight-line), and (let) CD (be) commensurable (in length) with AB. I say that CD is also a minor (straight-line).

For let the same things have been contrived (as in the former proposition). And since AE and EB are (straight-lines which are) incommensurable in square [Prop. 10.76], CF and FD are thus also (straight-lines which are) incommensurable in square [Prop. 10.13]. Therefore, since as AE is to EB, so CF (is) to FD[Props. 5.12, 5.16], thus also as the (square) on AE is to the (square) on EB, so the (square) on CF (is) to the (square) on FD [Prop. 6.22]. Thus, via composition, as the (sum of the squares) on AE and EB is to the (square) on EB, so the (sum of the squares) on CF and FD (is) to the (square) on FD [Prop. 5.18], [also alternately]. And the (square) on *BE* is commensurable with the (square) on DF [Prop. 10.104]. The sum of the squares on AEand EB (is) thus also commensurable with the sum of the squares on CF and FD [Prop. 5.16, 10.11]. And the sum of the (squares) on AE and EB is rational [Prop. 10.76]. Thus, the sum of the (squares) on CF and FD is also rational [Def. 10.4]. Again, since as the (square) on AE is to the (rectangle contained) by AE and EB, so the (square) on CF (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.], and the square on AE(is) commensurable with the square on CF, the (rectangle contained) by AE and EB is thus also commensurable with the (rectangle contained) by CF and FD. And the (rectangle contained) by AE and EB (is) medial [Prop. 10.76]. Thus, the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.]. CF and ρς'.

[°]Η τῆ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιούση σύμμετρος μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.



Έστω μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ καὶ τῆ ΑΒ σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Έστω γὰρ τῆ AB προσαρμόζουσα ἡ BE· αἱ AE, EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AE, EB τετραγώνων μέσον, τὸ δỉ ὑπỉ αὐτῶν ῥητόν. καὶ τὰ αὐτὰ κατεσκευάσθω. ὁμοίως δὴ δείξομεν τοῖς πρότερον, ὅτι αἱ ΓΖ, ΖΔ ἐν τῷ αὐτῷ λόγῳ εἰσὶ ταῖς AE, EB, καὶ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AE, EB τετραγώνων τῷ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων, τὸ δὲ ὑπὸ τῶν AE, EB τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον, τὸ δỉ ὑπỉ αὐτῶν ῥητόν.

Ἡ ΓΔ ἄρα μετὰ ἡητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.



Ή τῆ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ σύμμετρος καὶ αὐτὴ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.



Έστω μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ AB, καὶ τῇ

FD are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Thus, CD is a minor (straight-line) [Prop. 10.76]. (Which is) the very thing it was required to show.

Proposition 106

A (straight-line) commensurable (in length) with a (straight-line) which with a rational (area) makes a medial whole is a (straight-line) which with a rational (area) makes a medial whole.



Let AB be a (straight-line) which with a rational (area) makes a medial whole, and (let) CD (be) commensurable (in length) with AB. I say that CD is also a (straight-line) which with a rational (area) makes a medial (whole).

For let BE be an attachment to AB. Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on AE and EB medial, and the (rectangle contained) by them rational [Prop. 10.77]. And let the same construction have been made (as in the previous propositions). So, similarly to the previous (propositions), we can show that CF and FD are in the same ratio as AE and EB, and the sum of the squares on AE and EB is commensurable with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Hence, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on CF and FD medial, and the (rectangle contained) by them rational.

CD is thus a (straight-line) which with a rational (area) makes a medial whole [Prop. 10.77]. (Which is) the very thing it was required to show.

Proposition 107

A (straight-line) commensurable (in length) with a (straight-line) which with a medial (area) makes a medial whole is itself also a (straight-line) which with a medial (area) makes a medial whole.



Let AB be a (straight-line) which with a medial (area)

ΑΒ ἔστω σύμμετρος
 ή ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσ
ά ἐστιν.

Έστω γὰρ τῆ AB προσαρμόζουσα ἡ BE, καὶ τὰ αὐτὰ κατεσκευάσθω· ai AE, EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ² αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ² αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκέιμενον ἐκ τῶν ἀπ² αὐτῶν τετραγώνων τῷ ὑπ² αὐτῶν. καί εἰσιν, ὡς ἐδείχθη, αἱ AE, EB σύμμετροι ταῖς ΓΖ, ΖΔ, καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AE, EB τετραγώνων τῷ συγκείμενω ἐκ τῶν ἀπὸ τῶν ΛΕ, EB τετραγώνων τῷ συγκείμενω ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ τῶν AE, EB τῷ ὑπὸ τῶν ΓΖ, ΖΔ· καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ² αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ² ἀὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ² αὐτῶν [τετραγώνων] τῷ ὑπ² αὐτῶν.

Ή ΓΔ ἄρα μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρη΄.

᾿Απὸ ἑητοῦ μέσου ἀφαιρουμένου ἡ τὸ λοιπὸν χωρίον δυναμένη μία δύο ἀλόγων γίνεται ἤτοι ἀποτομὴ ἢ ἐλάσσων.



Άπὸ γὰρ ἑητοῦ τοῦ ΒΓ μέσον ἀφηρήσθω τὸ ΒΔ· λέγω, ὅτι ἡ τὸ λοιπὸν δυναμένη τὸ ΕΓ μία δύο ἀλόγων γίνεται ἤτοι ἀποτομὴ ἢ ἐλάσσων.

Έχχείσθω γὰρ ἑητὴ ἡ ZH, καὶ τῷ μὲν BΓ ἴσον παρὰ τὴν ZH παραβεβλήσθω ὀρθογώνιον παραλληλόγραμμον τὸ HΘ, τῷ δὲ ΔB ἴσον ἀφηρήσθω τὸ HK· λοιπὸν ἄρα τὸ EΓ ἴσον ἐστὶ τῷ ΛΘ. ἐπεὶ οῦν ἑητὸν μέν ἐστι τὸ BΓ, μέσον δὲ τὸ BΔ, ἴσον δὲ τὸ μὲν BΓ τῷ HΘ, τὸ δὲ BΔ τῷ HK, ἑητὸν μὲν ἄρα ἐστὶ τὸ HΘ, μέσον δὲ τὸ HK. καὶ παρὰ ἑητὴν τὴν ZH παράχειται· ἑητὴ μὲν ἄρα ἡ ZΘ καὶ σύμμετρος τῆ ZH μήχει, ἑητὴ δὲ ἡ ZK καὶ ἀσύμμετρος τῆ ZH μήχει· ἀσύμμετρος ἄρα makes a medial whole, and let CD be commensurable (in length) with AB. I say that CD is also a (straight-line) which with a medial (area) makes a medial whole.

For let BE be an attachment to AB. And let the same construction have been made (as in the previous propositions). Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.78]. And, as was shown (previously), AE and EB are commensurable (in length) with CF and FD(respectively), and the sum of the squares on AE and EB with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Thus, CF and FDare also (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the [squares] on them incommensurable with the (rectangle contained) by them.

Thus, CD is a (straight-line) which with a medial (area) makes a medial whole [Prop. 10.78]. (Which is) the very thing it was required to show.

Proposition 108

A medial (area) being subtracted from a rational (area), one of two irrational (straight-lines) arise (as) the square-root of the remaining area—either an apotome, or a minor (straight-line).



For let the medial (area) BD have been subtracted from the rational (area) BC. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either an apotome, or a minor (straight-line).

For let the rational (straight-line) FG have been laid out, and let the right-angled parallelogram GH, equal to BC, have been applied to FG, and let GK, equal to DB, have been subtracted (from GH). Thus, the remainder EC is equal to LH. Therefore, since BC is a rational (area), and BD a medial (area), and BC (is) equal to ἐστὶν ἡ ΖΘ τῆ ΖΚ μήχει. αἱ ΖΘ, ΖΚ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ, προσαρμόζουσα δὲ αὐτῆ ἡ ΚΖ. ἦτοι δὴ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἢ οὕ.

Δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου. καί ἐστιν ὅλη ἡ ΘZ σύμμετρος τῆ ἐκκειμένῃ ῥητῆ μήκει τῆ ZH· ἀποτομὴ ἄρα πρώτῃ ἐστὶν ἡ $K\Theta$. τὸ δ' ὑπὸ ἑητῆς καὶ ἀποτομῆς πρώτῃς περιεχόμενον ἡ δυναμένῃ ἀποτομή ἐστιν. ἡ ἄρα τὸ $\Lambda\Theta$, τουτέστι τὸ $E\Gamma$, δυναμένῃ ἀποτομή ἐστιν.

Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καί ἐστιν ὅλη ἡ ΖΘ σύμμετρος τῆ ἐκκειμένῃ ῥητῆ μήκει τῆ ΖΗ, ἀποτομὴ τετάρτῃ ἐστὶν ἡ ΚΘ. τὸ δ' ὑπὸ ἑῃτῆς καὶ ἀποτομῆς τετάρτῃς περιεχόμενον ἡ δυναμένῃ ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι.

ρθ΄.

Άπὸ μέσου ἑητοῦ ἀφαιρουμένου ἄλλαι δύο ἄλογοι γίνονται ἤτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα.

Άπὸ γὰρ μέσου τοῦ ΒΓ ἑητὸν ἀφηρήσθω τὸ ΒΔ. λέγω, ὅτι ἡ τὸ λοιπὸν τὸ ΕΓ δυναμένη μία δύο ἀλόγων γίνεται ἤτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσα.

Έκκείσθω γὰρ ἑητὴ ή ZH, καὶ παραβεβλήσθω ὁμοίως τὰ χωρία. ἔστι δὴ ἀκολούθως ἑητὴ μὲν ἡ ZΘ καὶ ἀσύμμετρος τῆ ZH μήκει, ἑητὴ δὲ ἡ KZ καὶ σύμμετρος τῆ ZH μήκει· αἱ ZΘ, ZK ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ǎρα ἐστὶν ἡ KΘ, προσαρμόζουσα δὲ ταύτῃ ἡ ZK. ἤτοι δὴ ἡ ΘZ τῆς ZK μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ ἢ τῷ ἀπὸ ἀσυμμέτρου. GH, and BD to GK, GH is thus a rational (area), and GK a medial (area). And they are applied to the rational (straight-line) FG. Thus, FH (is) rational, and commensurable in length with FG [Prop. 10.20], and FK (is) also rational, and incommensurable in length with FG [Prop. 10.22]. Thus, FH is incommensurable in length with FK [Prop. 10.13]. FH and FK are thus rational (straight-lines which are) commensurable in square only. Thus, KH is an apotome [Prop. 10.73], and KF an attachment to it. So, the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) either commensurable, or not (commensurable), (in length with HF).

First, let the square (on it) be (greater) by the (square) on (some straight-line which is) commensurable (in length with HF). And the whole of HF is commensurable in length with the (previously) laid down rational (straight-line) FG. Thus, KH is a first apotome [Def. 10.1]. And the square-root of an (area) contained by a rational (straight-line) and a first apotome is an apotome [Prop. 10.91]. Thus, the square-root of LH—that is to say, (of) EC—is an apotome.

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) incommensurable (in length) with (HF), and (since) the whole of FH is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fourth apotome [Prop. 10.14]. And the square-root of an (area) contained by a rational (straight-line) and a fourth apotome is a minor (straight-line) [Prop. 10.94]. (Which is) the very thing it was required to show.

Proposition 109

A rational (area) being subtracted from a medial (area), two other irrational (straight-lines) arise (as the square-root of the remaining area)—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (area) BD have been subtracted from the medial (area) BC. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (straight-line) FG be laid down, and let similar areas (to the preceding proposition) have been applied (to it). So, accordingly, FH is rational, and incommensurable in length with FG, and KF (is) also rational, and commensurable in length with FG. Thus, FH and FK are rational (straight-lines which are) com-



Eἰ μὲν οὖν ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῆ ἐκκειμένῃ ἑητῆ μήκει τῆ ΖΗ, ἀποτομὴ δευτέρα ἐστὶν ἡ KΘ. ἑητὴ δὲ ἡ ΖΗ· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένῃ μέσης ἀποτομὴ πρώτῃ ἐστίν.

Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου, καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῆ ἐκκειμένῃ ἑητῆ μήκει τῆ ΖΗ, ἀποτομὴ πέμπτη ἐστὶν ἡ ΚΘ· ὥστε ἡ τὸ ΕΓ δυναμένη μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρí.

Άπὸ μέσου μέσου ἀφαιρουμένου ἀσυμμέτρου τῷ ὅλῳ αἱ λοιπαὶ δύο ἄλογοι γίνονται ἤτοι μέσης ἀποτομὴ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Άφηρήσθω γὰρ ὡς ἐπὶ τῶν προχειμένων καταγραφῶν ἀπὸ μέσου τοῦ ΒΓ μέσον τὸ ΒΔ ἀσύμμετρον τῷ ὅλῷ· λέγω, ὅτι ἡ τὸ ΕΓ δυναμένη μία ἐστὶ δύο ἀλόγων ἤτοι μέσης ἀποτομὴ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.





Therefore, if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a second apotome [Def. 10.12]. And FG (is) rational. Hence, the square-root of LH—that is to say, (of) EC—is a first apotome of a medial (straight-line) [Prop. 10.92].

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) incommensurable (in length with HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fifth apotome [Def. 10.15]. Hence, the square-root of ECis that (straight-line) which with a rational (area) makes a medial whole [Prop. 10.95]. (Which is) the very thing it was required to show.

Proposition 110

A medial (area), incommensurable with the whole, being subtracted from a medial (area), the two remaining irrational (straight-lines) arise (as) the (square-root of the area)—either a second apotome of a medial (straightline), or that (straight-line) which with a medial (area) makes a medial whole.

For, as in the previous figures, let the medial (area) BD, incommensurable with the whole, have been subtracted from the medial (area) BC. I say that the squareroot of EC is one of two irrational (straight-lines) either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.



Έπει γὰρ μέσον ἐστιν ἑκάτερον τῶν ΒΓ, ΒΔ, καὶ ἀσύμμετρον τὸ ΒΓ τῷ ΒΔ, ἔσται ἀκολούθως ῥητὴ ἑκατέρα τῶν ΖΘ, ΖΚ καὶ ἀσύμμετρος τῆ ΖΗ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΒΓ τῷ ΒΔ, τουτέστι τὸ ΗΘ τῷ ΗΚ, ἀσύμμετρος καὶ ἡ ΘΖ τῆ ΖΚ· αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ [προσαρμόζουσα δὲ ἡ ΖΚ. ἤτοι δὴ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἢ τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ].

Εἰ μὲν δὴ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ οὐθετέρα τῶν ΖΘ, ΖΚ σύμμετρός ἐστι τῆ ἐκκειμέμνῃ ἑητῆ μήκει τῆ ΖΗ, ἀποτομὴ τρίτῃ ἐστὶν ἡ ΚΘ. ἑŋτὴ δὲ ἡ ΚΛ, τὸ δ' ὑπὸ ἑŋτῆς καὶ ἀποτομῆς τρίτῃς περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένῃ αὐτὸ ἅλογός ἐστιν, καλεῖται δὲ μέσῃς ἀποτομὴ δευτέρα· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένῃ μέσῃς ἀποτομή ἐστι δευτερά.

Eì δὲ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ [μήχει], καὶ οὐθετέρα τῶν ΘΖ, ΖΚ σύμμετρός ἐστι τῆ ΖΗ μήχει, ἀποτομὴ ἕχτη ἐστὶν ἡ ΚΘ. τὸ δ' ὑπὸ ἑητῆς καὶ ἀποτομῆς ἕχτης ἡ δυναμένη ἐστὶ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ἡ τὸ ΛΘ ἄρα, τουτέστι τὸ ΕΓ, δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν. ὅπερ ἔδει δεῖξαι.



For since BC and BD are each medial (areas), and BC (is) incommensurable with BD, accordingly, FH and FK will each be rational (straight-lines), and incommensurable in length with FG [Prop. 10.22]. And since BC is incommensurable with BD—that is to say, GH with GK—HF (is) also incommensurable (in length) with FK [Props. 6.1, 10.11]. Thus, FH and FK are rational (straight-lines which are) commensurable in square only. KH is thus as apotome [Prop. 10.73], [and FK an attachment (to it). So, the square on FH is greater than (the square on) FK either by the (square) on (some straight-line) incommensurable, (in length) with (FH).]

So, if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (FH), and (since) neither of FH and FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a third apotome [Def. 10.3]. And KL (is) rational. And the rectangle contained by a rational (straight-line) and a third apotome is irrational, and the square-root of it is that irrational (straight-line) called a second apotome of a medial (straight-line) [Prop. 10.93]. Hence, the squareroot of LH—that is to say, (of) EC—is a second apotome of a medial (straight-line).

And if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) incommensurable [in length] with (FH), and (since) neither of HF and FK is commensurable in length with FG, KHis a sixth apotome [Def. 10.16]. And the square-root of the (rectangle contained) by a rational (straight-line) and a sixth apotome is that (straight-line) which with a medial (area) makes a medial whole [Prop. 10.96]. Thus, the square-root of LH—that is to say, (of) EC—is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to



Έστω ἀποτομὴ ἡ AB· λέγω, ὅτι ἡ AB οὐκ ἔστιν ἡ αὐτὴ τῆ ἐκ δύο ὀνομάτων.

Εἰ γὰρ δυνατόν, ἔστω· καὶ ἐκκείσθω ἑητὴ ἡ ΔΓ, καὶ τῷ άπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΔΕ. ἐπεὶ οὖν ἀποτομή ἐστιν ἡ ΑΒ, άποτομή πρώτη έστιν ή ΔΕ. ἔστω αὐτῃ προσαρμόζουσα ή ΕΖ· αί ΔΖ, ΖΕ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ή ΔZ τῆς ZE μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῃ, καὶ ἡ ΔZ σύμμετρός ἐστι τῆ ἐκκειμένῃ ἑητῆ μήκει τῆ $\Delta \Gamma$. πάλιν, έπει έκ δύο όνομάτων έστιν ή AB, έκ δύο άρα όνομάτων πρώτη ἐστὶν ἡ ΔΕ. διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Η, καὶ ἔστω μεῖζον ὄνομα τὸ ΔH · αἱ ΔH , HE ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΗ τῆς ΗΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ τὸ μεῖζον ἡ ΔΗ σύμμετρός έστι τ
 ή έκκειμένη þητη μήκει τη $\Delta \Gamma.$ καὶ
 ή ΔZ ἄρα τη ΔH σύμμετρός έστι μήχει· χαὶ λοιπὴ ἄρα ἡ ΗΖ σύμμετρός ἐστι τῆ ΔZ μήχει. [ἐπεὶ οῦν σύμμετρός ἐστιν ἡ ΔZ τῆ HZ, ἑητὴ δέ ἐστιν ἡ ΔZ , ἑητὴ ἄρα ἐστὶ καὶ ἡ HZ. ἐπεὶ οὖν σύμμετρός έστιν ή ΔΖ τῆ ΗΖ μήχει] ἀσύμμετρος δὲ ἡ ΔΖ τῆ ΕΖ μήχει. ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΖΗ τῆ ΕΖ μήκει. αἱ ΗΖ, ΖΕ ἄρα ρηταί [εἰσι] δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ ΕΗ. ἀλλὰ καὶ ῥητή· ὅπερ ἐστὶν ἀδύνατον.

Ή ἄρα ἀποτομὴ οὐκ ἔστιν ἡ αὐτὴ τῆ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι. show.



Let AB be an apotome. I say that AB is not the same as a binomial.

For, if possible, let it be (the same). And let a rational (straight-line) DC be laid down. And let the rectangle CE, equal to the (square) on AB, have been applied to CD, producing DE as breadth. Therefore, since AB is an apotome, DE is a first apotome [Prop. 10.97]. Let EFbe an attachment to it. Thus, DF and FE are rational (straight-lines which are) commensurable in square only, and the square on DF is greater than (the square on) FEby the (square) on (some straight-line) commensurable (in length) with (DF), and DF is commensurable in length with the (previously) laid down rational (straightline) DC [Def. 10.10]. Again, since AB is a binomial, DE is thus a first binomial [Prop. 10.60]. Let (DE) have been divided into its (component) terms at G, and let DG be the greater term. Thus, DG and GE are rational (straight-lines which are) commensurable in square only, and the square on DG is greater than (the square on) GE by the (square) on (some straight-line) commensurable (in length) with (DG), and the greater (term) DGis commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.5]. Thus, DF is also commensurable in length with DG [Prop. 10.12]. The remainder GF is thus commensurable in length with DF [Prop. 10.15]. [Therefore, since DF is commensurable with GF, and DF is rational, GF is thus also rational. Therefore, since DF is commensurable in length with GF, DF (is) incommensurable in length with EF. Thus, FG is also incommensurable in length with EF[Prop. 10.13]. GF and FE [are] thus rational (straightlines which are) commensurable in square only. Thus,

EG is an apotome [Prop. 10.73]. But, (it is) also rational. The very thing is impossible.

Thus, an apotome is not the same as a binomial. (Which is) the very thing it was required to show.

[Corollary]

The apotome and the irrational (straight-lines) after it are neither the same as a medial (straight-line) nor (the same) as one another.

For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) incommensurable in length with the (straight-line) to which (the area) is applied [Prop. 10.22]. And the (square) on an apotome, applied to a rational (straight-line), produces as breadth a first apotome [Prop. 10.97]. And the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a second apotome [Prop. 10.98]. And the (square) on a second apotome of a medial (straight-line), applied to a rational (straightline), produces as breadth a third apotome [Prop. 10.99]. And (square) on a minor (straight-line), applied to a rational (straight-line), produces as breadth a fourth apotome [Prop. 10.100]. And (square) on that (straight-line) which with a rational (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a fifth apotome [Prop. 10.101]. And (square) on that (straight-line) which with a medial (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a sixth apotome [Prop. 10.102]. Therefore, since the aforementioned breadths differ from the first (breadth), and from one another-from the first, because it is rational, and from one another since they are not the same in order-clearly, the irrational (straightlines) themselves also differ from one another. And since it has been shown that an apotome is not the same as a binomial [Prop. 10.111], and (that) the (irrational straight-lines) after the apotome, being applied to a rational (straight-line), produce as breadth, each according to its own (order), apotomes, and (that) the (irrational straight-lines) after the binomial themselves also (produce as breadth), according (to their) order, binomials, the (irrational straight-lines) after the apotome are thus different, and the (irrational straight-lines) after the binomial (are also) different, so that there are, in order, 13 irrational (straight-lines) in all:

[Πόρισμα.]

Ή ἀποτομὴ καὶ αἱ μετ' αὐτὴν ἄλογοι οὕτε τῆ μέσῃ οὕτε ἀλλήλαις εἰσὶν αἱ αὐταί.

Τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρ' ἢν παράκειται, μήκει, τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην, τὸ δὲ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν, τὸ δὲ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην, τὸ δὲ ἀπὸ έλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν τετάρτην, τὸ δὲ ἀπὸ τῆς μετὰ ἑητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν πέμπτην, τὸ δὲ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν ἕκτην. ἐπεὶ οῦν τὰ εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητή ἐστιν, άλλήλων δὲ, ἐπεὶ τῆ τάξει οὐκ εἰσὶν αἱ αὐταί, δῆλον, ὡς καὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων. καὶ ἐπεὶ δέδεικται ή άποτομή ούχ οῦσα ή αὐτή τῆ ἐχ δύο ὀνομάτων, ποιοῦσι δὲ πλάτη παρὰ ἑητὴν παραβαλλόμεναι αἱ μετὰ τὴν ἀποτομὴν ἀποτομὰς ἀχολούθως ἑχάστη τῆ τάξει τῆ χαθ' αὑτήν, αἱ δὲ μετά την έκ δύο όνομάτων τας έκ δύο όνομάτων και αυταί τῆ τάξει ἀκολούθως, ἕτεραι ἄρα εἰσὶν αἱ μετὰ τὴν ἀποτομὴν καὶ ἕτεραι αἱ μετὰ τὴν ἐκ δύο ὀνομάτων, ὡς εἶναι τῆ τάξει πάσας ἀλόγους ιγ,

Μέσην,

Έχ δύο ἀνομάτων, Έχ δύο μέσων πρώτην, Έχ δύο μέσων δευτέραν, Μείζονα, Ῥητὸν χαὶ μέσον δυναμένην, Δύο μέσα δυναμένην, Ἀποτομήν, Μἑσης ἀποτομὴν πρώτην, Μἑσης ἀποτομὴν δευτέραν, Ἐλάσσονα, Μετὰ ἑητοῦ μέσον τὸ ὅλον ποιοῦσαν,

Μετὰ μέσου μέσον τὸ ὅλον ποιοῦσαν.

ριβ΄.

Τὸ ἀπὸ ἑητῆς παρὰ τὴν ἐκ δύο ὀνομάτων παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν, ῆς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι καὶ ἔτι ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ γινομένη ἀποτομὴ τὴν αὐτὴν ἕξει τάξιν τῆ ἐκ δύο ὀνομάτων.



Έστω ἡητὴ μὲν ἡ A, ἐκ δύο ἀνομάτων δὲ ἡ BΓ, ῆς μεῖζον ὄνομα ἔστω ἡ $\Delta\Gamma$, καὶ τῷ ἀπὸ τῆς A ἴσον ἔστω τὸ ὑπὸ τῶν BΓ, EΖ· λέγω, ὅτι ἡ EZ ἀποτομή ἐστιν, ῆς τὰ ἀνόματα σύμμετρά ἐστι τοῖς Γ Δ , Δ B, καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ EZ τὴν αὐτὴν ἕξει τάξιν τῆ BΓ.

Έστω γὰρ πάλιν τῷ ἀπὸ τῆς Α ἴσον τὸ ὑπὸ τῶν ΒΔ, H. ἐπεὶ οῦν τὸ ὑπὸ τῶν ΒΓ, ΕΖ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, H, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ Η πρὸς τὴν EZ. μείζων δὲ ἡ ΓΒ τῆς ΒΔ· μείζων ἄρα ἐστὶ καὶ ἡ Η τῆς EZ. ἔστω τῆ Η ἴση ἡ ΕΘ· ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν BΔ, οὕτως ἡ ΘΕ πρὸς τὴν ΕΖ· διελόντι ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΖ πρὸς τὴν ΖΕ. γεγονέτω ὡς ἡ ΘΖ πρὸς τὴν ΖΕ, οὕτως ἡ ΖΕ πρὸς τὴν ΚΕ· καὶ ὅλη ἄρα ἡ ΘΚ πρὸς ὅλην τὴν ΚΖ ἐστιν, ὡς ἡ ΖΚ πρὸς ΚΕ· ὡς γὰρ ἕν τῶν ἡγουμένων πρὸς ἕν τῶν ἑπομένων, οὕτως ἄπαντα τὰ ἡγούμενα πρὸς ἄπαντα τὰ ἑπόμενα. ὡς δὲ ἡ ΖΚ πρὸς ΚΖ, οὕτως ἐστὶν ἡ ΓΔ πρὸς τὴν ΔΒ· καὶ ὡς ἄρα ἡ ΘΚ πρὸς ΚΖ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΒ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΓΔ τῷ ἀπὸ τῆς ΔΒ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ τῷ Medial, Binomial, First bimedial, Second bimedial, Major, Square-root of a rational plus a medial (area), Square-root of (the sum of) two medial (areas), Apotome, First apotome of a medial, Second apotome of a medial, Minor,

That which with a rational (area) produces a medial whole,

That which with a medial (area) produces a medial whole.

Proposition 112[†]

The (square) on a rational (straight-line), applied to a binomial (straight-line), produces as breadth an apotome whose terms are commensurable (in length) with the terms of the binomial, and, furthermore, in the same ratio. Moreover, the created apotome will have the same order as the binomial.



Let A be a rational (straight-line), and BC a binomial (straight-line), of which let DC be the greater term. And let the (rectangle contained) by BC and EF be equal to the (square) on A. I say that EF is an apotome whose terms are commensurable (in length) with CD and DB, and in the same ratio, and, moreover, that EF will have the same order as BC.

For, again, let the (rectangle contained) by BD and G be equal to the (square) on A. Therefore, since the (rectangle contained) by BC and EF is equal to the (rectangle contained) by BD and G, thus as CB is to BD, so G (is) to EF [Prop. 6.16]. And CB (is) greater than BD. Thus, G is also greater than EF [Props. 5.16, 5.14]. Let EH be equal to G. Thus, as CB is to BD, so HE (is) to EF. Thus, via separation, as CD is to BD, so HF (is) to FE [Prop. 5.17]. Let it have been contrived that as HF (is) to FE, so FK (is) to KE. And, thus, the whole HK is to the whole KF, as FK (is) to KE. For as one of the leading (proportional magnitudes is) to one of the

ảnờ tῆς KZ. xaí ἐστιν ὡς τὸ ἀπὸ τῆς ΘK πρὸς τὸ ἀπὸ τῆς KZ, οὕτως ἡ ΘK πρὸς τὴν KE, ἐπεὶ aỉ τρεῖς aỉ ΘK, KZ, KE ἀνάλογόν εἰσιν. σύμμετρος ἄρα ἡ ΘK τῆ KE μήχει. ὅστε xaì ἡ ΘΕ τῆ EK σύμμετρός ἐστι μήχει. xaì ἐπεὶ τὸ ἀπὸ τῆς A ἴσον ἐστὶ τῷ ὑπὸ τῶν EΘ, BΔ, ἑŋτὸν δέ ἐστι τὸ ἀπὸ τῆς A, ἑŋτὸν ἄρα ἐστὶ xaì τὸ ὑπὸ τῶν EΘ, BΔ. xaì παρὰ ἑŋτὴν τὴν BΔ παράχειται· ἑŋτὴ ἄρα ἐστὶν ἡ EΘ xaì σύμμετρος τῆ BΔ μήχει. ὅστε xaì ἡ σύμμετρος aὐτῆ ἡ EK ἑŋτή ἐστι xaì σύμμετρος τῆ BΔ μήχει. ἐπεὶ οῦν ἐστιν ὡς ἡ ΓΔ πρὸς ΔB, οὕτως ἡ ZK πρὸς KE, ai δὲ ΓΔ, ΔB δυνάμει μόνον εἰσὶ σύμμετροι. ἑŋτὴ δἑ ἐστιν ἡ KE· ἑŋτὴ ἄρα ἐστὶ xaì ἡ ZK. ai ZK, KE ӑρα ἑŋταὶ δυνάμει μόνον εἰσὶ σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ EZ.

Ήτοι δὲ ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῃ ἢ τῷ ἀπὸ ἀσυμμέτρου.

Εἰ μὲν οὖν ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου [ἑαυτῆ], καὶ ἡ ΖΚ τῆς ΚΕ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΓΔ τῆ ἐκκειμένῃ ἑητῆ μήκει, καὶ ἡ ΖΚ· εἰ δὲ ἡ ΒΔ, καὶ ἡ ΚΕ· εἰ δὲ οὐδετέρα τῶν ΓΔ, ΔΒ, καὶ οὐδετέρα τῶν ΖΚ, ΚΕ.

Eì δὲ ἡ ΓΔ τῆς ΔB μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ ἡ ZK τῆς KE μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καὶ εἰ μὲν ἡ ΓΔ σύμμετρός ἑστι τῆ ἐκκειμένῃ ῥητῆ μήκει, καὶ ἡ ZK· εἰ δὲ ἡ BΔ, καὶ ἡ KE· εἰ δὲ οὐδετέρα τῶν ΓΔ, ΔB, καὶ οὐδετέρα τῶν ZK, KE· ὥστε ἀποτομή ἑστιν ἡ ZE, ῆς τὰ ὀνόματα τὰ ZK, KE σύμμετρά ἑστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς ΓΔ, ΔB καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ τὴν αὐτῆν τάξιν ἔχει τῆ BΓ· ὅπερ ἔδει δεῖξαι.

following, so all of the leading (magnitudes) are to all of the following [Prop. 5.12]. And as FK (is) to KE, so CDis to DB [Prop. 5.11]. And, thus, as HK (is) to KF, so CD is to DB [Prop. 5.11]. And the (square) on CD (is) commensurable with the (square) on DB [Prop. 10.36]. The (square) on HK is thus also commensurable with the (square) on KF [Props. 6.22, 10.11]. And as the (square) on HK is to the (square) on KF, so HK (is) to KE, since the three (straight-lines) HK, KF, and KEare proportional [Def. 5.9]. HK is thus commensurable in length with KE [Prop. 10.11]. Hence, HE is also commensurable in length with EK [Prop. 10.15]. And since the (square) on A is equal to the (rectangle contained) by EH and BD, and the (square) on A is rational, the (rectangle contained) by EH and BD is thus also rational. And it is applied to the rational (straight-line) BD. Thus, EH is rational, and commensurable in length with BD[Prop. 10.20]. And, hence, the (straight-line) commensurable (in length) with it, EK, is also rational [Def. 10.3], and commensurable in length with BD [Prop. 10.12]. Therefore, since as CD is to DB, so FK (is) to KE, and CD and DB are (straight-lines which are) commensurable in square only, FK and KE are also commensurable in square only [Prop. 10.11]. And KE is rational. Thus, FK is also rational. FK and KE are thus rational (straight-lines which are) commensurable in square only. Thus, EF is an apotome [Prop. 10.73].

And the square on CD is greater than (the square on) DB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (CD).

Therefore, if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) commensurable (in length) with [CD] then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) commensurable (in length) with (FK) [Prop. 10.14]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12]. And if BD (is commensurable), (so) also (is) KE [Prop. 10.12]. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE.

And if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) incommensurable (in length) with (CD) then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) incommensurable (in length) with (FK) [Prop. 10.14]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12]. And if BD (is commensurable), (so) also (is) KE

[Prop. 10.12]. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE. Hence, FE is an apotome whose terms, FK and KE, are commensurable (in length) with the terms, CD and DB, of the binomial, and in the same ratio. And (FE) has the same order as BC [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

[†] Heiberg considers this proposition, and the succeeding ones, to be relatively early interpolations into the original text.

ριγ΄.

Τὸ ἀπὸ ἑητῆς παρὰ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων, ἦς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἔτι δὲ ἡ γινομένη ἐκ δύο ὀνομάτων τὴν αὐτὴν τάξιν ἔχει τῇ ἀποτομῇ.



Έστω ἑητὴ μὲν ἡ Α, ἀποτομὴ δὲ ἡ ΒΔ, καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω τὸ ὑπὸ τῶν ΒΔ, ΚΘ, ὥστε τὸ ἀπὸ τῆς Α ῥητῆς παρὰ τὴν ΒΔ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ΚΘ· λέγω, ὅτι ἐκ δύο ὀνομάτων ἐστὶν ἡ ΚΘ, ῆς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ΒΔ ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΚΘ τὴν αὐτὴν ἔχει τάξιν τῇ ΒΔ.

Έστω γὰρ τῆ ΒΔ προσαρμόζουσα ἡ ΔΓ· αἱ ΒΓ, ΓΔ άρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω καὶ τὸ ὑπὸ τῶν ΒΓ, Η. ῥητὸν δὲ τὸ ἀπὸ τῆς Α· ἑητὸν ἄρα καὶ τὸ ὑπὸ τῶν ΒΓ, Η. καὶ παρὰ ἑητὴν τὴν ΒΓ παραβέβληται· ἑητὴ ἄρα ἐστὶν ἡ Η καὶ σύμμετρος τῆ ΒΓ μήχει. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΒΓ, Η ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, ΚΘ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΒ πρὸς ΒΔ, οὕτως ἡ ΚΘ πρὸς Η. μείζων δὲ ἡ ΒΓ τῆς ΒΔ· μείζων ἄρα καὶ ἡ ΚΘ τῆς Η. κείσθω τῆ Η ἴση ἡ ΚΕ΄ σύμμετρος ἄρα ἐστὶν ἡ ΚΕ τῆ ΒΓ μήκει. καὶ ἐπεί ἐστιν ὡς ἡ ΓΒ πρὸς ΒΔ, οὕτως ἡ ΘΚ πρὸς ΚΕ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ KΘ πρὸς ΘΕ. γεγονέτω ὡς ἡ KΘ πρὸς ΘΕ, οὕτως ἡ ΘΖ πρὸς ΖΕ· καὶ λοιπὴ ἄρα ἡ ΚΖ πρὸς ΖΘ ἐστιν, ώς ή ΚΘ πρός ΘΕ, τουτέστιν [ώς] ή ΒΓ πρός ΓΔ. αί δὲ ΒΓ, ΓΔ δυνάμει μόνον [εἰσὶ] σύμμετροι· καὶ αἱ ΚΖ, ΖΘ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι· καὶ ἐπεί ἐστιν ὡς ἡ ΚΘ πρὸς ΘΕ, ή ΚΖ πρὸς ΖΘ, ἀλλ' ὡς ή ΚΘ πρὸς ΘΕ, ή ΘΖ πρὸς ΖΕ, καὶ ὡς ἄρα ἡ ΚΖ πρὸς ΖΘ, ἡ ΘΖ πρὸς ΖΕ· ὥστε καὶ ώς ή πρώτη πρός την τρίτην, τὸ ἀπὸ τῆς πρώτης πρὸς τὸ άπὸ τῆς δευτέρας. καὶ ὡς ἄρα ἡ ΚΖ πρὸς ΖΕ, οὕτως τὸ ἀπὸ τῆς ΚΖ πρὸς τὸ ἀπὸ τῆς ΖΘ. σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΚΖ τῷ ἀπὸ τῆς ΖΘ· αἱ γὰρ ΚΖ, ΖΘ δυνάμει εἰσὶ σύμμετροι· σύμμετρος ἄρα ἐστὶ καὶ ἡ KZ τῇ ΖΕ μήκει· ὥστε ἡ KZ καὶ And BC and CD [are] commensurable in square only.

Proposition 113

The (square) on a rational (straight-line), applied to an apotome, produces as breadth a binomial whose terms are commensurable with the terms of the apotome, and in the same ratio. Moreover, the created binomial has the same order as the apotome.



Let A be a rational (straight-line), and BD an apotome. And let the (rectangle contained) by BD and KHbe equal to the (square) on A, such that the square on the rational (straight-line) A, applied to the apotome BD, produces KH as breadth. I say that KH is a binomial whose terms are commensurable with the terms of BD, and in the same ratio, and, moreover, that KH has the same order as BD.

For let DC be an attachment to BD. Thus, BC and CD are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let the (rectangle contained) by BC and G also be equal to the (square) on A. And the (square) on A (is) rational. The (rectangle contained) by BC and G (is) thus also rational. And it has been applied to the rational (straight-line) BC. Thus, G is rational, and commensurable in length with BC [Prop. 10.20]. Therefore, since the (rectangle contained) by BC and G is equal to the (rectangle contained) by BD and KH, thus, proportionally, as CB is to BD, so KH (is) to G [Prop. 6.16]. And BC (is) greater than BD. Thus, KH (is) also greater than G [Prop. 5.16, 5.14]. Let KE be made equal to G. KE is thus commensurable in length with BC. And since as CB is to BD, so HK (is) to KE, thus, via conversion, as BC (is) to CD, so KH (is) to HE [Prop. 5.19 corr.]. Let it have been contrived that as KH (is) to HE, so HF (is) to FE. And thus the remainder KF is to FH, as KH (is) to HE—that is to say, [as] BC (is) to CD [Prop. 5.19]. τῆ ΚΕ σύμμετρός [ἐστι] μήκει. ῥητὴ δέ ἐστιν ἡ ΚΕ καὶ σύμμετρος τῆ ΒΓ μήκει. ῥητὴ ἄρα καὶ ἡ ΚΖ καὶ σύμμετρος τῆ ΒΓ μήκει. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς ΓΔ, οὕτως ἡ ΚΖ πρὸς ΖΘ, ἐναλλὰξ ὡς ἡ ΒΓ πρὸς ΚΖ, οὕτως ἡ ΔΓ πρὸς ΖΘ. σύμμετρος δὲ ἡ ΒΓ τῆ ΚΖ· σύμμετρος ἄρα καὶ ἡ ΖΘ τῆ ΓΔ μήκει. αἱ ΒΓ, ΓΔ δὲ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ὀνομάτων ἐστὶν ἄρα ἡ ΚΘ.

Εἰ μὲν οὖν ἡ ΒΓ τῆς ΓΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆ, καὶ ἡ ΚΖ τῆς ΖΘ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΒΓ τῆ ἐκκειμένῃ ἑητῆ μήκει, καὶ ἡ KΖ, εἰ δὲ ἡ ΓΔ σύμμετρός ἐστι τῆ ἐκκειμένῃ ἑητῆ μήκει, καὶ ἡ ΖΘ, εἰ δὲ οὐδετέρα τῶν ΒΓ, ΓΔ, οὐδετέρα τῶν KZ, ΖΘ.

Εἰ δὲ ἡ ΒΓ τῆς ΓΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ ἡ ΚΖ τῆς ΖΘ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΒΓ τῆ ἐκκειμένῃ ἑητῆ μήκει, καὶ ἡ ΚΖ, εἰ δὲ ἡ ΓΔ, καὶ ἡ ΖΘ, εἰ δὲ οὐδετέρα τῶν ΒΓ, ΓΔ, οὐδετέρα τῶν ΚΖ, ΖΘ.

Έχ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΚΘ, ῆς τὰ ὀνόματα τὰ KZ, ΖΘ σύμμετρά [ἐστι] τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς ΒΓ, ΓΔ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΚΘ τῆ ΒΓ τὴν αὐτὴν ἕξει τάξιν. ὅπερ ἔδει δεῖξαι.

KF and *FH* are thus also commensurable in square only [Prop. 10.11]. And since as KH is to HE, (so) KF (is) to FH, but as KH (is) to HE, (so) HF (is) to FE, thus, also as KF (is) to FH, (so) HF (is) to FE [Prop. 5.11]. And hence as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9]. And thus as KF (is) to FE, so the (square) on KF (is) to the (square) on FH. And the (square) on KF is commensurable with the (square) on FH. For KF and FH are commensurable in square. Thus, KF is also commensurable in length with FE [Prop. 10.11]. Hence, KF [is] also commensurable in length with KE [Prop. 10.15]. And *KE* is rational, and commensurable in length with BC. Thus, KF (is) also rational, and commensurable in length with BC [Prop. 10.12]. And since as BC is to CD, (so) KF (is) to FH, alternately, as BC (is) to KF, so DC (is) to FH [Prop. 5.16]. And BC (is) commensurable (in length) with KF. Thus, FH (is) also commensurable in length with CD [Prop. 10.11]. And BCand CD are rational (straight-lines which are) commensurable in square only. KF and FH are thus also rational (straight-lines which are) commensurable in square only [Def. 10.3, Prop. 10.13]. Thus, KH is a binomial [Prop. 10.36].

Therefore, if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) commensurable (in length) with (BC), then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) commensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FF [Prop. 10.12]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

And if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) incommensurable (in length) with (BC) then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) incommensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable, (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

KH is thus a binomial whose terms, KF and FH, [are] commensurable (in length) with the terms, BC and CD, of the apotome, and in the same ratio. Moreover, ριδ΄.

Έλν χωρίον περιέχηται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ῆς τὰ ὀνόματα σύμμετρά τέ ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ῥητή ἐστιν.



Περιεχέσθω γὰρ χωρίον τὸ ὑπὸ τῶν AB, ΓΔ ὑπὸ ἀποτομῆς τῆς AB καὶ τῆς ἐκ δύο ὀνομάτων τῆς ΓΔ, ῆς μεῖζον ὄνομα ἔστω τὸ ΓΕ, καὶ ἔστω τὰ ὀνόματα τῆς ἐκ δύο ὀνομάτων τὰ ΓΕ, ΕΔ σύμμετρά τε τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς AZ, ZB καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔστω ἡ τὸ ὑπὸ τῶν AB, ΓΔ δυναμένη ἡ Η· λέγω, ὅτι ἑρτή ἑστιν ἡ Η.

Έκκείσθω γὰρ ἑητὴ ἡ Θ, καὶ τῷ ἀπὸ τῆς Θ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω πλάτος ποιοῦν τὴν ΚΛ· ἀποτομὴ ἄρα ἐστὶν ἡ $m K\Lambda$, ἦς τὰ ὀνόματα ἔστω τὰ $m KM, M\Lambda$ σύμμετρα τοῖς τῆς ἐκ δύο ὀνομάτων ἀνόμασι τοῖς ΓΕ, Ε Δ καὶ ἐν τῷ αὐτῷ λόγω. ἀλλὰ καὶ αἱ ΓΕ, ΕΔ σύμμετροί τέ εἰσι ταῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγω· ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΖΒ, οὕτως ἡ ΚΜ πρὸς ΜΛ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΚΜ, οὕτως ἡ ΒΖ πρὸς τὴν ΛΜ· καὶ λοιπὴ ἄρα ἡ ΑΒ πρός λοιπήν τήν ΚΛ έστιν ώς ή ΑΖ πρός ΚΜ. σύμμετρος δὲ ἡ ΑΖ τῆ ΚΜ· σύμμετρος ἄρα ἐστὶ καὶ ἡ ΑΒ τῆ ΚΛ. καί έστιν ώς ή AB πρὸς KΛ, οὕτως τὸ ὑπὸ τῶν ΓΔ, AB πρὸς τὸ ὑπὸ τῶν Γ Δ , Κ Λ · σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ὑπὸ τῶν ΓΔ, ΚΛ. ἴσον δὲ τὸ ὑπὸ τῶν ΓΔ, ΚΛ τῶ ἀπὸ τῆς Θ · σύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΓΔ, AB τῶ άπὸ τῆς Θ . τῷ δὲ ὑπὸ τῶν Γ Δ , AB ἴσον ἐστὶ τὸ ἀπὸ τῆς H· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Η τῷ ἀπὸ τῆς Θ. ῥητὸν δὲ τὸ ἀπὸ τῆς Θ· ἑητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς Η· ἑητὴ ἄρα έστιν ή Η. και δύναται τὸ ὑπὸ τῶν ΓΔ, ΑΒ.

Έὰν ἄρα χωρίον περιέχηται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ῆς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ἑητή ἐστιν. KH will have the same order as BC [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

Proposition 114

If an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome then the square-root of the area is a rational (straight-line).



For let an area, the (rectangle contained) by AB and CD, have been contained by the apotome AB, and the binomial CD, of which let the greater term be CE. And let the terms of the binomial, CE and ED, be commensurable with the terms of the apotome, AF and FB (respectively), and in the same ratio. And let the square-root of the (rectangle contained) by AB and CD be G. I say that G is a rational (straight-line).

For let the rational (straight-line) H be laid down. And let (some rectangle), equal to the (square) on H, have been applied to CD, producing KL as breadth. Thus, KL is an apotome, of which let the terms, KMand ML, be commensurable with the terms of the binomial, CE and ED (respectively), and in the same ratio [Prop. 10.112]. But, CE and ED are also commensurable with AF and FB (respectively), and in the same ratio. Thus, as AF is to FB, so KM (is) to ML. Thus, alternately, as AF is to KM, so BF (is) to LM [Prop. 5.16]. Thus, the remainder AB is also to the remainder KL as AF (is) to KM [Prop. 5.19]. And AF (is) commensurable with KM [Prop. 10.12]. AB is thus also commensurable with KL [Prop. 10.11]. And as AB is to KL, so the (rectangle contained) by CD and AB (is) to the (rectangle contained) by CD and KL [Prop. 6.1]. Thus, the (rectangle contained) by CD and AB is also commensurable with the (rectangle contained) by CD and KL [Prop. 10.11]. And the (rectangle contained) by CDand KL (is) equal to the (square) on H. Thus, the (rectangle contained) by CD and AB is commensurable with the (square) on H. And the (square) on G is equal to the (rectangle contained) by CD and AB. The (square) on G

Πόρισμα.

Καὶ γέγονεν ἡμῖν καὶ διὰ τούτου φανερόν, ὅτι δυνατόν ἐστι ἑητὸν χωρίον ὑπὸ ἀλόγων εὐθειῶν περιέχεσθαι. ὅπερ ἔδει δεῖξαι.

ριε΄.

Άπὸ μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιῷ τῶν πρότερον ἡ αὐτή.



Έστω μέση ή Α· λέγω, ὅτι ἀπὸ τῆς Α ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾶ τῶν πρότερον ἡ αὐτή.

Έχχείσθω ἡητὴ ἡ Β, χαὶ τῷ ὑπὸ τῶν Β, Α ἴσον ἔστω τὸ ἀπὸ τῆς Γ· ἄλογος ἄρα ἑστὶν ἡ Γ· τὸ γὰρ ὑπὸ ἀλόγου καὶ ὑητῆς ἄλογόν ἑστιν. καὶ οὐδεμιῷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾽ οὐδεμιῶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ μέσην. πάλιν δὴ τῷ ὑπὸ τῶν Β, Γ ἴσον ἔστω τὸ ἀπὸ τῆς Δ· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς Δ. ἄλογος ἄρα ἐστὶν ἡ Δ· καὶ οὐδεμιῷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾽ οὐδεμιῷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπὰ οὐδεμιῷ τῶν πρότερον και τὸ ἀπὸ τῆς Δ. ἔλογον ἔστω τὸ ἀπὸ τῆς Δ. ἄλογος ἄρα ἐστὶν ἡ Δ· καὶ οὐδεμιῷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾽ οὐδεμιῶς τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπὰ οὐδεμιῶς ψανερόν, ὅτι ἀπὸ τῆς μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμιῷ τῶν πρότερον ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

is thus commensurable with the (square) on H. And the (square) on H (is) rational. Thus, the (square) on G is also rational. G is thus rational. And it is the square-root of the (rectangle contained) by CD and AB.

Thus, if an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome, then the square-root of the area is a rational (straight-line).

Corollary

And it has also been made clear to us, through this, that it is possible for a rational area to be contained by irrational straight-lines. (Which is) the very thing it was required to show.

Proposition 115

An infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and none of them is the same as any of the preceding (straight-lines).



Let A be a medial (straight-line). I say that an infinite (series) of irrational (straight-lines) can be created from A, and that none of them is the same as any of the preceding (straight-lines).

Let the rational (straight-line) B be laid down. And let the (square) on C be equal to the (rectangle contained) by B and A. Thus, C is irrational [Def. 10.4]. For an (area contained) by an irrational and a rational (straight-line) is irrational [Prop. 10.20]. And (C is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces a medial (straight-line) as breadth. So, again, let the (square) on D be equal to the (rectangle contained) by B and C. Thus, the (square) on D is irrational [Prop. 10.20]. Dis thus irrational [Def. 10.4]. And (D is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces C as breadth. So, similarly, this arrangement being advanced to infinity, it is clear that an infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and that none of them is the same as any of the preceding (straight-lines). (Which is) the very thing it was required to show.

ELEMENTS BOOK 11

Elementary Stereometry

Οροι.

α΄. Στερεόν ἐστι τὸ μῆχος καὶ πλάτος καὶ βάθος ἔχον.

β'. Στερεοῦ δὲ πέρας ἐπιφάνεια.

γ'. Εὐθεῖα πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ [ὑποκειμένῳ] ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

δ΄. Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῆ κοινῆ τομῆ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ῶσιν.

ε'. Εὐθείας πρὸς ἐπίπεδον χλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῆ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρας τῆς εὐθείας εὐθεῖα ἐπιζευχθῆ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

γ΄. Ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῆ κοινῆ τομῆ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

ζ΄. Ἐπίπεδον πρὸς ἐπίπεδον ὑμοίως κεκλίσθαι λέγεται καὶ ἔτερον πρὸς ἔτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ῶσιν.

η΄. Παράλληλα ἐπίπεδά ἐστι τὰ ἀσύμπτωτα.

θ΄. Όμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τὸ πλῆθος.

ι΄. Ίσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

ια'. Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἀπτομένων ἀλλήλων καὶ μὴ ἐν τῆ αὐτῆ ἐπιφανεία οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἑνὶ σημείῳ συνισταμένων.

ιβ΄. Πυραμίς ἐστι σχῆμα στερεὸν ἐπιπέδοις περιχόμενον ἀπὸ ἑνὸς ἐπιπέδου πρὸς ἑνὶ σημείω συνεστώς.

ιγ΄. Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον, ῶν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστι καὶ παράλληλα, τὰ δὲ λοιπὰ παραλληλόγραμμα.

ιδ'. Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῆ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

ιε΄. Άξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ η̈ν τὸ ἡμικύκλιον στρέφεται.

ις'. Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

ιζ'. Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

ιη'. Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῆ, ὅθεν ἤρξατο

Definitions

1. A solid is a (figure) having length and breadth and depth.

2. The extremity of a solid (is) a surface.

3. A straight-line is at right-angles to a plane when it makes right-angles with all of the straight-lines joined to it which are also in the plane.

4. A plane is at right-angles to a(nother) plane when (all of) the straight-lines drawn in one of the planes, at right-angles to the common section of the planes, are at right-angles to the remaining plane.

5. The inclination of a straight-line to a plane is the angle contained by the drawn and standing (straight-lines), when a perpendicular is lead to the plane from the end of the (standing) straight-line raised (out of the plane), and a straight-line is (then) joined from the point (so) generated to the end of the (standing) straight-line (lying) in the plane.

6. The inclination of a plane to a(nother) plane is the acute angle contained by the (straight-lines), (one) in each of the planes, drawn at right-angles to the common segment (of the planes), at the same point.

7. A plane is said to have been similarly inclined to a plane, as another to another, when the aforementioned angles of inclination are equal to one another.

8. Parallel planes are those which do not meet (one another).

9. Similar solid figures are those contained by equal numbers of similar planes (which are similarly arranged).

10. But equal and similar solid figures are those contained by similar planes equal in number and in magnitude (which are similarly arranged).

11. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point.

12. A pyramid is a solid figure, contained by planes, (which is) constructed from one plane to one point.

13. A prism is a solid figure, contained by planes, of which the two opposite (planes) are equal, similar, and parallel, and the remaining (planes are) parallelograms.

14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.

15. And the axis of the sphere is the fixed straight-line about which the semicircle is turned.

φέρεσθαι, τὸ περιληφθὲν σχῆμα. κἂν μὲν ἡ μένουσα εὐθεῖα ἴση ἢ τῆ λοιπῆ [τῆ] περὶ τὴν ὀρθὴν περιφερομένῃ, ὀρθογώνιος ἔσται ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγώνιος, ἐὰν δὲ μείζων, ὀζυγώνιος.

ιθ΄. Ἄξων δὲ τοῦ χώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ η̈ν τὸ τρίγωνον στρέφεται.

κ'. Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

κα'. Κύλινδρός ἐστιν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῆ, ὅθεν ἦρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

κβ΄. Ἄξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἢν τὸ παραλληλόγραμμον στρέφεται.

κγ'. Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιαγομένων δύο πλευρῶν γραφόμενοι.

κδ'. Όμοιοι κῶνοι καὶ κύλινδροί εἰσιν, ῶν οἴ τε ἄξονες καὶ αἱ διάμετροι τῶν βάσεων ἀνάλογόν εἰσιν.

κε'. Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἑξ τετραγώνων ἴσων περιεχόμενον.

κτ΄. Όκτάεδρόν ἐστὶ σχῆμα στερεὸν ὑπὸ ὀκτὼ τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κζ΄. Εἰκοσάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κη'. Δωδεκάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλεύρων καὶ ἰσογωνίων περιεχόμενον.

α'.

Εὐθείας γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δέ τι ἐν μετεωροτέρῳ.

Εἰ γὰρ δυνατόν, εὐθείας γραμμῆς τῆς ΑΒΓ μέρος μέν τι τὸ ΑΒ ἔστω ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ, μέρος δέ τι τὸ ΒΓ ἐν μετεωροτέρῳ.

Έσται δή τις τῆ AB συνεχὴς εὐθεῖα ἐπ' εὐθείας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ. ἔστω ἡ BΔ· δύο ἄρα εὐθειῶν τῶν ABΓ, ABΔ κοινὸν τμῆμά ἐστιν ἡ AB· ὅπερ ἐστὶν ἀδύνατον, ἐπειδήπερ ἐὰν κέντρῳ τῷ B καὶ διαστήματι τῷ AB κύκλον γράψωμεν, αἱ διάμετροι ἀνίσους ἀπολήψονται τοῦ κύκλου 16. And the center of the sphere is the same as that of the semicircle.

17. And the diameter of the sphere is any straightline which is drawn through the center and terminated in both directions by the surface of the sphere.

18. A cone is the figure enclosed when, one of the sides of a right-angled triangle about the right-angle remaining (fixed), the triangle is carried around, and again established at the same (position) from which it began to be moved. And if the fixed straight-line is equal to the remaining (straight-line) about the right-angle, (which is) carried around, then the cone will be right-angled, and if less, obtuse-angled, and if greater, acute-angled.

19. And the axis of the cone is the fixed straight-line about which the triangle is turned.

20. And the base (of the cone is) the circle described by the (remaining) straight-line (about the right-angle which is) carried around (the axis).

21. A cylinder is the figure enclosed when, one of the sides of a right-angled parallelogram about the rightangle remaining (fixed), the parallelogram is carried around, and again established at the same (position) from which it began to be moved.

22. And the axis of the cylinder is the stationary straight-line about which the parallelogram is turned.

23. And the bases (of the cylinder are) the circles described by the two opposite sides (which are) carried around.

24. Similar cones and cylinders are those for which the axes and the diameters of the bases are proportional.

25. A cube is a solid figure contained by six equal squares.

26. An octahedron is a solid figure contained by eight equal and equilateral triangles.

27. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

28. A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

Proposition 1[†]

Some part of a straight-line cannot be in a reference plane, and some part in a more elevated (plane).

For, if possible, let some part, AB, of the straight-line ABC be in a reference plane, and some part, BC, in a more elevated (plane).

In the reference plane, there will be some straight-line continuous with, and straight-on to, AB.[‡] Let it be BD. Thus, AB is a common segment of the two (different) straight-lines ABC and ABD. The very thing is impossible, inasmuch as if we draw a circle with center B and

περιφερείας.



Εὐθείας ἄρα γραμμῆς μέρος μέν τι οὐχ ἔστιν ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν μετεωροτέρῳ· ὅπερ ἔδει δεῖξαι. radius *AB* then the diameters (*ABD* and *ABC*) will cut off unequal circumferences of the circle.



Thus, some part of a straight-line cannot be in a reference plane, and (some part) in a more elevated (plane). (Which is) the very thing it was required to show.

[†] The proofs of the first three propositions in this book are not at all rigorous. Hence, these three propositions should properly be regarded as additional axioms.

[‡] This assumption essentially presupposes the validity of the proposition under discussion.

β΄.

Έὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.



Δύο γὰρ εὐθεῖαι αἱ AB, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον. λέγω, ὅτι αἱ AB, ΓΔ ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

Εἰλήφθω γὰρ ἐπὶ τῶν ΕΓ, ΕΒ τυχόντα σημεῖα τὰ Ζ, Η, καὶ ἐπεζεύχθωσαν αἱ ΓΒ, ΖΗ, καὶ διήχθωσαν αἱ ΖΘ, ΗΚ· λέγω πρῶτον, ὅτι τὸ ΕΓΒ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. εἰ γάρ ἐστι τοῦ ΕΓΒ τριγώνου μέρος ἤτοι τὸ ΖΘΓ ἢ τὸ HBK ἐν τῷ ὑποκειμένῳ [ἐπιπέδῳ], τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ μιᾶς τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένῳ

Proposition 2

If two straight-lines cut one another then they are in one plane, and every triangle (formed using segments of both lines) is in one plane.



For let the two straight-lines AB and CD have cut one another at point E. I say that AB and CD are in one plane, and that every triangle (formed using segments of both lines) is in one plane.

For let the random points F and G have been taken on EC and EB (respectively). And let CB and FGhave been joined, and let FH and GK have been drawn across. I say, first of all, that triangle ECB is in one (reference) plane. For if part of triangle ECB, either FHC ἐπιπέδω, τὸ δὲ ἐν αλλω. εἰ δὲ τοῦ ΕΓΒ τριγώνου τὸ ΖΓΒΗ μέρος ἢ ἐν τῷ ὑποκειμένω ἐπιπέδω, τὸ δὲ λοιπὸν ἐν ἄλλω, ἔσται καὶ ἀμφοτέρων τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένω ἐπιπέδω, τὸ δὲ ἐν ἄλλω· ὅπερ ἄτοπον ἐδείχθη. τὸ ἄρα ΕΓΒ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδω. ἐν ῷ δἑ ἐστι τὸ ΕΓΒ τρίγωνον, ἐν τούτω καὶ ἑκατέρα τῶν ΕΓ, ΕΒ, ἐν ῷ δὲ ἑκατέρα τῶν ΕΓ, ΕΒ, ἐν τούτω καὶ αἱ ΑΒ, ΓΔ. αἱ ΑΒ, ΓΔ ἄρα εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδω, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδω. ὅπερ ἕδει δεῖξαι.

or *GBK*, is in the reference [plane], and the remainder in a different (plane) then a part of one the straight-lines EC and EB will also be in the reference plane, and (a part) in a different (plane). And if the part FCBG of triangle ECB is in the reference plane, and the remainder in a different (plane) then parts of both of the straightlines EC and EB will also be in the reference plane, and (parts) in a different (plane). The very thing was shown to be absurb [Prop. 11.1]. Thus, triangle ECBis in one plane. And in whichever (plane) triangle ECBis (found), in that (plane) EC and EB (will) each also (be found). And in whichever (plane) EC and EB (are) each (found), in that (plane) AB and CD (will) also (be found) [Prop. 11.1]. Thus, the straight-lines AB and CD are in one plane, and every triangle (formed using segments of both lines) is in one plane. (Which is) the very thing it was required to show.

Proposition 3

If two planes cut one another then their common section is a straight-line.



For let the two planes AB and BC cut one another, and let their common section be the line DB. I say that the line DB is straight.

For, if not, let the straight-line DEB have been joined from D to B in the plane AB, and the straight-line DFBin the plane BC. So two straight-lines, DEB and DFB, will have the same ends, and they will clearly enclose an area. The very thing (is) absurd. Thus, DEB and DFBare not straight-lines. So, similarly, we can show than no other straight-line can be joined from D to B except DB, the common section of the planes AB and BC.

Thus, if two planes cut one another then their common section is a straight-line. (Which is) the very thing it was required to show.

γ'.

Ἐἀν δύο ἐπίπεδα τεμνῆ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν.



Δύο γὰρ ἐπίπεδα τὰ AB, BΓ τεμνέτω ἄλληλα, χοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΔΒ γραμμή λέγω, ὅτι ἡ ΔΒ γραμμὴ εὐθεῖά ἐστιν.

Εἰ γὰρ μή, ἐπεζεύχθω ἀπὸ τοῦ Δ ἐπὶ τὸ B ἐν μὲν τῷ AB ἐπιπέδω εὐθεῖα ἡ ΔΕΒ, ἐν δὲ τῷ BΓ ἐπιπέδω εὐθεῖα ἡ ΔΖΒ. ἔσται δὴ δύο εὐθειῶν τῶν ΔΕΒ, ΔΖΒ τὰ αὐτὰ πέρατα, καὶ περιέξουσι δηλαδὴ χωρίον· ὅπερ ἄτοπον. οὕκ ἄρα αἰ ΔΕΒ, ΔΖΒ εὐθεῖαί εἰσιν. ὑμοίως δὴ δείζομεν, ὅτι οὐδὲ ἄλλη τις ἀπὸ τοῦ Δ ἐπὶ τὸ B ἐπιζευγνυμένη εὐθεῖα ἔσται πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Έὰν ἄρα δύο ἐπίπεδα τέμνῃ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν· ὅπερ ἔδει δεῖξαι. δ'.

Έαν εύθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῆ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



Εὐθεῖα γάρ τις ἡ ΕΖ δύο εὐθείαις ταῖς ΑΒ, ΓΔ τεμνούσαις ἀλλήλας κατὰ τὸ Ε σημεῖον ἀπὸ τοῦ Ε πρὸς ὀρθὰς ἐφεστάτω· λέγω, ὅτι ἡ ΕΖ καὶ τῷ διὰ τῶν ΑΒ, ΓΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Άπειλήφθωσαν γὰρ αἱ ΑΕ, ΕΒ, ΓΕ, ΕΔ ἴσαι ἀλλήλαις, καὶ διήχθω τις διὰ τοῦ Ε, ὡς ἔτυχεν, ἡ ΗΕΘ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΓΒ, καὶ ἔτι ἀπὸ τυχόντος τοῦ Ζ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΗ, ΖΔ, ΖΓ, ΖΘ, ΖΒ.

Καὶ ἐπεὶ δύο αἱ ΑΕ, ΕΔ δυσὶ ταῖς ΓΕ, ΕΒ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΑΔ βάσει τῇ ΓΒ ἴση έστίν, καὶ τὸ ΑΕΔ τρίγωνον τῷ ΓΕΒ τριγώνῳ ἴσον ἔσται· ώστε καὶ γωνία ἡ ὑπὸ ΔΑΕ γωνία τῆ ὑπὸ ΕΒΓ ἴση [ἐστίν]. ἔστι δὲ καὶ ἡ ὑπὸ ΑΕΗ γωνία τῇ ὑπὸ ΒΕΘ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΑΗΕ, ΒΕΘ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑχατέραν ἑχατέρα χαὶ μίαν πλευρὰν μιᾶ πλευρᾶ ίσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΑΕ τῆ ΕΒ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση άρα ή μὲν ΗΕ τῆ ΕΘ, ή δὲ ΑΗ τῆ ΒΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῆ ΕΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΖΕ, βάσις ἄρα ἡ ΖΑ βάσει τῆ ZB ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ZΓ τῆ Z Δ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ $A\Delta$ τῆ ΓB , ἔστι δὲ καὶ ἡ ZA τῆ ZBἴση, δύο δὴ αἱ ΖΑ, ΑΔ δυσὶ ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα έκατέρα· καὶ βάσις ἡ ΖΔ βάσει τῇ ΖΓ ἐδείχθη ἴση· καὶ γωνία ἄρα ἡ ὑπὸ $ZA\Delta$ γωνία τῆ ὑπὸ $ZB\Gamma$ ἴση ἐστίν. καὶ ἐπεὶ πάλιν έδείχθη ή ΑΗ τῆ ΒΘ ἴση, ἀλλὰ μὴν καὶ ή ΖΑ τῆ ΖΒ ἴση, δύο δη αί ZA, AH δυσὶ ταῖς ZB, BΘ ἴσαι εἰσίν. καὶ γωνία ἡ ὑπὸ ΖΑΗ ἐδείχθη ἴση τῆ ὑπὸ ΖΒΘ· βάσις ἄρα ἡ ΖΗ βάσει τῆ ΖΘ έστιν ἴση. καὶ ἐπεὶ πάλιν ἴση ἐδείχθη ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ή ΕΖ, δύο δη αί ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ βάσις ή ΖΗ βάσει τῆ ΖΘ ἴση· γωνία ἄρα ή ὑπὸ ΗΕΖ γωνία τῆ ὑπὸ ΘΕΖ ἴση ἐστίν. ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΗΕΖ, ΘΕΖ γωνιῶν. ή ΖΕ ἄρα πρὸς τὴν ΗΘ τυχόντως διὰ τοῦ Ε άχθεῖσαν ὀρθή ἐστιν. ὑμοίως δὴ δείξομεν, ὅτι ἡ ΖΕ καὶ

Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).



For let some straight-line EF have (been) set up at right-angles to two straight-lines, AB and CD, cutting one another at point E, at E. I say that EF is also at right-angles to the plane (passing) through AB and CD.

For let AE, EB, CE and ED have been cut off from (the two straight-lines so as to be) equal to one another. And let GEH have been drawn, at random, through E(in the plane passing through AB and CD). And let ADand CB have been joined. And, furthermore, let FA, FG, FD, FC, FH, and FB have been joined from the random (point) F (on EF).

For since the two (straight-lines) AE and ED are equal to the two (straight-lines) CE and EB, and they enclose equal angles [Prop. 1.15], the base AD is thus equal to the base CB, and triangle AED will be equal to triangle CEB [Prop. 1.4]. Hence, the angle DAE [is] equal to the angle EBC. And the angle AEG (is) also equal to the angle BEH [Prop. 1.15]. So AGE and BEH are two triangles having two angles equal to two angles, respectively, and one side equal to one side-(namely), those by the equal angles, AE and EB. Thus, they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, GE (is) equal to EH, and AG to BH. And since AE is equal to EB, and FE is common and at right-angles, the base FA is thus equal to the base FB [Prop. 1.4]. So, for the same (reasons), FC is also equal to FD. And since AD is equal to CB, and FA is also equal to FB, the two (straight-lines) FAand AD are equal to the two (straight-lines) FB and BC, respectively. And the base FD was shown (to be) equal to the base FC. Thus, the angle FAD is also equal to the angle FBC [Prop. 1.8]. And, again, since AG was shown (to be) equal to BH, but FA (is) also equal to πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ὑποκειμένω ἐπιπέδω ὀρθὰς ποιήσει γωνίας. εὐθεῖα δὲ πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ αὐτῷ ἐπιπέδῷ ὀρθὰς ποιῆ γωνίας ἡ ΖΕ ἄρα τῷ ὑποκειμένω ἐπιπέδῷ πρὸς ὀρθάς ἐστιν. τὸ δὲ ὑποκείμενον ἐπίπεδόν ἐστι τὸ διὰ τῶν AB, ΓΔ εὐθειῶν. ἡ ΖΕ ἄρα πρὸς ὀρθάς ἐστι τῷ διὰ τῶν AB, ΓΔ ἑπιπέδῷ.

Έαν ἄρα εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῆ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

ε΄.

Έὰν εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῆ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ.



Εὐθεῖα γάρ τις ἡ ΑΒ τρισὶν εὐθείαις ταῖς ΒΓ, ΒΔ, ΒΕ πρὸς ὀρθὰς ἐπὶ τῆς κατὰ τὸ Β ἁφῆς ἐφεστάτω· λέγω, ὅτι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδω.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστωσαν αἱ μὲν ΒΔ, ΒΕ ἐν τῷ ὑποκειμένῷ ἐπιπέδῷ, ἡ δὲ ΒΓ ἐν μετεωροτέρῷ, καὶ ἐκβεβλήσθω τὸ δὶα τῶν ΑΒ, ΒΓ ἐπίπεδον· κοινὴν δὴ τομὴν

FB, the two (straight-lines) FA and AG are equal to the two (straight-lines) FB and BH (respectively). And the angle FAG was shown (to be) equal to the angle FBH. Thus, the base FG is equal to the base FH [Prop. 1.4]. And, again, since GE was shown (to be) equal to EH, and EF (is) common, the two (straight-lines) GE and EF are equal to the two (straight-lines) HE and EF(respectively). And the base FG (is) equal to the base FH. Thus, the angle GEF is equal to the angle HEF[Prop. 1.8]. Each of the angles GEF and HEF (are) thus right-angles [Def. 1.10]. Thus, FE is at right-angles to GH, which was drawn at random through E (in the reference plane passing though AB and AC). So, similarly, we can show that FE will make right-angles with all straight-lines joined to it which are in the reference plane. And a straight-line is at right-angles to a plane when it makes right-angles with all straight-lines joined to it which are in the plane [Def. 11.3]. Thus, FE is at right-angles to the reference plane. And the reference plane is that (passing) through the straight-lines AB and CD. Thus, FE is at right-angles to the plane (passing) through AB and CD.

Thus, if a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both). (Which is) the very thing it was required to show.

Proposition 5

If a straight-line is set up at right-angles to three straight-lines cutting one another, at the common point of section, then the three straight-lines are in one plane.



For let some straight-line AB have been set up at right-angles to three straight-lines BC, BD, and BE, at the (common) point of section B. I say that BC, BD, and BE are in one plane.

For (if) not, and if possible, let BD and BE be in the reference plane, and BC in a more elevated (plane).

ποιήσει ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν BZ. ἐν ἐνὶ ἄρα εἰσὶν ἐπιπέδῳ τῷ διηγμένῳ διὰ τῶν AB, BΓ αἰ τρεῖς εὐθεῖαι αἱ AB, BΓ, BZ. καὶ ἐπεὶ ἡ AB ὀρθή ἐστι πρὸς ἑχατέραν τῶν BΔ, BE, καὶ τῷ διὰ τῶν BΔ, BE ἄρα ἐπιπέδῳ ὀρθή ἐστιν ἡ AB. τὸ δὲ διὰ τῶν BΔ, BE ἐπίπεδον τὸ ὑποχείμενον ἐστιν ἡ AB ἄρα ὀρθή ἐστι πρὸς τὸ ὑποχείμενον ἐπίπεδον. ὥστε καὶ πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ AB. ἄπτεται δὲ αὐτῆς ἡ BZ οὕσα ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ABZ γωνία ὀρθή ἐστιν. ὑπόχειται δὲ καὶ ἡ ὑπὸ ABΓ ὀρθή· ἴση ἄρα ἡ ὑπὸ ABZ γωνία τῆ ὑπὸ ABΓ. καί εἰσιν ἐν ἐνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον. οὐχ ἄρα ἡ BΓ εὐθεῖα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ. αἱ τρεῖς ἄρα εὐθεῖαι αἱ BΓ, BΔ, BE ἐν ἑνί εἰσιν ἐπιπέδῳ.

Έαν ἄρα εὐθεῖα τρισιν εὐθείαις ἁπτομέναις ἀλλήλων ἐπὶ τῆς ἁφῆς πρὸς ὀρθὰς ἐπισταθῆ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

ኖ'.

Έὰν δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ῶσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι.



Δύο γὰρ εὐθεῖαι αἱ AB, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῷ πρὸς ὀρθὰς ἔστωσαν· λέγω, ὅτι παράλληλός ἐστιν ἡ AB τῆ ΓΔ.

Συμβαλλέτωσαν γὰρ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ εὐθεῖα, καὶ ἤχθω τῆ ΒΔ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῆ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ.

Καὶ ἐπεὶ ἡ AB ὀρθή ἐστι πρὸς τὸ ὑποχείμενον ἐπίπεδον, καὶ πρὸς πάσας [ἄρα] τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ὑποχειμένῳ ἐπιπέδῷ ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ τῆς AB ἑχατέρα τῶν ΒΔ, BE οὖσα ἐν τῷ ὑπο-

And let the plane through AB and BC have been produced. So it will make a straight-line as a common section with the reference plane [Def. 11.3]. Let it make BF. Thus, the three straight-lines AB, BC, and BFare in one plane—(namely), that drawn through AB and BC. And since AB is at right-angles to each of BD and *BE*, *AB* is thus also at right-angles to the plane (passing) through BD and BE [Prop. 11.4]. And the plane (passing) through BD and BE is the reference plane. Thus, AB is at right-angles to the reference plane. Hence, ABwill also make right-angles with all straight-lines joined to it which are also in the reference plane [Def. 11.3]. And *BF*, which is in the reference plane, is joined to it. Thus, the angle ABF is a right-angle. And ABC was also assumed to be a right-angle. Thus, angle ABF (is) equal to ABC. And they are in one plane. The very thing is impossible. Thus, BC is not in a more elevated plane. Thus, the three straight-lines BC, BD, and BE are in one plane.

Thus, if a straight-line is set up at right-angles to three straight-lines cutting one another, at the (common) point of section, then the three straight-lines are in one plane. (Which is) the very thing it was required to show.

Proposition 6

If two straight-lines are at right-angles to the same plane then the straight-lines will be parallel.^{\dagger}



For let the two straight-lines AB and CD be at rightangles to a reference plane. I say that AB is parallel to CD.

For let them meet the reference plane at points B and D (respectively). And let the straight-line BD have been joined. And let DE have been drawn at right-angles to BD in the reference plane. And let DE be made equal to AB. And let BE, AE, and AD have been joined.

And since AB is at right-angles to the reference plane, it will [thus] also make right-angles with all straight-lines joined to it which are in the reference plane [Def. 11.3]. κειμένω ἐπιπέδω· ὀρθή ἄρα ἐστιν ἑκατέρα τῶν ὑπὸ ABΔ, ΑΒΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΔΒ, $\Gamma\Delta E$ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῃ ΔE , κοινὴ δ
ὲ ἡ BΔ, δύο δὴ αἱ AB, BΔ δυσὶ ταῖς EΔ, ΔB ἴσαι εἰσίν· καὶ γωνίας ὀρθὰς περιέχουσιν· βάσις ἄρα ἡ ΑΔ βάσει τῆ ΒΕ έστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, ἀλλὰ καὶ ή A Δ τῆ BE, δύο δὴ αί AB, BE δυσὶ ταῖς E Δ , Δ A ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνιά τῆ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΔΑ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς ἑκατέραν τῶν ${
m B}\Delta,\,\Delta\Gamma$ ὀρθή.
ἡ ${
m E}\Delta$ ἄρα τρισίν εὐθείαις ταῖς ΒΔ, ΔΑ, ΔΓ πρὸς ὀρθὰς ἐπὶ τῆς ἁφῆς έφέστηκεν· αί τρεῖς ἄρα εὐθεῖαι αί ${
m B}\Delta,\,\Delta {
m A},\,\Delta {\Gamma}$ ἐν ἑνί εἰσιν ἐπιπέδω. ἐν ῷ δὲ αἱ ΔB , ΔA , ἐν τούτω καὶ ἡ AB· πᾶν γὰρ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδω· αἱ ἄρα ΑΒ, ΒΔ, ΔΓ εὐθεῖαι έν ένί είσιν ἐπιπέδω. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ABΔ, ΒΔΓ γωνιῶν παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῆ ΓΔ.

Έαν ἄρα δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ῶσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

And *BD* and *BE*, which are in the reference plane, are each joined to AB. Thus, each of the angles ABD and ABE are right-angles. So, for the same (reasons), each of the angles CDB and CDE are also right-angles. And since AB is equal to DE, and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And they contain right-angles. Thus, the base AD is equal to the base BE [Prop. 1.4]. And since AB is equal to DE, and AD (is) also (equal) to BE, the two (straight-lines) ABand BE are thus equal to the two (straight-lines) EDand DA (respectively). And their base AE (is) common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. Thus, EDA (is) also a rightangle. ED is thus at right-angles to DA. And it is also at right-angles to each of BD and DC. Thus, ED is standing at right-angles to the three straight-lines BD, DA, and DC at the (common) point of section. Thus, the three straight-lines BD, DA, and DC are in one plane [Prop. 11.5]. And in which(ever) plane DB and DA (are found), in that (plane) AB (will) also (be found). For every triangle is in one plane [Prop. 11.2]. And each of the angles ABD and BDC is a right-angle. Thus, AB is parallel to CD [Prop. 1.28].

Thus, if two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. (Which is) the very thing it was required to show.

[†] In other words, the two straight-lines lie in the same plane, and never meet when produced in either direction.



Έὰν Ճσι δύο εὐθεῖαι παράλληλοι, ληφθῆ δὲ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.



Έστωσαν δύο εὐθεῖαι παράλληλοι αί AB, ΓΔ, καὶ εἰλήφθω ἐφ' ἑκατέρας αὐτῶν τυνχόντα σημεῖα τὰ Ε, Ζ· λέγω, ὅτι ἡ ἐπὶ τὰ Ε, Ζ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

Μή γάρ, ἀλλ' εἰ δυνατόν, ἔστω ἐν μετεωροτέρῳ ὡς ἡ EHZ, καὶ διήχθω διὰ τῆς EHZ ἐπίπεδον· τομὴν δὴ ποιήσει

Proposition 7

If there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines).



Let AB and CD be two parallel straight-lines, and let the random points E and F have been taken on each of them (respectively). I say that the straight-line joining points E and F is in the same (reference) plane as the parallel (straight-lines).

For (if) not, and if possible, let it be in a more elevated

ΣΤΟΙΧΕΙΩΝ ια'.

έν τῷ ὑποχειμένῳ ἐπιπέδῷ εὐθεῖαν. ποιείτω ὡς τὴν ΕΖ· δύο ἄρα εὐθεῖαι αἱ ΕΗΖ, ΕΖ χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. οὐχ ἄρα ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖαι ἐν μετεωροτέρῷ ἐστὶν ἐπιπέδῷ· ἐν τῷ διὰ τῶν ΑΒ, ΓΒ ἄρα παραλλήλων ἐστὶν ἐπιπέδῷ ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα.

Έὰν ἄρα ῶσι δύο εὐθεῖαι παράλληλοι, ληφθῆ δὲ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις· ὅπερ ἔδει δεῖξαι.

η'.

Έαν ῶσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ ἑτέρα αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ἦ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



Έστωσαν δύο εὐθεῖαι παράλληλοι αἱ AB, $\Gamma\Delta$, ἡ δὲ ἑτέρα αὐτῶν ἡ AB τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ ἡ λοιπὴ ἡ $\Gamma\Delta$ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Συμβαλλέτωσαν γὰρ αἱ AB, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῷ κατὰ τὰ B, Δ σημεῖα, καὶ ἐπεζέυχθω ἡ BΔ· αἱ AB, ΓΔ, BΔ ἄρα ἐν ἑνί εἰσιν ἐπιπέδῳ. ἤχθω τῆ BA πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῆ AB ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ BE, AE, AΔ.

Καὶ ἐπεὶ ἡ AB ὁρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἑστιν ἡ AB· ὀρθὴ ἄρα [ἐστὶν] ἑκατέρα τῶν ὑπὸ ABΔ, ABE γωνιῶν. καὶ ἐπεὶ εἰς παραλλήλους τὰς AB, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ BΔ, αἱ ἄρα ὑπὸ ABΔ, ΓΔΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ABΔ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΓΔB· ἡ ΓΔ ἄρα πρὸς τὴν BΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῷ ΔΕ, κοινὴ δὲ ἡ BΔ, (plane), such as EGF. And let a plane have been drawn through EGF. So it will make a straight cutting in the reference plane [Prop. 11.3]. Let it make EF. Thus, two straight-lines (with the same end-points), EGF and EF, will enclose an area. The very thing is impossible. Thus, the straight-line joining E to F is not in a more elevated plane. The straight-line joining E to F is thus in the plane through the parallel (straight-lines) AB and CD.

Thus, if there are two parallel straight-lines, and random points are taken on each of them, then the straightline joining the two points is in the same plane as the parallel (straight-lines). (Which is) the very thing it was required to show.

Proposition 8

If two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane.



Let AB and CD be two parallel straight-lines, and let one of them, AB, be at right-angles to a reference plane. I say that the remaining (one), CD, will also be at rightangles to the same plane.

For let AB and CD meet the reference plane at points B and D (respectively). And let BD have been joined. AB, CD, and BD are thus in one plane [Prop. 11.7]. Let DE have been drawn at right-angles to BD in the reference plane, and let DE be made equal to AB, and let BE, AE, and AD have been joined.

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are in the reference plane [Def. 11.3]. Thus, the angles ABD and ABE [are] each right-angles. And since the straight-line BD has met the parallel (straight-lines) AB and CD, the (sum of the) angles ABD and CDB is thus equal to two right-angles
δύο δη αί AB, B Δ δυσί ταῖς E Δ , Δ B ἴσαι εἰσίν καὶ γωνία ή ὑπὸ ΑΒΔ γωνία τῆ ὑπὸ ΕΔΒ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ή ΑΔ βάσει τῆ ΒΕ ἴση. καὶ ἐπεὶ ἴση ἐστὶν ή μèν AB τỹ ΔΕ,
 ή δὲ BE τỹ AΔ, δύο δỳ ai AB, BE δυσì ταῖς $\mathrm{E}\Delta,\,\Delta\mathrm{A}$ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα. καὶ βάσις αὐτῶν χοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνία τῆ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ABE· ὀρθὴ ἄρα καὶ ἡ ὑπὸ Ε ΔA · ή Ε Δ ἄρα πρὸς τὴν $A\Delta$ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς τὴν ΔB ὀρθή· ή EΔ ἄρα καὶ τῷ διὰ τῶν BΔ, ΔA ἐπιπέδῷ ὀρθή έστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ ούσας ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδω ὀρθὰς ποιήσει γωνίας ἡ $E\Delta$. ἐν δὲ τῷ διὰ τῶν $B\Delta A$ ἐπιπέδω ἐστὶν ἡ $\Delta \Gamma$, ἐπειδήπερ έν τῷ διὰ τῶν $B\Delta A$ ἐπιπέδῷ ἐστίν αί AB, $B\Delta$, ἐν ῷ δὲ αί ΑΒ, ΒΔ, ἐν τούτω ἐστὶ καὶ ἡ ΔΓ. ἡ ΕΔ ἄρα τῆ ΔΓ πρὸς ὀρθάς ἐστιν· ὥστε καὶ ἡ ΓΔ τῆ ΔΕ πρὸς ὀρθάς ἐστιν. ἔστι δὲ καὶ ἡ Γ Δ τῆ Β Δ πρὸς ὀρθάς. ἡ Γ Δ ἄρα δύο εὐθείαις τεμνούσαις ἀλλήλας ταῖς $\Delta \mathrm{E}, \Delta \mathrm{B}$ ἀπὸ τῆς κατὰ τὸ Δ τομῆς πρὸς ὀρθὰς ἐφέστηκεν· ὥστε ἡ Γ Δ καὶ τῷ διὰ τῶν ΔE , ΔB ἐπιπέδω πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν $\Delta ext{E}, \Delta ext{B}$ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΓΔ ἄρα τῷ ὑποκειμένω ἐπιπέδω πρὸς ὀρθάς ἐστιν.

Ἐὰν ἄρα ῶσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδω τινὶ πρὸς ὀρθὰς ἢ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδω πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

θ'.

Αἱ τῆ αὐτῆ εὐθεία παράλληλοι καὶ μὴ οὕσαι αὐτῆ ἐν τῷ αὐτῷ ἐπιπέδω καὶ ἀλλήλαις εἰσὶ παράλληλοι.



Έστω γὰρ ἑκατέρα τῶν ΑΒ, ΓΔ τῆ ΕΖ παράλληλος μὴ οὕσαι αὐτῆ ἐν τῷ αὐτῷ ἐπιπέδῷ· λέγω, ὅτι παράλληλός in the same plane as it. I say that AB is parallel to CD.

[Prop. 1.29]. And ABD (is) a right-angle. Thus, CDB (is) also a right-angle. CD is thus at right-angles to BD. And since AB is equal to DE, and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And angle ABD (is) equal to angle EDB. For each (is) a rightangle. Thus, the base AD (is) equal to the base BE[Prop. 1.4]. And since AB is equal to DE, and BE to AD, the two (sides) AB, BE are equal to the two (sides) ED, DA, respectively. And their base AE is common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. EDA (is) thus also a rightangle. Thus, ED is at right-angles to AD. And it is also at right-angles to DB. Thus, ED is also at right-angles to the plane through BD and DA [Prop. 11.4]. And ED will thus make right-angles with all of the straightlines joined to it which are also in the plane through BDA. And DC is in the plane through BDA, inasmuch as AB and BD are in the plane through BDA[Prop. 11.2], and in which(ever plane) AB and BD (are found), DC is also (found). Thus, ED is at right-angles to DC. Hence, CD is also at right-angles to DE. And CD is also at right-angles to BD. Thus, CD is standing at right-angles to two straight-lines, DE and DB, which meet one another, at the (point) of section, D. Hence, CD is also at right-angles to the plane through DE and DB [Prop. 11.4]. And the plane through DE and DB is the reference (plane). *CD* is thus at right-angles to the reference plane.

Thus, if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 9

(Straight-lines) parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another.



For let AB and CD each be parallel to EF, not being

Εἰλήφθω γὰρ ἐπὶ τῆς ΕΖ τυχὸν σημεῖον τὸ Η, καὶ ἀπ' αὐτοῦ τῆ ΕΖ ἐν μὲν τῷ διὰ τῶν ΕΖ, ΑΒ ἐπιπέδῳ πρὸς ὀρθὰς ἤχθω ἡ ΗΘ, ἐν δὲ τῷ διὰ τῶν ΖΕ, ΓΔ τῆ ΕΖ πάλιν πρὸς ὀρθὰς ἦχθω ἡ ΗΚ.

Καὶ ἐπεὶ ἡ ΕΖ πρὸς ἐxατέραν τῶν ΗΘ, ΗΚ ὀρθή ἐστιν, ἡ ΕΖ ἄρα καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν ἡ ΕΖ τῆ ΑΒ παράλληλος· καὶ ἡ ΑΒ ἄρα τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἑκατέρα ἄρα τῶν ΑΒ, ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθάς ὥσιν, παράλληλοί εἰσιν αἱ εὐθεῖαι· παράλληλος ἄρα ἐστιν ἡ ΑΒ τῆ ΓΔ· ὅπερ ἕδει δεῖζαι.

ι΄.

Ἐἀν δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ῶσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέζουσιν.



Δύο γὰρ εὐθεῖαι αἱ AB, BΓ ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΕ, EZ ἁπτομένας ἀλλήλων ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῷ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ.

Άπειλήφθωσαν γὰρ αί ΒΑ, ΒΓ, ΕΔ, ΕΖ ἴσαι ἀλλήλαις, καὶ ἐπεζεύχθωσαν αί ΑΔ, ΓΖ, ΒΕ, ΑΓ, ΔΖ.

Kai ἐπεὶ ἡ BA τῆ EΔ ἴση ἐστὶ xaì παράλληλος, xaì ἡ AΔ ἄρα τῆ BE ἴση ἐστὶ xaì παράλληλος. διὰ τὰ αὐτὰ δὴ xaì ἡ ΓΖ τῆ BE ἴση ἐστὶ xaì παράλληλος· ἑκατέρα ἄρα τῶν AΔ, ΓΖ τῆ BE ἴση ἐστὶ xaì παράλληλος· ἑκατέρα ἄρα τῶν AΔ, ΓΖ τῆ BE ἴση ἐστὶ xaì παράλληλος. ai δὲ τῆ αὐτῆ εὐθεία παράλληλοι xaì μὴ οὕσαι αὐτῆ ἐν τῷ αὐτῷ ἐπιπέδῷ xaì ἀλλήλαις εἰσὶ παράλληλοι· παράλληλος ἄρα ἐστὶν ἡ AΔ τῆ ΓΖ xaì ἴση. xaì ἐπιζευγνύουσιν αὐτὰς ai AΓ, ΔΖ· xaì ἡ AΓ ἄρα τῆ ΔΖ ἴση ἐστὶ xaì παράλληλος. xaì ἐπεὶ δύο ai AB, BΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, xaì βάσις ἡ AΓ βάσει τῆ ΔΖ ἴση, γωνία ἄρα ἡ ὑπὸ ABΓ γωνία τῆ ὑπὸ ΔΕΖ ἐστιν For let some point G have been taken at random on EF. And from it let GH have been drawn at right-angles to EF in the plane through EF and AB. And let GK have been drawn, again at right-angles to EF, in the plane through FE and CD.

And since EF is at right-angles to each of GH and GK, EF is thus also at right-angles to the plane through GH and GK [Prop. 11.4]. And EF is parallel to AB. Thus, AB is also at right-angles to the plane through HGK [Prop. 11.8]. So, for the same (reasons), CD is also at right-angles to the plane through HGK. Thus, AB and CD are each at right-angles to the plane through HGK. And if two straight-lines are at right-angles to the same plane then the straight-lines are parallel [Prop. 11.6]. Thus, AB is parallel to CD. (Which is) the very thing it was required to show.

Proposition 10

If two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles.



For let the two straight-lines joined to one another, AB and BC, be (respectively) parallel to the two straight-lines joined to one another, DE and EF, (but) not in the same plane. I say that angle ABC is equal to (angle) DEF.

For let BA, BC, ED, and EF have been cut off (so as to be, respectively) equal to one another. And let AD, CF, BE, AC, and DF have been joined.

And since BA is equal and parallel to ED, AD is thus also equal and parallel to BE [Prop. 1.33]. So, for the same reasons, CF is also equal and parallel to BE. Thus, AD and CF are each equal and parallel to BE. And straight-lines parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another [Prop. 11.9]. Thus, AD is parallel and equal to CF. And AC and DF join them. Thus, AC is also equal and ἴση.

Έὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ῶσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν. ὅπερ ἔδει δεῖξαι. parallel to DF [Prop. 1.33]. And since the two (straightlines) AB and BC are equal to the two (straight-lines) DE and EF (respectively), and the base AC (is) equal to the base DF, the angle ABC is thus equal to the (angle) DEF [Prop. 1.8].

Thus, if two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles. (Which is) the very thing it was required to show.

Proposition 11

To draw a perpendicular straight-line from a given raised point to a given plane.



Let A be the given raised point, and the given plane the reference (plane). So, it is required to draw a perpendicular straight-line from point A to the reference plane.

Let some random straight-line BC have been drawn across in the reference plane, and let the (straight-line) AD have been drawn from point A perpendicular to BC[Prop. 1.12]. If, therefore, AD is also perpendicular to the reference plane then that which was prescribed will have occurred. And, if not, let DE have been drawn in the reference plane from point D at right-angles to BC[Prop. 1.11], and let the (straight-line) AF have been drawn from A perpendicular to DE [Prop. 1.12], and let GH have been drawn through point F, parallel to BC[Prop. 1.31].

And since BC is at right-angles to each of DA and DE, BC is thus also at right-angles to the plane through EDA [Prop. 11.4]. And GH is parallel to it. And if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (straight-line) will also be at right-angles to the same plane [Prop. 11.8]. Thus, GH is also at right-angles to the plane through

ια΄.

Άπὸ τοῦ δοθέντος σημείου μετεώρου ἐπὶ τὸ δοθὲν ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.



Έστω τὸ μὲν δοθὲν σημεῖον μετέωρον τὸ Α, τὸ δὲ δοθὲν ἐπίπεδον τὸ ὑποχείμενον· δεῖ δὴ ἀπὸ τοῦ Α σημείου ἐπὶ τὸ ὑποχείμενον ἐπίπεδον χάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Διήχθω γάρ τις ἐν τῷ ὑποχειμένω ἐπιπέδω εὐθεῖα, ὡς ἔτυχεν, ἡ ΒΓ, καὶ ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ. εἰ μὲν οῦν ἡ ΑΔ κάθετός ἐστι καὶ ἐπὶ τὸ ὑποχείμενον ἐπίπεδον, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ οῦ, ἤχθω ἀπὸ τοῦ Δ σημείου τῆ ΒΓ ἐν τῷ ὑποχειμένω ἐπιπέδω πρὸς ὀρθὰς ἡ ΔΕ, καὶ ἤχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΔΕ κάθετος ἡ ΑΖ, καὶ διὰ τοῦ Ζ σημείου τῆ ΒΓ παράλληλος ἤχθω ἡ ΗΘ.

Καὶ ἐπεὶ ἡ ΒΓ ἑκατέρα τῶν ΔΑ, ΔΕ πρὸς ὀρθάς ἐστιν, ἡ ΒΓ ἄρα καὶ τῷ διὰ τῶν ΕΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν αὐτῆ παράλληλος ἡ ΗΘ· ἐὰν δὲ ῶσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ῆ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· καὶ ἡ ΗΘ ἄρα τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΗΘ. ἄπτεται δὲ αὐτῆς ἡ ΑΖ οῦσα ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ· ἡ ΗΘ ἄρα ὀθή ἐστι πρὸς τὴν ΖΑ· ὥστε καὶ ἡ ΖΑ ὀρθή ἐστι πρὸς τὴν ΘΗ. ἔστι δὲ ἡ ΑΖ καὶ πρὸς τὴν ΔΕ ὀρθή ἡ ΑΖ ἄρα πρὸς ἑκατέραν τῶν ΗΘ, ΔΕ ὀρθή ἐστιν. ἐἀν δὲ εὐθεῖα δυσὶν εὐθείαις τεμνούσαις ἀλλήλας ἐπὶ τῆς τομῆς πρὸς ὀρθὰς ἐπισταθῆ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ἡ ΖΑ ἄρα τῷ διὰ τῶν ΕΔ, ΗΘ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΕΔ, ΗΘ ἐπίπεδόν ἐστι τὸ ὑποκείμενον· ἡ ΑΖ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Άπὸ τοῦ ἄρα δοθέντος σημείου μετεώρου τοῦ Α ἐπὶ τὸ ὑποχείμενον ἐπίπεδον χάθετος εὐθεῖα γραμμὴ ἤχται ἡ ΑΖ· ὅπερ ἔδει ποιῆσαι. ED and DA. And GH is thus at right-angles to all of the straight-lines joined to it which are also in the plane through ED and AD [Def. 11.3]. And AF, which is in the plane through ED and DA, is joined to it. Thus, GH is at right-angles to FA. Hence, FA is also at right-angles to HG. And AF is also at right-angles to DE. Thus, AF is at right-angles to each of GH and DE. And if a straightline is set up at right-angles to two straight-lines cutting one another, at the point of section, then it will also be at right-angles to the plane through them [Prop. 11.4]. Thus, FA is at right-angles to the plane through ED and GH. And the plane through ED and GH is the reference (plane). Thus, AF is at right-angles to the reference plane.

Thus, the straight-line AF has been drawn from the given raised point A perpendicular to the reference plane. (Which is) the very thing it was required to do.

Proposition 12

To set up a straight-line at right-angles to a given plane from a given point in it.



Let the given plane be the reference (plane), and A a point in it. So, it is required to set up a straight-line at right-angles to the reference plane at point A.

Let some raised point B have been assumed, and let the perpendicular (straight-line) BC have been drawn from B to the reference plane [Prop. 11.11]. And let AD have been drawn from point A parallel to BC[Prop. 1.31].

Therefore, since AD and CB are two parallel straightlines, and one of them, BC, is at right-angles to the reference plane, the remaining (one) AD is thus also at rightangles to the reference plane [Prop. 11.8].

ιβ'.

Τῷ δοθέντι ἐπιπέδῷ ἀπὸ τοῦ πρὸς αὐτῷ δοθέντος σημείου πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.



Έστω τὸ μὲν δοθὲν ἐπίπεδον τὸ ὑποκείμενον, τὸ δὲ πρὸς αὐτῷ σημεῖον τὸ Α· δεῖ δὴ ἀπὸ τοῦ Α σημείου τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.

Νενοήσθω τι σημεῖον μετέωρον τὸ Β, καὶ ἀπὸ τοῦ Β ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος ἦχθω ἡ ΒΓ, καὶ διὰ τοῦ Α σημείου τῆ ΒΓ παράλληλος ἦχθω ἡ ΑΔ.

Έπεὶ οὒν δύο εὐθεῖαι παράλληλοί εἰσιν αἱ ΑΔ, ΓΒ, ἡ δὲ μία αὐτῶν ἡ ΒΓ τῷ ὑποχειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, χαὶ ἡ λοιπὴ ἄρα ἡ ΑΔ τῷ ὑποχειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστιν.

Τῷ ἄρα δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ σημείου τοῦ Α πρὸς ὀρθὰς ἀνέσταται ἡ ΑΔ· ὅπερ ἔδει ποιῆσαι. ıγ'.

Άπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς οὐκ ἀναστήσονται ἐπὶ τὰ αὐτὰ μέρη.



Εἰ γὰρ δυνατόν, ἀπὸ τοῦ αὐτοῦ σημείου τοῦ Α τῷ ὑποχειμένῳ ἐπιπέδῳ δύο εὐθεῖαι αἰ ΑΒ, ΒΓ πρὸς ὀρθὰς ἀνεστάτωσαν ἐπὶ τὰ αὐτὰ μέρη, καὶ διήχθω τὸ διὰ τῶν ΒΑ, ΑΓ ἐπὶπεδον· τομὴν δὴ ποιήσει διὰ τοῦ Α ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΔΑΕ· αἰ ἄρα ΑΒ, ΑΓ, ΔΑΕ εὐθεῖαι ἐν ἑνι εἰσιν ἐπιπέδῳ. καὶ ἐπεὶ ἡ ΓΑ τῷ ὑποχειμένῳ ἐπιπέδῷ πρὸς ὀρθάς ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ὑποχειμένῳ ἐπιπέδῷ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἡ ΔΑΕ οῦσα ἐν τῷ ὑποχειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΓΑΕ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΕ ὀρθή ἐστιν· ἴση ἄρα ἡ ὑπὸ ΓΑΕ τῆ ὑπὸ ΒΑΕ καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον.

Οὐχ ἄρα ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνασταθήσονται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

ιδ'.

Πρὸς ἂ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλα ἔσται τὰ ἐπίπεδα.

Εὐθεῖα γάρ τις ἡ ΑΒ πρὸς ἑκάτερον τῶν ΓΔ, ΕΖ ἐπιπέδων πρὸς ὀρθὰς ἔστω· λέγω, ὅτι παράλληλά ἐστι τὰ ἐπίπεδα.

Thus, AD has been set up at right-angles to the given plane, from the point in it, A. (Which is) the very thing it was required to do.

Proposition 13

Two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side.



For, if possible, let the two straight-lines AB and AC have been set up at the same point A at right-angles to the reference plane, on the same side. And let the plane through BA and AC have been drawn. So it will make a straight cutting (passing) through (point) A in the reference plane [Prop. 11.3]. Let it have made DAE. Thus, AB, AC, and DAE are straight-lines in one plane. And since CA is at right-angles to the reference plane, it will thus also make right-angles with all of the straight-lines joined to it which are also in the reference plane, is joined to it. Thus, angle CAE is a right-angle. So, for the same (reasons), BAE is also a right-angle. Thus, CAE (is) equal to BAE. And they are in one plane. The very thing is impossible.

Thus, two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side. (Which is) the very thing it was required to show.

Proposition 14

Planes to which the same straight-line is at rightangles will be parallel planes.

For let some straight-line AB be at right-angles to each of the planes CD and EF. I say that the planes are parallel.



Εἰ γὰρ μή, ἐκβαλλόμενα συμπεσοῦνται. συμπιπτέτωσαν ποιήσουσι δὴ κοινὴν τομὴν εὐθεῖαν. ποιείτωσαν τὴν ΗΘ, καὶ εἰλήφθω ἐπὶ τῆς ΗΘ τυχὸν σημεῖον τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΒΚ.

Καὶ ἐπεὶ ἡ AB ὀρθή ἐστι πρὸς τὸ EZ ἐπίπεδον, καὶ πρὸς τὴν BK ἄρα εὐθεῖαν οὕσαν ἐν τῷ EZ ἐκβληθέντι ἐπιπέδῳ ὀρθή ἐστιν ἡ AB· ἡ ἄρα ὑπὸ ABK γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ BAK ὀρθή ἐστιν. τριγώνου δὴ τοῦ ABK αἱ δύο γωνίαι αἱ ὑπὸ ABK, BAK δυσὶν ὀρθαῖς εἰσιν ἴσαι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΔ, EZ ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται· παράλληλα ἄρα ἐστὶ τὰ ΓΔ, EZ ἐπίπεδα.

Πρὸς ἂ ἐπίπεδα ἄρα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα[·] ὅπερ ἔδει δεῖξαι.

ιε΄.

Έαν δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ῶσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οῦσαι, παράλληλά ἐστι τὰ δι' αὐτῶν ἐπίπεδα.

Δύο γὰρ εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ AB, BΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΔΕ, ΕΖ ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῷ οῦσαι· λέγω, ὅτι ἐκβαλλόμενα τὰ διὰ τῶν AB, BΓ, ΔΕ, ΕΖ ἐπίπεδα οὐ συμπεσεῖται ἀλλήλοις.

Ήχθω γὰρ ἀπὸ τοῦ Β σημείου ἐπὶ τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον κάθετος ἡ ΒΗ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η σημεῖον, καὶ διὰ τοῦ Η τῆ μὲν ΕΔ παράλληλος ἤχθω ἡ ΗΘ, τῆ δὲ ΕΖ ἡ ΗΚ.



For, if not, being produced, they will meet. Let them have met. So they will make a straight-line as a common section [Prop. 11.3]. Let them have made GH. And let some random point K have been taken on GH. And let AK and BK have been joined.

And since AB is at right-angles to the plane EF, AB is thus also at right-angles to BK, which is a straight-line in the produced plane EF [Def. 11.3]. Thus, angle ABK is a right-angle. So, for the same (reasons), BAK is also a right-angle. So the (sum of the) two angles ABK and BAK in the triangle ABK is equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, planes CD and EF, being produced, will not meet. Planes CD and EF are thus parallel [Def. 11.8].

Thus, planes to which the same straight-line is at right-angles are parallel planes. (Which is) the very thing it was required to show.

Proposition 15

If two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another).

For let the two straight-lines joined to one another, AB and BC, be parallel to the two straight-lines joined to one another, DE and EF (respectively), not being in the same plane. I say that the planes through AB, BC and DE, EF will not meet one another (when) produced.

For let BG have been drawn from point B perpendicular to the plane through DE and EF [Prop. 11.11], and let it meet the plane at point G. And let GH have been drawn through G parallel to ED, and GK (parallel) to EF [Prop. 1.31].



Καὶ ἐπεὶ ἡ ΒΗ ὀρθή ἐστι πρὸς τὸ διὰ τῶν ΔΕ, ΕΖ έπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδω ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ αὐτῆς ἑκατέρα τῶν ΗΘ, ΗΚ οὖσα ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδω· ὀρθή ἄρα ἐστὶν ἑκατέρα τῶν ύπὸ ΒΗΘ, ΒΗΚ γωνιῶν. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΒΑ τῆ ΗΘ, αἱ ἄρα ὑπὸ ΗΒΑ, ΒΗΘ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθή δὲ ἡ ὑπὸ ΒΗΘ· ὀρθή ἄρα καὶ ἡ ὑπὸ ΗΒΑ· ἡ ΗΒ άρα τῆ ΒΑ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ ἡ ΗΒ καὶ τῆ ΒΓ ἐστι πρὸς ὀρθάς. ἐπεὶ οὖν εὐθεῖα ἡ ΗΒ δυσὶν εὐθείαις ταῖς ΒΑ, ΒΓ τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐφέστηχεν, ἡ ΗΒ ἄρα καὶ τῷ διὰ τῶν ΒΑ, ΒΓ ἐπιπέδω πρὸς ὀρθάς ἐστιν. [διὰ τὰ αὐτὰ δὴ ἡ ΒΗ χαὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδω πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΗΘ, ΗΚ ἐπίπεδόν ἐστι τὸ διὰ τῶν ΔΕ, ΕΖ· ἡ ΒΗ ἄρα τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδω έστὶ πρὸς ὀρθάς. ἐδείχθη δὲ ἡ ΗΒ καὶ τῷ διὰ τῶν ΑΒ, ΒΓ ἐπιπέδω πρὸς ὀρθάς]. πρὸς ἃ δὲ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή έστιν, παράλληλά έστι τὰ ἐπίπεδα· παράλληλον ἄρα ἐστὶ τὸ διὰ τῶν ΑΒ, ΒΓ ἐπίπεδον τῷ διὰ τῶν ΔΕ, ΕΖ.

Έλν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, παράλληλά ἐστι τὰ δι' αὐτῶν ἐπίπεδα. ὅπερ ἔδει δεῖξαι.

ເຈ່.

Ἐἀν δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Δύο γὰρ ἐπίπεδα παράλληλα τὰ AB, ΓΔ ὑπὸ ἐπιπέδου τοῦ ΕΖΗΘ τεμνέσθω, χοιναὶ δὲ αὐτῶν τομαὶ ἔστωσαν αἱ ΕΖ, ΗΘ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΕΖ τῆ ΗΘ.



And since BG is at right-angles to the plane through DE and EF, it will thus also make right-angles with all of the straight-lines joined to it, which are also in the plane through DE and EF [Def. 11.3]. And each of GH and GK, which are in the plane through DE and EF, are joined to it. Thus, each of the angles BGH and BGK are right-angles. And since BA is parallel to GH[Prop. 11.9], the (sum of the) angles GBA and BGH is equal to two right-angles [Prop. 1.29]. And BGH (is) a right-angle. GBA (is) thus also a right-angle. Thus, GB is at right-angles to BA. So, for the same (reasons), GB is also at right-angles to BC. Therefore, since the straight-line GB has been set up at right-angles to two straight-lines, BA and BC, cutting one another, GB is thus at right-angles to the plane through BA and BC[Prop. 11.4]. [So, for the same (reasons), BG is also at right-angles to the plane through GH and GK. And the plane through GH and GK is the (plane) through DE and EF. And it was also shown that GB is at rightangles to the plane through AB and BC.] And planes to which the same straight-line is at right-angles are parallel planes [Prop. 11.14]. Thus, the plane through ABand BC is parallel to the (plane) through DE and EF.

Thus, if two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another). (Which is) the very thing it was required to show.

Proposition 16

If two parallel planes are cut by some plane then their common sections are parallel.

For let the two parallel planes AB and CD have been cut by the plane EFGH. And let EF and GH be their common sections. I say that EF is parallel to GH.



Εἰ γὰρ μή, ἐκβαλλόμεναι αἰ ΕΖ, ΗΘ ἤτοι ἐπὶ τὰ Ζ, Θ μέρη ἢ ἐπὶ τὰ Ε, Η συμπεσοῦνται. ἐκβεβλήσθωσαν ὡς ἐπὶ τὰ Ζ, Θ μέρη καὶ συμπιπτέτωσαν πρότερον κατὰ τὸ Κ. καὶ ἐπεὶ ἡ ΕΖΚ ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ, καὶ πάντα ἄρα τὰ ἐπὶ τῆς ΕΖΚ σημεῖα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. ἐν δὲ τῶν ἐπὶ τῆς ΕΖΚ εὐθείας σημείων ἐστὶ τὸ Κ· τὸ Κ ἄρα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. διὰ τὰ αὐτὰ δὴ τὸ Κ καὶ ἐν τῷ ΓΔ ἐστιν ἐπιπέδῳ. τὰ ΑΒ, ΓΔ ἄρα ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται. οὐ συμπίπτουσι δὲ διὰ τὸ παράλληλα ὑποκεῖσθαι· οὐκ ἄρα αἱ ΕΖ, ΗΘ εὐθεῖαι ἐκβαλλόμεναι ἐπὶ τὰ Ζ, Θ μέρη συμπεσοῦνται. ὁμοίως δὴ δείξομεν, ὅτι αἱ ΕΖ, ΗΘ εὐθεῖαι οὐδέ ἐπὶ τὰ Ε, Η μέρη ἐκβαλλόμεναι συμπεσοῦνται. αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΕΖ τῇ ΗΘ.

Ἐἀν ἄρα δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. ὅπερ ἔδει δεῖξαι.

ιζ΄.

Έὰν δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπεδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται.

Δύο γὰρ εὐθεῖαι αἱ AB, ΓΔ ὑπὸ παραλλήλων ἐπιπέδων τῶν HΘ, ΚΛ, MN τεμνέσθωσαν κατὰ τὰ A, E, B, Γ, Ζ, Δ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ AE εὐθεῖα πρὸς τὴν EB, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΔ, ΑΔ, καὶ συμβαλλέτω ἡ ΑΔ τῷ ΚΛ ἐπιπέδῳ κατὰ τὸ Ξ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΞ, ΞΖ.

Καὶ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΚΛ, MN ὑπὸ ἐπιπέδου τοῦ ΕΒΔΞ τέμνεται, αἱ χοιναὶ αὐτῶν τομαὶ αἰ ΕΞ, ΒΔ παράλληλοί εἰσιν. διὰ τὰ αὐτὰ δὴ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΗΘ, ΚΛ ὑπὸ ἐπιπέδου τοῦ ΑΞΖΓ τέμνεται, αἱ χοιναὶ αὐτῶν τομαὶ αἱ ΑΓ, ΞΖ παράλληλοί εἰσιν. χαὶ ἐπεὶ τριγώνου τοῦ ΑΒΔ παρὰ μίαν τῶν πλευρῶν τὴν ΒΔ εὐθεῖα ῆχται ἡ ΕΞ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως



For, if not, being produced, EF and GH will meet either in the direction of F, H, or of E, G. Let them be produced, as in the direction of F, H, and let them, first of all, have met at K. And since EFK is in the plane AB, all of the points on EFK are thus also in the plane AB [Prop. 11.1]. And K is one of the points on EFK. Thus, K is in the plane AB. So, for the same (reasons), K is also in the plane CD. Thus, the planes AB and CD, being produced, will meet. But they do not meet, on account of being (initially) assumed (to be mutually) parallel. Thus, the straight-lines EF and GH, being produced in the direction of F, H, will not meet. So, similarly, we can show that the straight-lines EF and GH, being produced in the direction of E, G, will not meet either. And (straight-lines in one plane which), being produced, do not meet in either direction are parallel [Def. 1.23]. EF is thus parallel to GH.

Thus, if two parallel planes are cut by some plane then their common sections are parallel. (Which is) the very thing it was required to show.

Proposition 17

If two straight-lines are cut by parallel planes then they will be cut in the same ratios.

For let the two straight-lines AB and CD be cut by the parallel planes GH, KL, and MN at the points A, E, B, and C, F, D (respectively). I say that as the straight-line AE is to EB, so CF (is) to FD.

For let AC, BD, and AD have been joined, and let AD meet the plane KL at point O, and let EO and OF have been joined.

And since two parallel planes KL and MN are cut by the plane EBDO, their common sections EO and BDare parallel [Prop. 11.16]. So, for the same (reasons), since two parallel planes GH and KL are cut by the plane AOFC, their common sections AC and OF are parallel [Prop. 11.16]. And since the straight-line EOhas been drawn parallel to one of the sides BD of trianή ΑΞ πρὸς ΞΔ. πάλιν ἐπεὶ τριγώνου τοῦ ΑΔΓ παρὰ μίαν τῶν πλευρῶν τὴν ΑΓ εὐθεῖα ἦχται ἡ ΞΖ, ἀνάλογόν ἐστιν ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΓΖ πρὸς ΖΔ. ἐδείχθη δὲ καὶ ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΑΕ πρὸς ΕΒ· καὶ ὡς ἄρα ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ.



Έὰν ἄρα δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπέδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται· ὅπερ ἔδει δειξαι.

*ι*η'.

Έαν εύθεῖα ἐπιπέδω τινὶ πρὸς ὀρθὰς ἢ, καὶ πάντα τὰ δι' αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδω πρὸς ὀρθὰς ἔσται.

Εὐθεῖα γάρ τις ἡ AB τῷ ὑποχειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ πάντα τὰ διὰ τῆς AB ἐπίπεδα τῷ ὑποχειμένω ἐπιπέδω πρὸς ὀρθάς ἐστιν.

Έκβεβλήσθω γὰρ διὰ τῆς AB ἐπίπεδον τὸ ΔΕ, καὶ ἔστω κοινὴ τομὴ τοῦ ΔΕ ἐπιπέδου καὶ τοῦ ὑποκειμένου ἡ ΓΕ, καὶ εἰλήφθω ἐπὶ τῆς ΓΕ τυχὸν σημεῖον τὸ Ζ, καὶ ἀπὸ τοῦ Ζ τῆ ΓΕ πρὸς ὀρθὰς ἦχθω ἐν τῷ ΔΕ ἐπιπέδῳ ἡ ΖΗ.

Καὶ ἐπεὶ ἡ AB πρὸς τὸ ὑποχείμενον ἐπίπεδον ὀρθή ἐστιν, χαὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ὑποχειμένῳ ἐπιπέδῷ ὀρθή ἐστιν ἡ AB· ὥστε καὶ πρὸς τὴν ΓΕ ὀρθή ἐστιν· ἡ ἄρα ὑπὸ ABZ γωνία ὀρθή ἐστιν. ἔστι δὲ καὶ ἡ ὑπὸ HZB ὀρθὴ· παράλληλος ἄρα ἐστὶν ἡ AB τῆ ZH. ἡ δὲ AB τῷ ὑποχειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· καὶ ἡ ZH ἄρα τῷ ὑποχειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῆ κοινῆ τομῆ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ῶσιν. καὶ τῆ χοινῆ τομῆ τῶν ἐπιπέδων τῆ ΓΕ ἐν ἑνὶ τῶν ἐπιπέδων gle ABD, thus, proportionally, as AE is to EB, so AO (is) to OD [Prop. 6.2]. Again, since the straight-line OF has been drawn parallel to one of the sides AC of triangle ADC, proportionally, as AO is to OD, so CF (is) to FD [Prop. 6.2]. And it was also shown that as AO (is) to OD, so AE (is) to EB. And thus as AE (is) to EB, so CF (is) to FD [Prop. 5.11].



Thus, if two straight-lines are cut by parallel planes then they will be cut in the same ratios. (Which is) the very thing it was required to show.

Proposition 18

If a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at rightangles to the same plane.

For let some straight-line AB be at right-angles to a reference plane. I say that all of the planes (passing) through AB are also at right-angles to the reference plane.

For let the plane DE have been produced through AB. And let CE be the common section of the plane DE and the reference (plane). And let some random point F have been taken on CE. And let FG have been drawn from F, at right-angles to CE, in the plane DE [Prop. 1.11].

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straightlines joined to it which are also in the reference plane [Def. 11.3]. Hence, it is also at right-angles to CE. Thus, angle ABF is a right-angle. And GFB is also a rightangle. Thus, AB is parallel to FG [Prop. 1.28]. And ABis at right-angles to the reference plane. Thus, FG is also τῷ ΔΕ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΖΗ ἐδείχθη τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς· τὸ ἄρα ΔΕ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον. ὑμοίως δὴ δειχθήσεται καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα ὀρθὰ τυγχανοντα πρὸς τὸ ὑποκείμενον ἐπίπεδον.



Έὰν ἄρα εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ἦ, καὶ πάντα τὰ δι' αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

ıθ'.

Έὰν δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ἢ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



Δύο γὰρ ἐπίπεδα τὰ AB, BΓ τῷ ὑποχειμένῳ ἐπιπέδῷ πρὸς ὀρθὰς ἔστω, χοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ BΔ· λέγω, ὅτι ἡ BΔ τῷ ὑποχειμένῳ ἐπιπέδῷ πρὸς ὀρθάς ἐστιν.

at right-angles to the reference plane [Prop. 11.8]. And a plane is at right-angles to a(nother) plane when the straight-lines drawn at right-angles to the common section of the planes, (and lying) in one of the planes, are at right-angles to the remaining plane [Def. 11.4]. And FG, (which was) drawn at right-angles to the common section of the planes, CE, in one of the planes, DE, was shown to be at right-angles to the reference plane. Thus, plane DE is at right-angles to the reference (plane). So, similarly, it can be shown that all of the planes (passing) at random through AB (are) at right-angles to the reference plane.



Thus, if a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 19

If two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane.



For let the two planes AB and BC be at right-angles to a reference plane, and let their common section be BD. I say that BD is at right-angles to the reference

Mỳ γάρ, καὶ ἦχθωσαν ἀπὸ τοῦ Δ σημείου ἐν μὲν τῷ AB ἐπιπέδω τῆ $A\Delta$ εὐθεία πρὸς ὀρθὰς ἡ ΔE , ἐν δὲ τῷ $B\Gamma$ έπιπέδω τῆ ΓΔ πρὸς ὀρθὰς ἡ ΔΖ.

καὶ τῆ κοινῆ αὐτῶν τομῆ τῆ ΑΔ πρὸς ὀρθὰς ἐν τῷ AB ἐπιπέδω
 ήχται ή ΔΕ, ή ΔΕ ἄρα ὀρθή ἐστι πρὸς τὸ ύποχείμενον ἐπίπεδον. ὁμοίως δὴ δείξομεν, ὅτι χαὶ ἡ ΔΖ όρθή ἐστι πρὸς τὸ ὑποχείμενον ἐπίπεδον. ἀπὸ τοῦ αὐτοῦ ἄρα σημείου τοῦ Δ τῷ ὑποχειμένῳ ἐπιπέδῳ δύο εὐθεῖα πρὸς ὀρθὰς ἀνεσταμέναι εἰσὶν ἐπὶ τὰ αὐτὰ μέρη. ὅπερ ἐστὶν άδύνατον. οὐχ ἄρα τῷ ὑποχειμένω ἐπιπέδω ἀπὸ τοῦ Δ σημείου άνασταθήσεται πρός όρθὰς πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδω τινὶ πρὸς όρθὰς ἢ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς όρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

χ΄.

Ἐἀν στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.



 Σ τερεὰ γὰρ γωνία ή πρὸς τῷ A ὑπὸ τριῶν γωνιῶν έπιπέδων τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ περιεχέσθω· λέγω, ότι τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνιῶν δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.

Εἰ μὲν οὖν αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι δύο ὑποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν. εἰ δὲ οὔ, ἔστω μείζων ἡ ὑπὸ ΒΑΓ, καὶ συνεστάτω πρὸς τῆ AB εὐθεία καὶ τῷ πρὸς αὐτῆ σημεί
ώ τῷ Α τῆ ὑπὸ ΔAB γωνία ἐν τῷ διὰ τῶν ΒΑΓ ἐπιπέδω ἴση ἡ ὑπὸ ΒΑΕ, καὶ κείσθω τη A Δ ἴση ή AE, καὶ διὰ τοῦ E σημείου διαχθεῖσα ή ΒΕΓ τεμνέτω τὰς ΑΒ, ΑΓ εὐθείας κατὰ τὰ Β, Γ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΔΓ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῆ ΑΕ, κοινὴ δὲ ἡ ΑΒ, δύο δυσίν ἴσαι· καὶ γωνία ἡ ὑπὸ ΔΑΒ γωνία τῆ ὑπὸ ΒΑΕ ἴση· βάσις ἄρα
ἡ ΔB βάσει τῆ BE ἐστιν ἴση. καὶ ἐπεὶ δύο
 αἱ $B\Delta,$ ΔΓ τῆς ΒΓ μείζονές εἰσιν, ὤν ἡ ΔΒ τῆ ΒΕ ἐδείχϑη ἴση,

plane.

For (if) not, let DE also have been drawn from point D, in the plane AB, at right-angles to the straight-line Kαὶ ἐπεὶ τὸ AB ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποχείμενον, AD, and DF, in the plane BC, at right-angles to CD.

> And since the plane AB is at right-angles to the reference (plane), and DE has been drawn at right-angles to their common section AD, in the plane AB, DE is thus at right-angles to the reference plane [Def. 11.4]. So, similarly, we can show that DF is also at right-angles to the reference plane. Thus, two (different) straight-lines are set up, at the same point D, at right-angles to the reference plane, on the same side. The very thing is impossible [Prop. 11.13]. Thus, no (other straight-line) except the common section DB of the planes AB and BC can be set up at point *D*, at right-angles to the reference plane.

> Thus, if two planes cutting one another are at rightangles to some plane then their common section will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 20

If a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way).



For let the solid angle A have been contained by the three plane angles BAC, CAD, and DAB. I say that (the sum of) any two of the angles BAC, CAD, and DAB is greater than the remaining (one), (the angles) being taken up in any (possible way).

For if the angles *BAC*, *CAD*, and *DAB* are equal to one another then (it is) clear that (the sum of) any two is greater than the remaining (one). But, if not, let BACbe greater (than CAD or DAB). And let (angle) BAE, equal to the angle DAB, have been constructed in the plane through BAC, on the straight-line AB, at the point A on it. And let AE be made equal to AD. And BEC being drawn across through point E, let it cut the straightlines AB and AC at points B and C (respectively). And let DB and DC have been joined.

And since DA is equal to AE, and AB (is) common,

λοιπὴ ἄρα ἡ ΔΓ λοιπῆς τῆς ΕΓ μείζων ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῆ ΑΕ, κοινὴ δὲ ἡ ΑΓ, καὶ βάσις ἡ ΔΓ βάσεως τῆς ΕΓ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνάις τῆς ὑπὸ ΕΑΓ μείζων ἐστίν. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΔΑΒ τῆ ὑπὸ ΒΑΕ ἴση· αἱ ἄρα ὑπὸ ΔΑΒ, ΔΑΓ τῆς ὑπὸ ΒΑΓ μείζονές εἰσιν. ὑμοίως δὴ δείζομεν, ὅτι καὶ αἱ λοιπαὶ σύνδυο λαμβανόμεναι τῆς λοιπῆς μείζονές εἰσιν.

Έλν ἄρα στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὑποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

κα΄.

Άπασα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται.



Έστω στερεὰ γωνία ἡ πρὸς τῷ Α περιεχομένη ὑπὸ ἐπιπέδων γωνιῶν τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ· λέγω, ὅτι αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Εἰλήφθω γὰρ ἐφ' ἑκάστης τῶν ΑΒ, ΑΓ, ΑΔ τυχόντα σημεῖα τὰ Β, Γ, Δ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΓΔ, ΔΒ. καὶ ἐπεὶ στερεὰ γωνία ἡ πρὸς τῷ Β ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται τῶν ὑπὸ ΓΒΑ, ΑΒΔ, ΓΒΔ, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν· αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΔ τῆς ὑπὸ ΓΒΔ μείζονές εἰσιν· αἱ ἄρα ὑπὸ ΓΔΑ, ΑΔΒ τῆς ὑπὸ ΓΔΑ τῆς ὑπὸ ΓΔΑ τῆς ὑπὸ ΓΔΑ μείζονές εἰσιν· αἱ ἕξ ἄρα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ τῆς ὑπὸ ΓΔΑ, ΑΔΒ τῆς ὑπὸ ΓΔΑ μείζονές εἰσιν· αἱ ἕξ ἄρα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ τριῶν τῶν ὑπὸ ΓΒΔ, ΒΓΑ, ΓΔΒ μείζονές εἰσιν· αἱ ἕξ ἄρα αὶ ὑπὸ ΓΒΔ, ΒΔΓ, ΒΓΑ δυσὶν ὀρθῶς ἴσαι εἰσίν· αὶ ἕξ ἄρα αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ τριγώνων αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, αἱ ἔρα τῶν τριῶν τριγώνων ἐννέα γωνίαι αἱ ὑπὸ

the two (straight-lines AD and AB are) equal to the two (straight-lines EA and AB, respectively). And angle DAB (is) equal to angle BAE. Thus, the base DBis equal to the base BE [Prop. 1.4]. And since the (sum of the) two (straight-lines) BD and DC is greater than BC [Prop. 1.20], of which DB was shown (to be) equal to BE, the remainder DC is thus greater than the remainder EC. And since DA is equal to AE, but AC(is) common, and the base DC is greater than the base EC, the angle DAC is thus greater than the angle EAC[Prop. 1.25]. And DAB was also shown (to be) equal to BAE. Thus, (the sum of) DAB and DAC is greater than BAC. So, similarly, we can also show that the remaining (angles), being taken in pairs, are greater than the remaining (one).

Thus, if a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way). (Which is) the very thing it was required to show.

Proposition 21

Any solid angle is contained by plane angles (whose sum is) less [than] four right-angles.^{\dagger}



Let the solid angle A be contained by the plane angles BAC, CAD, and DAB. I say that (the sum of) BAC, CAD, and DAB is less than four right-angles.

For let the random points B, C, and D have been taken on each of (the straight-lines) AB, AC, and AD(respectively). And let BC, CD, and DB have been joined. And since the solid angle at B is contained by the three plane angles CBA, ABD, and CBD, (the sum of) any two is greater than the remaining (one) [Prop. 11.20]. Thus, (the sum of) CBA and ABD is greater than CBD. So, for the same (reasons), (the sum of) BCA and ACD is also greater than BCD, and (the sum of) CDA and ADB is greater than CDB. Thus, the (sum of the) six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than the (sum of the) three (angles) CBD, BCD, and CDB. But, the (sum of the) three (angles) CBD, BDC, and BCD is equal to two ΓΒΑ, ΑΓΒ, ΒΑΓ, ΑΓΔ, ΓΔΑ, ΓΑΔ, ΑΔΒ, ΔΒΑ, ΒΑΔ ξξ ὀρθαῖς ἴσαι εἰσίν, ῶν αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ, ΔΒΑ ξξ γωνίαι δύο ὀρθῶν εἰσι μείζονες λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τρεῖς [γωνίαι] περιέχουσαι τὴν στερεὰν γωνίαν τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Άπασα ἄρα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἤ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται· ὅπερ ἔδει δεῖξαι. right-angles [Prop. 1.32]. Thus, the (sum of the) six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, and *ADB* is greater than two right-angles. And since the (sum of the) three angles of each of the triangles *ABC*, *ACD*, and *ADB* is equal to two right-angles, the (sum of the) nine angles *CBA*, *ACB*, *BAC*, *ACD*, *CDA*, *CAD*, *ADB*, *DBA*, and *BAD* of the three triangles is equal to six right-angles, of which the (sum of the) six angles *ABC*, *BCA*, *ACD*, *CDA*, *ADB*, *BAC*, *ACD*, *CDA*, *ADB*, *BAC*, *ACD*, *CDA*, *ADB*, *BAC*, *ACD*, *CDA*, *ADB*, and *DBA* is greater than two right-angles. Thus, the (sum of the) remaining three [angles] *BAC*, *CAD*, and *DAB*, containing the solid angle, is less than four right-angles.

Thus, any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. (Which is) the very thing it was required to show.

[†] This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalization to a solid angle contained by more than three plane angles is straightforward.

xβ'.

Έὰν ῶσι τρεῖς γωνίαι ἐπίπεδοι, ῶν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, περιέχωσι δὲ αὐτὰς ἴσαι εὐθεῖαι, δυνατόν ἐστιν ἐχ τῶν ἐπιζευγνυουσῶν τὰς ἴσας εὐθείας τρίγωνον συστήσασθαι.



Έστωσαν τρεῖς γωνίαι ἐπίπεδοι αί ὑπὸ ABΓ, ΔΕΖ, HΘK, ῶν αί δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αί μὲν ὑπὸ ABΓ, ΔΕΖ τῆς ὑπὸ HΘK, αἱ δὲ ὑπὸ ΔΕΖ, HΘK τῆς ὑπὸ ABΓ, καὶ ἔτι αἱ ὑπὸ HΘK, ABΓ τῆς ὑπὸ ΔΕΖ, καὶ ἔστωσαν ἴσαι αἱ AB, BΓ, ΔΕ, EZ, HΘ, ΘΚ εὐθεῖαι, καὶ ἐπεζεύχθωσαν αἱ AΓ, ΔΖ, HK λέγω, ὅτι δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς AΓ, ΔΖ, HK τρίγωνον συστήσασθαι, τουτέστιν ὅτι τῶν AΓ, ΔΖ, HK δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν.

Εἰ μὲν οὖν αἱ ὑπὸ ABΓ, ΔΕΖ, ΗΘΚ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι καὶ τῶν AΓ, ΔΖ, ΗΚ ἴσων γινομένων δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς AΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. εἰ δὲ οὐ, ἔστωσαν ἄνισοι, καὶ συνεστάτω πρὸς τῆ ΘΚ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Θ τῆ ὑπὸ ABΓ γωνία ἴση ἡ ὑπὸ ΚΘΛ· καὶ κείσθω μιῷ τῶν AB, BΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἴση ἡ ΘΛ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΗΛ. καὶ ἐπεὶ δύο αἱ AB, BΓ δυσὶ ταῖς KΘ, ΘΛ ἴσαι εἰσίν, καὶ γωνία ἡ πρὸς τῷ Β γωνία τῆ ὑπὸ ΚΘΛ ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῆ ΚΛ ἴση. καὶ ἐπεὶ αἱ ὑπὸ ABΓ, ΗΘΚ τῆς

Proposition 22

If there are three plane angles, of which (the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way), and if equal straight-lines contain them, then it is possible to construct a triangle from (the straight-lines created by) joining the (ends of the) equal straight-lines.



Let ABC, DEF, and GHK be three plane angles, of which the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way)—(that is), ABC and DEF (greater) than GHK, DEF and GHK (greater) than ABC, and, further, GHKand ABC (greater) than DEF. And let AB, BC, DE, EF, GH, and HK be equal straight-lines. And let AC, DF, and GK have been joined. I say that that it is possible to construct a triangle out of (straight-lines) equal to AC, DF, and GK—that is to say, that (the sum of) any two of AC, DF, and GK is greater than the remaining (one).

Now, if the angles ABC, DEF, and GHK are equal to one another then (it is) clear that, (with) AC, DF, and GK also becoming equal, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. And if not, let them be unequal, and let KHL, equal to angle ABC, have been constructed on the straight-line HK, at the point H on it. And let HL be made equal to ύπὸ ΔΕΖ μείζονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΒΓ τῆ ὑπὸ ΚΘΛ, ἡ ἄρα ὑπὸ ΗΘΛ τῆς ὑπὸ ΔΕΖ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΗΘ, ΘΛ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΗΘΛ γωνίας τῆς ὑπὸ ΔΕΖ μείζων, βάσις ἄρα ἡ ΗΛ βάσεως τῆς ΔΖ μείζων ἐστίν. ἀλλὰ αἱ ΗΚ, ΚΛ τῆς ΗΛ μείζονές εἰσιν. πολλῷ ἄρα αἱ ΗΚ, ΚΛ τῆς ΔΖ μείζονές εἰσιν. ἴση δὲ ἡ ΚΛ τῆ ΑΓ· αἱ ΑΓ, ΗΚ ἄρα τῆς λοιπῆς τῆς ΔΖ μείζονές εἰσιν. ὁμοίως δὴ δείζομεν, ὅτι καὶ αἱ μὲν ΑΓ, ΔΖ τῆς ΗΚ μείζονές εἰσιν, καὶ ἕτι αἱ ΔΖ, ΗΚ τῆς ΑΓ μείζονές εἰσιν. δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι· ὅπερ ἕδει δεῖξαι.

*κ*γ΄.

Έκ τριῶν γωνιῶν ἐπιπέδων, ῶν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, στερεὰν γωνίαν συστήσασθαι δεῖ δὴ τὰς τρεῖς τεσσάρων ὀρθῶν ἐλάσςονας εἶναι.



Έστωσαν αί δοθεϊσαι τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ABΓ, ΔΕΖ, ΗΘΚ, ῶν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, ἔτι δὲ αἰ τρεῖς τεσσάρων ὀρθῶν ἐλάσσονες· δεῖ δὴ ἐχ τῶν ἴσων ταῖς ὑπὸ ABΓ, ΔΕΖ, ΗΘΚ στερεὰν γωνίαν συστήσασθαι.

Άπειλήφθωσαν ίσαι αί AB, BΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, καὶ ἐπεζεύχθωσαν αί ΑΓ, ΔΖ, ΗΚ· δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. συνεστάτω τὸ ΛΜΝ, ὥστε ἴσην εἶναι τὴν μὲν ΑΓ τῆ ΛΜ, τὴν δὲ ΔΖ τῆ ΜΝ, καὶ ἔτι τὴν ΗΚ τῆ ΝΛ, καὶ περιγεγράφθω περὶ τὸ ΛΜΝ τρίγωνον κύκλος ὁ ΛΜΝ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον καὶ ἔστω τὸ Ξ, καὶ ἐπεζεύχθωσαν αἱ ΛΞ, ΜΞ, ΝΞ·

one of AB, BC, DE, EF, GH, and HK. And let KLand GL have been joined. And since the two (straightlines) AB and BC are equal to the two (straight-lines) KH and HL (respectively), and the angle at B (is) equal to KHL, the base AC is thus equal to the base KL[Prop. 1.4]. And since (the sum of) ABC and GHKis greater than DEF, and ABC equal to KHL, GHLis thus greater than DEF. And since the two (straightlines) GH and HL are equal to the two (straight-lines) DE and EF (respectively), and angle GHL (is) greater than DEF, the base GL is thus greater than the base DF[Prop. 1.24]. But, (the sum of) GK and KL is greater than GL [Prop. 1.20]. Thus, (the sum of) GK and KL is much greater than DF. And KL (is) equal to AC. Thus, (the sum of) AC and GK is greater than the remaining (straight-line) DF. So, similarly, we can show that (the sum of) AC and DF is greater than GK, and, further, that (the sum of) DF and GK is greater than AC. Thus, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. (Which is) the very thing it was required to show.

Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].



Let ABC, DEF, and GHK be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to ABC, DEF, and GHK.

Let AB, BC, DE, EF, GH, and HK be cut off (so as to be) equal (to one another). And let AC, DF, and GK have been joined. It is, thus, possible to construct a triangle from (straight-lines) equal to AC, DF, and GK[Prop. 11.22]. Let (such a triangle), LMN, have be constructed, such that AC is equal to LM, DF to MN, and, further, GK to NL. And let the circle LMN have been circumscribed about triangle LMN [Prop. 4.5]. And let



Λέγω, ὅτι ἡ ΑΒ μείζων ἐστὶ τῆς ΛΞ. εἰ γὰρ μή, ἤτοι ίση ἐστὶν ἡ ΑΒ τῆ ΑΞ ἢ ἐλάττων. ἔστω πρότερον ἴση. καὶ έπεὶ ἴση ἐστὶν ἡ ΑΒ τῆ ΛΞ, ἀλλὰ ἡ μὲν ΑΒ τῆ ΒΓ ἐστιν ἴση, ἡ δ
ὲ ΞΛ τỹ ΞΜ, δύο δὴ αἰ AB, BΓ δύο ταῖς ΛΞ, ΞΜ ίσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ βάσις ἡ ΑΓ βάσει τῆ ΛΜ ύπόχειται ίση· γωνία άρα ή ύπὸ ΑΒΓ γωνία τῆ ὑπὸ ΛΞΜ ἐστιν ἴση. διὰ τὰ
 αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΔEZ τ
 ỹ ὑπὸ MEN έστιν ἴση, καὶ ἔτι ἡ ὑπὸ ΗΘΚ τῆ ὑπὸ ΝΞΛ· αἱ ἄρα τρεῖς αἱ ύπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι τρισὶ ταῖς ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ είσιν ίσαι. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τέτταρσιν ὀρθαῖς εἰσιν ἴσαι· καὶ αἱ τρεῖς ἄρα αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τέτταρσιν ὀρθαῖς ἴσαι εἰσίν. ὑπόχεινται δὲ χαὶ τεσσάρων ὀρθῶν ἐλάσσονες· ὅπερ ἄτοπον. οὐκ ἄρα ἡ AB τῆ $\Lambda \Xi$ ἴση ἐστίν. λέγω δή, ὅτι ούδὲ ἐλάττων ἐστὶν ἡ AB τῆς ΛΞ. εἰ γὰρ δυνατόν, ἔστω· καὶ κείσθω τῆ μὲν ΑΒ ἴση ἡ ΞΟ, τῆ δὲ ΒΓ ἴση ἡ ΞΠ, καὶ ἐπεζεύχθω ἡ ΟΠ. καὶ ἐπεὶ ἴση ἐστὶν ή ΑΒ τῆ ΒΓ, ἴση ἐστὶ καὶ ἡ ΞΟ τῆ ΞΠ· ὥστε καὶ λοιπὴ ἡ ΛΟ τῆ ΠΜ ἐστιν ἴση. παράλληλος ἄρα ἐστὶν ἡ ΛΜ τῆ ΟΠ, καὶ ἰσογώνιον τὸ ΛΜΞ τῷ ΟΠΞ· ἔστιν ἄρα ὡς ἡ ΞΛ πρὸς ΛM , οὕτως ἡ ΞΟ πρὸς ΟΠ· ἐναλλὰξ ὡς ἡ $\Lambda Ξ$ πρὸς ΞΟ, οὕτως
ἡ ΛΜ πρὸς ΟΠ. μείζων δὲ ἡ ΛΞ τῆς ΞΟ· μείζων ἄρα καὶ ἡ ΛΜ τῆς ΟΠ. ἀλλὰ ἡ ΛΜ κεῖται τῆ ΑΓ ἴση· καὶ ἡ ΑΓ άρα τῆς ΟΠ μείζων ἐστίν. ἐπεὶ οὖν δύο αἱ AB, BΓ δυσὶ ταῖς ΟΞ, ΞΠ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσεως τῆς ΟΠ μείζων έστίν, γωνία ἄρα ή ὑπὸ ΑΒΓ γωνίας τῆς ὑπὸ ΟΞΠ μεῖζων έστίν. ὑμοίως δὴ δείξομεν, ὅτι καὶ ἡ μὲν ὑπὸ ΔΕΖ τῆς ὑπὸ ΜΞΝ μείζων ἐστίν, ἡ δὲ ὑπὸ ΗΘΚ τῆς ὑπὸ ΝΞΛ. αἱ ἄρα τρεῖς γωνίαι αἱ ὑπὸ ABΓ, ΔΕΖ, ΗΘΚ τριῶν τῶν ὑπὸ ΛΞΜ, MΞN, NΞΛ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ ABΓ, ΔΕΖ, HΘK τεσσάρων όρθῶν ἐλάσσονες ὑπόχεινται· πολλῷ ἄρα αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν. ἀλλὰ καὶ ἴσαι· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ ΑΒ ἐλάσσων ἐστὶ τῆς $\Lambda \Xi$. ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἡ AB τῆς $\Lambda \Xi$.

Άνεστάτω δὴ ἀπὸ τοῦ Ξ σημείου τῷ τοῦ ΛΜΝ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΞΡ, καὶ ῷ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τετράγωνον τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ

its center have been found, and let it be (at) *O*. And let *LO*, *MO*, and *NO* have been joined.



I say that AB is greater than LO. For, if not, AB is either equal to, or less than, LO. Let it, first of all, be equal. And since AB is equal to LO, but AB is equal to BC, and OL to OM, so the two (straight-lines) AB and BC are equal to the two (straight-lines) LO and OM, respectively. And the base AC was assumed (to be) equal to the base LM. Thus, angle ABC is equal to angle LOM [Prop. 1.8]. So, for the same (reasons), DEF is also equal to MON, and, further, GHK to NOL. Thus, the three angles ABC, DEF, and GHK are equal to the three angles LOM, MON, and NOL, respectively. But, the (sum of the) three angles LOM, MON, and NOL is equal to four right-angles. Thus, the (sum of the) three angles ABC, DEF, and GHK is also equal to four rightangles. And it was also assumed (to be) less than four right-angles. The very thing (is) absurd. Thus, AB is not equal to LO. So, I say that AB is not less than LOeither. For, if possible, let it be (less). And let OP be made equal to AB, and OQ equal to BC, and let PQhave been joined. And since AB is equal to BC, OPis also equal to OQ. Hence, the remainder LP is also equal to (the remainder) QM. LM is thus parallel to PQ[Prop. 6.2], and (triangle) LMO (is) equiangular with (triangle) PQO [Prop. 1.29]. Thus, as OL is to LM, so OP (is) to PQ [Prop. 6.4]. Alternately, as LO (is) to OP, so LM (is) to PQ [Prop. 5.16]. And LO (is) greater than *OP*. Thus, LM (is) also greater than PQ [Prop. 5.14]. But LM was made equal to AC. Thus, AC is also greater than PQ. Therefore, since the two (straight-lines) ABand BC are equal to the two (straight-lines) PO and OQ(respectively), and the base AC is greater than the base PQ, the angle ABC is thus greater than the angle POQ[Prop. 1.25]. So, similarly, we can show that DEF is also greater than MON, and GHK than NOL. Thus, the (sum of the) three angles ABC, DEF, and GHK is greater than the (sum of the) three angles LOM, MON, τῆς ΞΡ, καὶ ἐπεζεύχθωσαν αἱ ΡΛ, ΡΜ, ΡΝ.

Καὶ ἐπεὶ ἡ ΡΞ ὀρθὴ ἐστι πρὸς τὸ τοῦ ΛΜΝ κύκλου έπίπεδον, καὶ πρὸς ἑκάστην ἄρα τῶν ΛΞ, ΜΞ, ΝΞ ὀρθή έστιν ή ΡΞ. καὶ ἐπεὶ ἴση ἐστὶν ή ΛΞ τῆ ΞΜ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΞΡ, βάσις ἄρα ἡ ΡΛ βάσει τὴ ΡΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΡΝ ἑκατέρα τῶν ΡΛ, ΡΜ ἐστιν ἴση· αί τρεῖς ἄρα αί ΡΛ, ΡΜ, ΡΝ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ $\ddot{\omega}$ με
ῖζόν ἐστι τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς AΞ, ἐχείν ω ἴσον ύπόχειται τὸ ἀπὸ τῆς ΞΡ, τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς άπὸ τῶν ΛΞ, ΞΡ. τοῖς δὲ ἀπὸ τῶν ΛΞ, ΞΡ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΛΡ· ὀρθὴ γὰρ ἡ ὑπὸ ΛΞΡ· τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς PA· ἴση ἄρα ἡ AB τῆ PA. ἀλλὰ τῃ μὲν AB ἴση ἐστὶν ἑκάστη τῶν $B\Gamma$, ΔE , EZ, $H\Theta$, ΘK , τῆ δὲ $P\Lambda$ ἴση έκατέρα τῶν PM, PN· ἑκάστη ἄρα τῶν AB, BΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἑκάστη τῶν ΡΛ, ΡΜ, ΡΝ ἴση ἐστίν. καὶ ἐπεὶ δύο αί ΛΡ, ΡΜ δυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΛΜ βάσει τῆ ΑΓ ὑπόκειται ἴση, γωνία ἄρα ἡ ὑπὸ ΛΡΜ γωνία τῆ ὑπὸ ΑΒΓ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΜΡΝ τῆ ὑπὸ ΔΕΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΛΡΝ τῆ ὑπὸ ΗΘΚ.

Έκ τριῶν ἄρα γωνιῶν ἐπιπέδων τῶν ὑπὸ ΛΡΜ, ΜΡΝ, ΛΡΝ, αἴ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, στερεὰ γωνία συνέσταται ἡ πρὸς τῷ Ρ περιεχομένη ὑπὸ τῶν ΛΡΜ, MPN, ΛΡΝ γωνιῶν· ὅπερ ἔδει ποιῆσαι. and *NOL*. But, (the sum of) *ABC*, *DEF*, and *GHK* was assumed (to be) less than four right-angles. Thus, (the sum of) LOM, MON, and NOL is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, *AB* is not less than *LO*. And it was shown (to be) not equal either. Thus, *AB* (is) greater than *LO*.

So let OR have been set up at point O at rightangles to the plane of circle LMN [Prop. 11.12]. And let the (square) on OR be equal to that (area) by which the square on AB is greater than the (square) on LO[Prop. 11.23 lem.]. And let RL, RM, and RN have been joined.

And since RO is at right-angles to the plane of circle LMN, RO is thus also at right-angles to each of LO, MO, and NO. And since LO is equal to OM, and ORis common and at right-angles, the base RL is thus equal to the base RM [Prop. 1.4]. So, for the same (reasons), RN is also equal to each of RL and RM. Thus, the three (straight-lines) RL, RM, and RN are equal to one another. And since the (square) on OR was assumed to be equal to that (area) by which the (square) on AB is greater than the (square) on LO, the (square) on ABis thus equal to the (sum of the squares) on LO and OR. And the (square) on LR is equal to the (sum of the squares) on LO and OR. For LOR (is) a right-angle [Prop. 1.47]. Thus, the (square) on AB is equal to the (square) on RL. Thus, AB (is) equal to RL. But, each of BC, DE, EF, GH, and HK is equal to AB, and each of RM and RN equal to RL. Thus, each of AB, BC, DE, EF, GH, and HK is equal to each of RL, RM, and RN. And since the two (straight-lines) LR and RMare equal to the two (straight-lines) AB and BC (respectively), and the base LM was assumed (to be) equal to the base AC, the angle LRM is thus equal to the angle ABC [Prop. 1.8]. So, for the same (reasons), MRN is also equal to DEF, and LRN to GHK.

Thus, the solid angle R, contained by the angles LRM, MRN, and LRN, has been constructed out of the three plane angles LRM, MRN, and LRN, which are equal to the three given (plane angles) ABC, DEF, and GHK (respectively). (Which is) the very thing it was required to do.



Όν δὲ τρόπον, ῷ μεῖζόν ἑστι τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς ΛΞ, ἑκείνῳ ἴσον λαβεῖν ἔστι τὸ ἀπὸ τῆς ΞΡ, δείξομεν οὕτως. ἐκκείσθωσαν αἱ AB, ΛΞ εὐθεῖαι, καὶ ἔστω μείζων ἡ AB, καὶ γεγράφθω ἐπ᾽ αὐτῆς ἡμικύκλιον τὸ ABΓ, καὶ εἰς τὸ ABΓ ἡμικύκλιον ἐνηρμόσθω τῆ ΛΞ εὐθεία μὴ μείζονι οὕσῃ τῆς AB διαμέτρου ἴση ἡ AΓ, καὶ ἐπεζεύχθω ἡ ΓB. ἐπεὶ οὕν ἐν ἡμικυκλίῳ τῷ AΓB γωνία ἐστὶν ἡ ὑπὸ AΓB, ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ AΓB. τὸ ἄρα ἀπὸ τῆς AB ἴσον ἐστὶ τοῖς ἀπὸ τῶν AΓ, ΓB. ὥστε τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς AΓ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓB. ἴση δὲ ἡ AΓ τῆ ΛΞ. τὸ ἄρα ἀπὸ τῆς AB τοῦ ἀπὸ τῆς ΛΞ μεῖζόν ἐστι τῷ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς ΛΞ μεῖζον τῷ ἀπὸ τῆς ΞΡ. ὅπερ προέκειτο ποιῆσαι.

χδ'.

Έὰν στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.



Στερεὸν γὰρ τὸ ΓΔΘΗ ὑπὸ παραλλήλων ἐπιπέδων περιεχέσθω τῶν ΑΓ, ΗΖ, ΑΘ, ΔΖ, ΒΖ, ΑΕ· λέγω, ὅτι τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

Έπεὶ γὰρ δύο ἐπίπεδα παράλληλα τὰ BH, ΓΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αί κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῆ ΔΓ. πάλιν, ἑπεὶ δύο ἐπίπεδα παράλληλα τὰ BZ, ΑΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αί κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.





And we can demonstrate, thusly, in which manner to take the (square) on OR equal to that (area) by which the (square) on AB is greater than the (square) on LO. Let the straight-lines AB and LO be set out, and let ABbe greater, and let the semicircle ABC have been drawn around it. And let AC, equal to the straight-line LO, which is not greater than the diameter AB, have been inserted into the semicircle ABC [Prop. 4.1]. And let CB have been joined. Therefore, since the angle ACBis in the semicircle ACB, ACB is thus a right-angle [Prop. 3.31]. Thus, the (square) on AB is equal to the (sum of the) squares on AC and CB [Prop. 1.47]. Hence, the (square) on AB is greater than the (square) on ACby the (square) on CB. And AC (is) equal to LO. Thus, the (square) on AB is greater than the (square) on LO by the (square) on CB. Therefore, if we take OR equal to BC then the (square) on AB will be greater than the (square) on LO by the (square) on OR. (Which is) the very thing it was prescribed to do.

Proposition 24

If a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic.



For let the solid (figure) CDHG have been contained by the parallel planes AC, GF, and AH, DF, and BF, AE. I say that its opposite planes are both equal and parallelogrammic.

For since the two parallel planes BG and CE are cut by the plane AC, their common sections are parallel [Prop. 11.16]. Thus, AB is parallel to DC. Again, since the two parallel planes BF and AE are cut by the plane

παράλληλος ἄρα ἐστὶν ἡ ΒΓ τῆ ΑΔ. ἐδείχθη δὲ καὶ ἡ ΑΒ τῆ ΔΓ παράλληλος· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν ΔΖ, ΖΗ, ΗΒ, ΒΖ, ΑΕ παραλληλόγραμμόν ἐστιν.

Ἐπεζεύχθωσαν αἱ ΑΘ, ΔΖ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ μὲν AB τῆ ΔΓ, ἡ δὲ BΘ τῆ ΓΖ, δύο δὴ αἱ AB, BΘ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΓ, ΓΖ ἁπτομένας ἀλλήλων εἰσὶν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῷ· ἴσας ἄρα γωνίας περιέξουσιν· ἴση ἄρα ἡ ὑπὸ ABΘ γωνία τῆ ὑπὸ ΔΓΖ. καὶ ἐπεὶ δύο αἱ AB, BΘ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ABΘ γωνία τῆ ὑπὸ ΔΓΖ ἐστιν ἴση, βάσις ἄρα ἡ ΑΘ βάσει τῆ ΔΖ ἐστιν ἴση, καὶ τὸ ABΘ τρίγωνον τῷ ΔΓΖ τριγώνῷ ἴσον ἐστίν. καί ἐστι τοῦ μὲν ABΘ διπλάσιον τὸ BH παραλληλόγραμμον, τοῦ δὲ ΔΓΖ διπλάσιον τὸ ΓΕ παραλληλόγραμμον· ἴσον ἄρα τὸ BH παραλληλόγραμμον τῷ ΓΕ παραλληλογράμμῷ· ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ μὲν ΑΓ τῷ HZ ἐστιν ἴσον, τὸ δὲ AE τῶ BZ.

Έὰν ἄρα στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν· ὅπερ ἔδει δεῖξαι. AC, their common sections are parallel [Prop. 11.16]. Thus, BC is parallel to AD. And AB was also shown (to be) parallel to DC. Thus, AC is a parallelogram. So, similarly, we can also show that DF, FG, GB, BF, and AEare each parallelograms.

Let AH and DF have been joined. And since AB is parallel to DC, and BH to CF, so the two (straight-lines) joining one another, AB and BH, are parallel to the two straight-lines joining one another, DC and CF (respectively), not (being) in the same plane. Thus, they will contain equal angles [Prop. 11.10]. Thus, angle ABH(is) equal to (angle) DCF. And since the two (straightlines) AB and BH are equal to the two (straight-lines) DC and CF (respectively) [Prop. 1.34], and angle ABHis equal to angle DCF, the base AH is thus equal to the base DF, and triangle ABH is equal to triangle DCF[Prop. 1.4]. And parallelogram BG is double (triangle) ABH, and parallelogram CE double (triangle) DCF[Prop. 1.34]. Thus, parallelogram BG (is) equal to parallelogram CE. So, similarly, we can show that AC is also equal to GF, and AE to BF.

Thus, if a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic. (Which is) the very thing it was required to show.

Proposition 25

If a parallelipiped solid is cut by a plane which is parallel to the opposite planes (of the parallelipiped) then as the base (is) to the base, so the solid will be to the solid.



For let the parallelipiped solid ABCD have been cut by the plane FG which is parallel to the opposite planes RA and DH. I say that as the base AEFV (is) to the base EHCF, so the solid ABFU (is) to the solid EGCD.

For let AH have been produced in each direction. And let any number whatsoever (of lengths), AK and KL, be made equal to AE, and any number whatsoever (of lengths), HM and MN, equal to EH. And let the parallelograms LP, KV, HW, and MS have been completed,

κε΄.

Έὰν στερεὸν παραλληλεπίπεδον ἐπιπέδω τμηθῆ παραλλήλῷ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ἡ βάσις πρὸς τὴν βάσιν, οὕτως τὸ στερεὸν πρὸς τὸ στερεόν.



Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒΓΔ ἐπιπέδῳ τῷ ΖΗ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΡΑ, ΔΘ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕΖΦ βάσις πρὸς τὴν ΕΘΓΖ βάσιν, οὕτως τὸ ΑΒΖΥ στερεὸν πρὸς τὸ ΕΗΓΔ στερεόν.

Έκβεβλήσθω γὰρ ἡ ΑΘ ἐφ' ἑκάτερα τὰ μέρη, καὶ κείσθωσαν τῆ μὲν ΑΕ ἴσαι ὁσαιδηποτοῦν αἱ ΑΚ, ΚΛ, τῆ δὲ ΕΘ ἴσαι ὁσαιδηποτοῦν αἱ ΘΜ, ΜΝ, καὶ συμπεπληρώσθω τὰ ΛΟ, ΚΦ, ΘΧ, ΜΣ παραλληλόγραμμα καὶ τὰ ΛΠ, ΚΡ,

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΛΚ, ΚΑ, ΑΕ εὐθεῖαι ἀλλήλαις, ἴσα έστὶ καὶ τὰ μὲν ΛΟ, ΚΦ, ΑΖ παραλληλόγραμμα ἀλλήλοις, τὰ δὲ KΞ, KB, AH ἀλλήλοις καὶ ἔτι τὰ Λ $\Psi,$ KΠ, AP ἀλλήλοις[.] άπεναντίον γάρ. διὰ τὰ αὐτὰ δὴ καὶ τὰ μὲν ΕΓ, ΘΧ, ΜΣ παραλληλόγραμμα ίσα εἰσὶν ἀλλήλοις, τὰ δὲ ΘΗ, ΘΙ, ΙΝ ίσα εἰσὶν ἀλλήλοις, καὶ ἔτι τὰ $\Delta\Theta$, MΩ, NT· τρία ἄρα ἐπίπεδα τῶν ΛΠ, ΚΡ, ΑΥ στερεῶν τρισὶν ἐπιπέδοις ἐστὶν ἴσα. ἀλλὰ τὰ τρία τρισὶ τοῖς ἀπεναντίον ἐστὶν ἴσα· τὰ ἄρα τρία στερεὰ τὰ ΛΠ, ΚΡ, ΑΥ ἴσα ἀλλήλοις ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ τὰ τρία στερεὰ τὰ $\rm E\Delta, \ \Delta M, \ MT$ ἴσα ἀλλήλοις ἐστίν· ὑσαπλασίων ἄρα ἐστὶν ἡ ΛΖ βάσις τῆς ΑΖ βάσεως, τοσαυταπλάσιόν έστι και τὸ ΛΥ στερεὸν τοῦ ΑΥ στερεοῦ. διὰ τὰ αὐτὰ δὴ ὑσαπλασίων ἐστὶν ἡ ΝΖ βάσις τῆς ΖΘ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΝΥ στερεὸν τοῦ ΘΥ στερεοῦ. καὶ εἰ ἴση ἐστὶν ἡ ΛΖ βάσις τῇ ΝΖ βάσει, ἴσον ἐστὶ καὶ τὸ ΛΥ στερεὸν τῷ ΝΥ στερεῷ, καὶ εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΝΖ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ στερεοῦ, και ει έλλείπει, έλλείπει. τεσσάρων δη όντων μεγεθῶν, δύο μέν βάσεων τῶν ΑΖ, ΖΘ, δύο δὲ στερεῶν τῶν ΑΥ, ΥΘ, εἴληπται ἰσάχις πολλαπλάσια τῆς μὲν ΑΖ βάσεως χαὶ τοῦ ΑΥ στερεοῦ ἤ τε ΛΖ βάσις καὶ τὸ ΛΥ στερεόν, τῆς δὲ ΘΖ βάσεως καὶ τοῦ ΘΥ στερεοῦ ἤ τε ΝΖ βάσις καὶ τὸ ΝΥ στερεόν, καὶ δέδεικται, ὅτι εἱ ὑπερέχει ἡ ΛΖ βάσις τῆς ΖΝ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ [στερεοῦ], καὶ εἰ ἴση, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. ἔστιν ἄρα ὡς ἡ ΑΖ βάσις πρός την ΖΘ βάσιν, οὕτως τὸ ΑΥ στερεὸν πρὸς τὸ ΥΘ στερεόν ὅπερ ἔδει δεῖξαι.

and the solids LQ, KR, DM, and MT.

And since the straight-lines LK, KA, and AE are equal to one another, the parallelograms LP, KV, and AF are also equal to one another, and KO, KB, and AG(are equal) to one another, and, further, LX, KQ, and AR (are equal) to one another. For (they are) opposite [Prop. 11.24]. So, for the same (reasons), the parallelograms EC, HW, and MS are also equal to one another, and HG, HI, and IN are equal to one another, and, further, DH, MY, and NT (are equal to one another). Thus, three planes of (one of) the solids LQ, KR, and AU are equal to the (corresponding) three planes (of the others). But, the three planes (in one of the soilds) are equal to the three opposite planes [Prop. 11.24]. Thus, the three solids LQ, KR, and AU are equal to one another [Def. 11.10]. So, for the same (reasons), the three solids ED, DM, and MT are also equal to one another. Thus, as many multiples as the base LF is of the base AF, so many multiples is the solid LU also of the the solid AU. So, for the same (reasons), as many multiples as the base NF is of the base FH, so many multiples is the solid NUalso of the solid HU. And if the base LF is equal to the base NF then the solid LU is also equal to the solid NU.[†] And if the base LF exceeds the base NF then the solid LU also exceeds the solid NU. And if (LF) is less than (NF) then (LU) is (also) less than (NU). So, there are four magnitudes, the two bases AF and FH, and the two solids AU and UH, and equal multiples have been taken of the base AF and the solid AU— (namely), the base LF and the solid LU—and of the base HF and the solid HU—(namely), the base NF and the solid NU. And it has been shown that if the base LF exceeds the base FNthen the solid LU also exceeds the [solid] NU, and if (LF is) equal (to FN) then (LU is) equal (to NU), and if (LF is) less than (FN) then (LU is) less than (NU). Thus, as the base AF is to the base FH, so the solid AU(is) to the solid UH [Def. 5.5]. (Which is) the very thing it was required to show.

[†] Here, Euclid assumes that $LF \stackrel{>}{=} NF$ implies $LU \stackrel{>}{=} NU$. This is easily demonstrated.

χς'.

Πρὸς τῆ δοθείση εὐθεία καὶ τῷ πρὸς αὐτῆ σημείῳ τῆ δοθείση στερεᾶ γωνία ἴσην στερεὰν γωνίαν συστήσασθαι.

Έστω ή μὲν δοθεῖσα εὐθεῖα ή AB, τὸ δὲ πρὸς αὐτῆ δοθὲν σημεῖον τὸ A, ή δὲ δοθεῖσα στερεὰ γωνία ή πρὸς τῷ Δ περιεχομένη ὑπὸ τῶν ὑπὸ ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν ἐπιπέδων· δεῖ δὴ πρὸς τῆ AB εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ A τῆ πρὸς τῷ Δ στερεῷ γωνία ἴσην στερεὰν γωνίαν συστήσασθαι.

Proposition 26

To construct a solid angle equal to a given solid angle on a given straight-line, and at a given point on it.

Let AB be the given straight-line, and A the given point on it, and D the given solid angle, contained by the plane angles EDC, EDF, and FDC. So, it is necessary to construct a solid angle equal to the solid angle D on the straight-line AB, and at the point A on it.



Εἰλήφθω γὰρ ἐπὶ τῆς ΔΖ τυχὸν σημεῖον τὸ Ζ, καὶ ἦχθω ἀπὸ τοῦ Ζ ἐπὶ τὸ διὰ τῶν ΕΔ, ΔΓ ἐπίπεδον κάθετος ἡ ΖΗ, καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΔΗ, καὶ συνεστάτω πρὸς τῆ ΑΒ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείῳ τῷ Α τῆ μὲν ὑπὸ ΕΔΓ γωνία ἴση ἡ ὑπὸ ΒΑΛ, τῆ δὲ ὑπὸ ΕΔΗ ἴση ἡ ὑπὸ ΒΑΚ, καὶ κείσθω τῆ ΔΗ ἴση ἡ ΑΚ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ διὰ τῶν ΒΑΛ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΚΘ, καὶ κείσθω ἴση τῆ ΗΖ ἡ ΚΘ, καὶ ἐπεζεύχθω ἡ ΘΑ· λέγω, ὅτι ἡ πρὸς τῷ Α στερεὰ γωνία περιεχομένη ὑπὸ τῶν ΒΑΛ, ΒΑΘ, ΘΑΛ γωνιῶν ἴση ἐστὶ τῆ πρὸς τῷ Δ στερεῷ γωνία τῆ περιεχομένη ὑπὸ τῶν ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν.

Άπειλήφθωσαν γὰρ ἴσαι αἱ ΑΒ, ΔΕ, καὶ ἐπεζεύχθωσαν αί ΘΒ, ΚΒ, ΖΕ, ΗΕ. και ἐπει ἡ ΖΗ ὀρθή ἐστι πρὸς τὸ ύποχείμενον ἐπίπεδον, χαὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας όρθη άρα έστιν έχατέρα τῶν ὑπὸ ΖΗΔ, ΖΗΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΘΚΑ, ΘΚΒ γωνιῶν ὀρθή ἐστιν. καὶ ἐπεὶ δύο αἱ ΚΑ, ΑΒ δύο ταῖς ΗΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ή ΚΒ βάσει τῆ ΗΕ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΚΘ τῆ ΗΖ ἴση· καὶ γωνίας ὀρθὰς περιέχουσιν· ἴση άρα καὶ ἡ ΘΒ τῆ ΖΕ. πάλιν ἐπεὶ δύο αἱ ΑΚ, ΚΘ δυσὶ ταῖς ΔH , HZ ἴσαι εἰσίν, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ή ΑΘ βάσει τῆ Ζ Δ ἴση ἐστίν. ἔστι δὲ καὶ ή AB τῆ $\Delta \mathrm{E}$ ἴση $^{\cdot}$ δύο δη αί ΘA , AB δύο ταῖς ΔZ , ΔE ἴσαι εἰσίν. καὶ βάσις ἡ ΘΒ βάσει τῆ ΖΕ ἴση· γωνία ἄρα ἡ ὑπὸ ΒΑΘ γωνία τῆ ὑπὸ $\rm E\Delta Z$ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ $\Theta A\Lambda$ τῆ ὑπὸ $\rm Z\Delta \Gamma$ έστιν ίση.
 έστι δὲ καὶ ἡ ὑπὸ ${\rm BAA}$ τῆ ὑπὸ ${\rm E}\Delta\Gamma$ ίση.

Πρὸς ἄρα τῆ δοθείση εὐθεία τῆ AB xaì τῷ πρὸς αὐτῆ σημεί
ώ τῷ A τῆ δοθείση στερεᾶ γωνία τῆ πρὸς τῷ Δ ἴση συνέσταται· ὅπερ ἕδει ποιῆσαι.



For let some random point F have been taken on DF, and let FG have been drawn from F perpendicular to the plane through ED and DC [Prop. 11.11], and let it meet the plane at G, and let DG have been joined. And let BAL, equal to the angle EDC, and BAK, equal to EDG, have been constructed on the straight-line AB at the point A on it [Prop. 1.23]. And let AK be made equal to DG. And let KH have been set up at the point K at right-angles to the plane through BAL [Prop. 11.12]. And let KH be made equal to GF. And let HA have been joined. I say that the solid angle at A, contained by the (plane) angles BAL, BAH, and HAL, is equal to the solid angle at D, contained by the (plane) angles EDC, EDF, and FDC.

For let AB and DE have been cut off (so as to be) equal, and let HB, KB, FE, and GE have been joined. And since FG is at right-angles to the reference plane (EDC), it will also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Thus, the angles FGD and FGE are right-angles. So, for the same (reasons), the angles HKA and HKB are also right-angles. And since the two (straight-lines) KA and AB are equal to the two (straight-lines) GD and DE, respectively, and they contain equal angles, the base KB is thus equal to the base GE [Prop. 1.4]. And KH is also equal to GF. And they contain right-angles (with the respective bases). Thus, HB (is) also equal to FE [Prop. 1.4]. Again, since the two (straight-lines) AK and KH are equal to the two (straight-lines) DG and GF (respectively), and they contain right-angles, the base AH is thus equal to the base FD [Prop. 1.4]. And AB (is) also equal to DE. So, the two (straight-lines) HA and AB are equal to the two (straight-lines) DF and DE (respectively). And the base HB (is) equal to the base FE. Thus, the angle BAH is equal to the angle EDF [Prop. 1.8]. So, for the same (reasons), HAL is also equal to FDC. And BAL is also equal to EDC.

Thus, (a solid angle) has been constructed, equal to the given solid angle at D, on the given straight-line AB,

χζ΄.

Άπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι στερεῷ παραλληλεπιπέδῷ ὅμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Έστω ή μὲν δοθεῖσα εὐθεῖα ή AB, τὸ δὲ δοθὲν στερεὸν παραλληλεπίπεδον τὸ ΓΔ· δεῖ δὴ ἀπὸ τῆς δοθείσης εὐθείας τῆς AB τῷ δοθέντι στερεῷ παραλληλεπιπέδω τῷ ΓΔ ὄμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Συνεστάτω γὰρ πρὸς τῆ AB εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ A τῆ πρὸς τῷ Γ στερεῷ γωνία ἴση ἡ περιεχομένη ὑπὸ τῶν ΒΑΘ, ΘΑΚ, ΚΑΒ, ὥστε ἴσην εἶναι τὴν μὲν ὑπὸ ΒΑΘ γωνίαν τῆ ὑπὸ ΕΓΖ, τὴν δὲ ὑπὸ ΒΑΚ τῆ ὑπὸ ΕΓΗ, τὴν δὲ ὑπὸ ΚΑΘ τῆ ὑπὸ ΗΓΖ[.] καὶ γεγονέτω ὡς μὲν ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, ὡς δὲ ἡ ΗΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΚΑ πρὸς τὴν ΑΘ. καὶ δι' ἴσου ἄρα ἐστὶν ὡς ἡ ΕΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΘ. καὶ συμπεπληρώσθω τὸ ΘΒ παραλληλόγραμμον καὶ τὸ ΑΛ στερεόν.



Καὶ ἐπεί ἐστιν ὡς ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΕΓΗ, ΒΑΚ αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΗΕ παραλληλόγραμμον τῷ ΚΒ παραλληλογράμμω. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΘ παραλληλόγραμμον τῷ ΗΖ παραλληλογράμμῷ ὅμοιόν ἐστι καὶ ἔτι τὸ ΖΕ τῷ ΘΒ· τρία ἄρα παραλληλόγραμμα τοῦ ΓΔ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΑΛ στερεοῦ ὅμοιά ἐστιν. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι ∞μοια. ὅλον ἄρα τὸ ΓΔ στερεὸν ὅλῷ τῷ ΑΛ στερεῷ ὅμοιόν ἐστιν.

Άπὸ τῆς δοθείσης ἄρα εὐθείας τῆς AB τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὅμοιόν τε καὶ ὑμοίως κείμενον ἀναγέγραπται τὸ ΑΛ· ὅπερ ἔδει ποιῆσαι.

at the given point A on it. (Which is) the very thing it was required to do.

Proposition 27

To describe a parallelepiped solid similar, and similarly laid out, to a given parallelepiped solid on a given straight-line.

Let the given straight-line be AB, and the given parallelepiped solid CD. So, it is necessary to describe a parallelepiped solid similar, and similarly laid out, to the given parallelepiped solid CD on the given straight-line AB.

For, let a (solid angle) contained by the (plane angles) BAH, HAK, and KAB have been constructed, equal to solid angle at C, on the straight-line AB at the point A on it [Prop. 11.26], such that angle BAH is equal to ECF, and BAK to ECG, and KAH to GCF. And let it have been contrived that as EC (is) to CG, so BA (is) to AK, and as GC (is) to CF, so KA (is) to AH [Prop. 6.12]. And thus, via equality, as EC is to CF, so BA (is) to AH [Prop. 5.22]. And let the parallelogram HB have been completed, and the solid AL.



And since as EC is to CG, so BA (is) to AK, and the sides about the equal angles ECG and BAK are (thus) proportional, the parallelogram GE is thus similar to the parallelogram KB. So, for the same (reasons), the parallelogram KH is also similar to the parallelogram GF, and, further, FE (is similar) to HB. Thus, three of the parallelograms of solid CD are similar to three of the parallelograms of solid AL. But, the (former) three are equal and similar to the three opposite, and the (latter) three are equal and similar to the three opposite. Thus, the whole solid CD is similar to the whole solid AL [Def. 11.9].

Thus, AL, similar, and similarly laid out, to the given parallelepiped solid CD, has been described on the given straight-lines AB. (Which is) the very thing it was required to do.

×η'.

Έὰν στερεὸν παραλληλεπίπεδον ἐπιπέδω τμηθῆ κατὰ

Proposition 28

If a parallelepiped solid is cut by a plane (passing)

τὸ στερεὸν ὑπὸ τοῦ ἐπιπέδου.



Στερεόν γὰρ παραλληλεπίπεδον τὸ ΑΒ ἐπιπέδω τῶ ΓΔΕΖ τετμήσθω κατά τὰς διαγωνίους τῶν ἀπεναντίον έπιπέδων τὰς ΓΖ, ΔΕ· λέγω, ὅτι δίγα τμηθήσεται τὸ ΑΒ στερεὸν ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου.

Έπεὶ γὰρ ἴσον ἐστὶ τὸ μὲν ΓΗΖ τρίγωνον τῷ ΓΖΒ τριγώνω, τὸ δὲ ΑΔΕ τῷ ΔΕΘ, ἔστι δὲ καὶ τὸ μὲν ΓΑ παραλληλόγραμμον τῷ ΕΒ ἴσον· ἀπεναντίον γάρ· τὸ δὲ ΗΕ τῷ ΓΘ, καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΓΗΖ, ΑΔΕ, τριῶν δὲ παραλληλογράμμων τῶν ΗΕ, ΑΓ, ΓΕ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένω ύπὸ δύο μὲν τριγώνων τῶν ΓΖΒ, ΔΕΘ, τριῶν δὲ παραλληλογράμμων τῶν ΓΘ, ΒΕ, ΓΕ· ὑπὸ γὰρ ἴσων ἐπιπέδων περιέχονται τῷ τε πλήθει καὶ τῷ μεγέθει. ὥστε ὅλον τὸ ΑΒ στερεὸν δίχα τέτμηται ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου· ὅπερ έδει δεῖξαι.

τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων, δίχα τμηθήσεται through the diagonals of (a pair of) opposite planes then the solid will be cut in half by the plane.



For let the parallelepiped solid AB have been cut by the plane *CDEF* (passing) through the diagonals of the opposite planes CF and DE.[†] I say that the solid AB will be cut in half by the plane CDEF.

For since triangle CGF is equal to triangle CFB, and ADE (is equal) to DEH [Prop. 1.34], and parallelogram CA is also equal to EB—for (they are) opposite [Prop. 11.24]—and GE (equal) to CH, thus the prism contained by the two triangles CGF and ADE, and the three parallelograms GE, AC, and CE, is also equal to the prism contained by the two triangles CFB and DEH, and the three parallelograms CH, BE, and CE. For they are contained by planes (which are) equal in number and in magnitude [Def. 11.10].^{\ddagger} Thus, the whole of solid ABis cut in half by the plane *CDEF*. (Which is) the very thing it was required to show.

[†] Here, it is assumed that the two diagonals lie in the same plane. The proof is easily supplied.

[‡] However, strictly speaking, the prisms are not similarly arranged, being mirror images of one another.

×ϑ'.

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.



Έστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλλη-

Proposition 29

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are on the same straight-lines, are equal to one another.



For let the parallelepiped solids CM and CN be on

λεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΖ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ ἐπὶ τῶν αὐτῶν εὐθειῶν ἔστωσαν τῶν ΖΝ, ΔΚ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Έπεὶ γὰρ παραλληλόγραμμόν ἐστιν ἑκάτερον τῶν ΓΘ, ΓΚ, ἴση ἐστὶν ἡ ΓΒ ἑκατέρα τῶν $\Delta\Theta$, ΕΚ· ὥστε καὶ ἡ ΔΘ τῆ ΕΚ ἐστιν ἴση. κοινὴ ἀφηρήσθω ἡ ΕΘ· λοιπὴ ἄρα ή ΔΕ λοιπῆ τῆ ΘΚ ἐστιν ἴση. ὤστε καὶ τὸ μὲν ΔΓΕ τρίγωνον τῷ ΘΒΚ τριγώνω ἴσον ἐστίν, τὸ δὲ ΔΗ παραλληλόγραμμον τῷ ΘΝ παραλληλογράμμω. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΖΗ τρίγωνον τῷ ΜΛΝ τριγώνῳ ἴσον ἐστίν. ἔστι δὲ καὶ τὸ μὲν ΓΖ παραλληλόγραμμον τῷ ΒΜ παραλληλογράμμω ίσον, τὸ δὲ ΓΗ τῷ BN· ἀπεναντίον γάρ· καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΑΖΗ, ΔΓΕ, τριῶν δὲ παραλληλογράμμων τῶν ΑΔ, ΔΗ, ΓΗ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένω ὑπὸ δύο μὲν τριγώνων τῶν ΜΛΝ, ΘΒΚ, τριῶν δὲ παραλληλογράμμων τῶν ΒΜ, ΘΝ, BN. κοινόν προσκείσθω τὸ στερεὸν, οὕ βάσις μὲν τὸ AB παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΘΜ· ὅλον ἄρα τὸ ΓΜ στερεόν παραλληλεπίπεδον ὅλω τῷ ΓΝ στερεῷ παραλληλεπιπέδω ίσον ἐστίν.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι. the same base AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AG, AF, LM, LN, CD, CE, BH, and BK, be on the same straight-lines, FN and DK. I say that solid CM is equal to solid CN.

For since CH and CK are each parallelograms, CBis equal to each of DH and EK [Prop. 1.34]. Hence, DH is also equal to EK. Let EH have been subtracted from both. Thus, the remainder DE is equal to the remainder HK. Hence, triangle DCE is also equal to triangle HBK [Props. 1.4, 1.8], and parallelogram DG to parallelogram HN [Prop. 1.36]. So, for the same (reasons), traingle AFG is also equal to triangle MLN. And parallelogram CF is also equal to parallelogram BM, and CG to BN [Prop. 11.24]. For they are opposite. Thus, the prism contained by the two triangles AFG and DCE, and the three parallelograms AD, DG, and CG, is equal to the prism contained by the two triangles MLNand HBK, and the three parallelograms BM, HN, and BN. Let the solid whose base (is) parallelogram AB, and (whose) opposite (face is) GEHM, have been added to both (prisms). Thus, the whole parallelepiped solid CMis equal to the whole parallelepiped solid CN.

Thus, parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up (are) on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

Proposition 30

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straightlines, are equal to one another.



λ'.

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐϑειῶν, ἴσα ἀλλήλοις ἐστίν.



Έστω ἐπὶ τῆς αὐτῆς βάσεως τῆς AB στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι αἱ AZ, AH, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ μὴ ἔστωσαν ἐπὶ τῶν

αὐτῶν εὐθειῶν· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐκβεβλήσθωσαν γὰρ αἱ ΝΚ, ΔΘ καὶ συμπιπτέτωσαν άλλήλαις κατὰ τὸ Ρ, καὶ ἔτι ἐκβεβλήσθωσαν αἱ ZM, ΗΕ έπι τὰ Ο, Π, και ἐπεζεύχθωσαν αί ΑΞ, ΛΟ, ΓΠ, ΒΡ. ἴσον δή ἐστι τὸ ΓΜ στερεόν, οὕ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΖΔΘΜ, τῶ ΓΟ στερεῶ, οὕ βάσις μέν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ύπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΞ, ΛΜ, ΛΟ, ΓΔ, ΓΠ, ΒΘ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΖΟ, ΔΡ. άλλὰ τὸ ΓΟ στερεόν, οὕ βάσις μέν ἐστι τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ, ἴσον ἐστὶ τῷ ΓΝ στερεῷ, οὐ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΚΝ· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς είσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΞ, ΓΕ, ΓΠ, ΛΝ, ΛΟ, ΒΚ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εύθειῶν τῶν ΗΠ, ΝΡ. ὥστε καὶ τὸ ΓΜ στερεὸν ἴσον ἐστὶ τῷ ΓΝ στερεῷ.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐϑειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λα΄.

Τὰ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν.

Έστω ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα τὰ ΑΕ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος. λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ.

Έστωσαν δὴ πρότερον αἱ ἐφεστηκυῖαι αἱ ΘΚ, ΒΕ, ΑΗ, ΛΜ, ΟΠ, ΔΖ, ΓΞ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν, καὶ ἐκβεβλήσθω ἐπ' εὐθείας τῆ ΓΡ εὐθεῖα ἡ ΡΤ, καὶ συνεστάτω πρὸς τῆ ΡΤ εὐθεία καὶ τῷ πρὸς αὐτῆ σημείω τῷ Ρ τῆ ὑπὸ ΑΛΒ γωνία ἴση ἡ ὑπὸ ΤΡΥ, καὶ κείσθω τῆ μὲν ΑΛ ἴση ἡ ΡΤ, τῆ δὲ ΛΒ ἴση ἡ ΡΥ, καὶ συμπεπληρώσθω ἤ τε ΡΧ βάσις καὶ τὸ ΨΥ στερεόν. LM, LN, CD, CE, BH, and BK, not be on the same straight-lines. I say that the solid CM is equal to the solid CN.

For let NK and DH have been produced, and let them have joined one another at R. And, further, let FMand GE have been produced to P and Q (respectively). And let AO, LP, CQ, and BR have been joined. So, solid CM, whose base (is) parallelogram ACBL, and opposite (face) FDHM, is equal to solid CP, whose base (is) parallelogram ACBL, and opposite (face) OQRP. For they are on the same base, ACBL, and (have) the same height, and the (ends of the straight-lines) standing up in them, AF, AO, LM, LP, CD, CQ, BH, and BR, are on the same straight-lines, FP and DR [Prop. 11.29]. But, solid CP, whose base is parallelogram ACBL, and opposite (face) OQRP, is equal to solid CN, whose base (is) parallelogram ACBL, and opposite (face) GEKN. For, again, they are on the same base, ACBL, and (have) the same height, and the (ends of the straight-lines) standing up in them, AG, AO, CE, CQ, LN, LP, BK, and BR, are on the same straight-lines, GQ and NR[Prop. 11.29]. Hence, solid *CM* is also equal to solid CN.

Thus, parallelepiped solids (which are) on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

Proposition 31

Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

Let the parallelepiped solids AE and CF be on the equal bases AB and CD (respectively), and (have) the same height. I say that solid AE is equal to solid CF.

So, let the (straight-lines) standing up, HK, BE, AG, LM, PQ, DF, CO, and RS, first of all, be at right-angles to the bases AB and CD. And let RT have been produced in a straight-line with CR. And let (angle) TRU, equal to angle ALB, have been constructed on the straight-line RT, at the point R on it [Prop. 1.23]. And let RT be made equal to AL, and RU to LB. And let the base RW, and the solid XU, have been completed.



Καὶ ἐπεὶ δύο αἱ ΤΡ, ΡΥ δυσὶ ταῖς ΑΛ, ΛΒ ἴσαι εἰσίν, καὶ γωνίας ἴσας περιέχουσιν, ἴσον ἄρα καὶ ὅμοιον τὸ ΡΧ παραλληλόγραμμον τῷ ΘΛ παραλληλογράμμω. καὶ ἐπεὶ πάλιν ἴση μὲν ἡ ΑΛ τῆ ΡΤ, ἡ δὲ ΛΜ τῆ ΡΣ, καὶ γωνίας όρθὰς περιέχουσιν, ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ $P\Psi$ παραλληλόγραμμον τῷ ΑΜ παραλληλογράμμω. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΛE τῷ $\Sigma \Upsilon$ ἴσον τέ ἐστι καὶ ὅμοιον· τρία ἄρα παραλληλόγραμμα τοῦ ΑΕ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΨΥ στερεοῦ ἴσα τέ ἐστι καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὄμοια, τὰ δὲ τρία τρισί τοῖς ἀπεναντίον· ὅλον ἄρα τὸ ΑΕ στερεὸν παραλληλεπίπεδον όλω τῷ ΨΥ στερεῷ παραλληλεπιπέδω ἴσον ἐστίν. διήχθωσαν αί ΔΡ, ΧΥ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ω, καὶ διὰ τοῦ Τ τῆ ΔΩ παράλληλος ἤχθω ἡ αΤϠ, καὶ ἐκβεβλήσθω ή Ο Δ κατὰ τὸ α, καὶ συμπεπληρώσθω τὰ $\Omega\Psi$, ΡΙ στερεά.
 ἴσον δή ἐστι τὸ $\Psi \Omega$ στερεόν, ο
ὖ βάσις μέν έστι τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ Ωη, τῷ ΨΥ στερεῷ, οὕ βάσις μὲν τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ
 ΥΦ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΨ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι αἱ ΡΩ, ΡΥ, Tλ, TX, Σ γ, Σ õ, Ψ μ, Ψ Φ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν $\Omega X,$
ς Φ. ἀλλὰ τὸ $\Psi \Upsilon$ στερεὸν τῷ ΑΕ ἐστιν ἴσον· καὶ τὸ
 $\Psi \Omega$ άρα στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΡΥΧΤ παραλληλόγραμμον τῷ ΩΤ παραλληλογράμμω. ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς $\rm PT$ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΡΤ, ΩΧ· ἀλλὰ τὸ ΡΥΧΤ τῷ ΓΔ ἐστιν ίσον, ἐπεὶ καὶ τῷ ΑΒ, καὶ τὸ ΩΤ ἄρα παραλληλόγραμμον



О

Ο

С



And since the two (straight-lines) TR and RU are equal to the two (straight-lines) AL and LB (respectively), and they contain equal angles, parallelogram RW is thus equal and similar to parallelogram HL[Prop. 6.14]. And, again, since AL is equal to RT, and LM to RS, and they contain right-angles, parallelogram RX is thus equal and similar to parallelogram AM [Prop. 6.14]. So, for the same (reasons), LE is also equal and similar to SU. Thus, three parallelograms of solid AE are equal and similar to three parallelograms of solid XU. But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid AE is equal to the whole parallelepiped solid XU[Def. 11.10]. Let DR and WU have been drawn across, and let them have met one another at Y. And let aTbhave been drawn through T parallel to DY. And let PDhave been produced to a. And let the solids YX and RI have been completed. So, solid XY, whose base is parallelogram RX, and opposite (face) Yc, is equal to solid XU, whose base (is) parallelogram RX, and opposite (face) UV. For they are on the same base RX, and (have) the same height, and the (ends of the straightlines) standing up in them, RY, RU, Tb, TW, Se, Sd, Xc and XV, are on the same straight-lines, YW and eV [Prop. 11.29]. But, solid XU is equal to AE. Thus, τῷ ΓΔ ἐστιν ἴσον. ἄλλο δὲ τὸ ΔΤ· ἔστιν ἄρα ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΙ ἐπιπέδωις, ἔστιν ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ βάσιν, οὕτως τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν. διὰ τὰ αὐτὰ δή, ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΩΙ ἐπιπέδω τῷ ΡΨ τέτμηται παραλλήλῷ ὄντι τοῖς ἀπεναντίον ἐπιπέδωις, ἔστιν ὡς ἡ ΩΤ βάσις πρὸς τὴν ΤΛ βάσιν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἀλλ' ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ· καὶ ὡς ἄρα τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἐκάτερον ἄρα τῶν ΓΖ, ΩΨ στερεῶν πρὸς τὸ ΡΙ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΓΖ στερεὸν τῷ ΩΨ στερεῷ. ἀλλὰ τὸ ΩΨ τῷ ΑΕ ἐδείχϑη ἴσον· καὶ τὸ ΑΕ ἄρα τῷ ΓΖ ἐστιν ἴσον.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηχυῖαι αἱ ΑΗ, ΘΚ, ΒΕ, ΛΜ, ΓΞ, ΟΠ, ΔΖ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν· λέγω πάλιν, ὅτι ἴσον τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ. ἤχθωσαν γὰρ ἀπὸ τῶν Κ, Ε, Η, Μ, Π, Ζ, Ξ, Σ σημείων ἐπὶ τὸ ὑποχείμενον ἐπίπεδον κάθετοι αἱ ΚΝ, ΕΤ, ΗΥ, ΜΦ, ΠΧ, ΖΨ, ΞΩ, ΣΙ, καὶ συμβαλλέτωσαν τῷ ἐπιπέδῷ κατὰ τὰ Ν, Τ, Υ, Φ, Χ, Ψ, Ω, Ι σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΝΤ, ΝΥ, ΥΦ, ΤΦ, ΧΨ, XΩ, ΩΙ, ΙΨ. ἴσον δή ἐστι τὸ ΚΦ στερεὸν τῷ ΠΙ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΚΜ, ΠΣ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι πρὸς ὀρθάς εἰσι ταῖς βάσεσιν. ἀλλὰ τὸ μὲν ΚΦ στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον, τὸ δὲ ΠΙ τῷ ΓΖ' ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι καὶ ὑπὸ τὸ αὐτὸ ὕψος, ῶν αἱ ἐφεστῶσαι οὕχ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν. καὶ τὸ ΑΕ ἄρα στερεὸν τῷ ΓΖ στερεῷ ἐστιν ἴσον.

Τὰ ἄρα ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι. solid XY is also equal to solid AE. And since parallelogram RUWT is equal to parallelogram YT. For they are on the same base RT, and between the same parallels RT and YW [Prop. 1.35]. But, RUWT is equal to CD, since (it is) also (equal) to AB. Parallelogram YT is thus also equal to CD. And DT is another (parallelogram). Thus, as base CD is to DT, so YT (is) to DT [Prop. 5.7]. And since the parallelepiped solid CIhas been cut by the plane RF, which is parallel to the opposite planes (of CI), as base CD is to base DT, so solid CF (is) to solid RI [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane RX, which is parallel to the opposite planes (of YI), as base YT is to base TD, so solid YX (is) to solid RI [Prop. 11.25]. But, as base CD (is) to DT, so YT (is) to DT. And, thus, as solid CF (is) to solid RI, so solid YX (is) to solid RI. Thus, solids CF and YXeach have the same ratio to RI [Prop. 5.11]. Thus, solid CF is equal to solid YX [Prop. 5.9]. But, YX was show (to be) equal to AE. Thus, AE is also equal to CF.



And so let the (straight-lines) standing up, AG, HK, BE, LM, CO, PQ, DF, and RS, not be at right-angles to the bases AB and CD. Again, I say that solid AE(is) equal to solid CF. For let KN, ET, GU, MV, QW, FX, OY, and SI have been drawn from points K, E, G, M, Q, F, O, and S (respectively) perpendicular to the reference plane (*i.e.*, the plane of the bases AB and CD), and let them have met the plane at points N, T, U, V, W, X, Y, and I (respectively). And let NT, NU, UV, TV, WX, WY, YI, and IX have been joined. So solid KV is equal to solid QI. For they are on the equal bases KM and QS, and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid KV is equal to solid AE, and QI to CF. For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid AE is also equal to solid CF.

Thus, parallelepiped solids which are on equal bases,

λβ΄.

Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



Έστω ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα τὰ AB, ΓΔ· λέγω, ὅτι τὰ AB, ΓΔ στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις, τουτέστιν ὅτι ἐστὶν ὡς ἡ AE βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ AB στερεὸν πρὸς τὸ ΓΔ στερεόν.

Παραβεβλήσθω γὰρ παρὰ τὴν ΖΗ τῷ ΑΕ ἴσον τὸ ΖΘ, καὶ ἀπὸ βάσεως μὲν τῆς ΖΘ, ὕψους δὲ τοῦ αὐτοῦ τῷ ΓΔ στερεὸν παραλληλεπίπεδον συμπεπληρώσθω τὸ ΗΚ. ἴσον δή ἐστι τὸ ΑΒ στερεὸν τῷ ΗΚ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΑΕ, ΖΘ καὶ ὑπὸ τὸ αὐτο ὕψος. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΚ ἐπιπέδω τῷ ΔΗ τέτμηται παραλλήλῷ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ἡ ΓΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΔΘ στερεόν. ἴση δὲ ἡ μὲν ΖΘ βάσις τῆ ΑΕ βάσει, τὸ δὲ ΗΚ στερεὸν τῷ ΑΒ στερεῷ· ἔστιν ἄρα καὶ ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Τὰ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

λγ΄.

Τὰ ὅμοια στερεὰ παραλληλεπίπεδα πρὸς ἄλληλα ἐν τριπλασίονι λόγω εἰσὶ τῶν ὁμολόγων πλευρῶν.

Έστω ὄμοια στερεὰ παραλληλεπίπεδα τὰ AB, ΓΔ, ὑμόλογος δὲ ἔστω ἡ ΑΕ τῆ ΓΖ· λέγω, ὅτι τὸ AB στερεὸν πρὸς τὸ ΓΔ στερεὸν τριπλασίονα λόγον ἔχει, ἤπερ ἡ ΑΕ πρὸς τὴν ΓΖ.

Ἐκβεβλήσθωσαν γὰρ ἐπ' εὐθείας ταῖς ΑΕ, ΗΕ, ΘΕ αἰ ΕΚ, ΕΛ, ΕΜ, καὶ κείσθω τῆ μὲν ΓΖ ἴση ἡ ΕΚ, τῆ δὲ ΖΝ ἴση ἡ ΕΛ, καὶ ἔτι τῆ ΖΡ ἴση ἡ ΕΜ, καὶ συμπεπληρώσθω τὸ ΚΛ παραλληλόγραμμον καὶ τὸ ΚΟ στερεόν. and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

Proposition 32

Parallelepiped solids which (have) the same height are to one another as their bases.



Let AB and CD be parallelepiped solids (having) the same height. I say that the parallelepiped solids AB and CD are to one another as their bases. That is to say, as base AE is to base CF, so solid AB (is) to solid CD.

For let FH, equal to AE, have been applied to FG (in the angle FGH equal to angle LCG) [Prop. 1.45]. And let the parallelepiped solid GK, (having) the same height as CD, have been completed on the base FH. So solid AB is equal to solid GK. For they are on the equal bases AE and FH, and (have) the same height [Prop. 11.31]. And since the parallelepiped solid CK has been cut by the plane DG, which is parallel to the opposite planes (of CK), thus as the base CF is to the base FH, so the solid CD (is) to the solid DH [Prop. 11.25]. And base FH (is) equal to base AE, and solid GK to solid AB. And thus as base AE is to base CF, so solid AB (is) to solid CD.

Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 33

Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let AB and CD be similar parallelepiped solids, and let AE correspond to CF. I say that solid AB has to solid CD the cubed ratio that AE (has) to CF.

For let EK, EL, and EM have been produced in a straight-line with AE, GE, and HE (respectively). And let EK be made equal to CF, and EL equal to FN, and, further, EM equal to FR. And let the parallelogram KL have been completed, and the solid KP.



Καὶ ἐπεὶ δύο αἱ ΚΕ, ΕΛ δυσὶ ταῖς ΓΖ, ΖΝ ἴσαι εἰσίν, άλλὰ καὶ γωνία ἡ ὑπὸ ΚΕΛ γωνία τῇ ὑπὸ ΓΖΝ ἐστιν ἴση, έπειδήπερ καὶ ἡ ὑπὸ ΑΕΗ τῆ ὑπὸ ΓΖΝ ἐστιν ἴση διὰ τὴν όμοιότητα τῶν ΑΒ, ΓΔ στερεῶν, ἴσον ἄρα ἐστὶ [χαὶ ὅμοιον] τὸ ΚΛ παραλληλόγραμμον τῷ ΓΝ παραλληλογράμμω. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΜ παραλληλόγραμμον ἴσον ἐστὶ καὶ ὄμοιον τῷ ΓΡ [παραλληλογράμμω] καὶ ἔτι τὸ ΕΟ τῷ ΔΖ· τρία ἄρα παραλληλόγραμμα τοῦ ΚΟ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΓΔ στερεοῦ ἴσα ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μέν τρία τρισί τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισί τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια· ὅλον ἄρα τὸ ΚΟ στερεόν όλω τῶ ΓΔ στερεῶ ἴσον ἐστὶ καὶ ὅμοιον. συμπεπληρώσθω τὸ ΗΚ παραλληλόγραμμον, καὶ ἀπὸ βάσεων μὲν τῶν ΗΚ, ΚΛ παραλληλόγραμμων, ὕψους δὲ τοῦ αὐτοῦ τῷ ΑΒ στερεὰ συμπεπληρώσθω τὰ ΕΞ, ΛΠ. καὶ ἐπεὶ διὰ τὴν όμοιότητα τῶν ΑΒ, ΓΔ στερεῶν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΝ, καὶ ἡ ΕΘ πρὸς τὴν ΖΡ, ἴση δὲ ἡ μὲν ΓΖ τῆ ΕΚ, ἡ δὲ ΖΝ τῆ ΕΛ, ἡ δὲ ΖΡ τῆ ΕΜ, ἔστιν ἄρα ὡς ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΛ καὶ ἡ ΘΕ πρὸς τὴν ΕΜ. ἀλλ' ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως τὸ ΑΗ [παραλληλόγραμμον] πρός τὸ ΗΚ παραλληλόγραμμον, ώς δὲ ἡ ΗΕ πρὸς τὴν ΕΛ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ, ὡς δὲ ἡ ΘΕ πρὸς ΕΜ, οὕτως τὸ ΠΕ πρὸς τὸ ΚΜ· καὶ ὡς ἄρα τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ καὶ τὸ ΠΕ πρὸς τὸ ΚΜ. ἀλλ' ὡς μὲν τὸ ΑΗ πρὸς τὸ ΗΚ, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ στερεόν, ὡς δὲ τὸ ΗΚ πρὸς τὸ ΚΛ, οὕτως τὸ ΞΕ στερεὸν πρὸς τὸ ΠΛ στερεόν, ώς δὲ τὸ ΠΕ πρὸς τὸ ΚΜ, οὕτως τὸ ΠΛ στερεὸν πρός τὸ ΚΟ στερεόν καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ, οὕτως τὸ ΕΞ πρὸς τὸ ΠΛ καὶ τὸ ΠΛ πρὸς τὸ ΚΟ. έὰν δὲ τέσσαρα μεγέθη κατὰ τὸ συνεχὲς ἀνάλογον ἢ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὸ δεύτερον· τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ τὸ ΑΒ πρὸς τὸ ΕΞ. ἀλλ' ὡς τὸ ΑΒ πρὸς τὸ ΕΞ, οὕτως τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ καὶ ἡ ΑΕ εύθεῖα πρὸς τὴν ΕΚ· ὥστε καὶ τὸ ΑΒ στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΕ πρὸς τὴν ΕΚ. ἴσον δὲ τὸ [μὲν] ΚΟ στερεὸν τῷ ΓΔ στερεῷ, ἡ δὲ ΕΚ εὐθεῖα τῆ ΓΖ· καὶ τὸ AB ẳρα στερεὸν πρὸς τὸ ΓΔ στερεὸν τρι-



And since the two (straight-lines) KE and EL are equal to the two (straight-lines) CF and FN, but angle KEL is also equal to angle CFN, inasmuch as AEG is also equal to CFN, on account of the similarity of the solids AB and CD, parallelogram KL is thus equal [and similar] to parallelogram CN. So, for the same (reasons), parallelogram KM is also equal and similar to [parallelogram] CR, and, further, EP to DF. Thus, three parallelograms of solid KP are equal and similar to three parallelograms of solid CD. But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid KP is equal and similar to the whole of solid CD [Def. 11.10]. Let parallelogram GK have been completed. And let the the solids EO and LQ, with bases the parallelograms GKand KL (respectively), and with the same height as AB, have been completed. And since, on account of the similarity of solids AB and CD, as AE is to CF, so EG (is) to FN, and EH to FR [Defs. 6.1, 11.9], and CF (is) equal to EK, and FN to EL, and FR to EM, thus as AE is to EK, so GE (is) to EL, and HE to EM. But, as AE (is) to EK, so [parallelogram] AG (is) to parallelogram GK, and as GE (is) to EL, so GK (is) to KL, and as HE (is) to EM, so QE (is) to KM [Prop. 6.1]. And thus as parallelogram AG (is) to GK, so GK (is) to KL, and QE (is) to KM. But, as AG (is) to GK, so solid AB (is) to solid EO, and as GK (is) to KL, so solid OE (is) to solid QL, and as QE (is) to KM, so solid QL(is) to solid KP [Prop. 11.32]. And, thus, as solid ABis to EO, so EO (is) to QL, and QL to KP. And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid AB has to KP the cubed ratio which AB (has) to EO. But, as AB (is) to EO, so parallelogram AG (is) to GK, and the straight-line AEto EK [Prop. 6.1]. Hence, solid AB also has to KP the cubed ratio that AE (has) to EK. And solid KP (is)

πλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος αὐτοῦ πλευρὰ ἡ ΑΕ πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΓΖ.



Τὰ ἄρα ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· ὅπερ ἔδει δεῖξαι.

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι ἐἀν τέσσαρες εὐθεῖαι ἀνάλογον ῶσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τετάρτην, οὕτω τὸ ἀπὸ τῆς πρώτης στερεὸν παραλληλεπίπεδον πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἐπείπερ καὶ ἡ πρώτη πρὸς τὴν τετάρτην τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὴν δευτέραν.

$\lambda\delta'.$

Τῶν ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ῶν στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα.

Έστω ίσα στερεὰ παραλληλεπίπεδα τὰ AB, ΓΔ· λέγω, ὅτι τῶν AB, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος.

^{*}Εστωσαν γὰρ πρότερον αἱ ἐφεστηχυῖαι αἱ AH, EZ, AB, ΘK, ΓΜ, ΝΞ, ΟΔ, ΠΡ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν· λέγω, ὅτι ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν AH.

Εἰ μὲν οῦν ἴση ἐστὶν ἡ ΕΘ βάσιν τῆ ΝΠ βάσει, ἔστι δὲ καὶ τὸ AB στερεὸν τῷ ΓΔ στερεῷ ἴσον, ἔσται καὶ ἡ ΓΜ τῆ AH ἴση. τὰ γὰρ ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις. καὶ ἔσται ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ, οὕτως ἡ ΓΜ πρὸς τὴν AH, καὶ φανερόν, ὅτι equal to solid CD, and straight-line EK to CF. Thus, solid AB also has to solid CD the cubed ratio which its corresponding side AE (has) to the corresponding side CF.



Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

Proposition 34[†]

The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let AB and CD be equal parallelepiped solids. I say that the bases of the parallelepiped solids AB and CDare reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

For, first of all, let the (straight-lines) standing up, AG, EF, LB, HK, CM, NO, PD, and QR, be at rightangles to their bases. I say that as base EH is to base NQ, so CM (is) to AG.

Therefore, if base EH is equal to base NQ, and solid AB is also equal to solid CD, CM will also be equal to AG. For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base



τῶν AB, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἰ βάσεις τοῖς ὕψεσιν.

Μὴ ἔστω δὴ ἴση ἡ ΕΘ βάσις τῆ ΝΠ βάσει, ἀλλ' ἔστω μείζων ή ΕΘ. ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον· μείζων ἄρα ἐστὶ καὶ ἡ ΓΜ τῆς ΑΗ. κείσθω οὖν τῆ ΑΗ ἴση ἡ ΓΤ, καὶ συμπεπληρώσθω ἀπὸ βάσεως μὲν τῆς ΝΠ, ὕψους δὲ τοῦ ΓΤ, στερεὸν παραλληλεπίπεδον τὸ ΦΓ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἔξωθεν δὲ τὸ $\Gamma \Phi$, τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ώς τὸ AB στερε
ὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ Γ $\!\Delta$ στερεόν πρός τὸ ΓΦ στερεόν. ἀλλ' ὡς μὲν τὸ ΑΒ στερεόν πρὸς τὸ $\Gamma \Phi$ στερεόν, οὕτως ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν· ίσοϋψῆ γὰρ τὰ ΑΒ, ΓΦ στερεά ὡς δὲ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΜΠ βάσις πρὸς τὴν ΤΠ βάσιν καὶ ἡ ΓΜ πρὸς τὴν ΓΤ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΓΤ. ἴση δὲ ἡ ΓΤ τῆ ΑΗ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρός τὴν ΑΗ. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων άντιπεπόνθασιν αί βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν AB, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αί βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ AB στερεὸν τῷ ΓΔ στερεῷ.

Έστωσαν [γὰρ] πάλιν αἱ ἐφεστηχυῖαι πρὸς ὀρθὰς ταῖς βάσεσιν. καὶ εἰ μὲν ἴση ἐστὶν ἡ ΕΘ βάσις τῆ ΝΠ βάσει, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴσον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τῷ τοῦ ΑΒ στερεοῦ ὕψει. τὰ δὲ ἐπὶ ἴσων βάσεων στερεά παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Μὴ ἔστω δὴ ἡ ΕΘ βάσις τῆ ΝΠ [βάσει] ἴση, ἀλλ' ἔστω μείζων ἡ ΕΘ· μεῖζον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τοῦ τοῦ AB στερεοῦ ὕψους, τουτέστιν ἡ ΓΜ τῆς AH. κείσθω τῆ AH ἴση πάλιν ἡ ΓΤ, καὶ συμπεπληρώσθω ὁμοίως τὸ ΓΦ στερεόν. ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν AH, ἴση δὲ ἡ AH τῆ ΓΤ, EH (is) to NQ, so CM will be to AG. And (so it is) clear that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.



So let base EH not be equal to base NQ, but let EHbe greater. And solid AB is also equal to solid CD. Thus, CM is also greater than AG. Therefore, let CT be made equal to AG. And let the parallelepiped solid VC have been completed on the base NQ, with height CT. And since solid AB is equal to solid CD, and CV (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid AB is to solid CV, so solid CD (is) to solid CV. But, as solid AB(is) to solid CV, so base EH (is) to base NQ. For the solids AB and CV (are) of equal height [Prop. 11.32]. And as solid CD (is) to solid CV, so base MQ (is) to base TQ [Prop. 11.25], and CM to CT [Prop. 6.1]. And, thus, as base EH is to base NQ, so MC (is) to AG. And CT(is) equal to AG. And thus as base EH (is) to base NQ, so MC (is) to AG. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepipid solids ABand CD be reciprocally proportional to their heights, and let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD. [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base EH is equal to base NQ, and as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, the height of solid CD is thus also equal to the height of solid AB. And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid AB is equal to solid CD.

So, let base EH not be equal to [base] NQ, but let EH be greater. Thus, the height of solid CD is also greater than the height of solid AB, that is to say CM (greater) than AG. Let CT again be made equal to AG, and let the solid CV have been similarly completed. Since as base EH is to base NQ, so MC (is) to AG,

ἕστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΓΤ. ἀλλ' ὡς μὲν ἡ ΕΘ [βάσις] πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ AB στερεὸν πρὸς τὸ ΓΦ στερεόν ἰσοϋψῆ γάρ ἐστι τὰ AB, ΓΦ στερεά· ὡς δὲ ἡ ΓΜ πρὸς τὴν ΓΤ, οὕτως ἤ τε ΜΠ βάσις πρὸς τὴν ΠΤ βάσιν καὶ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. καὶ ὡς ἄρα τὸ AB στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν ἑκάτερον ἄρα τῶν AB, ΓΔ πρὸς τὸ ΓΦ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ AB στερεὸν τῷ ΓΔ στερεῷ.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΖΕ, ΒΛ, ΗΑ, ΚΘ, ΞΝ, ΔΟ, ΜΓ, ΡΠ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν, καὶ ἤχθωσαν ἀπὸ τῶν Ζ, Η, Β, Κ, Ξ, Μ, Ρ, Δ σημείων ἐπὶ τὰ διὰ τῶν ΕΘ, ΝΠ ἐπίπεδα κάθετοι καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω, ς, καὶ συμπεπληρώσθω τὰ ΖΦ, ΞΩ στερεά· λέγω, ὅτι καὶ οὕτως ἴσων ὄντων τῶν ΑΒ, ΓΔ στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσιν πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Έπεὶ ἴσον ἐστὶ τὸ AB στερεὸν τῷ ΓΔ στερεῷ, ἀλλὰ τὸ μὲν AB τῷ BT ἐστιν ἴσον· ἐπί τε γὰρ τῆς αὐτῆς βάσεὡς εἰσι τῆς ZK καὶ ὑπὸ τὸ αὐτὸ ὕψος· τὸ δὲ ΓΔ στερεὸν τῷ ΔΨ ἐστιν ἴσον· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεὡς εἰσι τῆς PΞ καὶ ὑπὸ τὸ αὐτὸ ὕψος· καὶ τὸ BT ἄρα στερεὸν τῷ ΔΨ στερεῷ ἴσον ἐστίν. ἔστιν ἄρα ὡς ἡ ZK βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ BT στερεοῦ ὕψος. ἴση δὲ ἡ μὲν ZK βάσις τῆ ΕΘ βάσις πρὸς τὴν NΠ βάσει· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὸ τοῦ BT στερεοῦ ὕψος. τὰ δ' αὐτὰ ὕψη ἐστὶ τῶν ΔΨ, BT στερεῶν καὶ τῶν ΔΓ, BA· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν NΠ

and AG (is) equal to CT, thus as base EH (is) to base NQ, so CM (is) to CT. But, as [base] EH (is) to base NQ, so solid AB (is) to solid CV. For solids AB and CV are of equal heights [Prop. 11.32]. And as CM (is) to CT, so (is) base MQ to base QT [Prop. 6.1], and solid CD to solid CV [Prop. 11.25]. And thus as solid AB (is) to solid CV, so solid CD (is) to solid CV. Thus, AB and CD each have the same ratio to CV. Thus, solid AB is equal to solid CD [Prop. 5.9].



So, let the (straight-lines) standing up, FE, BL, GA, KH, ON, DP, MC, and RQ, not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through EH and NQ from points F, G, B, K, O, M, R, and D, and let them have joined the planes at (points) S, T, U, V, W, X, Y, and a (respectively). And let the solids FV and OY have been completed. In this case, also, I say that the solids AB and CD being equal, their bases are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

Since solid AB is equal to solid CD, but AB is equal to BT. For they are on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid CD is equal is equal to DX. For, again, they are on the same base RO, and (have) the same height [Props. 11.29, 11.30]. Solid BT is thus also equal to solid DX. Thus, as base FK (is) to base OR, so the height of solid DX (is) to the height of solid BT (see first part of proposition). And base FK(is) equal to base EH, and base OR to NQ. Thus, as base EH is to base NQ, so the height of solid DX (is) to βάσιν, οὕτως τὸ τοῦ ΔΓ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος. τῶν AB, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν AB, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἰ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ AB στερεὸν τῷ ΓΔ στερεῷ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος, ἴση δὲ ἡ μὲν ΕΘ βάσις τῆ ZK βάσει, ἡ δὲ ΝΠ τῆ ΞΡ, ἔστιν ἄρα ὡς ἡ ZK βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος. τὰ δ' αὐτὰ ὕψη ἐστὶ τῶν AB, ΓΔ στερεῶν καὶ τῶν BT, ΔΨ· ἔστιν ἄρα ὡς ἡ ZK βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ BT στερεοῦ ὕψος. τῶν BT, ΔΨ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἴσον ἄρα ἐστὶ τὸ BT στερεὸν τῷ ΔΨ στερεῷ. ἀλλὰ τὸ μὲν BT τῷ BA ἴσον ἐστίν· ἐπί τε γὰρ τῆς αὐτῆς βάσεως [εἰσι] τῆς ZK καὶ ὑπὸ τὸ αὐτὸ ὕψος. τὸ δὲ ΔΨ στερεὸν τῷ ΔΓ στερεῷ ἰσον ἐστίν. καὶ τὸ AB ἄρα στερεὸν τῷ ΓΔ στερεῷ ἐστιν ἴσον· ὅπερ ἔδει δεῖξαι. the height of solid BT. And solids DX, BT are the same height as (solids) DC, BA (respectively). Thus, as base EH is to base NQ, so the height of solid DC (is) to the height of solid AB. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids AB and CD be reciprocally proportional to their heights, and (so) let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD.

For, with the same construction (as before), since as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, and base EH (is) equal to base FK, and NQ to OR, thus as base FK is to base OR, so the height of solid CD (is) to the height of solid AB. And solids AB, CD are the same height as (solids) BT, DX (respectively). Thus, as base FK is to base OR, so the height of solid DX (is) to the height of solid BT. Thus, the bases of the parallelepiped solids BT and DXare reciprocally proportional to their heights. Thus, solid BT is equal to solid DX (see first part of proposition). But, BT is equal to BA. For [they are] on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid DX is equal to solid DC [Props. 11.29, 11.30]. Thus, solid AB is also equal to solid CD. (Which is) the very thing it was required to show.

[†] This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

λε΄.

Έαν ῶσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπὶ δὲ τῶν κορυφῶν αὐτῶν μετέωροι εὐθεῖαι ἐπισταθῶσιν ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἐκατέραν ἐκατέρα, ἐπὶ δὲ τῶν μετεώρων ληφθῆ τυχόντα σημεῖα, καὶ ἀπ᾽ αὐτῶν ἐπὶ τὰ ἐπίπεδα, ἐν οῖς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, κάθετοι ἀχθῶσιν, ἀπὸ δὲ τῶν γενομένων σημείων ἐν τοῖς ἐπιπέδοις ἐπὶ τὰς ἐξ ἀρχῆς γωνίας ἐπιζευχθῶσιν εὐθεῖαι, ἴσας γωνίας περιέξουσι μετὰ τῶν μετεώρων.

Έστωσαν δύο γωνίαι εὐθύγραμμοι ἴσαι αί ὑπὸ ΒΑΓ, ΕΔΖ, ἀπὸ δὲ τῶν Α, Δ σημείων μετέωροι εὐθεῖαι ἐφεστάτωσαν αἱ ΑΗ, ΔΜ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρα, τὴν μὲν ὑπὸ ΜΔΕ τῆ ὑπὸ HAB, τὴν δὲ ὑπὸ ΜΔΖ τῆ ὑπὸ HAΓ, καὶ εἰλήφθω ἐπὶ τῶν ΑΗ, ΔΜ τυχόντα σημεῖα τὰ Η, Μ, καὶ ἤχθωσαν ἀπὸ τῶν Η, Μ σημείων ἐπὶ τὰ διὰ τῶν ΒΑΓ, ΕΔΖ ἐπίπεδα κάθετοι αἱ ΗΛ, ΜΝ, καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Λ, Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΑ, ΝΔ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΗΑΛ γωνία τῆ ὑπὸ ΜΔΝ γωνία.

Proposition 35

If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straightlines).

Let BAC and EDF be two equal rectilinear angles. And let the raised straight-lines AG and DM have been stood on points A and D, containing equal angles respectively with the original straight-lines. (That is) MDE(equal) to GAB, and MDF (to) GAC. And let the random points G and M have been taken on AG and DM(respectively). And let the GL and MN have been drawn from points G and M perpendicular to the planes through



Κείσθω τῆ ΔΜ ἴση ἡ ΑΘ, καὶ ἤχθω διὰ τοῦ Θ σημείου τῆ ΗΛ παράλληλος ἡ ΘΚ. ἡ δὲ ΗΛ κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον· καὶ ἡ ΘΚ ἄρα κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον. ἤχθωσαν ἀπὸ τῶν Κ, Ν σημείων έπὶ τὰς ΑΓ, ΔΖ, ΑΒ, ΔΕ εὐθείας κάθετοι αἱ ΚΓ, ΝΖ, ΚΒ, ΝΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΓ, ΓΒ, ΜΖ, ΖΕ. ἐπεὶ τὸ ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ίσα ἐστὶ τὰ ἀπὸ τῶν ΚΓ, ΓΑ, καὶ τὸ ἀπὸ τῆς ΘΑ ἄρα ἴσον έστι τοῖς ἀπὸ τῶν ΘΚ, ΚΓ, ΓΑ. τοῖς δὲ ἀπὸ τῶν ΘΚ, ΚΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς $\Theta\Gamma^{\cdot}$ τὸ ἄρα ἀπὸ τῆς ΘA ἴσον ἐστὶ τοῖς άπὸ τῶν ΘΓ, ΓΑ. ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΘΓΑ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΔΖΜ γωνία ὀρθή ἐστιν. ἴση ἄρα ἐστὶν ή ὑπὸ ΑΓΘ γωνία τῆ ὑπὸ ΔΖΜ. ἔστι δὲ καὶ ἡ ὑπὸ ΘΑΓ τῆ ὑπὸ MΔZ ἴση. δύο δὴ τρίγων
ά ἐστι τὰ MΔZ, ΘΑΓ δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρα καὶ μίαν πλευράν μια πλευρα ίσην την υποτείνουσαν υπό μίαν τῶν ίσων γωνιῶν τὴν ΘΑ τῆ ΜΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει ἑκατέραν ἑκαρέρα. ἴση ἄρα έστιν ή ΑΓ τῆ ΔΖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ή ΑΒ τῆ ΔE ἐστιν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν $A\Gamma$ τῃ ΔZ , ἡ δὲ AB tỹ ΔE , δύο δη αί ΓΑ, AB δυσὶ ταῖς $Z\Delta$, ΔE ἴσαι εἰσίν. άλλὰ καὶ γωνία ἡ ὑπὸ ΓΑΒ γωνία τῆ ὑπὸ ΖΔΕ ἐστιν ἴση· βάσις ἄρα ή ΒΓ βάσει τῆ ΕΖ ἴση ἐστὶ καὶ τὸ τρίγωνον τῷ τριγώνω καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις. ἴση ἄρα ἡ ύπὸ ΑΓΒ γωνία τῆ ὑπὸ ΔΖΕ. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΓΚ όρθῆ τῆ ὑπὸ ΔΖΝ ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΓΚ λοιπῆ τῆ ύπὸ EZN ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΒΚ τῆ ὑπὸ ΖΕΝ ἐστιν ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΒΓΚ, ΕΖΝ [τὰς] δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρα καὶ μίαν πλευράν μια πλευρα ίσην την πρός ταις ίσαις γωνίαις τὴν ΒΓ τῆ ΕΖ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἐστὶν ἡ ΓΚ τῆ ΖΝ. ἔστι δὲ

BAC and EDF (respectively). And let them have joined the planes at points L and N (respectively). And let LAand ND have been joined. I say that angle GAL is equal to angle MDN.



Let AH be made equal to DM. And let HK have been drawn through point H parallel to GL. And GL is perpendicular to the plane through BAC. Thus, HK is also perpendicular to the plane through BAC [Prop. 11.8]. And let KC, NF, KB, and NE have been drawn from points K and N perpendicular to the straight-lines AC, DF, AB, and DE. And let HC, CB, MF, and FE have been joined. Since the (square) on HA is equal to the (sum of the squares) on HK and KA [Prop. 1.47], and the (sum of the squares) on KC and CA is equal to the (square) on KA [Prop. 1.47], thus the (square) on HAis equal to the (sum of the squares) on HK, KC, and CA. And the (square) on HC is equal to the (sum of the squares) on HK and KC [Prop. 1.47]. Thus, the (square) on HA is equal to the (sum of the squares) on HC and CA. Thus, angle HCA is a right-angle [Prop. 1.48]. So, for the same (reasons), angle DFMis also a right-angle. Thus, angle ACH is equal to (angle) DFM. And HAC is also equal to MDF. So, MDFand HAC are two triangles having two angles equal to two angles, respectively, and one side equal to one side-(namely), that subtending one of the equal angles ---(that is), HA (equal) to MD. Thus, they will also have the remaining sides equal to the remaining sides, respectively [Prop. 1.26]. Thus, AC is equal to DF. So, similarly, we can show that AB is also equal to DE. Therefore, since AC is equal to DF, and AB to DE, so the two (straightlines) CA and AB are equal to the two (straight-lines) FD and DE (respectively). But, angle CAB is also equal to angle FDE. Thus, base BC is equal to base EF, and triangle (ACB) to triangle (DFE), and the remaining angles to the remaining angles (respectively) [Prop. 1.4].

xaì ή AΓ τῆ ΔZ ἴση· δύο δὴ aἱ AΓ, ΓΚ δυσὶ ταῖς ΔZ , ZN ἴσαι εἰσίν· xaì ὀρθὰς γωνίας περιέχουσιν. βάσις ἄρα ἡ AK βάσει τῆ ΔN ἴση ἐστίν. xaì ἐπεὶ ἴση ἐστὶν ἡ AΘ τῆ ΔM , ἴσον ἐστὶ xaì τὸ ἀπὸ τῆς AΘ τῷ ἀπὸ τῆς ΔM . ἀλλὰ τῷ μὲν ἀπὸ τῆς AΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν AK, KΘ· ὀρθὴ γὰρ ἡ ὑπὸ AKΘ· τῷ δὲ ἀπὸ τῆς ΔM ἴσα τὰ ἀπὸ τῶν ΔN , NM· ὀρθὴ γὰρ ἡ ὑπὸ ΔNM · τὰ ἄρα ἀπὸ τῶν AK, KΘ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΔN , NM, ῶν τὸ ἀπὸ τῆς AK ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔN · λοιπὸν ἄρα τὸ ἀπὸ τῆς KΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς MΛ· λοιπὸν ἄρα τὸ ἀπὸ τῆς KΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς MΔ, ΔN ἴσαι εἰσὶν ἑxατέρα ἑxατέρα, xaì βάσις ἡ ΘΚ βάσει τῆ MN ἐδείχθη ἴση, γωνία ἄρα ἡ ὑπὸ ΘΑΚ γωνία τῆ ὑπὸ M ΔN ἐστιν ἴση.

Έαν ἄρα ῶσι δύο γωνίαι ἐπίπεδοι ἴσαι καὶ τὰ ἑξῆς τῆς προτάσεως [ὅπερ ἔδει δεῖξαι].

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι, ἐἀν ῶσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπισταθῶσι δὲ ἐπ' αὐτῶν μετέωροι εὐθεῖαι ἴσαι ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑχατέραν ἑχατέρα, αἱ ἀπ' αὐτῶν χάθετοι ἀγόμεναι ἐπὶ τὰ ἐπίπεδα, ἐν οῖς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, ἴσαι ἀλλήλαις εἰσίν. ὅπερ ἔδει δεῖξαι.

λτ΄.

Έὰν τρεῖς εὐθεῖαι ἀνάλογον ὥσιν, τὸ ἐκ τῶν τριῶν στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης στερεῷ παραλληλεπιπέδῷ ἰσοπλεύρῷ μέν, ἰσογωνίῷ δὲ τῷ προειρημένῷ. Thus, angle ACB (is) equal to DFE. And the right-angle ACK is also equal to the right-angle DFN. Thus, the remainder BCK is equal to the remainder EFN. So, for the same (reasons), CBK is also equal to FEN. So, BCK and EFN are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that by the equal angles—(that is), BC (equal) to EF. Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus, CK is equal to FN. And AC (is) also equal to DF. So, the two (straight-lines) AC and CK are equal to the two (straight-lines) DF and FN (respectively). And they enclose right-angles. Thus, base AK is equal to base DN [Prop. 1.4]. And since AH is equal to DM, the (square) on AH is also equal to the (square) on DM. But, the the (sum of the squares) on AK and KHis equal to the (square) on AH. For angle AKH (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on DN and NM (is) equal to the square on DM. For angle DNM (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AK and KH is equal to the (sum of the squares) on DN and NM, of which the (square) on AK is equal to the (square) on DN. Thus, the remaining (square) on KH is equal to the (square) on NM. Thus, HK (is) equal to MN. And since the two (straight-lines) HA and AK are equal to the two (straight-lines) MDand DN, respectively, and base HK was shown (to be) equal to base MN, angle HAK is thus equal to angle *MDN* [Prop. 1.8].

Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

Corollary

So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).



Έστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ A, B, Γ, ὡς ἡ A πρὸς τὴν B, οὕτως ἡ B πρὸς τὴν Γ· λέγω, ὅτι τὸ ἐκ τῶν A, B, Γ στερεὸν ἴσον ἐστὶ τῷ ἀπὸ τῆς B στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίω δὲ τῷ προειρημένω.

Έκκείσθω στερεὰ γωνία ἡ πρὸς τῷ Ε περιεχομένη ὑπὸ τῶν ὑπὸ ΔΕΗ, ΗΕΖ, ΖΕΔ, καὶ κείσθω τῆ μὲν Β ἴση ἑκάστη τῶν ΔΕ, ΗΕ, ΕΖ, καὶ συμπεπληρώσθω τὸ ΕΚ στερεὸν παραλληλεπίπεδον, τῆ δὲ Α ἴση ἡ ΛΜ, καὶ συνεστάτω πρὸς τῆ ΛM εὐθεία καὶ τῷ πρὸς αὐτῆ σημείῳ τῷ Λ τῆ πρὸς τῷ Ε στερεᾶ γωνία ἴση στερεὰ γωνία ἡ περειχομένη ὑπὸ τῶν ΝΛΞ, ΞΛΜ, ΜΛΝ, καὶ κείσθω τῆ μὲν Β ἴση ἡ ΛΞ, τῆ δὲ Γ ἴση ἡ ΛΝ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ, ἴση δὲ ἡ μὲν Α τῆ ΛΜ, ἡ δὲ Β ἑκατέρα τῶν ΛΞ, $E\Delta$, ή δ
έ Γ τῆ ΛΝ, ἔστιν ἄρα ὡς ή ΛΜ πρὸς τὴν EZ, οὕτως ή ΔΕ πρός τὴν ΛΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΝΛΜ, ΔEZ αί πλευραὶ ἀντιπεπόνθασιν· ἴσον ἄρα ἐστὶ τὸ MN παραλληλόγραμμον τῷ ΔΖ παραλληλογραμάμμω. καὶ ἐπεὶ δύο γωνίαι ἐπίπεδοι εὐθύγραμμοι ἴσαι εἰσὶν αἱ ὑπὸ ΔΕΖ, ΝΛΜ, καὶ ἐπ' αὐτῶν μετέωροι εὐθεῖαι ἐφεστᾶσιν αἱ ΛΞ, ΕΗ ἴσαι τε άλλήλαις καὶ ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εύθειῶν ἑκατέραν ἑκατέρα, αἱ ἄρα ἀπὸ τῶν Η, Ξ σημείων κάθετοι ἀγόμεναι ἐπὶ τὰ διὰ τῶν ΝΛΜ, ΔΕΖ ἐπίπεδα ἴσαι άλλήλαις εἰσίν ὥστε τὰ ΛΘ, ΕΚ στερεὰ ὑπὸ τὸ αὐτὸ ὕψος έστίν. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ $\Theta\Lambda$ στερεὸν τῷ ΕΚ στερεῷ. καί ἐστι τὸ μὲν ΛΘ τὸ ἐκ τῶν Α, B, Γ στερεόν, τὸ δὲ ΕΚ τὸ ἀπὸ τῆς Β στερεόν· τὸ ἄρα ἐκ τῶν Α, Β, Γ στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρω μέν, ἰσογωνίω δὲ τῷ προειρημένω· ὄπερ ἔδει δεῖξαι.



Let A, B, and C be three (continuously) proportional straight-lines, (such that) as A (is) to B, so B (is) to C. I say that the (parallelepiped) solid (formed) from A, B, and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid).

Let the solid angle at E, contained by DEG, GEF, and FED, be set out. And let DE, GE, and EF each be made equal to B. And let the parallelepiped solid EK have been completed. And (let) LM (be made) equal to A. And let the solid angle contained by NLO, OLM, and MLN have been constructed on the straightline LM, and at the point L on it, (so as to be) equal to the solid angle E [Prop. 11.23]. And let LO be made equal to B, and LN equal to C. And since as A (is) to B, so B (is) to C, and A (is) equal to LM, and Bto each of LO and ED, and C to LN, thus as LM (is) to EF, so DE (is) to LN. And (so) the sides around the equal angles NLM and DEF are reciprocally proportional. Thus, parallelogram MN is equal to parallelogram DF [Prop. 6.14]. And since the two plane rectilinear angles DEF and NLM are equal, and the raised straight-lines stood on them (at their apexes), LO and EG, are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points G and Oto the planes through NLM and DEF (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids LH and EK (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid HL is equal to solid EK. And LH is the solid (formed) from A, B, and C, and EK the solid on B. Thus, the parallelepiped solid (formed) from A, B, and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

λζ΄.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν

Proposition 37[†]

If four straight-lines are proportional then the similar,

στερεὰ παραλληλεπίπεδα ὄμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἔσται· καὶ ἐὰν τὰ ἀπ' αὐτῶν στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἢ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.

 $\begin{array}{c|c} & & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & &$

Έστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἰ AB, ΓΔ, ΕΖ, HΘ, ὡς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν HΘ, καὶ ἀναγεγράφθωσαν ἀπὸ τῶν AB, ΓΔ, ΕΖ, ΗΘ ὅμοιά τε καὶ ὁμοίως κείμενα στερεὰ παραλληλεπίπεδα τὰ KA, ΛΓ, ΜΕ, NH· λέγω, ὅτι ἐστὶν ὡς τὸ KA πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ NH.

Έπεὶ γὰρ ὅμοιόν ἐστι τὸ ΚΑ στερεὸν παραλληλεπίπεδον τῷ ΛΓ, τὸ ΚΑ ἄρα πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΓΔ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΜΕ πρὸς τὸ NH τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΗΘ. καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ. καὶ ὡς ἄρα τὸ ΑΚ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ NH.

Άλλὰ δὴ ἔστω ὡς τὸ ΑΚ στερεὸν πρὸς τὸ ΛΓ στερεόν, οὕτως τὸ ΜΕ στερεὸν πρὸς τὸ ΝΗ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒ εὐθεῖα πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Έπεὶ γὰρ πάλιν τὸ ΚΑ πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἦπερ ἡ ΑΒ πρὸς τὴν ΓΔ, ἔχει δὲ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἦπερ ἡ ΕΖ πρὸς τὴν ΗΘ, καί ἐστιν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ NH, καὶ ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Έαν αঁρα τέσσαρες εύθεῖαι ἀνάλογον ῶσι καὶ τὰ ἑξῆς τῆς προτάσεως· ὅπερ ἔδει δεῖξαι.



and similarly described, parallelepiped solids on them

will also be proportional. And if the similar, and similarly

described, parallelepiped solids on them are proportional

then the straight-lines themselves will be proportional.

Let AB, CD, EF, and GH, be four proportional straight-lines, (such that) as AB (is) to CD, so EF (is) to GH. And let the similar, and similarly laid out, parallelepiped solids KA, LC, ME and NG have been described on AB, CD, EF, and GH (respectively). I say that as KA is to LC, so ME (is) to NG.

For since the parallelepiped solid KA is similar to LC, KA thus has to LC the cubed ratio that AB (has) to CD [Prop. 11.33]. So, for the same (reasons), ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33]. And since as AB is to CD, so EF (is) to GH, thus, also, as AK (is) to LC, so ME (is) to NG.

And so let solid AK be to solid LC, as solid ME (is) to NG. I say that as straight-line AB is to CD, so EF (is) to GH.

For, again, since KA has to LC the cubed ratio that AB (has) to CD [Prop. 11.33], and ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33], and as KA is to LC, so ME (is) to NG, thus, also, as AB (is) to CD, so EF (is) to GH.

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show.

[†] This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and vice versa.

λη'.

Έὰν κύβου τῶν ἀπεναντίον ἐπιπέδων αί πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῆ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας.

Proposition 38

If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.


Κύβου γὰρ τοῦ AZ τῶν ἀπεναντίον ἐπιπέδων τῶν ΓΖ, AΘ ai πλευραὶ δίχα τετμήσθωσαν κατὰ τὰ Κ, Λ, Μ, Ν, Ξ, Π, Ο, Ρ σημεῖα, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβεβλήσθω τὰ KN, ΞΡ, κοινὴ δὲ τομὴ τῶν ἐπιπέδων ἔστω ἡ ΥΣ, τοῦ δὲ AZ κύβου διαγώνιος ἡ ΔΗ. λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ΥΤ τῆ ΤΣ, ἡ δὲ ΔΤ τῆ TH.

Ἐπεζεύχθωσαν γὰρ αί ΔΥ, ΥΕ, ΒΣ, ΣΗ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΔΞ τῇ ΟΕ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΔΞΥ, ΥΟΕ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΞ τῆ ΟΕ, ἡ δὲ ΞΥ τῆ ΥΟ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ή ΔΥ τῆ ΥΕ ἐστιν ἴση, καὶ τὸ ΔΞΥ τρίγωνον τῷ ΟΥΕ τριγώνω ἐστὶν ἴσον καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι· ἴση ἄρα ἡ ὑπὸ ΞΥΔ γωνία τῃ ὑπὸ ΟΥΕ γωνία. διὰ δὴ τοῦτο εὐθεῖά ἐστιν ἡ ΔΥΕ. διὰ τὰ αὐτὰ δὴ καὶ ΒΣΗ εὐθεῖά ἐστιν, καὶ ἴση ἡ ΒΣ τῆ ΣΗ. καὶ ἐπεὶ ἡ ΓΑ τῆ ΔΒ ἴση ἐστὶ καὶ παράλληλος, ἀλλὰ ἡ ΓΑ καὶ τῆ EH ἴση τέ έστι καὶ παράλληλος, καὶ ἡ ΔΒ ἄρα τῆ ΕΗ ἴση τέ ἐστι καὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΔΕ, BH· παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῆ BH. ἴση ἄρα ἡ μὲν ὑπὸ ΕΔΤ γωνία τῆ ὑπὸ ΒΗΤ· ἐναλλὰξ γάρ· ἡ δὲ ὑπὸ ΔΤΥ τῆ ύπὸ ΗΤΣ. δύο δὴ τρίγωνά ἐστι τὰ ΔΤΥ, ΗΤΣ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα χαὶ μίαν πλευρὰν μιặ πλευρα ίσην την ύποτείνουσαν ύπο μίαν τῶν ίσων γωνιῶν τὴν ΔΥ τῆ ΗΣ· ἡμίσειαι γάρ εἰσι τῶν ΔΕ, ΒΗ· καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει. ἴση ẳρα ἡ μέν ΔΤ τῆ ΤΗ, ἡ δὲ ΥΤ τῆ ΤΣ.

Έαν ἄρα χύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῆ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας· ὅπερ ἔδει δεῖξαι.



For let the opposite planes CF and AH of the cube AF have been cut in half at the points K, L, M, N, O, Q, P, and R. And let the planes KN and OR have been produced through the pieces. And let US be the common section of the planes, and DG the diameter of cube AF. I say that UT is equal to TS, and DT to TG.

For let DU, UE, BS, and SG have been joined. And since DO is parallel to PE, the alternate angles DOU and UPE are equal to one another [Prop. 1.29]. And since DO is equal to PE, and OU to UP, and they contain equal angles, base DU is thus equal to base UE, and triangle DOU is equal to triangle PUE, and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle OUD (is) equal to angle PUE. So, for this (reason), DUE is a straight-line [Prop. 1.14]. So, for the same (reason), BSG is also a straight-line, and BSequal to SG. And since CA is equal and parallel to DB, but CA is also equal and parallel to EG, DB is thus also equal and parallel to EG [Prop. 11.9]. And the straightlines DE and BG join them. DE is thus parallel to BG[Prop. 1.33]. Thus, angle EDT (is) equal to BGT. For (they are) alternate [Prop. 1.29]. And (angle) DTU (is equal) to GTS [Prop. 1.15]. So, DTU and GTS are two triangles having two angles equal to two angles, and one side equal to one side—(namely), that subtended by one of the equal angles—(that is), DU (equal) to GS. For they are halves of DE and BG (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, DT (is) equal to TG, and UT to TS.

Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

λθ΄.

Έὰν ἢ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῆ βάσιν παpaλληλόγpaμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ἢ τὸ παpaλληλόγpaμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα.



Έστω δύο πρίσματα ἰσοϋψῆ τὰ ΑΒΓΔΕΖ, ΗΘΚΛΜΝ, καὶ τὸ μὲν ἐχέτω βάσιν τὸ ΑΖ παραλληλόγραμμον, τὸ δὲ τὸ ΗΘΚ τρίγωνον, διπλάσιον δὲ ἔστω τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Συμπεπληρώσθω γὰρ τὰ ΑΞ, ΗΟ στερεά. ἐπεὶ διπλάσιόν ἐστι τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου, ἔστι δὲ καὶ τὸ ΘΚ παραλληλόγραμμον διπλάσιον τοῦ ΗΘΚ τριγώνου, ἴσον ἄρα ἐστὶ τὸ ΑΖ παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμω. τὰ δὲ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΞ στερεὸν τῷ ΗΟ στερεῷ. καί ἐστι τοῦ μὲν ΑΞ στερεοῦ ἤμισυ τὸ ΑΒΓΔΕΖ πρίσμα, τοῦ δὲ ΗΟ στερεοῦ ἤμισυ τὸ ΗΘΚΛΜΝ πρίσματι.

Έὰν ἄρα ἢ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῆ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ἢ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔστὶ τὰ πρίσματα· ὅπερ ἔδει δεῖξαι.

Proposition 39

If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.



Let ABCDEF and GHKLMN be two equal height prisms, and let the former have the parallelogram AF, and the latter the triangle GHK, as a base. And let parallelogram AF be twice triangle GHK. I say that prism ABCDEF is equal to prism GHKLMN.

For let the solids AO and GP have been completed. Since parallelogram AF is double triangle GHK, and parallelogram HK is also double triangle GHK[Prop. 1.34], parallelogram AF is thus equal to parallelogram HK. And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid AO is equal to solid GP. And prism ABCDEF is half of solid AO, and prism GHKLMN half of solid GP [Prop. 11.28]. Prism ABCDEF is thus equal to prism GHKLMN.

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.

ELEMENTS BOOK 12

Proportional Stereometry †

[†]The novel feature of this book is the use of the so-called *method of exhaustion* (see Prop. 10.1), a precursor to integration which is generally attributed to Eudoxus of Cnidus.

α΄.

Τὰ ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἄλληλά ἐστιν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.



Έστωσαν χύχλοι οἱ ABΓ, ZHΘ, χαὶ ἐν αὐτοῖς ὅμοια πολύγωνα ἔστω τὰ ABΓΔΕ, ZHΘΚΛ, διάμετροι δὲ τῶν χύχλων ἔστωσαν BM, HN· λέγω, ὅτι ἐστὶν ὡς τὸ ἀπὸ τῆς BM τετράγωνον πρὸς τὸ ἀπὸ τῆς HN τετράγωνον, οὕτως τὸ ABΓΔΕ πολύγωνον πρὸς τὸ ZHΘΚΛ πολύγωνον.

Ἐπεζεύχθωσαν γὰρ αί ΒΕ, ΑΜ, ΗΛ, ΖΝ. καὶ ἐπεὶ δμοιον τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνω, ἴση ἐστὶ ϰαὶ ἡ ὑπὸ ΒΑΕ γωνία τῆ ὑπὸ ΗΖΛ, ϰαί ἐστιν ὡς ἡ ΒΑ πρός την ΑΕ, οὕτως ή ΗΖ πρός την ΖΛ. δύο δη τρίγωνά έστι τὰ ΒΑΕ, ΗΖΛ μίαν γωνίαν μιᾶ γωνία ἴσην ἕχοντα τὴν ὑπὸ ΒΑΕ τῆ ὑπὸ ΗΖΛ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνω. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῆ ὑπὸ ΖΛΗ. ἀλλ ή μεν ύπο ΑΕΒ τῆ ύπο ΑΜΒ έστιν ἴση ἐπὶ γὰρ τῆς αὐτῆς περιφερείας βεβήχασιν ή δὲ ὑπὸ ΖΛΗ τῆ ὑπὸ ΖΝΗ καὶ ή ύπὸ AMB ἄρα τῆ ὑπὸ ZNH ἐστιν ἴση. ἔστι δὲ καὶ ὀρθὴ ή ὑπὸ BAM ὀρθῆ τῆ ὑπὸ HZN ἴση· καὶ ἡ λοιπὴ ἄρα τῆ λοιπῆ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ABM τρίγωνον τῷ ZHN τρίγωνω. ἀνάλογον ἄρα ἐστὶν ὡς ἡ BM πρὸς τὴν ΗΝ, οὕτως ἡ ΒΑ πρὸς τὴν ΗΖ. ἀλλὰ τοῦ μὲν τῆς ΒΜ πρός την ΗΝ λόγον διπλασίων έστιν ὁ τοῦ ἀπὸ τῆς ΒΜ τετραγώνου πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, τοῦ δὲ τῆς BA πρός τὴν HZ διπλασίων ἐστὶν ὁ τοῦ ΑΒΓΔΕ πολυγώνου πρός τὸ ΖΗΘΚΛ πολύγωνον καὶ ὡς ἄρα τὸ ἑπὸ τῆς ΒΜ τετράγωνον πρός τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρός τὸ ΖΗΘΚΛ πολύγωνον.

Τὰ ἄρα ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἄλληλά ἐστιν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

β'.

Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Έστωσαν κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ, διάμετροι δὲ αὐτῶν

Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).



Let ABC and FGH be circles, and let ABCDE and FGHKL be similar polygons (inscribed) in them (respectively), and let BM and GN be the diameters of the circles (respectively). I say that as the square on BM is to the square on GN, so polygon ABCDE (is) to polygon FGHKL.

For let BE, AM, GL, and FN have been joined. And since polygon ABCDE (is) similar to polygon FGHKL, angle BAE is also equal to (angle) GFL, and as BAis to AE, so GF (is) to FL [Def. 6.1]. So, BAE and GFL are two triangles having one angle equal to one angle, (namely), BAE (equal) to GFL, and the sides around the equal angles proportional. Triangle ABE is thus equiangular with triangle FGL [Prop. 6.6]. Thus, angle AEB is equal to (angle) FLG. But, AEB is equal to AMB, and FLG to FNG, for they stand on the same circumference [Prop. 3.27]. Thus, AMB is also equal to FNG. And the right-angle BAM is also equal to the right-angle GFN [Prop. 3.31]. Thus, the remaining (angle) is also equal to the remaining (angle) [Prop. 1.32]. Thus, triangle ABM is equiangular with triangle FGN. Thus, proportionally, as BM is to GN, so BA (is) to GF[Prop. 6.4]. But, the (ratio) of the square on BM to the square on GN is the square of the ratio of BM to GN, and the (ratio) of polygon ABCDE to polygon FGHKLis the square of the (ratio) of BA to GF [Prop. 6.20]. And, thus, as the square on BM (is) to the square on GN, so polygon ABCDE (is) to polygon FGHKL.

Thus, similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles). (Which is) the very thing it was required to show.

Proposition 2

Circles are to one another as the squares on (their) diameters.

Let *ABCD* and *EFGH* be circles, and [let] *BD* and

[ἕστωσαν] αί $B\Delta$, $Z\Theta$ · λέγω, ὅτι ἐστὶν ὡς ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον, οὕτως τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$ τετράγωνον.



Εἰ γὰρ μή ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, ἔσται ώς τὸ ἀπὸ τῆς $\mathrm{B}\Delta$ πρὸς τὸ ἀπὸ τῆς $\mathrm{Z}\Theta,$ οὕτως ὁ ΑΒΓΔ χύχλος ήτοι πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ χύχλου χωρίον ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Σ. και ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ. τὸ δὴ ἐγγεγραμμένον τετράγωνον μεῖζόν ἐστιν ἢ τὸ ήμισυ τοῦ ΕΖΗΘ χύχλου, ἐπειδήπερ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων ἐφαπτομένας [εὐθείας] τοῦ χύχλου ἀγάγωμεν, τοῦ περιγραφομένου περί τὸν κύκλον τετραγώνου ἤμισύ ἐστι τὸ ΕΖΗΘ τετράγωνον, τοῦ δὲ περιγραφέντος τετραγώνου έλάττων έστιν ὁ κύκλος· ὥστε τὸ ΕΖΗΘ ἐγγεγραμμένον τετράγωνον μεϊζόν έστι τοῦ ἡμίσεως τοῦ ΕΖΗΘ κύκλου. τετμήσθωσαν δίχα αί ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι κατά τὰ Κ, Λ, Μ, Ν σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΚ, ΚΖ, ZΛ, ΛΗ, ΗΜ, MΘ, ΘΝ, ΝΕ[.] καὶ ἕκαστον ἄρα τῶν EKZ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεϊζόν ἐστιν ἢ τὸ ἤμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Κ, Λ, Μ, Ν σημείων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν καὶ ἀναπληρώσωμεν τὰ ἐπὶ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ εὐθειῶν παραλληλόγραμμα, ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων ήμισυ έσται τοῦ καθ' ἑαυτὸ παραλληλογράμμου, άλλὰ τὸ καθ' ἑαυτὸ τμῆμα ἔλαττόν ἐστι τοῦ παραλληλογράμμου· ώστε ἕχαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ καθ' ἑαυτὸ τμήματος τοῦ χύχλου. τέμνοντες δη τὰς ὑπολειπομένας περιφερείας δίγα καὶ ἐπιζευγνύντες εὐθείας καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κύκλου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ἦ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. FH [be] their diameters. I say that as circle ABCD is to circle EFGH, so the square on BD (is) to the square on FH.



For if the circle ABCD is not to the (circle) EFGH, as the square on BD (is) to the (square) on FH, then as the (square) on BD (is) to the (square) on FH, so circle ABCD will be to some area either less than, or greater than, circle *EFGH*. Let it, first of all, be (in that ratio) to (some) lesser (area), S. And let the square EFGH have been inscribed in circle EFGH [Prop. 4.6]. So the inscribed square is greater than half of circle EFGH, inasmuch as if we draw tangents to the circle through the points E, F, G, and H, then square EFGH is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square EFGH is greater than half of circle EFGH. Let the circumferences EF, FG, GH, and HE have been cut in half at points K, L, M, and N (respectively), and let EK, KF, FL, LG, GM, MH, HN, and NE have been joined. And, thus, each of the triangles EKF, FLG, GMH, and HNE is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points K, L, M, and N, and complete the parallelograms on the straight-lines EF, FG, GH, and HE, then each of the triangles EKF, FLG, GMH, and HNE will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles EKF, FLG, GMH, and HNE is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (evenέδείχθη γὰρ ἐν τῷ πρώτω θεωρήματι τοῦ δεκάτου βιβλίου, ότι δύο μεγεθῶν ἀνίσων ἐχχειμένων, ἐὰν ἀπὸ τοῦ μείζονος άφαιρεθή μεῖζον ή τὸ ήμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἤμισυ, ϰαὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, δ ἕσται ἕλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους. λελείφθω ούν, καὶ ἔστω τὰ ἐπὶ τῶν ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ τμήματα τοῦ ΕΖΗΘ χύχλου ἐλάττονα τῆς ὑπεροχῆς, ἤ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. λοιπὸν ἄρα τὸ ΕΚΖΛΗΜΘΝ πολύγωνον με
ῖζόν ἐστι τοῦ Σ χωρίου. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΚΖ-ΛΗΜΘΝ πολυγώνω όμοιον πολύγωνον τὸ ΑΞΒΟΓΠΔΡ· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ${
m B}\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. ἀλλὰ καὶ ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρός τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον· καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖ-ΛΗΜΘΝ πολύγωνον· ἐναλλάξ ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρός τὸ ἐν αὐτῷ πολύγωνον, οὕτως τὸ Σ χωρίον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. μείζων δὲ ὁ ΑΒΓΔ κύκλος τοῦ ἐν αὐτῷ πολυγώνου· μεῖζον ἄρα καὶ τὸ Σ χωρίον τοῦ ΕΚΖΛΗΜΘΝ πολυγώνου. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν άδύνατον. ούκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ἑ $AB\Gamma\Delta$ χύχλος πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ώς τὸ ἀπὸ ΖΘ πρὸς τὸ ἀπὸ ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Λέγω δή, ὅτι οὐδὲ ὡς τὸ ἀπὸ τῆς $B\Delta$ πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς μεῖζόν τι τοῦ $EZH\Theta$ κύκλου χωρίον.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Σ. ἀνάπαλιν ἄρα [ἐστὶν] ὡς τὸ ἀπὸ τῆς ΖΘ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΔΒ, οὕτως τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον. ἀλλ' ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἕλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΖΘ πρὸς τὸ ἀπὸ τῆς ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἕλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἀδύνατον ἑδείχϑη. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ἐδείχϑη δέ, ὅτι οὐδὲ πρὸς ἕλασσον· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ ἀπὸ τῆς

Οἱ ἄρα κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

tually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle EFGHexceeds the area S. For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude [Prop. 10.1]. Therefore, let the (segments) have been left, and let the (sum of the) segments of the circle EFGH on EK, KF, FL, LG, GM, MH, HN, and NEbe less than the excess by which circle EFGH exceeds area S. Thus, the remaining polygon EKFLGMHN is greater than area S. And let the polygon AOBPCQDR, similar to the polygon EKFLGMHN, have been inscribed in circle ABCD. Thus, as the square on BD is to the square on FH, so polygon AOBPCQDR (is) to polygon EKFLGMHN [Prop. 12.1]. But, also, as the square on BD (is) to the square on FH, so circle ABCD(is) to area S. And, thus, as circle ABCD (is) to area S, so polygon AOBPGQDR (is) to polygon EKFLGMHN[Prop. 5.11]. Thus, alternately, as circle ABCD (is) to the polygon (inscribed) within it, so area S (is) to polygon *EKFLGMHN* [Prop. 5.16]. And circle *ABCD* (is) greater than the polygon (inscribed) within it. Thus, area S is also greater than polygon EKFLGMHN. But, (it is) also less. The very thing is impossible. Thus, the square on BD is not to the (square) on FH, as circle ABCD (is) to some area less than circle EFGH. So, similarly, we can show that the (square) on FH (is) not to the (square) on BD as circle EFGH (is) to some area less than circle ABCD either.

So, I say that neither (is) the (square) on BD to the (square) on FH, as circle ABCD (is) to some area greater than circle EFGH.

For, if possible, let it be (in that ratio) to (some) greater (area), S. Thus, inversely, as the square on FH [is] to the (square) on DB, so area S (is) to circle ABCD [Prop. 5.7 corr.]. But, as area S (is) to circle ABCD, so circle EFGH (is) to some area less than circle ABCD (see lemma). And, thus, as the (square) on FH (is) to the (square) on BD, so circle EFGH (is) to some area less than circle ABCD [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on BD is to the (square) on FH, so circle ABCD (is) not to some area greater than circle EFGH. And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on BD is to the (square) on FH, so circle ABCD (is) to ci

Thus, circles are to one another as the squares on

Λῆμμα.

Λέγω δή, ὅτι τοῦ Σ χωρίου μείζονος ὄντος τοῦ ΕΖΗΘ κύκλου ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἕλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Γεγονέτω γὰρ ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον. λέγω, ὅτι ἔλαττόν ἐστι τὸ Τ χωρίον τοῦ ΑΒΓΔ κύκλου. ἐπεὶ γάρ ἐστιν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον, ἐναλλάξ ἐστιν ὡς τὸ Σ χωρίον πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Τ χωρίον. μεῖζον δὲ τὸ Σ χωρίον τοῦ ΕΖΗΘ κύκλου· μείζων ἄρα καὶ ὁ ΑΒΓΔ κύκλος τοῦ Τ χωρίου. ὥστε ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἕλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἔδει δεῖξαι.

γ'.

Πᾶσα πυραμὶς τρίγωνον ἔχουσα βάσιν διαιρεῖται εἰς δύο πυραμίδας ἴσας τε καὶ ὁμοίας ἀλλήλαις καὶ [ὁμοίας] τῆ ὅλῆ τριγώνους ἐχουσας βάσεις καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.



Έστω πυραμίς, ῆς βάσις μέν ἐστι τὸ ABΓ τρίγωνον, χορυφὴ δὲ τὸ Δ σημεῖον· λέγω, ὅτι ἡ ABΓΔ πυραμὶς διαιρεῖται εἰς δύο πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἐχούσας καὶ ὁμοίας τῆ ὅλῆ καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἤμισυ τῆς ὅλης πυραμίδος.

Τετμήσθωσαν γὰρ αί AB, BΓ, ΓΑ, ΑΔ, ΔΒ, ΔΓ δίχα κατὰ τὰ Ε, Ζ, Η, Θ, Κ, Λ σημεῖα, καὶ ἐπεζεύχθωσαν αἰ ΘΕ, ΕΗ, ΗΘ, ΘΚ, ΚΛ, ΛΘ, ΚΖ, ΖΗ. ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῆ ΕΒ, ἡ δὲ ΑΘ τῆ ΔΘ, παράλληλος ἄρα ἐστὶν ἡ ΕΘ τῆ ΔΒ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΚ τῆ ΑΒ παράλληλός ἐστιν. (their) diameters. (Which is) the very thing it was required to show.

Lemma

So, I say that, area S being greater than circle EFGH, as area S is to circle ABCD, so circle EFGH (is) to some area less than circle ABCD.

For let it have been contrived that as area S (is) to circle ABCD, so circle EFGH (is) to area T. I say that area T is less than circle ABCD. For since as area S is to circle ABCD, so circle EFGH (is) to area T, alternately, as area S is to circle EFGH, so circle ABCD (is) to area T [Prop. 5.16]. And area S (is) greater than circle EFGH. Thus, circle ABCD (is) also greater than area T [Prop. 5.14]. Hence, as area S is to circle ABCD, so circle EFGH (is) to some area less than circle ABCD. (Which is) the very thing it was required to show.

Proposition 3

Any pyramid having a triangular base is divided into two pyramids having triangular bases (which are) equal, similar to one another, and [similar] to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.



Let there be a pyramid whose base is triangle ABC, and (whose) apex (is) point D. I say that pyramid ABCD is divided into two pyramids having triangular bases (which are) equal to one another, and similar to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.

For let AB, BC, CA, AD, DB, and DC have been cut in half at points E, F, G, H, K, and L (respectively). And let HE, EG, GH, HK, KL, LH, KF, and FG have been joined. Since AE is equal to EB, and AH to DH,

παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΕΒΚ· ἴση ἄρα ἐστὶν ἡ ΘΚ τῆ ΕΒ. ἀλλὰ ἡ ΕΒ τῆ ΕΑ ἐστιν ἴση· καὶ ἡ ΑΕ ἄρα τῆ ΘΚ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΑΘ τῆ Θ Δ ἴση· δύο δὴ αἱ EA, AΘ δυσὶ ταῖς ΚΘ, ΘΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρα καὶ γωνία ἡ ύπὸ ΕΑΘ γωνία τῆ ὑπὸ ΚΘΔ ἴση· βάσις ἄρα ἡ ΕΘ βάσει τῆ $\mathrm{K}\Delta$ ἐστιν ἴση. ἴσον ἄρα καὶ
 ὅμοιόν ἐστι τὸ $\mathrm{AE}\Theta$ τρίγωνον τῷ ΘΚΔ τριγώνω. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΘΗ τρίγωνον τῷ $\Theta \Lambda \Delta$ τριγώνῳ ἴσον τέ ἐστι καὶ ὅμοιον. καὶ ἐπεὶ δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΕΘ, ΘΗ παρὰ δύο εὐθείας άπτομένας άλλήλων τὰς ΚΔ, ΔΛ εἰσιν οὐκ ἐν τῷ αὐτῷ έπιπέδω ούσαι, ίσας γωνίας περιέξουσιν. ίση ἄρα έστιν ή ὑπὸ $E\Theta H$ γωνία τῆ ὑπὸ $K\Delta\Lambda$ γωνία. καὶ ἐπεὶ δύο εὐθεῖαι αἱ $E\Theta$, Θ H δυσὶ ταῖς K Δ , Δ Λ ἴσαι εἰσὶν ἑκατέρα εκατέρα, καὶ γωνία ή ὑπὸ ΕΘΗ γωνία τῆ ὑπὸ Κ $\Delta\Lambda$ ἐστιν ἴση, βάσις ẳρα ἡ ΕΗ βάσει τῆ ΚΛ [ἐστιν] ἴση· ἴσον ἄρα καὶ ὄμοιόν ἐστι τὸ ΕΘΗ τρίγωνον τῷ ΚΔΛ τριγώνω. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΕΗ τρίγωνον τῷ ΘΚΛ τριγώνω ἴσον τε καὶ ὄμοιόν ἐστιν. ἡ ἄρα πυραμίς, ής βάσις μέν έστι τὸ ΑΕΗ τρίγωνον, χορυφή δὲ τὸ Θ σημεῖον, ἴση χαὶ ὁμοία ἐστὶ πυραμίδι, ἤς βάσις μέν ἐστι τὸ $\Theta K\Lambda$ τρίγωνον, χορυφή δὲ τὸ Δ σημεῖον. χαὶ ἐπεὶ τριγώνου τοῦ ΑΔΒ παρὰ μίαν τῶν πλευρῶν τὴν ΑΒ ἦχται ἡ ΘΚ, ίσογώνιόν ἐστι τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνω, καὶ τὰς πλευρὰς ἀνάλογον ἔχουσιν· ὅμοιον ἄρα ἐστὶ τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνω. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν $\Delta B\Gamma$ τρίγωνον τῷ $\Delta K\Lambda$ τριγώνῳ ὅμοιόν ἐστιν, τὸ δὲ $A\Delta\Gamma$ τῷ ΔΛΘ. καὶ ἐπεὶ δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΒΑ, ΑΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΚΘ, ΘΛ εἰσιν ούχ έν τῶ αὐτῶ ἐπιπέδω, ἴσας γωνίας περιέξουσιν. ἴση ἄρα έστιν ἡ ὑπὸ ΒΑΓ γωνία τῆ ὑπὸ ΚΘΛ. καί ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΛ· ὄμοιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΘΚΛ τριγώνω. καὶ πυραμὶς ἄρα, ῆς βάσις μέν ἐστι τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, όμοία ἐστὶ πυραμίδι, ἤς βάσις μέν ἐστι τὸ ΘΚΛ τρίγωνον, κορυφή δὲ τὸ Δ σημεῖον. ἀλλὰ πυραμίς, ῆς βάσις μέν [ἐστι] τὸ Θ ΚΛ τρίγωνον, χορυφὴ δὲ τὸ Δ σημεῖον, ὑμοία ἐδείχϑη πυραμίδι, ής βάσις μέν έστι τὸ ΑΕΗ τρίγωνον, χορυφή δὲ τὸ Θ σημεῖον. ἑκατέρα ἄρα τῶν ΑΕΗΘ, ΘΚΛΔ πυραμίδων όμοία ἐστὶ τῆ ὅλη τῆ ΑΒΓΔ πυραμίδι.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΖ τῆ ΖΓ, διπλάσιόν ἐστι τὸ EBZH παραλληλόγραμμον τοῦ ΗΖΓ τριγώνου. καὶ ἐπεὶ, ἐὰν ἤ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ἤ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἐστὶ τὰ πρίσματα, ἴσον ἄρα ἐστὶ τὸ πρίσμα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν BKZ, EΘH, τριῶν δὲ παραλληλογράμμων τῶν EBZH, EBKΘ, ΘΚΖΗ τῷ πρισματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν HZΓ, ΘΚΛ, τριῶν δὲ παραλληλογράμμων τῶν ΚΖΓΛ, ΛΓΗΘ, ΘΚΖΗ. καὶ φανερόν, ὅτι ἐκάτρον τῶν πρισμάτων, οὖ τε βάσις τὸ EBZH παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, καὶ οῦ βάσις τὸ HZΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον, μεῖζόν ἐστιν ἑκατέρας

EH is thus parallel to DB [Prop. 6.2]. So, for the same (reasons), HK is also parallel to AB. Thus, HEBK is a parallelogram. Thus, HK is equal to EB [Prop. 1.34]. But, EB is equal to EA. Thus, AE is also equal to HK. And AH is also equal to HD. So the two (straight-lines) EA and AH are equal to the two (straight-lines) KHand HD, respectively. And angle EAH (is) equal to angle KHD [Prop. 1.29]. Thus, base EH is equal to base KD [Prop. 1.4]. Thus, triangle AEH is equal and similar to triangle HKD [Prop. 1.4]. So, for the same (reasons), triangle AHG is also equal and similar to triangle HLD. And since EH and HG are two straight-lines joining one another (which are respectively) parallel to two straight-lines joining one another, KD and DL, not being in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle EHG is equal to angle KDL. And since the two straight-lines EH and HG are equal to the two straight-lines KD and DL, respectively, and angle EHG is equal to angle KDL, base EG [is] thus equal to base KL [Prop. 1.4]. Thus, triangle EHG is equal and similar to triangle KDL. So, for the same (reasons), triangle AEG is also equal and similar to triangle HKL. Thus, the pyramid whose base is triangle AEG, and apex the point H, is equal and similar to the pyramid whose base is triangle HKL, and apex the point D[Def. 11.10]. And since HK has been drawn parallel to one of the sides, AB, of triangle ADB, triangle ADB is equiangular to triangle DHK [Prop. 1.29], and they have proportional sides. Thus, triangle ADB is similar to triangle DHK [Def. 6.1]. So, for the same (reasons), triangle DBC is also similar to triangle DKL, and ADC to DLH. And since two straight-lines joining one another, BA and AC, are parallel to two straight-lines joining one another, KH and HL, not in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle BAC is equal to (angle) KHL. And as BA is to AC, so KH (is) to HL. Thus, triangle ABC is similar to triangle HKL[Prop. 6.6]. And, thus, the pyramid whose base is triangle ABC, and apex the point D, is similar to the pyramid whose base is triangle HKL, and apex the point D[Def. 11.9]. But, the pyramid whose base [is] triangle HKL, and apex the point D, was shown (to be) similar to the pyramid whose base is triangle AEG, and apex the point H. Thus, each of the pyramids AEGH and HKLDis similar to the whole pyramid ABCD.

And since BF is equal to FC, parallelogram EBFG is double triangle GFC [Prop. 1.41]. And since, if two prisms (have) equal heights, and the former has a parallelogram as a base, and the latter a triangle, and the parallelogram (is) double the triangle, then the prisms are equal [Prop. 11.39], the prism contained by the two

τῶν πυραμίδων, ὧν βάσεις μὲν τὰ ΑΕΗ, ΘΚΛ τρίγωνα, κορυφαί, δὲ τὰ Θ , Δ σημεῖα, ἐπειδήπερ [καί] ἐὰν ἐπιζεύξωμεν τὰς ΕΖ, ΕΚ εὐθείας, τὸ μὲν πρίσμα, οὕ βάσις τὸ EBZH παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, μεῖζόν ἐστι τῆς πυραμίδος, ἤς βάσις τὸ ΕΒΖ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον. ἀλλ' ἡ πυραμίς, ἦς βάσις τὸ ΕΒΖ τρίγωνον, κορυφή δὲ τὸ Κ σημεῖον, ἴση ἐστὶ πυραμίδι, ἤς βάσις τὸ ΑΕΗ τρίγωνον, χορυφή δὲ τὸ Θ σημεῖον ὑπὸ γὰρ ἴσων καὶ ὁμοίων ἐπιπέδων περιέχονται. ὥστε καὶ τὸ πρίσμα, οὕ βάσις μέν τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εύθεῖα, μεῖζόν ἐστι πυραμίδος, ἤς βάσις μὲν τὸ ΑΕΗ τρίγωνον, χορυφή δὲ τὸ Θ σημεῖον. ἴσον δὲ τὸ μὲν πρίσμα, οὕ βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εύθεῖα, τῷ πρίσματι, οὕ βάσις μὲν τὸ ΗΖΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον. ἡ δὲ πυραμίς, ῆς βάσις τὸ ΑΕΗ τρίγωνον, κορυφή δὲ τὸ Θ σημεῖον, ἴση ἐστὶ πυραμίδι, ῆς βάσις τὸ ΘΚΛ τρίγωνον, χορυφὴ δὲ τὸ Δ σημεῖον. τὰ άρα εἰρημένα δύο πρίσματα μείζονά ἐστι τῶν εἰρημένων δύο πυραμίδων, ῶν βάσεις μὲν τὰ ΑΕΗ, ΘΚΛ τρίγωνα, χορυφαὶ δὲ τὰ Θ , Δ σημεῖα.

Η ἄρα ὅλη πυραμίς, ῆς βάσις τὸ ABΓ τρίγωνον, xoρυφὴ δὲ τὸ Δ σημεῖον, διήρηται εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις [καὶ ὁμοίας τῆ ὅλη] καὶ εἰς δύο πρίσματα ἴσα, καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος· ὅπερ ἔδει δεῖξαι.

δ'.

Έαν ῶσι δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις, διαιρεθῆ δὲ ἑκατέρα αὐτῶν εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῆ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα, ἔσται ὡς ἡ τῆς μιᾶς πυραμίδος βάσις πρὸς τὴν τῆς ἑτέρας πυραμίδος βάσιν, οὕτως τὰ ἐν τῆ μιᾶ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῆ ἑτάρα πυραμίδι πρίσματα πάντα ἰσοπληθῆ.

Έστωσαν δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις τὰς ΑΒΓ, ΔΕΖ, χορυφὰς δὲ τὰ Η, Θ σημεῖα, καὶ διηρήσθω ἑκατέρα αὐτῶν εἶς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῆ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· λέγω,

triangles BKF and EHG, and the three parallelograms EBFG, EBKH, and HKFG, is thus equal to the prism contained by the two triangles GFC and HKL, and the three parallelograms KFCL, LCGH, and HKFG. And (it is) clear that each of the prisms whose base (is) parallelogram *EBFG*, and opposite (side) straight-line HK, and whose base (is) triangle GFC, and opposite (plane) triangle HKL, is greater than each of the pyramids whose bases are triangles AEG and HKL, and apexes the points H and D (respectively), inasmuch as, if we [also] join the straight-lines EF and EK then the prism whose base (is) parallelogram EBFG, and opposite (side) straight-line HK, is greater than the pyramid whose base (is) triangle EBF, and apex the point K. But the pyramid whose base (is) triangle EBF, and apex the point K, is equal to the pyramid whose base is triangle AEG, and apex point H. For they are contained by equal and similar planes. And, hence, the prism whose base (is) parallelogram EBFG, and opposite (side) straightline HK, is greater than the pyramid whose base (is) triangle AEG, and apex the point H. And the prism whose base is parallelogram EBFG, and opposite (side) straight-line HK, (is) equal to the prism whose base (is) triangle GFC, and opposite (plane) triangle HKL. And the pyramid whose base (is) triangle AEG, and apex the point H, is equal to the pyramid whose base (is) triangle HKL, and apex the point D. Thus, the (sum of the) aforementioned two prisms is greater than the (sum of the) aforementioned two pyramids, whose bases (are) triangles AEG and HKL, and apexes the points H and D (respectively).

Thus, the whole pyramid, whose base (is) triangle ABC, and apex the point D, has been divided into two pyramids (which are) equal to one another [and similar to the whole], and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid. (Which is) the very thing it was required to show.

Proposition 4

If there are two pyramids with the same height, having trianglular bases, and each of them is divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms then as the base of one pyramid (is) to the base of the other pyramid, so (the sum of) all the prisms in one pyramid will be to (the sum of all) the equal number of prisms in the other pyramid.

Let there be two pyramids with the same height, having the triangular bases ABC and DEF, (with) apexes the points G and H (respectively). And let each of them have been divided into two pyramids equal to one anότι ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῆ ΑΒΓΗ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῆ ΔΕΖΘ πυραμίδι πρίσματα ἰσοπληθῆ.



Έπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΒΞ τῆ ΞΓ, ἡ δὲ ΑΛ τῆ ΛΓ, παράλληλος ἄρα ἐστὶν ἡ ΛΞ τῆ ΑΒ καὶ ὅμοιον τὸ ΑΒΓ τρίγωνον τῷ $\Lambda \Xi \Gamma$ τριγώνω. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΔEZ τρίγωνον τῷ ΡΦΖ τριγώνῳ ὄμοιόν ἐστιν. καὶ ἐπεὶ διπλασίων ἐστὶν ἡ μὲν ΒΓ τῆς ΓΞ, ἡ δὲ ΕΖ τῆς ΖΦ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΞ, οὕτως ἡ ΕΖ πρὸς τὴν ΖΦ. καὶ άναγέγραπται άπὸ μὲν τῶν ΒΓ, ΓΞ ὄμοιά τε καὶ ὁμοίως χείμενα εὐθύγραμμα τὰ ABΓ, ΛΞΓ, ἀπὸ δὲ τῶν EZ, ZΦ δμοιά τε καὶ ὁμοίως κείμενα [εὐθύγραμμα] τὰ ΔΕΖ, ΡΦΖ· έστιν άρα ώς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΛΞΓ τρίγωνον, οὕτως τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον ἐναλλὰξ άρα ἐστὶν ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ [τρίγωνον], ούτως τὸ $\Lambda \Xi \Gamma$ [τρίγωνον] πρὸς τὸ $P\Phi Z$ τρίγωνον. ἀλλ ώς τὸ $\Lambda \Xi \Gamma$ τρίγωνον πρὸς τὸ $P\Phi Z$ τρίγωνον, οὕτως τὸ πρίσμα, οὕ βάσις μέν [ἐστι] τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὕ βάσις μὲν τὸ ΡΦΖ τρίγωνον, άπεναντίον δὲ τὸ ΣΤΥ· καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον, οὕτως τὸ πρίσμα, οὕ βάσις μὲν τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὕ βάσις μέν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ. ὡς δὲ τὰ εἰρημένα πρίσματα πρὸς ἄλληλα, οὕτως τὸ πρίσμα, οῦ βάσις μέν τὸ ΚΒΞΛ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΟΜ εὐθεῖα, πρὸς τὸ πρίσμα, οὖ βάσις μὲν τὸ ΠΕΦΡ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΣΤ εὐθεῖα. καὶ τὰ δύο ἄρα πρίσματα, οὕ τε βάσις μὲν τὸ ΚΒΞΛ παραλληλόγραμμον, άπεναντίον δὲ ἡ OM, καὶ οὕ βάσις μὲν τὸ ΛΞΓ, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὰ πρίσματα, οὕ τε βάσις μὲν τὸ ΠΕΦΡ, ἀπεναντίον δὲ ἡ ΣΤ εὐθεῖα, καὶ οὕ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ. καὶ ὡς ἄρα ἡ ΑΒΓ βάσις πρός την ΔΕΖ βάσιν, οὕτως τὰ εἰρημένα δύο πρίσματα πρός τὰ εἰρημένα δύο πρίσματα.

Καὶ ὁμοίως, ἐἀν διαιρεθῶσιν αἱ ΟΜΝΗ, ΣΤΥΘ πυραμίδες εἴς τε δύο πρίσματα καὶ δύο πυραμίδας, ἔσται ὡς ἡ other, and similar to the whole, and into two equal prisms [Prop. 12.3]. I say that as base ABC is to base DEF, so (the sum of) all the prisms in pyramid ABCG (is) to (the sum of) all the equal number of prisms in pyramid DEFH.



For since BO is equal to OC, and AL to LC, LO is thus parallel to AB, and triangle ABC similar to triangle LOC [Prop. 12.3]. So, for the same (reasons), triangle DEF is also similar to triangle RVF. And since BC is double CO, and EF (double) FV, thus as BC (is) to CO, so EF (is) to FV. And the similar, and similarly laid out, rectilinear (figures) ABC and LOC have been described on BC and CO (respectively), and the similar, and similarly laid out, [rectilinear] (figures) DEF and RVF on EF and FV (respectively). Thus, as triangle ABC is to triangle LOC, so triangle DEF (is) to triangle RVF [Prop. 6.22]. Thus, alternately, as triangle ABC is to [triangle] DEF, so [triangle] LOC (is) to triangle RVF [Prop. 5.16]. But, as triangle LOC (is) to triangle RVF, so the prism whose base [is] triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) triangle RVF, and opposite (plane) STU(see lemma). And, thus, as triangle ABC (is) to triangle DEF, so the prism whose base (is) triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) triangle RVF, and opposite (plane) STU. And as the aforementioned prisms (are) to one another, so the prism whose base (is) parallelogram *KBOL*, and opposite (side) straight-line PM, (is) to the prism whose base (is) parallelogram QEVR, and opposite (side) straightline ST [Props. 11.39, 12.3]. Thus, also, (is) the (sum of the) two prisms-that whose base (is) parallelogram KBOL, and opposite (side) PM, and that whose base (is) LOC, and opposite (plane) PMN—to (the sum of) the (two) prisms—that whose base (is) QEVR, and opposite (side) straight-line ST, and that whose base (is) triangle RVF, and opposite (plane) STU [Prop. 5.12]. And, thus, as base ABC (is) to base DEF, so the (sum ΟΜΝ βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως τὰ ἐν τῆ ΟΜΝΗ πυραμίδι δύο πρίσματα πρὸς τὰ ἐν τῆ ΣΤΥΘ πυραμίδι δύο πρίσματα. ἀλλ' ὡς ἡ ΟΜΝ βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν. ἴσον γὰρ ἑxάτερον τῶν ΟΜΝ, ΣΤΥ τριγώνων ἑxατέρω τῶν ΛΞΓ, ΡΦΖ. xαὶ ὡς ἄρα ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ τέσσαρα πρίσματα πρὸς τὰ τέσσαρα πρίσματα. ὑμοίως δὲ xἂν τὰς ὑπολειπομένας πυραμίδας διέλωμεν εἴς τε δύο πυραμίδας xαὶ εἰς δύο πρίσματα, ἔσται ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῆ ΑΒΓΗ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῆ ΔΕΖΘ πυραμίδι πρίσματα πάντα ἰσοπληθῆ· ὅπερ ἕδει δεῖξαι.

Λῆμμα.

Ότι δέ ἐστιν ὡς τὸ ΛΞΓ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον, οὕτως τὸ πρίσμα, οὕ βάσις τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οῦ βάσις μὲν τὸ ΡΦΖ [τρίγωνον], ἀπεναντίον δὲ τὸ ΣΤΥ, οὕτω δεικτέον.

Ἐπὶ γὰρ τῆς αὐτῆς καταγραφῆς νενοήσθωσαν ἀπὸ τῶν Η, Θ κάθετοι ἐπὶ τὰ ΑΒΓ, ΔΕΖ ἐπίπεδα, ἴσαι δηλαδὴ τυγχάνουσαι διὰ τὸ ἰσοϋψεῖς ὑποχεῖσθαι τὰς πυραμίδας. καὶ ἐπεὶ δύο εὐθεῖαι ἥ τε ΗΓ καὶ ἡ ἀπὸ τοῦ Η κάθετος ύπὸ παραλλήλων ἐπιπέδων τῶν ΑΒΓ, ΟΜΝ τέμνονται, εἰς τούς αὐτοὺς λόγους τμηθήσονται. καὶ τέτμηται ἡ ΗΓ δίχα ύπὸ τοῦ OMN ἐπιπέδου κατὰ τὸ Ν· καὶ ἡ ἀπὸ τοῦ Η ἄρα κάθετος ἐπὶ τὸ ΑΒΓ ἐπίπεδον δίγα τμηθήσεται ὑπὸ τοῦ ΟΜΝ ἐπιπέδου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἀπὸ τοῦ Θ κάθετος ἐπὶ τὸ ΔΕΖ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ ΣΤΥ ἐπιπέδου. καί εἰσιν ἴσαι αἱ ἀπὸ τῶν Η, Θ κάθετοι ἐπὶ τὰ ΑΒΓ, ΔΕΖ ἐπίπεδα· ἴσαι ἄρα καὶ αἱ ἀπὸ τῶν ΟΜΝ, ΣΤΥ τριγώνων έπι τὰ ΑΒΓ, ΔΕΖ κάθετοι. ἰσοϋψῆ ἄρα [ἐστι] τὰ πρίσματα, ῶν βάσεις μέν εἰσι τὰ ΛΞΓ, ΡΦΖ τρίγωνα, ἀπεναντίον δὲ τὰ ΟΜΝ, ΣΤΥ. ὥστε καὶ τὰ στερεὰ παραλληλεπίπεδα τὰ άπὸ τῶν εἰρημένων πρισμάτων ἀναγραφόμενα ἰσοϋψῆ καὶ πρὸς ἄλληλά [εἰσιν] ὡς αἱ βάσεις· καὶ τὰ ἡμίση ἄρα ἐστὶν ώς ή ΛΞΓ βάσις πρός την ΡΦΖ βάσιν, οὕτως τὰ εἰρημένα πρίσματα πρὸς ἄλληλα· ὅπερ ἔδει δεῖξαι.

of the first) aforementioned two prisms (is) to the (sum of the second) aforementioned two prisms.

And, similarly, if pyramids PMNG and STUH are divided into two prisms, and two pyramids, as base PMN (is) to base STU, so (the sum of) the two prisms in pyramid PMNG will be to (the sum of) the two prisms in pyramid STUH. But, as base PMN (is) to base STU, so base ABC (is) to base DEF. For the triangles PMN and STU (are) equal to LOC and RVF, respectively. And, thus, as base ABC (is) to base DEF, so (the sum of) the four prisms (is) to (the sum of) the four prisms [Prop. 5.12]. So, similarly, even if we divide the pyramids left behind into two pyramids and into two prisms, as base ABC (is) to base DEF, so (the sum of) all the prisms in pyramid ABCG will be to (the sum of) all the equal number of prisms in pyramid DEFH. (Which is) the very thing it was required to show.

Lemma

And one may show, as follows, that as triangle LOC is to triangle RVF, so the prism whose base (is) triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) [triangle] RVF, and opposite (plane) STU.

For, in the same figure, let perpendiculars have been conceived (drawn) from (points) G and H to the planes ABC and DEF (respectively). These clearly turn out to be equal, on account of the pyramids being assumed (to be) of equal height. And since two straight-lines, GCand the perpendicular from G, are cut by the parallel planes ABC and PMN they will be cut in the same ratios [Prop. 11.17]. And GC was cut in half by the plane PMN at N. Thus, the perpendicular from G to the plane ABC will also be cut in half by the plane PMN. So, for the same (reasons), the perpendicular from H to the plane DEF will also be cut in half by the plane STU. And the perpendiculars from G and H to the planes ABC and *DEF* (respectively) are equal. Thus, the perpendiculars from the triangles PMN and STU to ABC and DEF (respectively, are) also equal. Thus, the prisms whose bases are triangles LOC and RVF, and opposite (sides) PMN and STU (respectively), [are] of equal height. And, hence, the parallelepiped solids described on the aforementioned prisms [are] of equal height and (are) to one another as their bases [Prop. 11.32]. Likewise, the halves (of the solids) [Prop. 11.28]. Thus, as base LOC is to base RVF, so the aforementioned prisms (are) to one another. (Which is) the very thing it was required to show.

ε΄.

Αἱ ὑπὸ τὸ αὐτὸ ὕψος οῦσαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.



Έστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ῶν βάσεις μὲν τὰ ABΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα.

Εἰ γὰρ μή ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα, ἔσται ὡς ή ΑΒΓ βάσις πρός την ΔΕΖ βάσιν, οὕτως ή ΑΒΓΗ πυραμὶς ήτοι πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεὸν ἢ πρὸς μεϊζον. ἔστω πρότερον πρός ἔλασσον τὸ Χ, καὶ διῃρἡσθω ή ΔΕΖΘ πυραμίς εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ όμοίας τῆ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· τὰ δὴ δύο πρίσαμτα μείζονά ἐστιν ἢ τὸ ἤμισυ τῆς ὅλης πυραμίδος. καὶ πάλιν αἱ ἐκ τῆς διαιρέσεως γινόμεναι πυραμίδες ὑμοίως διῃρήσθωσαν, χαὶ τοῦτο ἀεὶ γινέσθω, ἕως οῦ λειφθῶσί τινες πυραμίδες ἀπὸ τῆς ΔΕΖΘ πυραμίδος, αἴ εἰσιν ἐλάττονες τῆς ὑπεροχῆς, ἦ ύπερέχει ή ΔΕΖΘ πυραμίς τοῦ Χ στερεοῦ. λελείφθωσαν καὶ έστωσαν λόγου ένεκεν αί ΔΠΡΣ, ΣΤΥΘ· λοιπὰ ἄρα τὰ ἐν τῆ ΔΕΖΘ πυραμίδι πρίσματα μείζονά ἐστι τοῦ Χ στερεοῦ. διηρήσθω καὶ ἡ ΑΒΓΗ πυραμὶς ὁμοίως καὶ ἰσοπληθῶς τῆ $\Delta EZ\Theta$ πυραμίδι· ἔστιν ἄρα ὡς ἡ ABΓ βάσις πρὸς τὴν ΔEZ βάσιν, οὕτως τὰ ἐν τῆ ΑΒΓΗ πυραμίδι πρίσματα πρὸς τὰ ἐν τῆ ΔΕΖΘ πυραμίδι πρίσματα, ἀλλὰ καὶ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὸ Χ στερεόν καὶ ὡς ἄρα ἡ ΑΒΓΗ πυραμὶς πρὸς τὸ Χ στερεόν, οὕτως τὰ ἐν τῆ ΑΒΓΗ πυραμίδι πρίσματα πρὸς τὰ ἐν τῆ ΔΕΖΘ πυραμίδι πρίσματα έναλλὰξ ἄρα ὡς ἡ ΑΒΓΗ πυραμὶς πρὸς τὰ ἐν αὐτῆ πρίσματα, οὕτως τὸ Χ στερεὸν πρὸς τὰ ἐν τῆ ΔΕΖΘ πυραμίδι πρίσματα. μείζων δὲ ἡ ΑΒΓΗ πυραμὶς τῶν έν αὐτῆ πρισμάτων· μεῖζον ἄρα καὶ τὸ Χ στερεὸν τῶν ἐν τῆ ΔΕΖΘ πυραμίδι πρισμάτων. ἀλλὰ καὶ ἕλαττον· ὅπερ ἐστὶν άδύνατον. ούχ ἄρα ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως
ἡ ABΓΗ πυραμὶς πρὸς ἔλασσόν τι τῆς $\Delta EZ\Theta$ πυραμίδος στερεόν. ὁμοίως δὴ δειχθήσεται, ὅτι οὐδὲ ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἕλαττόν τι τῆς ΑΒΓΗ πυραμίδος στερεόν.

Λέγω δή, ὅτι οὐχ ἔστιν οὐδὲ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς μεῖζόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν.

Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the triangles ABC and DEF, and apexes the points G and H (respectively). I say that as base ABC is to base DEF, so pyramid ABCG (is) to pyramid DEFH.

For if base ABC is not to base DEF, as pyramid ABCG (is) to pyramid DEFH, then base ABC will be to base DEF, as pyramid ABCG (is) to some solid either less than, or greater than, pyramid DEFH. Let it, first of all, be (in this ratio) to (some) lesser (solid), W. And let pyramid DEFH have been divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms. So, the (sum of the) two prisms is greater than half of the whole pyramid [Prop. 12.3]. And, again, let the pyramids generated by the division have been similarly divided, and let this be done continually until some pyramids are left from pyramid DEFHwhich (when added together) are less than the excess by which pyramid DEFH exceeds the solid W [Prop. 10.1]. Let them have been left, and, for the sake of argument, let them be DQRS and STUH. Thus, the (sum of the) remaining prisms within pyramid DEFH is greater than solid W. Let pyramid ABCG also have been divided similarly, and a similar number of times, as pyramid DEFH. Thus, as base ABC is to base DEF, so the (sum of the) prisms within pyramid ABCG (is) to the (sum of the) prisms within pyramid DEFH [Prop. 12.4]. But, also, as base ABC (is) to base DEF, so pyramid ABCG (is) to solid W. And, thus, as pyramid ABCG (is) to solid W, so the (sum of the) prisms within pyramid ABCG(is) to the (sum of the) prisms within pyramid DEFH[Prop. 5.11]. Thus, alternately, as pyramid *ABCG* (is) to the (sum of the) prisms within it, so solid W (is) to the (sum of the) prisms within pyramid DEFH [Prop. 5.16]. And pyramid *ABCG* (is) greater than the (sum of the) prisms within it. Thus, solid W (is) also greater than the (sum of the) prisms within pyramid *DEFH* [Prop. 5.14]. But, (it is) also less. This very thing is impossible. Thus, as base ABC is to base DEF, so pyramid ABCG (is) Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Χ· ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως τὸ Χ στερεὸν πρὸς τὴν ΑΒΓΗ πυραμίδα. ὡς δὲ τὸ Χ στερεὸν πρὸς τὴν ΑΒΓΗ πυραμίδα, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλασσόν τι τῆς ΑΒΓΗ πυραμίδος, ὡς ἔμπροσθεν ἐδείχθη· καὶ ὡς ἄρα ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλασσόν τι τῆς ΑΒΓΗ πυραμίδος· ὅπερ ἄτοπον ἐδείχθη. οὐχ ἄρα ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς μεῖζόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἕλασσον. ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα· ὅπερ ἕδει δεῖξαι.

র্ণ.

Ai ὑπὸ τὸ αὐτὸ ὕψος οῦσαι πυραμίδες καὶ πολυγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.

Ν

Κ

Θ

Η

М

В

Г



Έπεζεύχθωσαν γὰρ αἱ ΑΓ, ΑΔ, ΖΘ, ΖΚ. ἐπεὶ οὖν δύο πυραμίδες εἰσιν αἱ ΑΒΓΜ, ΑΓΔΜ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, πρὸς ἀλλήλας εἰσιν ὡς αἱ βάσεις ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΑΓΔ βάσιν, οὕτως ἡ ΑΒΓΜ πυραμὶς πρὸς τὴν ΑΓΔΜ πυραμίδα. καὶ συνθέντι ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΑΓΔ βάσιν, οὕτως ἡ ΑΒΓΔΜ

not to some solid less than pyramid DEFH. So, similarly, we can show that base DEF is not to base ABC, as pyramid DEFH (is) to some solid less than pyramid ABCG either.

So, I say that neither is base ABC to base DEF, as pyramid ABCG (is) to some solid greater than pyramid DEFH.

For, if possible, let it be (in this ratio) to some greater (solid), W. Thus, inversely, as base DEF (is) to base ABC, so solid W (is) to pyramid ABCG [Prop. 5.7. corr.]. And as solid W (is) to pyramid ABCG, so pyramid DEFH (is) to some (solid) less than pyramid ABCG, as shown before [Prop. 12.2 lem.]. And, thus, as base DEF (is) to base ABC, so pyramid DEFH (is) to some (solid) less than pyramid ABCG [Prop. 5.11]. The very thing was shown (to be) absurd. Thus, base ABC is not to base DEF, as pyramid ABCG (is) to some solid greater than pyramid DEFH. And, it was shown that neither (is it in this ratio) to a lesser (solid). Thus, as base ABC is to base DEF, so pyramid ABCG (is) to yramid DEFH. (Which is) the very thing it was required to show.

Proposition 6

Pyramids which are of the same height, and have polygonal bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the polygons ABCDE and FGHKL, and apexes the points M and N (respectively). I say that as base ABCDE is to base FGHKL, so pyramid ABCDEM (is) to pyramid FGHKLN.

For let AC, AD, FH, and FK have been joined. Therefore, since ABCM and ACDM are two pyramids having triangular bases and equal height, they are to one another as their bases [Prop. 12.5]. Thus, as base ABC is to base ACD, so pyramid ABCM (is) to pyramid ACDM. And, via composition, as base ABCD

πυραμίς πρός την ΑΓΔΜ πυραμίδα. ἀλλὰ καὶ ὡς ἡ ΑΓΔ βάσις πρός τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΓΔΜ πυραμὶς πρός τὴν ΑΔΕΜ πυραμίδα. δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν $A\Delta E$ βάσιν, οὕτως ἡ $AB\Gamma\Delta M$ πυραμὶς πρὸς τὴν $A\Delta EM$ πυραμίδα. καὶ συνθέντι πάλιν, ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ή ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. ὑμοίως δὴ δειχθήσεται, ὅτι καὶ ὡς ἡ ΖΗΘΚΛ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως καὶ ἡ ΖΗΘΚΛΝ πυραμὶς πρός την ΖΗΘΝ πυραμίδα. και έπει δύο πυραμίδες είσιν αί ΑΔΕΜ, ΖΗΘΝ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, έστιν άρα ώς ή ΑΔΕ βάσις πρός την ΖΗΘ βάσιν, οὕτως ή ΑΔΕΜ πυραμίς πρός την ΖΗΘΝ πυραμίδα. ἀλλ' ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΑΒΓΔΕ βάσιν, οὕτως ῆν ἡ ΑΔΕΜ πυραμίς πρός την ΑΒΓΔΕΜ πυραμίδα. και δι' ίσου άρα ώς ή ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμίς πρός τὴν ΖΗΘΝ πυραμίδα. ἀλλὰ μὴν καὶ ὡς ἡ ΖΗΘ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἦν καὶ ἡ ΖΗΘΝ πυραμίς πρός την ΖΗΘΚΛΝ πυραμίδα, και δι' ίσου άρα ώς ή ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμίς πρός την ΖΗΘΚΛΝ πυραμίδα. ὅπερ ἔδει δεῖξαι.

ζ.

Πᾶν πρίσμα τρίγωνον ἔχον βάσιν διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἐχούσας.



Έστω πρίσμα, οὕ βάσις μὲν τὸ ABΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ· λέγω, ὅτι τὸ ΑΒΓΔΕΖ πρίσμα διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἐχούσας βάσεις.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΔ, ΕΓ, ΓΔ. ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΑΒΕΔ, διάμετρος δὲ αὐτὸῦ ἐστιν ἡ ΒΔ, ἴσον ἄρα ἐστι τὸ ΑΒΔ τρίγωνον τῷ ΕΒΔ τρίγωνῳ·

(is) to base ACD, so pyramid ABCDM (is) to pyramid ACDM [Prop. 5.18]. But, as base ACD (is) to base ADE, so pyramid ACDM (is) also to pyramid ADEM [Prop. 12.5]. Thus, via equality, as base ABCD (is) to base ADE, so pyramid ABCDM (is) to pyramid ADEM [Prop. 5.22]. And, again, via composition, as base ABCDE (is) to base ADE, so pyramid ABCDEM (is) to pyramid ADEM [Prop. 5.18]. So, similarly, it can also be shown that as base FGHKL (is) to base FGH, so pyramid FGHKLN (is) also to pyramid FGHN. And since ADEM and FGHN are two pyramids having triangular bases and equal height, thus as base ADE (is) to base FGH, so pyramid ADEM (is) to pyramid FGHN[Prop. 12.5]. But, as base ADE (is) to base ABCDE, so pyramid ADEM (was) to pyramid ABCDEM. Thus, via equality, as base ABCDE (is) to base FGH, so pyramid ABCDEM (is) also to pyramid FGHN [Prop. 5.22]. But, furthermore, as base FGH (is) to base FGHKL, so pyramid FGHN was also to pyramid FGHKLN. Thus, via equality, as base ABCDE (is) to base FGHKL, so pyramid ABCDEM (is) also to pyramid FGHKLN[Prop. 5.22]. (Which is) the very thing it was required to show.

Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to one another.



Let there be a prism whose base (is) triangle ABC, and opposite (plane) DEF. I say that prism ABCDEF is divided into three pyramids having triangular bases (which are) equal to one another.

For let BD, EC, and CD have been joined. Since ABED is a parallelogram, and BD is its diagonal, triangle ABD is thus equal to triangle EBD [Prop. 1.34].

καὶ ἡ πυραμὶς ἄρα, ἦς βάσις μὲν τὸ ΑΒΔ τρίγωνον, κορυφή δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἤς βάσις μέν έστι τὸ ΔΕΒ τρίγωνον, χορυφή δὲ τὸ Γ σημεῖον. ἀλλὰ ή πυραμίς, ής βάσις μέν ἐστι τὸ ΔΕΒ τρίγωνον, χορυφή δὲ τὸ Γ σημεῖον, ἡ αὐτή ἐστι πυραμίδι, ἤς βάσις μέν ἐστι τὸ ΕΒΓ τρίγωνον, χορυφή δὲ τὸ Δ σημεῖον. ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχεται. καὶ πυραμὶς ἄρα, ἤς βάσις μέν έστι τὸ ΑΒΔ τρίγωνον, χορυφή δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ής βάσις μέν έστι τὸ ΕΒΓ τρίγωνον, χορυφή δὲ τὸ Δ σημεῖον. πάλιν, ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΖΓΒΕ, διάμετρος δέ ἐστιν αὐτοῦ ἡ ΓΕ, ἴσον ἐστὶ τὸ ΓΕΖ τρίγωνον τῷ ΓΒΕ τριγώνω. καὶ πυραμὶς ἄρα, ῆς βάσις μέν έστι τὸ BFE τρίγωνον, χορυφή δὲ τὸ Δ σημεῖον, ἴση έστι πυραμίδι, ής βάσις μέν έστι το ΕΓΖ τρίγωνον, χορυφή δὲ τὸ Δ σημεῖον.
ἡ δὲ πυραμίς, ἦς βάσις μέν ἐστι τὸ BFE τρίγωνον, χορυφή δὲ τὸ Δ σημεῖον, ἴση ἐδείχθη πυραμίδι, ῆς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον καὶ πυραμὶς ἄρα, ἦς βάσις μέν ἐστι τὸ ΓΕΖ τρίγωνον, κορυφή δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἤς βάσις μέν [ἐστι] τὸ ΑΒΔ τρίγωνον, χορυφὴ δὲ τὸ Γ σημεῖον διήρηται άρα τὸ ΑΒΓΔΕΖ πρίσμα εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους έχούσας βάσεις.

Καὶ ἐπεὶ πυραμίς, ῆς βάσις μέν ἐστι τὸ ABΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτή ἐστι πυραμίδι, ῆς βάσις τὸ ΓΑΒ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχονται· ἡ δὲ πυραμίς, ῆς βάσις τὸ ABΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, τρίτον ἐδείχθη τοῦ πρίσματος, οῦ βάσις τὸ ABΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ, καὶ ἡ πυραμὶς ἄρα, ῆς βάσις τὸ ABΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, τρίτον ἐστὶ τοῦ πρίσματος τοῦ ἔχοντος βάσις τὴν αὐτὴν τὸ ABΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ.

Πόρισμα.

Έχ δὴ τούτου φανερόν, ὅτι πᾶσα πυραμὶς τρίτον μέρος ἐστὶ τοῦ πρίσματος τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῆ χαὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

η'.

Αἱ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγω εἰσὶ τῶν ὁμολόγων πλευρῶν.

Έστωσαν ὄμοιαι καὶ ὁμοίως κείμεναι πυραμίδες, ῶν βάσεις μέν εἰσι τὰ ΑΒΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. And, thus, the pyramid whose base (is) triangle ABD, and apex the point C, is equal to the pyramid whose base is triangle DEB, and apex the point C [Prop. 12.5]. But, the pyramid whose base is triangle DEB, and apex the point C, is the same as the pyramid whose base is triangle EBC, and apex the point D. For they are contained by the same planes. And, thus, the pyramid whose base is ABD, and apex the point C, is equal to the pyramid whose base is EBC and apex the point D. Again, since FCBE is a parallelogram, and CE is its diagonal, triangle CEF is equal to triangle CBE [Prop. 1.34]. And, thus, the pyramid whose base is triangle BCE, and apex the point D, is equal to the pyramid whose base is triangle ECF, and apex the point D [Prop. 12.5]. And the pyramid whose base is triangle BCE, and apex the point D, was shown (to be) equal to the pyramid whose base is triangle ABD, and apex the point C. Thus, the pyramid whose base is triangle CEF, and apex the point D, is also equal to the pyramid whose base [is] triangle ABD, and apex the point C. Thus, the prism ABCDEF has been divided into three pyramids having triangular bases (which are) equal to one another.

And since the pyramid whose base is triangle ABD, and apex the point C, is the same as the pyramid whose base is triangle CAB, and apex the point D. For they are contained by the same planes. And the pyramid whose base (is) triangle ABD, and apex the point C, was shown (to be) a third of the prism whose base is triangle ABC, and opposite (plane) DEF, thus the pyramid whose base is triangle ABC, and apex the point D, is also a third of the pyramid having the same base, triangle ABC, and opposite (plane) DEF.

Corollary

And, from this, (it is) clear that any pyramid is the third part of the prism having the same base as it, and an equal height. (Which is) the very thing it was required to show.

Proposition 8

Similar pyramids which also have triangular bases are in the cubed ratio of their corresponding sides.

Let there be similar, and similarly laid out, pyramids whose bases are triangles ABC and DEF, and apexes the points G and H (respectively). I say that pyramid ABCG has to pyramid DEFH the cubed ratio of that BC (has) to EF.



Συμπεπληρώσθω γὰρ τὰ ΒΗΜΛ, ΕΘΠΟ στερεὰ παραλληλεπίπεδα.
 καὶ ἐπεὶ ὁμοία ἐστὶν ἡ ABFH πυραμὶς τ
ῆ $\Delta EZ\Theta$ πυραμίδι, ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΑΒΓ γωνία τῆ ὑπὸ ΔΕΖ γωνία, ή δὲ ὑπὸ ΗΒΓ τῆ ὑπὸ ΘΕΖ, ή δὲ ὑπὸ ΑΒΗ τῆ ὑπὸ $\Delta E\Theta$, καί ἐστιν ὡς ἡ AB πρὸς τὴν ΔE , οὕτως ἡ BΓ πρὸς τὴν ΕΖ, καὶ ἡ ΒΗ πρὸς τὴν ΕΘ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΔE , οὕτως ἡ $B\Gamma$ πρὸς τὴν EZ, καὶ περὶ ἴσας γωνίας αί πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ BM παραλληλόγραμμον τῷ ΕΠ παραλληλογράμμω. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΒΝ τῷ ΕΡ ὄμοιόν ἐστι, τὸ δὲ ΒΚ τῷ ΕΞ· τὰ τρία άρα τὰ MB, BK, BN τρισὶ τοῖς ΕΠ, ΕΞ, ΕΡ ὅμοιά ἐστιν. άλλὰ τὰ μὲν τρία τὰ MB, BK, BN τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστιν, τὰ δὲ τρία τὰ ΕΠ, ΕΞ, ΕΡ τρισὶ τοῖς άπεναντίον ίσα τε καὶ ὅμοιά ἐστιν. τὰ ΒΗΜΛ, ΕΘΠΟ ἄρα στερεὰ ὑπὸ ὁμοίων ἐπιπέδων ἴσων τὸ πλῆθος περιέχεται. όμοιον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν τῷ ΕΘΠΟ στερεῷ. τὰ δὲ ὄμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὑμολόγων πλευρῶν. τὸ ΒΗΜΛ ἄρα στερεὸν πρὸς τὸ ΕΘΠΟ στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ὑμόλογος πλευρά ή ΒΓ πρός την δμόλογον πλευράν την ΕΖ. ώς δε τὸ ΒΗΜΛ στερεόν πρός τὸ ΕΘΠΟ στερεόν, οὕτως ή ΑΒΓΗ πυραμίς πρός την ΔΕΖΘ πυραμίδα, ἐπειδήπερ ή πυραμίς ἕκτον μέρος ἐστὶ τοῦ στερεοῦ διὰ τὸ καὶ τὸ πρίσμα ἤμισυ ον τοῦ στερεοῦ παραλληλεπιπέδου τριπλάσιον εἶναι τῆς πυραμίδος. και ή ΑΒΓΗ ἄρα πυραμίς πρός την ΔΕΖΘ πυραμίδα τριπλασίονα λόγον έχει ήπερ ή ΒΓ πρός την ΕΖὄπερ ἔδει δεῖξαι.

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι καὶ αἱ πολυγώνους ἔχουσαι βάσεις ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγω εἰσὶ τῶν ὁμολόγων πλευρῶν. διαιρεθεισῶν γὰρ αὐτῶν εἰς τὰς ἐν αὐταῖς πυραμίδας τριγώνους βάσεις ἐχούσας τῷ καὶ τὰ ὅμοια πολύγωνα τῶν βάσεων εἰς ὅμοια τρίγωνα διαιρεῖσθαι καὶ ἴσα τῷ πλήθει καὶ ὁμόλογα τοῖς ὅλοις ἔσται



For let the parallelepiped solids BGML and EHQPhave been completed. And since pyramid ABCG is similar to pyramid *DEFH*, angle *ABC* is thus equal to angle DEF, and GBC to HEF, and ABG to DEH. And as AB is to DE, so BC (is) to EF, and BG to EH [Def. 11.9]. And since as AB is to DE, so BC (is) to EF, and (so) the sides around equal angles are proportional, parallelogram BM is thus similar to paralleleogram EQ. So, for the same (reasons), BN is also similar to ER, and BK to EO. Thus, the three (parallelograms) MB, BK, and BN are similar to the three (parallelograms) EQ, *EO*, *ER* (respectively). But, the three (parallelograms) MB, BK, and BN are (both) equal and similar to the three opposite (parallelograms), and the three (parallelograms) EQ, EO, and ER are (both) equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the solids BGML and EHQP are contained by equal numbers of similar (and similarly laid out) planes. Thus, solid BGML is similar to solid EHQP [Def. 11.9]. And similar parallelepiped solids are in the cubed ratio of corresponding sides [Prop. 11.33]. Thus, solid BGML has to solid EHQP the cubed ratio that the corresponding side BC (has) to the corresponding side EF. And as solid BGML (is) to solid EHQP, so pyramid ABCG (is) to pyramid DEFH, inasmuch as the pyramid is the sixth part of the solid, on account of the prism, being half of the parallelepiped solid [Prop. 11.28], also being three times the pyramid [Prop. 12.7]. Thus, pyramid ABCG also has to pyramid DEFH the cubed ratio that BC (has) to EF. (Which is) the very thing it was required to show.

Corollary

So, from this, (it is) also clear that similar pyramids having polygonal bases (are) to one another as the cubed ratio of their corresponding sides. For, dividing them into the pyramids (contained) within them which have triangular bases, with the similar polygons of the bases also being divided into similar triangles (which are)

ΣΤΟΙΧΕΙΩΝ ιβ'.

ώς [ή] ἐν τῆ ἑτέρα μία πυραμὶς τρίγωνον ἔχουσα βάσιν πρὸς τὴν ἐν τῆ ἑτέρα μίαν πυραμίδα τρίγωνον ἔχουσαν βάσιν, οὕτως καὶ ἄπασαι αἱ ἐν τῆ ἑτέρα πυραμίδι πυραμίδες τριγώνους ἔχουσαι βάσεις πρὸς τὰς ἐν τῆ ἑτέρα πυραμίδι πυραμίδας τριγώνους βάσεις ἐχούσας, τουτέστιν αὐτὴ ἡ πολύγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν πολύγωνον βάσιν ἔχουσαν πυραμίδα. ἡ δὲ τρίγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν τρίγωνον βάσιν ἔχουσαν ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγον πλευρῶν. καὶ ἡ πολύγωνον ἄρα βάσιν ἔχουσα πρὸς τὴν ὁμοίαν βάσιν ἔχουσαν τριπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

θ'.

Τῶν ἴσων πυραμίδων καὶ τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ῶν πυραμίδων τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσιν ἐκεῖναι.



Έστωσαν γὰρ ἴσαι πυραμίδες τριγώνους βάσεις ἔχουσαι τὰς ΑΒΓ, ΔΕΖ, χορυφὰς δὲ τὰ Η, Θ σημεῖα λέγω, ὅτι τῶν ΑΒΓΗ, ΔΕΖΘ πυραμίδων ἀντιπεπόνθασιν αἰ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος.

Συμπεπληρώσθω γὰρ τὰ ΒΗΜΛ, ΕΘΠΟ στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῆ ΔΕΖΘ πυραμίδι, καί ἐστι τῆς μὲν ΑΒΓΗ πυραμίδος ἑξαπλάσιον τὸ ΒΗΜΛ στερεόν, τῆς δὲ ΔΕΖΘ πυραμίδος ἑξαπλάσιον τὸ ΕΘΠΟ στερεόν, ἴσον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν τῷ ΕΘΠΟ στερεῷ. τῶν δὲ ἴσων στερεῶν παραλληλεπιπώδων both equal in number, and corresponding, to the wholes [Prop. 6.20]. As one pyramid having a triangular base in the former (pyramid having a polygonal base is) to one pyramid having a triangular base in the latter (pyramid having a polygonal base), so (the sum of) all the pyramids having triangular bases in the former pyramid will also be to (the sum of) all the pyramids having triangular bases in the latter pyramid [Prop. 5.12]-that is to say, the (former) pyramid itself having a polygonal base to the (latter) pyramid having a polygonal base. And a pyramid having a triangular base is to a (pyramid) having a triangular base in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, a (pyramid) having a polygonal base also has to to a (pyramid) having a similar base the cubed ratio of a (corresponding) side to a (corresponding) side.

Proposition 9

The bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids which have triangular bases whose bases are reciprocally proportional to their heights are equal.



For let there be (two) equal pyramids having the triangular bases ABC and DEF, and apexes the points Gand H (respectively). I say that the bases of the pyramids ABCG and DEFH are reciprocally proportional to their heights, and (so) that as base ABC is to base DEF, so the height of pyramid DEFH (is) to the height of pyramid ABCG.

For let the parallelepiped solids BGML and EHQP have been completed. And since pyramid ABCG is equal to pyramid DEFH, and solid BGML is six times pyramid ABCG (see previous proposition), and solid EHQP (is) six times pyramid DEFH, solid BGML is

άντιπεπόνθασιν αί βάσεις τοῖς ὕψεσιν· ἔστιν ἄρα ὡς ἡ BM βάσις πρὸς τὴν ΕΠ βάσιν, οὕτως τὸ τοῦ ΕΘΠΟ στερεοῦ ὕψος πρὸς τὸ τοῦ BHMΛ στερεοῦ ὕψος. ἀλλ' ὡς ἡ BM βάσις πρὸς τὴν ΕΠ, οὕτως τὸ ABΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον. καὶ ὡς ἄρα τὸ ABΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον, οὕτως τὸ τοῦ ΕΘΠΟ στερεοῦ ὕψος πρὸς τὸ τοῦ BHMΛ στερεοῦ ὕψος. ἀλλὰ τὸ μὲν τοῦ ΕΘΠΟ στερεοῦ ὕψος τὸ αὐτὸ ἐστι τῷ τῆς ΔΕΖΘ πυραμίδος ὕψει, τὸ δὲ τοῦ BHMΛ στερεοῦ ὕψος τὸ αὐτό ἐστι τῷ τῆς ABΓH πυραμίδος ὕψει· ἔστιν ἄρα ὡς ἡ ABΓ βάσις πρὸς τὸ τῆς ABΓH πυραμίδος ὕψος. τῶν ABΓH, ΔΕΖΘ ἄρα πυραμίδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Άλλὰ δὴ τῶν ΑΒΓΗ, ΔΕΖΘ πυραμίδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος· λέγω, ὅτι ἴση ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ABΓ βάσις πρὸς τὴν ΔEZ βάσιν, οὕτως τὸ τῆς $\Delta EZ\Theta$ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος, ἀλλ' ὡς ή ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ ΒΜ παραλληλόγραμμον πρός τὸ ΕΠ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ ΒΜ παραλληλόγραμμον πρὸς τὸ ΕΠ παραλληλόγραμμον, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος. ἀλλὰ τὸ [μὲν] τῆς ΔΕΖΘ πυραμίδος ὕψος τὸ αὐτό ἐστι τῷ τοῦ ΕΘΠΟ παραλληλεπιπέδου ὕψει, τὸ δὲ τῆς ΑΒΓΗ πυραμίδος ὕψος τὸ αὐτό ἐστι τῷ τοῦ ΒΗΜΛ παραλληλεπιπέδου ὕψει· ἔστιν ἄρα ὡς ἡ ΒΜ βάσις πρὸς τήν ΕΠ βάσιν, οὕτως τὸ τοῦ ΕΘΠΟ παραλληλεπιπέδου ύψος πρὸς τὸ τοῦ ΒΗΜΛ παραλληλεπιπέδου ὕψος. ὧν δὲ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐχεῖνα· ἴσον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεόν παραλληλεπίπεδον τῷ ΕΘΠΟ στερεῷ παραλληλεπιπέδω. καί ἐστι τοῦ μὲν ΒΗΜΛ ἕκτον μέρος ἡ ΑΒΓΗ πυραμίς, τοῦ δὲ ΕΘΠΟ παραλληλεπιπέδου ἕκτον μέρος ἡ $\Delta EZ\Theta$ πυραμίς· ἴση ἄρα ἡ ABFH πυραμὶς τῃ $\Delta EZ\Theta$ πυραμίδι.

Τῶν ἄρα ἴσων πυραμίδων καὶ τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ῶν πυραμίδων τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσὶν ἐκεῖναι· ὅπερ ἔδει δεῖξαι.

ι΄.

Πᾶς κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον.

Έχέτω γὰρ κῶνος κυλίνδρῷ βάσιν τε τὴν αὐτὴν τὸν

thus equal to solid EHQP. And the bases of equal parallelepiped solids are reciprocally proportional to their heights [Prop. 11.34]. Thus, as base BM is to base EQ, so the height of solid EHQP (is) to the height of solid BGML. But, as base BM (is) to base EQ, so triangle ABC (is) to triangle DEF [Prop. 1.34]. And, thus, as triangle ABC (is) to triangle DEF, so the height of solid EHQP (is) to the height of solid BGML [Prop. 5.11]. But, the height of solid EHQP is the same as the height of pyramid DEFH, and the height of solid BGML is the same as the height of pyramid ABCG. Thus, as base ABC is to base DEF, so the height of pyramid DEFH(is) to the height of pyramid ABCG. Thus, the bases of pyramids ABCG and DEFH are reciprocally proportional to their heights.

And so, let the bases of pyramids ABCG and DEFH be reciprocally proportional to their heights, and (thus) let base ABC be to base DEF, as the height of pyramid DEFH (is) to the height of pyramid ABCG. I say that pyramid ABCG is equal to pyramid DEFH.

For, with the same construction, since as base ABCis to base DEF, so the height of pyramid DEFH (is) to the height of pyramid ABCG, but as base ABC (is) to base DEF, so parallelogram BM (is) to parallelogram EQ [Prop. 1.34], thus as parallelogram BM (is) to parallelogram EQ, so the height of pyramid DEFH (is) also to the height of pyramid ABCG [Prop. 5.11]. But, the height of pyramid DEFH is the same as the height of parallelepiped EHQP, and the height of pyramid ABCGis the same as the height of parallelepiped *BGML*. Thus, as base BM is to base EQ, so the height of parallelepiped EHQP (is) to the height of parallelepiped BGML. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal [Prop. 11.34]. Thus, the parallelepiped solid *BGML* is equal to the parallelepiped solid EHQP. And pyramid ABCG is a sixth part of BGML, and pyramid DEFH a sixth part of parallelepiped EHQP. Thus, pyramid ABCG is equal to pyramid *DEFH*.

Thus, the bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids having triangular bases whose bases are reciprocally proportional to their heights are equal. (Which is) the very thing it was required to show.

Proposition 10

Every cone is the third part of the cylinder which has the same base as it, and an equal height.

For let there be a cone (with) the same base as a cylin-

ΑΒΓΔ κύκλον καὶ ὕψος ἴσον· λέγω, ὅτι ὁ κῶνος τοῦ κυλίνδρου τρίτον ἐστὶ μέρος, τουτέστιν ὅτι ὁ κύλινδρος τοῦ κώνου τριπλασίων ἐστίν.



Εἰ γὰρ μή ἐστιν ὁ κύλινδρος τοῦ κώνου τριπλασίων, έσται ὁ κύλινδρος τοῦ κώνου ἤτοι μείζων ἢ τριπλασίων ἢ ἐλάσσων ἢ τριπλασίων. ἔστω πρότερον μείζων ἢ τριπλασίων, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ δὴ ΑΒΓΔ τετράγωνον μείζόν ἐστιν ἢ τὸ ήμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πρίσμα ἰσοϋψὲς τῷ χυλίνδρῳ. τὸ δὴ ἀνιστάμενον πρίσμα μεϊζόν ἐστιν ἢ τὸ ἤμισυ τοῦ χυλίνδου, ἐπειδήπερ κἂν περί τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψωμεν, τὸ ἐγγεγραμμένον εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον ημισύ ἐστι τοῦ περιγεγραμμένου· καί ἐστι τὰ ἀπ' αὐτῶν άνιστάμενα στερεὰ παραλληλεπίπεδα πρίσματα ἰσοϋψῆ· τὰ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς άλληλά ἐστιν ὡς αἱ βάσεις· καὶ τὸ ἐπὶ τοῦ ΑΒΓΔ ἄρα τετραγώνου ἀνασταθέν πρίσμα ἤμισύ ἐστι τοῦ ἀνασταθέντος πρίσματος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου. χαί ἐστιν ὁ χύλινδρος ἐλάττων τοῦ πρίσματος τοῦ ἀνατραθέντος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· τὸ ἄρα πρίσμα τὸ ἀνασταθὲν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ἰσοϋψὲς τῷ κυλίνδρῳ μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ χυλίνδρου. τετμήσθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατά τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ έπεζεύχθωσαν αί ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μειζόν έστιν η τὸ ήμισυ τοῦ χαθ' ἑαυτὸ τηήματος τοῦ ΑΒΓΔ χύχλου, ώς ἔμπροσθεν ἐδείχνυμεν. ἀνεστάτω ἐφ' έκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα ίσοϋψη τῷ χυλίνδρω· καὶ ἕχαστον ἄρα τῶν ἀνασταθέντων πρισμάτων μεῖζόν ἐστιν ἢ τὸ ἤμισυ μέρος τοῦ χαθ' ἑαυτὸ τμήματος τοῦ χυλίνδρου, ἐπειδήπερ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων παραλλήλους ταῖς ΑΒ, ΒΓ, ΓΔ, ΔΑ ἀγάγωμεν, καὶ συμπληρώσωμεν τὰ ἐπὶ τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ παραλder, (namely) the circle *ABCD*, and an equal height. I say that the cone is the third part of the cylinder—that is to say, that the cylinder is three times the cone.



For if the cylinder is not three times the cone then the cylinder will be either more than three times, or less than three times, (the cone). Let it, first of all, be more than three times (the cone). And let the square ABCD have been inscribed in circle ABCD [Prop. 4.6]. So, square ABCD is more than half of circle ABCD [Prop. 12.2]. And let a prism of equal height to the cylinder have been set up on square ABCD. So, the prism set up is more than half of the cylinder, inasmuch as if we also circumscribe a square around circle ABCD [Prop. 4.7] then the square inscribed in circle ABCD is half of the circumscribed (square). And the solids set up on them are parallelepiped prisms of equal height. And parallelepiped solids having the same height are to one another as their bases [Prop. 11.32]. And, thus, the prism set up on square ABCD is half of the prism set up on the square circumscribed about circle ABCD. And the cylinder is less than the prism set up on the square circumscribed about circle ABCD. Thus, the prism set up on square ABCD of the same height as the cylinder is more than half of the cylinder. Let the circumferences AB, BC, CD, and DA have been cut in half at points E, F, G, and H. And let AE, EB, BF, FC, CG, GD, DH, and HA have been joined. And thus each of the triangles AEB, BFC, CGD, and DHA is more than half of the segment of circle ABCD about it, as was shown previously [Prop. 12.2]. Let prisms of equal height to the cylinder have been set up on each of the triangles AEB, BFC, CGD, and DHA. And each of the prisms set up is greater than the half part of the segment of the cylinder about it-inasmuch as if we draw (straight-lines) parallel to AB, BC, CD, and DA through points E, F, G, and H ληλόγραμμα, καὶ ἀπ' αὐτῶν ἀναστήσωμεν στερεὰ παραλληλεπίπεδα ἰσοϋψῆ τῷ χυλίνδρω, ἑχάσου τῶν ἀνασταθέντων ήμίση ἐστὶ τὰ πρίσματα τὰ ἐπὶ τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων καί έστι τὰ τοῦ κυλίνδρου τμήματα ἐλάττονα τῶν ἀνασταθέντων στερεῶν παραλληλεπιπέδων. ὥστε καὶ τὰ ἐπὶ τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα μείζονά έστιν η τὸ ήμισυ τῶν καθ' ἑαυτὰ τοῦ κυλίνδρου τμημάτων. τέμνοντες δη τας υπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ' ἑκάσου τῶν τριγώνων πρίσματα ἰσοϋψῆ τῷ κυλίνδρω καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κυλίνδρου, α έσται έλάττονα τῆς ὑπεροχῆς, ἤ ὑπερέχει ὁ κυλίνδρος τοῦ τριπλασίου τοῦ χώνου. λελείφθω, χαὶ ἔστω τὰ ΑΕ, EB, BZ, ZΓ, ΓΗ, ΗΔ, $\Delta\Theta$, Θ Α· λοιπὸν ἄρα τὸ πρίσμα, οῦ βάσις μέν τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ χυλίνδρῷ, μεῖζόν ἐστὶν ἢ τριπλάσιον τοῦ χώνου. ἀλλὰ τὸ πρίσμα, οὕ βάσις μὲν ἐστὶ τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ύψος δὲ τὸ αὐτὸ τῷ χυλίνδρω, τριπλάσιόν ἐστι τῆς πυραμίδος, ής βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον. κορυφή δὲ ἡ αὐτὴ τῷ κώνω· καὶ ἡ πυραμὶς ἄρα, ῆς βάσις μέν [ἐστι] τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ χώνω, μείζων ἐστὶ τοῦ χώνου τοῦ βάσιν ἔχοντες τὸν ΑΒΓΔ κύκλον. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ' αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὁ κύλινδρος τοῦ χώνου μεῖζων ἢ τριπλάσιος.

Λέγω δή, ὅτι οὐδὲ ἐλάττων ἐστὶν ἢ τριπλάσιος ὁ κύλινδρος τοῦ κώνου.

Εἰ γὰρ δυνατόν, ἔστω ἐλάττων ἢ τριπλάσιος ὁ κύλινδρος τοῦ χώνου· ἀνάπαλιν ἄρα ὁ χῶνος τοῦ χυλίνδρου μεῖζων έστιν ἢ τρίτον μέρος. ἐγγεγράφθω δὴ εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ ΑΒΓΔ ἄρα τετράγωνον μεῖζόν έστιν ἢ τὸ ἤμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πυραμίς την αὐτην κορυφην ἔχουσα τῷ κώνω. ή ἄρα άνασταθεῖσα πυραμίς μείζων ἐστίν ἢ τὸ ἤμισυ μέρος τοῦ χώνου, ἐπειδήπερ, ὡς ἕμπροσθεν ἐδείχνυμεν, ότι ἐὰν περὶ τὸν κύκλον τετράγωνον περιγράψωμεν, ἔσται τὸ ΑΒΓΔ τετράγωνον ἤμισυ τοῦ περὶ τὸν κύκλον περιγεγραμμένου τετραγώνου. και έαν από τῶν τετραγώνων στερεὰ παραλληλεπίπεδα ἀναστήσωμεν ἰσοϋψῆ τῷ κώνω, ἂ καὶ καλεῖται πρίσματα, ἔσται τὸ ἀνασταθὲν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ήμισυ τοῦ ἀνασταθέντος ἀπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τετραγώνου. πρός ἄλληλα γάρ εἰσιν ώς αί βάσεις. ὥστε καὶ τὰ τρίτα καὶ πυραμὶς ἄρα, ῆς βάσις τὸ ΑΒΓΔ τετράγωνον, ἤμισύ ἐστι τῆς πυραμίδος τῆς άνασταθείσης άπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τετραγώνου. καί έστι μείζων ή πυραμίς ή άνασταθεῖσα ἀπὸ τοῦ περὶ τὸν χύχλον τετραγώνου τοῦ χώνου· ἐμπεριέγει γὰρ αὐτόν. ἡ ἄρα πυραμὶς, ἦς βάσις τὸ ΑΒΓΔ τετράγωνον, κορυφή δὲ ή αὐτή τῷ κώνω, μείζων ἐστὶν ἢ τὸ ἤμισυ τοῦ κώνου. τετμήσθωσαν αί ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατὰ τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ

(respectively), and complete the parallelograms on AB, BC, CD, and DA, and set up parallelepiped solids of equal height to the cylinder on them, then the prisms on triangles AEB, BFC, CGD, and DHA are each half of the set up (parallelepipeds). And the segments of the cylinder are less than the set up parallelepiped solids. Hence, the prisms on triangles AEB, BFC, CGD, and DHA are also greater than half of the segments of the cylinder about them. So (if) the remaining circumferences are cut in half, and straight-lines are joined, and prisms of equal height to the cylinder are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cylinder whose (sum) is less than the excess by which the cylinder exceeds three times the cone [Prop. 10.1]. Let them have been left, and let them be AE, EB, BF, FC, CG, GD, DH, and HA. Thus, the remaining prism whose base (is) polygon AEBFCGDH, and height the same as the cylinder, is greater than three times the cone. But, the prism whose base is polygon AEBFCGDH, and height the same as the cylinder, is three times the pyramid whose base is polygon *AEBFCGDH*, and apex the same as the cone [Prop. 12.7 corr.]. And thus the pyramid whose base [is] polygon AEBFCGDH, and apex the same as the cone, is greater than the cone having (as) base circle ABCD. But (it is) also less. For it is encompassed by it. The very thing (is) impossible. Thus, the cylinder is not more than three times the cone.

So, I say that neither (is) the cylinder less than three times the cone.

For, if possible, let the cylinder be less than three times the cone. Thus, inversely, the cone is greater than the third part of the cylinder. So, let the square ABCD have been inscribed in circle ABCD [Prop. 4.6]. Thus, square ABCD is greater than half of circle ABCD. And let a pyramid having the same apex as the cone have been set up on square *ABCD*. Thus, the pyramid set up is greater than the half part of the cone, inasmuch as we showed previously that if we circumscribe a square about the circle [Prop. 4.7] then the square *ABCD* will be half of the square circumscribed about the circle [Prop. 12.2]. And if we set up on the squares parallelepiped solids-which are also called prisms-of the same height as the cone, then the (prism) set up on square ABCD will be half of the (prism) set up on the square circumscribed about the circle. For they are to one another as their bases [Prop. 11.32]. Hence, (the same) also (goes for) the thirds. Thus, the pyramid whose base is square ABCD is half of the pyramid set up on the square circumscribed about the circle [Prop. 12.7 corr.]. And the pyramid set up on the square circumscribed about the circle is greater

AE, EB, BZ, ZΓ, ΓΗ, ΗΔ, $\Delta\Theta$, Θ A· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἤμισυ μέρος του καθ' ἑαυτό τμήματος τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτωσαν ἐφ' ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πυραμίδες την αυτήν χορυφήν έχουσαι τῷ χώνω. καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων κατὰ τὸν αὐτὸν τρόπον μείζων ἐστὶν ἢ τὸ ἤμισυ μέρος τοῦ καθ έαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ άνιστάντες έφ' ἑχάστου τῶν τριγώνων πυραμίδα τὴν αὐτὴν κορυφήν έχουσαν τῷ κώνω καὶ τοῦτο ἀεὶ ποιοῦτες καταλείψομέν τινα ἀποτμήματα τοῦ χώνου, ἃ ἔσται ἐλάττονα τῆς ὑπεροχῆς, ἤ ὑπερέχει ὁ κῶνος τοῦ τρίτου μέρους τοῦ κυλίνδρου. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· λοιπὴ ἄρα ἡ πυραμίς, ἤς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μείζων ἐστίν ἢ τρίτον μέρος τοῦ χυλίνδρου. ἀλλ' ἡ πυραμίς, ῆς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, χορυφὴ δὲ ή αυτή τῷ χώνω, τρίτον ἐστὶ μέρος τοῦ πρίσματος, οὐ βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ χυλίνδρω· τὸ ẳρα πρίσμα, ο
ὕ βάσις μέν ἐστι τὸ $\rm AEBZ\Gamma H\Delta\Theta$ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ χυλίνδρῳ, μεῖζόν ἐστι τοῦ κυλίνδρου, οὕ βάσις ἐστὶν ὁ ΑΒΓΔ κύκλος. ἀλλὰ καὶ ἔλαττον έμπεριέχεται γὰρ ὑπ' αὐτοῦ. ὅπερ ἐστὶν ἀδύνατον. οὐχ άρα ὁ κύλινδρος τοῦ κώνου ἐλάττων ἐστὶν ἢ τριπλάσιος. έδείχθη δέ, ὅτι οὐδὲ μείζων ἢ τριπλάσιος. τριπλάσιος ἄρα ὁ χύλινδρος τοῦ χώνου. ὥστε ὁ χῶνος τρίτον ἐστὶ μέρος τοῦ χυλίνδρου.

Πᾶς ἄρα κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

ια'.

Οἱ ὑπο τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις.

Έστωσαν ὑπὸ τὸ αὐτὸ ὕψος κῶνοι καὶ κύλινδροι, ῶν βάσεις μὲν [εἰσιν] οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, ἄξονες δὲ οἱ ΚΛ, ΜΝ, διάμετροι δὲ τῶν βάσεων αἱ ΑΓ, ΕΗ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

than the cone. For it encompasses it. Thus, the pyramid whose base is square *ABCD*, and apex the same as the cone, is greater than half of the cone. Let the circumferences AB, BC, CD, and DA have been cut in half at points E, F, G, and H (respectively). And let AE, EB, BF, FC, CG, GD, DH, and HA have been joined. And, thus, each of the triangles AEB, BFC, CGD, and DHA is greater than the half part of the segment of circle ABCD about it [Prop. 12.2]. And let pyramids having the same apex as the cone have been set up on each of the triangles AEB, BFC, CGD, and DHA. And, thus, in the same way, each of the pyramids set up is more than the half part of the segment of the cone about it. So, (if) the remaining circumferences are cut in half, and straightlines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which the cone exceeds the third part of the cylinder [Prop. 10.1]. Let them have been left, and let them be the (segments) on AE, EB, BF, FC, CG, GD, DH, and HA. Thus, the remaining pyramid whose base is polygon AEBFCGDH, and apex the same as the cone, is greater than the third part of the cylinder. But, the pyramid whose base is polygon AEBFCGDH, and apex the same as the cone, is the third part of the prism whose base is polygon AEBFCGDH, and height the same as the cylinder [Prop. 12.7 corr.]. Thus, the prism whose base is polygon AEBFCGDH, and height the same as the cylinder, is greater than the cylinder whose base is circle ABCD. But, (it is) also less. For it is encompassed by it. The very thing is impossible. Thus, the cylinder is not less than three times the cone. And it was shown that neither (is it) greater than three times (the cone). Thus, the cylinder (is) three times the cone. Hence, the cone is the third part of the cylinder.

Thus, every cone is the third part of the cylinder which has the same base as it, and an equal height. (Which is) the very thing it was required to show.

Proposition 11

Cones and cylinders having the same height are to one another as their bases.

Let there be cones and cylinders of the same height whose bases [are] the circles ABCD and EFGH, axes KL and MN, and diameters of the bases AC and EG (respectively). I say that as circle ABCD is to circle EFGH, so cone AL (is) to cone EN.



Εἰ γὰρ μή, ἔσται ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος ἤτοι πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεόν η πρός μείζον. ἔστω πρότερον πρός ἔλασσον τὸ Ξ, καὶ ῷ ἔλασσόν ἐστι τὸ Ξ στερεὸν τοῦ ΕΝ κώνου, έχείνω ίσον έστω τὸ Ψ στερεόν· ὁ ΕΝ χῶνος ἄρα ίσος ἐστὶ τοῖς Ξ, Ψ στερεοῖς. ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ κύκλου. ἀνεστάτω ἀπὸ τοῦ ΕΖΗΘ τετραγώνου πυραμίς ίσοϋψής τῷ χώνω. ή ἄρα ἀνασταθεῖσα πυραμίς μείζων έστιν η τὸ ήμισυ τοῦ κώνου, ἐπειδήπερ ἐἀν περιγράψωμεν περί τὸν χύχλον τετράγωνον, χαὶ ἀπ' αὐτοῦ άναστήσωμεν πυραμίδα ἰσοϋψῆ τῷ κώνω, ἡ ἐγγραφεῖσα πυραμὶς ἥμισύ ἐστι τῆς περιγραφείσης. πρὸς ἀλλήλας γάρ εἰσιν ώς αἱ βάσεις· ἐλάττων δὲ ὁ κῶνος τῆς περιγραφείσης πυραμίδος. τετμήσθωσαν αί ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίγα κατά τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύγθωσαν αἱ ΘΟ, ΟΕ, ΕΠ, ΠΖ, ΖΡ, ΡΗ, ΗΣ, ΣΘ. ἕκαστον ἄρα τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΖΘ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἤμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου. ἀνεστάτω ἐφ' έχάστου τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ τριγώνων πυραμίς ίσοϋψής τῷ κώνω· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων έστιν ή τὸ ήμισυ τοῦ καθ' ἑαυτὴν τμήματος τοῦ χώνου. τέμνοντες δη τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐπὶ ἑκάστου τῶν τριγώνων πυραμίδας ἰσοϋψεῖς τῷ κώνῳ καὶ ἀεὶ τοῦτο ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἂ ἔσται ἐλάσσονα τοῦ Ψ στερεοῦ. λελείφ ϑ ω, καὶ ἔστω τὰ έπι τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ· λοιπὴ ἄρα ἡ πυραμίς, ῆς βάσις τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ χώνω, μείζων έστι τοῦ Ξ στερεοῦ. ἐγγεγράφθω χαὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΘΟΕΠΖΡΗΣ πολυγώνω ὄμοιόν τε καὶ όμοίως χείμενον πολύγωνον τὸ ΔΤΑΥΒΦΓΧ, χαὶ ἀνεστάτω έπ' αὐτοῦ πυραμὶς ἰσοϋψὴς τῷ ΑΛ κώνω. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως τὸ Δ TAYBΦΓX πολύγωνον πρός τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, ὡς δὲ τὸ άπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρός τὸ ΘΟΕΠΖΡΗΣ πολύγωνον. ὡς δὲ ὁ ΑΒΓΔ κύκλος πρός τόν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρός τὸ Ξ στερεόν, ώς δὲ τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟ-ΕΠΖΡΗΣ πολύγωνον, οὕτως ἡ πυραμίς, ἦς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφή δὲ τὸ Λ σημεῖον, πρὸς



For if not, then as circle ABCD (is) to circle EFGH, so cone AL will be to some solid either less than, or greater than, cone EN. Let it, first of all, be (in this ratio) to (some) lesser (solid), O. And let solid X be equal to that (magnitude) by which solid O is less than cone EN. Thus, cone EN is equal to (the sum of) solids Oand X. Let the square EFGH have been inscribed in circle EFGH [Prop. 4.6]. Thus, the square is greater than half of the circle [Prop. 12.2]. Let a pyramid of the same height as the cone have been set up on square *EFGH*. Thus, the pyramid set up is greater than half of the cone, inasmuch as, if we circumscribe a square about the circle [Prop. 4.7], and set up on it a pyramid of the same height as the cone, then the inscribed pyramid is half of the circumscribed pyramid. For they are to one another as their bases [Prop. 12.6]. And the cone (is) less than the circumscribed pyramid. Let the circumferences EF, FG, GH, and HE have been cut in half at points P, Q, R, and S. And let HP, PE, EQ, QF, FR, RG, GS, and SH have been joined. Thus, each of the triangles HPE, EQF, FRG, and GSH is greater than half of the segment of the circle about it [Prop. 12.2]. Let pyramids of the same height as the cone have been set up on each of the triangles HPE, EQF, FRG, and GSH. And, thus, each of the pyramids set up is greater than half of the segment of the cone about it [Prop. 12.10]. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids of equal height to the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone (the sum of) which is less than solid X [Prop. 10.1]. Let them have been left, and let them be the (segments) on HPE, EQF, FRG, and GSH. Thus, the remaining pyramid whose base is polygon *HPEQFRGS*, and height the same as the cone, is greater than solid O [Prop. 6.18]. And let the polygon DTAUBVCW, similar, and similarly laid out, to polygon HPEQFRGS, have been inscribed in circle ABCD. And on it let a pyramid of the same height as cone AL have been set up. Therefore, since as the (square) on AC is to the (square) on EG, so polygon DTAUBVCW (is) to polygon HPEQFRGS [Prop. 12.1], and as the (square) on AC (is) to the (square) on EG, so circle ABCD (is) τὴν πυραμίδα, ῆς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον. καὶ ὡς ἄρα ὁ ΑΛ κῶνος πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμίς, ῆς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν πυραμίδα, ῆς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ ΑΛ κῶνος πρὸς τὴν ἐν αὐτῷ πυραμίδα, οὕτως τὸ Ξ στερεὸν πρὸς τὴν ἐν τῷ ΕΝ κώνῷ πυραμίδα. μείζων δὲ ὁ ΑΛ κῶνος τῆς ἐν αὐτῷ πυραμίδος· μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς ἐν τῷ ΕΝ κώνῷ πυραμίδος. ἀλλὰ καὶ ἔλασσον· ὅπερ ἄτοπον. οὐx ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεόν. ὁμοίως δὲ δείξομεν, ὅτι οὐδέ ἐστιν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἕλασσόν τι τοῦ

Λέγω δή, ὅτι οὐδέ ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΝ κώνου στερεόν.

Εἰ γὰρ δυνατόν, ἕστω πρὸς μεῖζον τὸ Ξ· ἀνάπαλιν ἄρα ἐστὶν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως τὸ Ξ στερεὸν πρὸς τὸν ΑΛ κῶνον. ἀλλ' ὡς τὸ Ξ στερεὸν πρὸς τὸν ΑΛ κῶνον, οὕτως ὁ ΕΝ κῶνος πρὸς ἕλασσόν τι τοῦ ΑΛ κώνου στερεόν· καὶ ὡς ἄρα ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἕλασσόν τι τοῦ ΑΛ κώνου στερεόν· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΝ κώνου στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἕλασσον· ἔστιν ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

³Αλλ' ὡς ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλασίων γὰρ ἑκάτερος ἑκατέρου. καὶ ὡς ӑρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως οἱ ἐπ' αὐτῶν ἰσοϋψεῖς.

Οἱ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

ıβ'.

Οἱ ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν τριπλασίονι λόγω εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων.

Έστωσαν ὄμοιοι κῶνοι καὶ κύλινδροι, ῶν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ τῶν βάσεων αἰ ΒΔ, ΖΘ, ἄξονες δὲ τῶν κώνων καὶ κυλίνδρων οἱ ΚΛ, MN· λέγω,

to circle EFGH [Prop. 12.2], thus as circle ABCD (is) to circle EFGH, so polygon DTAUBVCW also (is) to polygon HPEQFRGS. And as circle ABCD (is) to circle EFGH, so cone AL (is) to solid O. And as polygon DTAUBVCW (is) to polygon HPEQFRGS, so the pyramid whose base is polygon DTAUBVCW, and apex the point L, (is) to the pyramid whose base is polygon HPEQFRGS, and apex the point N [Prop. 12.6]. And, thus, as cone AL (is) to solid O, so the pyramid whose base is DTAUBVCW, and apex the point L, (is) to the pyramid whose base is polygon *HPEQFRGS*, and apex the point N [Prop. 5.11]. Thus, alternately, as cone ALis to the pyramid within it, so solid O (is) to the pyramid within cone EN [Prop. 5.16]. But, cone AL (is) greater than the pyramid within it. Thus, solid O (is) also greater than the pyramid within cone EN [Prop. 5.14]. But, (it is) also less. The very thing (is) absurd. Thus, circle ABCD is not to circle EFGH, as cone AL (is) to some solid less than cone EN. So, similarly, we can show that neither is circle EFGH to circle ABCD, as cone EN (is) to some solid less than cone AL.

So, I say that neither is circle ABCD to circle EFGH, as cone AL (is) to some solid greater than cone EN.

For, if possible, let it be (in this ratio) to (some) greater (solid), O. Thus, inversely, as circle EFGH is to circle ABCD, so solid O (is) to cone AL [Prop. 5.7 corr.]. But, as solid O (is) to cone AL, so cone EN (is) to some solid less than cone AL [Prop. 12.2 lem.]. And, thus, as circle EFGH (is) to circle ABCD, so cone EN (is) to some solid less than cone AL. The very thing was shown (to be) impossible. Thus, circle ABCD is not to circle EFGH, as cone AL (is) to some solid greater than cone EN. And, it was shown that neither (is it in this ratio) to (some) lesser (solid). Thus, as circle ABCD is to circle EFGH, so cone AL (is) to cone EN.

But, as the cone (is) to the cone, (so) the cylinder (is) to the cylinder. For each (is) three times each [Prop. 12.10]. Thus, circle ABCD (is) also to circle EFGH, as (the ratio of the cylinders) on them (having) the same height.

Thus, cones and cylinders having the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 12

Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

Let there be similar cones and cylinders of which the bases (are) the circles ABCD and EFGH, the diameters of the bases (are) BD and FH, and the axes of the cones

ὅτι ὁ κῶνος, οὖ βάσις μέν [ἐστιν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸν κῶνον, οὕ βάσις μέν [ἐστιν] ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.



Εἰ γὰρ μὴ ἔχει ὁ ΑΒΓΔΛ κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον πριπλασίονα λόγον ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ, ἕξει ό ΑΒΓΔΛ κῶνος ἢ πρὸς ἔλασσόν τι τοῦ ΕΖΗΘΝ κώνου στερεόν τριπλασίονα λόγον ἢ πρὸς μεῖζον. ἐχέτω πρότερον πρὸς ἔλασσον τὸ Ξ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα ΕΖΗΘ τετράγωνον μεῖζόν έστιν ἢ τὸ ἤμισυ τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐπὶ τοῦ ΕΖΗΘ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ χώνω. ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ χώνου. τετμήσθωσαν δὴ αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίγα κατά τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύχθωσαν αί ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ. καὶ έκαστον ἄρα τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων μεῖζόν έστιν ἢ τὸ ἤμισυ μέρος τοῦ καθ' ἑαυτὸ τμήματος τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐφ' ἑκάστου τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων πυραμίς την αυτήν κορυφήν έχουσα τῶ κώνω· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων έστιν η τὸ ήμισυ μέρος τοῦ χαθ' ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δη τας υπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ' ἑκάστου τῶν τριγώνων πυραμίδας την αὐτην κορυφην ἐχούσας τῷ κώνῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, & έσται έλάσσονα τῆς ὑπεροχῆς, ἤ ὑπερέχει ὁ ΕΖΗΘΝ χῶνος τοῦ Ξ στερεοῦ. λελείφθω, χαὶ ἔστω τὰ ἐπὶ τῶν ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ· λοιπὴ ἄρα ἡ πυραμίς, ής βάσις μέν έστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, χορυφή δὲ τὸ Ν σημεῖον, μείζων ἐστὶ τοῦ Ξ στερεοῦ. έγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΟΖΠΗΡΘΣ πολυγώνω ὄμοιόν τε καὶ ὁμοίως κείμενον πολύγωνον τὸ ΑΤΒΥΓΦΔΧ, καὶ ἀνεστάτω ἐπὶ τοῦ ΑΤΒΥΓΦΔΧ πολυγώνου πυραμίς την αὐτην χορυφην ἔχουσα τῷ χώνῳ, καὶ τῶν μὲν περιεχόντων τὴν πυραμίδα, ἤς βάσις μέν ἐστι τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφή δὲ τὸ Λ σημεῖον, ἕν τρίγωνον ἔστω τὸ ΛΒΤ, τῶν δὲ περειγόντων τὴν πυραμίδα, ής βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, and cylinders (are) KL and MN (respectively). I say that the cone whose base [is] circle ABCD, and apex the point L, has to the cone whose base [is] circle EFGH, and apex the point N, the cubed ratio that BD (has) to FH.



For if cone *ABCDL* does not have to cone *EFGHN* the cubed ratio that BD (has) to FH then cone ABCDL will have the cubed ratio to some solid either less than, or greater than, cone EFGHN. Let it, first of all, have (such a ratio) to (some) lesser (solid), O. And let the square *EFGH* have been inscribed in circle *EFGH* [Prop. 4.6]. Thus, square EFGH is greater than half of circle EFGH[Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on square EFGH. Thus, the pyramid set up is greater than the half part of the cone [Prop. 12.10]. So, let the circumferences EF, FG, GH, and HE have been cut in half at points P, Q, R, and S (respectively). And let EP, PF, FQ, QG, GR, RH, HS, and SE have been joined. And, thus, each of the triangles EPF, FQG, GRH, and HSE is greater than the half part of the segment of circle EFGH about it [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on each of the triangles EPF, FQG, GRH, and HSE. And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone EFGHN exceeds solid O [Prop. 10.1]. Let them have been left, and let them be the (segments) on EP, PF, FQ, QG, GR, RH, HS, and SE. Thus, the remaining pyramid whose base is polygon *EPFQGRHS*, and apex the point N, is greater than solid O. And let the polygon ATBUCVDW, similar, and similarly laid out, to polygon EPFQGRHS, have been inscribed in circle ABCD [Prop. 6.18]. And let a pyramid having the same apex as the cone have been set up on polygon ATBUCVDW.

κορυφή δὲ τὸ Ν σημεῖον, ἕν τρίγωνον ἔστω τὸ NZO, καὶ ἐπεζεύχθωσαν αἱ KT, MO. καὶ ἐπεὶ ὅμοιός ἐστιν ὁ ΑΒΓΔΛ κῶνος τῷ ΕΖΗΘΝ κώνω, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ZΘ, οὕτως ὁ ΚΛ ẳξων πρὸς τὸν MN ẳξονα.
ὡς δὲ ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΒΚ πρὸς τὴν ΖΜ· καὶ ὡς ἄρα ἡ ΒΚ πρός τὴν ΖΜ, οὕτως ἡ ΚΛ πρός τὴν ΜΝ. καὶ ἐναλλὰξ ὡς ή ΒΚ πρός τὴν ΚΛ, οὕτως ή ΖΜ πρός τὴν ΜΝ. καὶ περὶ ίσας γωνίας τὰς ὑπὸ ΒΚΛ, ΖΜΝ αἱ πλευραὶ ἀνάλογόν εἰσιν· δμοιον ἄρα ἐστὶ τὸ ΒΚΛ τρίγωνον τῷ ΖΜΝ τριγώνω. πάλιν, έπεί έστιν ώς ή ΒΚ πρός την ΚΤ, οὕτως ή ΖΜ πρός την ΜΟ, και περι ίσας γωνίας τὰς ὑπὸ ΒΚΤ, ΖΜΟ, ἐπειδήπερ, δ μέρος ἐστὶν ἡ ὑπὸ ΒΚΤ γωνία τῶν πρὸς τῷ Κ κέντρω τεσσάρων ὀρθῶν, τὸ αὐτὸ μέρος ἐστὶ καὶ ἡ ὑπὸ ΖΜΟ γωνία τῶν πρὸς τῷ Μ κέντρῳ τεσσάρων ὀρθῶν· ἐπεὶ οὖν περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστι τὸ ΒΚΤ τρίγωνον τῷ ΖΜΟ τριγώνω. πάλιν, ἐπεὶ ἐδείχθη ὡς ἡ ΒΚ πρός τὴν ΚΛ, οὕτως ἡ ΖΜ πρός τὴν ΜΝ, ἴση δὲ ἡ μὲν ΒΚ τῆ ΚΤ, ἡ δὲ ΖΜ τῆ ΟΜ, ἔστιν ἄρα ὡς ἡ ΤΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΟΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΤΚΛ, ΟΜΝ· ὀρθαὶ γάρ· αἱ πλευραὶ ἀνάλογόν εἰσιν· δμοιον ἄρα ἐστὶ τὸ ΛΚΤ τρίγωνον τῷ NMO τριγώνω. καὶ έπει διὰ τὴν ὁμοιότητα τῶν ΛΚΒ, ΝΜΖ τριγώνων ἐστιν ώς ή ΛΒ πρός την ΒΚ, οὕτως ή ΝΖ πρός την ΖΜ, διὰ δὲ τὴν ὑμοιότητα τῶν ΒΚΤ, ΖΜΟ τριγώνων ἐστὶν ὡς ἡ KB πρὸς τὴν BT, οὕτως ἡ MZ πρὸς τὴν ZO, δι' ἴσου ἄρα ὡς ή ΛΒ πρός την ΒΤ, οὕτως ή ΝΖ πρός την ΖΟ. πάλιν, ἐπεὶ διὰ τὴν ομοιότητα τῶν ΛΤΚ, ΝΟΜ τριγώνων ἐστὶν ὡς ἡ ΑΤ πρός την ΤΚ, ούτως ή ΝΟ πρός την ΟΜ, διά δὲ την όμοιότητα τῶν ΤΚΒ, ΟΜΖ τριγώνων ἐστὶν ὡς ἡ ΚΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΜΟ πρὸς τὴν ΟΖ, δι' ἴσου ἄρα ὡς ἡ ΛΤ πρός τὴν ΤΒ, οὕτως ἡ ΝΟ πρός τὴν ΟΖ. ἐδείχθη δὲ καὶ ώς ή ΤΒ πρός την ΒΛ, οὕτως ή ΟΖ πρός την ΖΝ. δι ἴσου ἄρα ὡς ἡ ΤΛ πρὸς τὴν ΛΒ, οὕτως ἡ ΟΝ πρὸς τὴν NΖ. τῶν ΛΤΒ, ΝΟΖ ἄρα τριγώνων ἀνάλογόν εἰσιν αἱ πλευραί· ίσογώνια ἄρα ἐστὶ τὰ ΛΤΒ, NOZ τρίγωνα· ὥστε καὶ ὅμοια. καὶ πυραμὶς ἄρα, ῆς βάσις μὲν τὸ ΒΚΤ τρίγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ὑμοία ἐστὶ πυραμίδι, ῆς βάσις μὲν τὸ ΖΜΟ τρίγωνον, κορυφή δὲ τὸ Ν σημεῖον ὑπὸ γὰρ ὅμοίων έπιπέδων περιέχονται ίσων τὸ πλῆθος. αἱ δὲ ὄμοιαι πυραμίδες και τριγώνους έχουσαι βάσεις έν τριπλασίονι λόγω είσι τῶν ὁμολόγων πλευρῶν. ἡ ἄρα ΒΚΤΛ πυραμίς πρὸς τὴν ΖΜΟΝ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΚ πρὸς τὴν ZM. ὑμοίως δὴ ἐπιζευγνύντες ἀπὸ τῶν A, X, Δ , Φ , Γ , Υ ἐπὶ τὸ Κ εὐθείας καὶ ἀπὸ τῶν Ε, Σ, Θ, Ρ, Η, Π ἐπὶ τὸ Μ καὶ άνιστάντες έφ' έχάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφήν έχούσας τοῖς κώνοις δείξομεν, ὅτι καὶ ἑκάστη τῶν όμοταγῶν πυραμίδων πρὸς ἑχάστην ὁμοταγῆ πυραμίδα τριπλασίονα λόγον ἕξει ἤπερ ἡ ΒΚ ὁμόλογος πλευρὰ πρὸς τὴν ΖΜ ὑμόλογον πλευράν, τουτέστιν ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. καὶ ὡς ἕν τῶν ἡγουμένων πρὸς ἕν τῶν ἑπομένων, οὕτως ăπαντα τὰ ἡγούμενα πρὸς ἄπαντα τὰ ἑπόμενα· ἔστιν ἄρα

And let LBT be one of the triangles containing the pyramid whose base is polygon ATBUCVDW, and apex the point L. And let NFP be one of the triangles containing the pyramid whose base is triangle EPFQGRHS, and apex the point N. And let KT and MP have been joined. And since cone *ABCDL* is similar to cone *EFGHN*, thus as BD is to FH, so axis KL (is) to axis MN [Def. 11.24]. And as BD (is) to FH, so BK (is) to FM. And, thus, as BK (is) to FM, so KL (is) to MN. And, alternately, as BK (is) to KL, so FM (is) to MN [Prop. 5.16]. And the sides around the equal angles BKL and FMN are proportional. Thus, triangle BKL is similar to triangle FMN [Prop. 6.6]. Again, since as BK (is) to KT, so FM (is) to MP, and (they are) about the equal angles BKT and FMP, inasmuch as whatever part angle BKTis of the four right-angles at the center K, angle FMP is also the same part of the four right-angles at the center M. Therefore, since the sides about equal angles are proportional, triangle BKT is thus similar to traingle FMP [Prop. 6.6]. Again, since it was shown that as BK (is) to KL, so FM (is) to MN, and BK (is) equal to KT, and FM to PM, thus as TK (is) to KL, so PM (is) to MN. And the sides about the equal angles TKL and PMN—for (they are both) right-angles—are proportional. Thus, triangle LKT (is) similar to triangle *NMP* [Prop. 6.6]. And since, on account of the similarity of triangles LKB and NMF, as LB (is) to BK, so NF(is) to FM, and, on account of the similarity of triangles BKT and FMP, as KB (is) to BT, so MF (is) to FP[Def. 6.1], thus, via equality, as LB (is) to BT, so NF(is) to FP [Prop. 5.22]. Again, since, on account of the similarity of triangles LTK and NPM, as LT (is) to TK, so NP (is) to PM, and, on account of the similarity of triangles TKB and PMF, as KT (is) to TB, so MP (is) to PF, thus, via equality, as LT (is) to TB, so NP (is) to PF [Prop. 5.22]. And it was shown that as TB (is) to BL, so PF (is) to FN. Thus, via equality, as TL (is) to LB, so PN (is) to NF [Prop. 5.22]. Thus, the sides of triangles LTB and NPF are proportional. Thus, triangles *LTB* and *NPF* are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle BKT, and apex the point L, is similar to the pyramid whose base is triangle FMP, and apex the point N. For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid BKTL has to pyramid FMPN the cubed ratio that BK(has) to FM. So, similarly, joining straight-lines from (points) A, W, D, V, C, and U to (center) K, and from (points) E, S, H, R, G, and Q to (center) M, and set-

καὶ ὡς ἡ ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα, οὕτως ή ὅλη πυραμίς, ής βάσις τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφή δὲ τὸ Λ σημεῖον, πρὸς τὴν ὅλην πυραμίδα, ῆς βάσις μέν τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, χορυφή δὲ τὸ Ν σημεῖον. ώστε καὶ πυραμίς, ἦς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ, κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἦς βάσις [μὲν] τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφή δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον έχει ήπερ ή ΒΔ πρός την ΖΘ. ὑπόκειται δὲ καὶ ὁ κῶνος, οὕ βάσις [μέν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸ Ξ στερεὸν τριπλασίονα λόγον ἔχων ἤπερ ἡ $\mathrm{B}\Delta$ πρὸς τὴν ΖΘ· ἔστιν ἄρα ὡς ὁ κῶνος, οὕ βάσις μέν ἐστιν ὁ ΑΒΓΔ κύκλος, κορυφή δὲ τὸ Λ , πρὸς τὸ Ξ στερεόν, οὕτως ή πυραμίς, ής βάσις μέν τὸ ΑΤΒΥΓΦΔΧ [πολύγωνον], χορυφή δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἦς βάσις μέν ἐστι τὸ ΕΟΖ-ΠΗΡΘΣ πολύγωνον, κορυφή δὲ τὸ Ν· ἐναλλὰξ ἄρα, ὡς ὁ κῶνος, οὕ βάσις μὲν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὴν ἐν αὐτῷ πυραμίδα, ἦς βάσις μὲν τὸ $ATB\Upsilon \Gamma \Phi \Delta X$ πολύγωνον, χορυφή δὲ τὸ Λ , οὕτως τὸ Ξ [στερεὸν] πρὸς τὴν πυραμίδα, ής βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφή δὲ τὸ Ν. μείζων δὲ ὁ εἰρημένος κῶνος τῆς ἐν αὐτῷ πυραμίδος έμπεριέχει γάρ αὐτὴν. μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς πυραμίδος, ἦς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, χορυφή δὲ τὸ Ν. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κῶνος, οὖ βάσις ὁ ${
m AB}\Gamma\Delta$ κύκλος, κορυφή δὲ τὸ Λ [σημεῖον], πρὸς ἔλαττόν τι τοῦ κώνου στερεόν, οὕ βάσις μὲν ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὑμοίως δὴ δείξομεν, ὄτι οὐδὲ ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ χώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ.

Λέγω δή, ὅτι οὐδὲ ὁ $ABΓ\Delta\Lambda$ κῶνος πρὸς μεῖζόν τι τοῦ $EZH\ThetaN$ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ $B\Delta$ πρὸς τὴν ZΘ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μεῖζον τὸ Ξ. ἀνάπαλιν ἄρα τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον τριπλασίονα λόγον ἔχει ἦπερ ἡ ΖΘ πρὸς τὴν ΒΔ. ὡς δὲ τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνου, οὕτως ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεόν. καὶ ὁ ΕΖΗΘΝ ἄρα κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἦπερ ἡ ΖΘ πρὸς τὴν ΒΔ· ὅπερ ἀδύνατον ἐδείχϑη. οὐκ ἄρα ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἦπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ἐδείχϑη δέ, ὅτι οὐδὲ πρὸς ἕλαττον. ὁ ΑΒΓΔΛ ἄρα κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον τριπλασίονα λόγον ἔχει ἦπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

 $\Omega \varsigma$ δὲ ὁ xῶνος πρὸς τὸν xῶνον, ὁ xύλινδρος πρὸς τὸν xúλινδρον· τριπλάσιος γὰρ ὁ xύλινδρος τοῦ xώνου ὁ ἐπὶ τῆς αὐτῆς βάσεως τῷ xώνῳ xαὶ ἰσοῦψὴς αὐτῷ. xαὶ ὁ xύλινδρος ӑρα πρὸς τὸν xύλινδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ $B\Delta$ πρὸς τὴν ZΘ.

Οἱ ἄρα ὄμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν

ting up pyramids having the same apexes as the cones on each of the triangles (so formed), we can also show that each of the pyramids (on base ABCD taken) in order will have to each of the pyramids (on base EFGH taken) in order the cubed ratio that the corresponding side BK (has) to the corresponding side FM—that is to say, that BD (has) to FH. And (for two sets of proportional magnitudes) as one of the leading (magnitudes is) to one of the following, so (the sum of) all of the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. And, thus, as pyramid BKTL (is) to pyramid FMPN, so the whole pyramid whose base is polygon ATBUCVDW, and apex the point L, (is) to the whole pyramid whose base is polygon *EPFQGRHS*, and apex the point N. And, hence, the pyramid whose base is polygon ATBUCVDW, and apex the point L, has to the pyramid whose base is polygon *EPFQGRHS*, and apex the point N, the cubed ratio that BD (has) to FH. And it was also assumed that the cone whose base is circle ABCD, and apex the point L, has to solid O the cubed ratio that BD (has) to FH. Thus, as the cone whose base is circle ABCD, and apex the point L, is to solid O, so the pyramid whose base (is) [polygon] ATBUCVDW, and apex the point L, (is) to the pyramid whose base is polygon *EPFQGRHS*, and apex the point N. Thus, alternately, as the cone whose base (is) circle ABCD, and apex the point L, (is) to the pyramid within it whose base (is) the polygon ATBUCVDW, and apex the point L, so the [solid] O (is) to the pyramid whose base is polygon EPFQGRHS, and apex the point N [Prop. 5.16]. And the aforementioned cone (is) greater than the pyramid within it. For it encompasses it. Thus, solid O (is) also greater than the pyramid whose base is polygon EPFQGRHS, and apex the point N. But, (it is) also less. The very thing is impossible. Thus, the cone whose base (is) circle ABCD, and apex the [point] L, does not have to some solid less than the cone whose base (is) circle EFGH, and apex the point N, the cubed ratio that BD (has) to EH. So, similarly, we can show that neither does cone EFGHN have to some solid less than cone ABCDL the cubed ratio that FH (has) to BD.

So, I say that neither does cone ABCDL have to some solid greater than cone EFGHN the cubed ratio that BD (has) to FH.

For, if possible, let it have (such a ratio) to a greater (solid), O. Thus, inversely, solid O has to cone ABCDL the cubed ratio that FH (has) to BD [Prop. 5.7 corr.]. And as solid O (is) to cone ABCDL, so cone EFGHN (is) to some solid less than cone ABCDL [12.2 lem.]. Thus, cone EFGHN also has to some solid less than cone ABCDL the cubed ratio that FH (has) to BD. The very

τριπλασίονι λόγω εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων· ὅπερ ἔδει δεῖξαι.

ιγ΄.

Έὰν χύλινδρος ἐπιπέδω τμηθῆ παραλλήλω ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ὁ χύλινδρος πρὸς τὸν χύλινδρον, οὕτως ὁ ἄξων πρὸς τὸν ἄξονα.



Κύλινδρος γὰρ ὁ ΑΔ ἐπιπέδω τῷ ΗΘ τετμήσθω παραλλήλω ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΑΒ, ΓΔ, καὶ συμβαλλέτω τῷ ἄξονι τὸ ΗΘ ἐπίπεδον κατὰ τὸ Κ σημεῖον· λέγω, ὅτι ἐστὶν ὡς ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον, οὕτως ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα.

Έκβεβλήσθω γὰρ ὁ ΕΖ ἄξων ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Λ, Μ σημεῖα, καὶ ἐκκείσθωσαν τῷ ΕΚ ἄξονι ἴσοι ὁσοιδηποτοῦν οἱ ΕΝ, ΝΛ, τῷ δὲ ΖΚ ἴσοι ὁσοιδηποτοῦν οἱ ΖΞ, ΞΜ, καὶ νοείσθω ὁ ἐπὶ τοῦ ΛΜ ἄξονος κύλινδρος ὁ ΟΧ, οὕ βάσεις οἱ ΟΠ, ΦΧ κύκλοι. καὶ ἐκβεβλήσθω διὰ τῶν Ν, Ξ σημείων ἐπίπεδα παράλληλα τοῖς ΑΒ, ΓΔ καὶ ταῖς βάσεσι τοῦ ΟΧ κυλίνδρου καὶ ποιείτωσαν τοὺς ΡΣ, ΤΥ κύκλους περί τὰ Ν, Ξ κέντρα. καὶ ἐπεὶ οἱ ΛΝ, ΝΕ, ΕΚ άξονες ίσοι εἰσὶν ἀλλήλοις, οἱ ἄρα ΠΡ, ΡΒ, ΒΗ χύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσιν αἱ βάσεις· ίσοι άρα καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἀλλήλοις. επεὶ οὖν οί ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλήθος τῷ πλήθει, ὑσαπλασίων ἄρα ὑ ΚΛ ἄξων τοῦ ΕΚ άξονος, τοσαυταπλασίων ἔσται καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΒ κυλίνδρου. διὰ τὰ αὐτὰ δὴ καὶ ὑσαπλασίων ἐστὶν ὁ ΜΚ ἄξων τοῦ ΚΖ ἄξονος, τοσαυταπλασίων ἐστὶ καὶ ὁ ΧΗ κύλινδρος τοῦ ΗΔ χυλίνδρου. χαὶ εἰ μὲν ἴσος ἐστὶν ὁ ΚΛ ἄξων τῷ ΚΜ ἄξονι, ἴσος ἔσται καὶ ὁ ΠΗ κύλινδρος τῷ ΗΧ κυλίνδρῳ,

thing was shown (to be) impossible. Thus, cone ABCDL does not have to some solid greater than cone EFGHN the cubed ratio than BD (has) to FH. And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone ABCDL has to cone EFGHN the cubed ratio that BD (has) to FG.

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that BD (has) to FH.

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.

Proposition 13

If a cylinder is cut by a plane which is parallel to the opposite planes (of the cylinder) then as the cylinder (is) to the cylinder, so the axis will be to the axis.



For let the cylinder AD have been cut by the plane GH which is parallel to the opposite planes (of the cylinder), AB and CD. And let the plane GH have met the axis at point K. I say that as cylinder BG is to cylinder GD, so axis EK (is) to axis KF.

For let axis EF have been produced in each direction to points L and M. And let any number whatsoever (of lengths), EN and NL, equal to axis EK, be set out (on the axis *EL*), and any number whatsoever (of lengths), FO and OM, equal to (axis) FK, (on the axis KM). And let the cylinder PW, whose bases (are) the circles PQ and VW, have been conceived on axis LM. And let planes parallel to AB, CD, and the bases of cylinder PW, have been produced through points N and O, and let them have made the circles RS and TU around the centers N and O (respectively). And since axes LN, NE, and EK are equal to one another, the cylinders QR, RB, and BG are to one another as their bases [Prop. 12.11]. But the bases are equal. Thus, the cylinders QR, RB, and BG (are) also equal to one another. Therefore, since the axes LN, NE, and EK are equal to one another, and the cylinders QR, RB, and BG are also equal to one another, and the number (of the former) is equal to the number (of the latter), thus as many multiples as axis KL

εἰ δὲ μείζων ὁ ἄξων τοῦ ἄξονος, μείζων καὶ ὁ κύλινδρος τοῦ κυλίνδρου, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ μεγεθῶν ὄντων, ἀξόνων μὲν τῶν ΕΚ, ΚΖ, κυλίνδρων δὲ τῶν BH, HΔ, εἴληπται ἰσάκις πολλαπλάσια, τοῦ μὲν ΕΚ ἄξονος καὶ τοῦ BH κυλίνδρου ὅ τε ΛΚ ἄξων καὶ ὁ ΠΗ κύλινδρος, τοῦ δὲ KZ ἄξονες καὶ τοῦ HΔ κυλίνδρου ὅ τε KM ἄζων καὶ ὁ HX κύλινδρος, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ὁ KΛ ἄζων τοῦ KM ἄξονος, ὑπερέχει καὶ ὁ ΠΗ κύλινδρος τοῦ HX κυλίνδρου, καὶ εἰ ἴσος, ἴσος, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα ὡς ὁ ΕΚ ἄξων πρὸς τὸν KZ ἄξονα, οὕτως ὁ BH κύλινδρος πρὸς τὸν HΔ κύλινδρον· ὅπερ ἕδει δεῖζαι.

ιδ'.

Οἱ ἐπὶ ἴσων βάσεων ὄντες κῶνοι καὶ κύλινδροι πρὸς αλλήλους εἰσὶν ὡς τὰ ὕψη.



Έστωσαν γὰρ ἐπὶ ἴσων βάσεων τῶν AB, ΓΔ κύκλων κύλινδροι οἱ EB, ZΔ· λέγω, ὅτι ἐστὶν ὡς ὁ EB κύλινδρος πρὸς τὸν ZΔ κύλινδρον, οὕτως ὁ HΘ ἄξων πρὸς τὸν ΚΛ ἄξονα.

Έχβεβλήσθω γὰρ ὁ ΚΛ ἄξων ἐπὶ τὸ Ν σημεῖον, καὶ κείσθω τῷ ΗΘ ἄξονι ἴσος ὁ ΛΝ, καὶ περὶ ἄξονα τὸν ΛΝ κύλινδρος νενοήσθω ὁ ΓΜ. ἐπεὶ οῦν οἱ ΕΒ, ΓΜ κύλινδροι ὑπὸ τὸ αὐτὸ ὕψος εἰσίν, πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσίν αἱ βάσεις ἀλλήλαις· ἴσοι ἄρα εἰσὶ καὶ οἱ ΕΒ, ΓΜ κύλινδροι. καὶ ἐπεὶ κύλινδρος ὁ ΖΜ ἐπιπέδῳ τέτμηται τῷ ΓΔ παραλλήλῷ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ὁ ΓΜ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΛΝ ἄξων πρὸς τὸν ΚΛ ἄξονα. ἴσος δέ ἐστιν ὁ μὲν ΓΜ κύλινδρος τῷ ΕΒ κυλίνδρο, ἡ δὲ ΛΝ ἄξων τῷ ΗΘ ἄξονι· ἔστιν ἄρα ὡς ἱ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα. ὡς δὲ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ is of axis EK, so many multiples is cylinder QG also of cylinder GB. And so, for the same (reasons), as many multiples as axis MK is of axis KF, so many multiples is cylinder WG also of cylinder GD. And if axis KL is equal to axis KM then cylinder QG will also be equal to cylinder GW, and if the axis (is) greater than the axis then the cylinder (will also be) greater than the cylinder, and if (the axis is) less then (the cylinder will also be) less. So, there are four magnitudes—the axes EK and KF, and the cylinders BG and GD—and equal multiples have been taken of axis EK and cylinder BG—(namely), axis LK and cylinder QG—and of axis KF and cylinder GD—(namely), axis KM and cylinder GW. And it has been shown that if axis KL exceeds axis KM then cylinder QG also exceeds cylinder GW, and if (the axes are) equal then (the cylinders are) equal, and if (KL is) less then (QG is) less. Thus, as axis EK is to axis KF, so cylinder BG (is) to cylinder GD [Def. 5.5]. (Which is) the very thing it was required to show.

Proposition 14

Cones and cylinders which are on equal bases are to one another as their heights.



For let EB and FD be cylinders on equal bases, (namely) the circles AB and CD (respectively). I say that as cylinder EB is to cylinder FD, so axis GH (is) to axis KL.

For let the axis KL have been produced to point N. And let LN be made equal to axis GH. And let the cylinder CM have been conceived about axis LN. Therefore, since cylinders EB and CM have the same height they are to one another as their bases [Prop. 12.11]. And the bases are equal to one another. Thus, cylinders EB and CM are also equal to one another. And since cylinder FM has been cut by the plane CD, which is parallel to its opposite planes, thus as cylinder CM is to cylinder FD, so axis LN (is) to axis KL [Prop. 12.13]. And cylinder CM is equal to cylinder EB, and axis LN to axis GH. Thus, as cylinder EB is to cylinder FD, so axis GH (is) κύλινδρον, ούτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον. καὶ ὡς ἄρα ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον καὶ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

ιε΄.

Τῶν ἴσων κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ῶν κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσοι εἰσὶν ἐκεῖνοι.



Έστωσαν ίσοι κῶνοι καὶ κύλινδροι, ῶν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ αὐτῶν αἰ ΑΓ, ΕΗ, ἄξονες δὲ οἱ ΚΛ, ΜΝ, οἴτινες καὶ ὕψη εἰσὶ τῶν κώνων ἢ κυλίνδρων, καὶ συμπεπληρώσθωσαν οἱ ΑΞ, ΕΟ κύλινδροι. λέγω, ὅτι τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἑστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος.

Τὸ γὰρ ΛΚ ὕψος τῷ ΜΝ ὕψει ἤτοι ἴσον ἐστὶν ἢ οὕ. έστω πρότερον ίσον. έστι δὲ καὶ ὁ ΑΞ κύλινδρος τῷ ΕΟ χυλίνδρω ἴσος. οἱ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ χύλινδροι πρός άλλήλους είσιν ώς αί βάσεις. ἴση ἄρα χαί ή ΑΒΓΔ βάσις τῆ ΕΖΗΘ βάσει. ὥστε καὶ ἀντιπέπονθεν, ώς ή ΑΒΓΔ βάσις πρός την ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ύψος πρὸς τὸ ΚΛ ὕψος. ἀλλὰ δὴ μὴ ἔστω τὸ ΛΚ ὕψος τῷ ΜΝ ἴσον, ἀλλ' ἔστω μεῖζον τὸ ΜΝ, καὶ ἀφηρήσθω ἀπὸ τοῦ ΜΝ ὕψους τῷ ΚΛ ἴσον τὸ ΠΝ, καὶ διὰ τοῦ Π σημείου τετμήσθω ὁ ΕΟ κύλινδρος ἐπιπέδω τῷ ΤΥΣ παραλλήλω τοῖς τῶν ΕΖΗΘ, ΡΟ κύκλων ἐπιπέδοις, καὶ ἀπὸ βάσεως μὲν τοῦ ΕΖΗΘ κύκλου, ὕψους δὲ τοῦ ΝΠ κύλινδρος νενοήσθω ό ΕΣ. καί ἐπεὶ ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρω, ἔστιν ἄρα ώς
 ό $A\Xi$ κύλινδρος πρὸς τὸν ΕΣ κυλίνδρον, οὕτως ό ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον. ἀλλ
 ὡς μὲν ὁ ΑΞ κύλινδρος πρός τὸν ΕΣ κύλινδρον, οὕτως ἡ ΑΒΓΔ βάσις πρὸς τὴν EZH Θ · ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσὶν οἱ AΞ, EΣ κύλινδροι· ώς δὲ ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος· ὁ γὰρ ΕΟ χύλινδρος ἐπιπέδω τέτμηται παραλλήλω όντι τοῖς ἀπεναντίον ἐπιπέδοις. ἔστιν ἄρα καὶ ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἴσον δὲ τὸ ΠΝ ὕψος τῷ ΚΛ ὕψει· ἔστιν ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, ούτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος. τῶν ἄρα ΑΞ, ΕΟ χυλίνδρων άντιπεπόνθασιν αί βάσεις τοῖς ὕψεσιν.

to axis KL. And as cylinder EB (is) to cylinder FD, so cone ABG (is) to cone CDK [Prop. 12.10]. Thus, also, as axis GH (is) to axis KL, so cone ABG (is) to cone CDK, and cylinder EB to cylinder FD. (Which is) the very thing it was required to show.

Proposition 15

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.



Let there be equal cones and cylinders whose bases are the circles ABCD and EFGH, and the diameters of (the bases) AC and EG, and (whose) axes (are) KLand MN, which are also the heights of the cones and cylinders (respectively). And let the cylinders AO and EP have been completed. I say that the bases of cylinders AO and EP are reciprocally proportional to their heights, and (so) as base ABCD is to base EFGH, so height MN(is) to height KL.

For height LK is either equal to height MN, or not. Let it, first of all, be equal. And cylinder AO is also equal to cylinder EP. And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base ABCD (is) also equal to base EFGH. And, hence, reciprocally, as base ABCD (is) to base EFGH, so height MN (is) to height KL. And so, let height LKnot be equal to MN, but let MN be greater. And let QN, equal to KL, have been cut off from height MN. And let the cylinder EP have been cut, through point Q, by the plane TUS (which is) parallel to the planes of the circles EFGH and RP. And let cylinder ES have been conceived, with base the circle EFGH, and height NQ. And since cylinder AO is equal to cylinder EP, thus, as cylinder AO (is) to cylinder ES, so cylinder EP (is) to cylinder ES [Prop. 5.7]. But, as cylinder AO (is) to cylinder ES, so base ABCD (is) to base EFGH. For cylinders AO and ES (have) the same height [Prop. 12.11]. And as cylinder EP (is) to (cylinder) ES, so height MN (is) to height QN. For cylinder EP has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base ABCD is to base EFGH, so height MN (is) to height QN [Prop. 5.11]. And height QN

Άλλὰ δὴ τῶν ΑΞ, ΕΟ χυλίνδρων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, xαὶ ἔστω ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος· λέγω, ὅτι ἴσος ἐστὶν ὁ ΑΞ χύλινδρος τῷ ΕΟ χυλίνδρῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ἐπεί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος, ἴσον δὲ τὸ ΚΛ ὕψος τῷ ΠΝ ὕψει, ἔσται ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἀλλ' ὡς μὲν ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσίν· ὡς δὲ τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ [ὕψος], οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κυλίνδρος πρὸς τὸν ΕΣ κύλινδρος τῷ ΕΟ κυλίνδρος πρὸς τὸν ΕΣ. ἴσος ἄρα ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ. ὡσαύτως δὲ καὶ ἐπὶ τῶν κώνων· ὅπερ ἔδει δεῖξαι.

ເຈ′.

Δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων εἰς τὸν μείζονα κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ἐλάσσονος κύκλου.



Έστωσαν οἱ δοθέντες δύο κύκλοι οἱ ABFΔ, ΕΖΗΘ περὶ τὸ αὐτὸ κέντρον τὸ Κ· δεῖ δὴ εἰς τὸν μείζονα κύκλον τὸν ABFΔ πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ΕΖΗΘ κύκλου.

Ήχθω γὰρ διὰ τοῦ Κ κέντρου εὐθεῖα ἡ ΒΚΔ, καὶ ἀπὸ τοῦ Η σημείου τῆ ΒΔ εὐθεία πρὸς ὀρθὰς ἤχθω ἡ ΗΑ καὶ διήχθω ἐπὶ τὸ Γ· ἡ ΑΓ ἄρα ἐφάπτεται τοῦ ΕΖΗΘ κύκλου. τέμνοντες δὴ τὴν ΒΑΔ περιφέρειαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομεν περιφέρειαν ἐλάσσονα τῆς ΑΔ. λελείφθω, καὶ ἔστω ἡ ΛΔ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΒΔ κάθετος ἤχθω ἡ ΛΜ καὶ διήχθω (is) equal to height KL. Thus, as base ABCD is to base EFGH, so height MN (is) to height KL. Thus, the bases of cylinders AO and EP are reciprocally proportional to their heights.

And, so, let the bases of cylinders AO and EP be reciprocally proportional to their heights, and (thus) let base ABCD be to base EFGH, as height MN (is) to height KL. I say that cylinder AO is equal to cylinder EP.

For, with the same construction, since as base ABCD is to base EFGH, so height MN (is) to height KL, and height KL (is) equal to height QN, thus, as base ABCD (is) to base EFGH, so height MN will be to height QN. But, as base ABCD (is) to base EFGH, so cylinder AO (is) to cylinder ES. For they are the same height [Prop. 12.11]. And as height MN (is) to [height] QN, so cylinder EP (is) to cylinder ES [Prop. 12.13]. Thus, as cylinder AO is to cylinder ES, so cylinder EP (is) to (cylinder) ES [Prop. 5.11]. Thus, cylinder AO (is) equal to cylinder EP [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.

Proposition 16

There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.



Let ABCD and EFGH be the given two circles, about the same center, K. So, it is necessary to inscribe an equilateral and even-sided polygon in the greater circle ABCD, not touching circle EFGH.

Let the straight-line BKD have been drawn through the center K. And let GA have been drawn, at rightangles to the straight-line BD, through point G, and let it have been drawn through to C. Thus, AC touches circle EFGH [Prop. 3.16 corr.]. So, (by) cutting circumference BAD in half, and the half of it in half, and doing this continually, we will (eventually) leave a circumference less ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΔ, ΔΝ· ἴση ἄρα ἐστὶν ἡ ΛΔ τῆ ΔΝ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΛΝ τῆ ΑΓ, ἡ δὲ ΑΓ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου, ἡ ΛΝ ἄρα οὐκ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου· πολλῷ ἄρα αἱ ΛΔ, ΔΝ οὐκ ἐφάπτονται τοῦ ΕΖΗΘ κύκλου. ἐὰν δὴ τῆ ΛΔ εὐθεία ἴσας κατὰ τὸ συνεχὲς ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ κύκλον, ἐγγραφήσεται εἰς τὸν ΑΒΓΔ κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΕΖΗΘ· ὅπερ ἔδει ποιῆσαι.

than AD [Prop. 10.1]. Let it have been left, and let it be LD. And let LM have been drawn, from L, perpendicular to BD, and let it have been drawn through to N. And let LD and DN have been joined. Thus, LD is equal to DN [Props. 3.3, 1.4]. And since LN is parallel to AC [Prop. 1.28], and AC touches circle EFGH, LN thus does not touch circle EFGH. Thus, even more so, LD and DN do not touch circle EFGH. And if we continuously insert (straight-lines) equal to straight-line LD into circle ABCD [Prop. 4.1] then an equilateral and evensided polygon, not touching the lesser circle EFGH, will have been inscribed in circle ABCD.[†] (Which is) the very thing it was required to do.

 † Note that the chord of the polygon, LN, does not touch the inner circle either.

ιζ΄.

Δύο σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.



Νενοήσθωσαν δύο σφαῖραι περὶ τὸ αὐτὸ κέντρον τὸ Α· δεῖ δὴ εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Τετμήσθωσαν αἱ σφαῖραι ἐπιπέδω τινὶ διὰ τοῦ κέντρου ἔσονται δὴ αἱ τομαὶ κύκλοι, ἐπειδήπερ μενούσης τῆς διαμέτρου καὶ περιφερομένου τοῦ ἡμικυκλίου ἐγιγνετο ἡ σφαῖρα· ὥστε καὶ καθ' οἴας ἂν θέσεως ἐπινοήσωμεν τὸ ἡμικύκλιον, τὸ δι' αὐτοῦ ἐκβαλλόμενον ἐπίπεδον ποιήσει ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας κύκλον. καὶ φανερόν, ὅτι καὶ μέγιστον, ἐπειδήπερ ἡ διάμετρος τῆς σφαίρας, ἤτις

Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.



Let two spheres have been conceived about the same center, A. So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres have been cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we conceive (of for) the semi-circle, the plane produced through it will make a

έστι και τοῦ ἡμικυκλίου διάμετρος δηλαδή και τοῦ κύκλου, μείζων ἐστὶ πασῶν τῶν εἰς τὸν χύχλον ἢ τὴν σφαῖραν διαγομένων [εὐθειῶν]. ἔστω οὖν ἐν μὲν τῃ μείζονι σφαίρα κύκλος
ό ${\rm BF}\Delta {\rm E},$ ἐν δὲ τῆ ἐλάσσονι σφαίρα κύκλος
ό ${\rm ZH}\Theta,$ καὶ ἤχθωσαν αὐτῶν δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΒΔ, ΓΕ, καὶ δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων τῶν ΒΓΔΕ, ΖΗΘ εἰς τὸν μείζονα κύκλον τὸν ΒΓΔΕ πολύγωνον ισόπλευρον και αρτιόπλευρον έγγεγράφθω μή ψαῦον τοῦ έλάσσονος χύχλου τοῦ ΖΗΘ, οὕ πλευραὶ ἔστωσαν ἐν τῷ ΒΕ τεταρτημορίω αί ΒΚ, ΚΛ, ΛΜ, ΜΕ, και ἐπιζευχθεῖσα ἡ ΚΑ διήχθω ἐπὶ τὸ Ν, καὶ ἀνεστάτω ἀπὸ τοῦ Α σημείου τῷ τοῦ ΒΓΔΕ κύκλου ἐπιπέδω πρὸς ὀρθὰς ἡ ΑΞ καὶ συμβαλλέτω τῆ ἐπιφανεία τῆς σφαίρας κατὰ τὸ Ξ, καὶ διὰ τῆς ΑΞ καὶ έκατέρας τῶν ΒΔ, ΚΝ ἐπίπεδα ἐκβεβλήσθω· ποιήσουσι δὴ διὰ τὰ εἰρημένα ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας μεγίστους κύκλους. ποιείτωσαν, ῶν ἡμικύκλια ἔστω ἐπὶ τῶν ΒΔ, ΚΝ διαμέτρων τὰ ΒΞΔ, ΚΞΝ. καὶ ἐπεὶ ἡ ΞΑ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, καὶ πάντα ἄρα τὰ διὰ τῆς ΞΑ ἐπίπεδά ἐστιν ὀρθὰ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον· ώστε καὶ τὰ ΒΞΔ, ΚΞΝ ἡμικύκλια ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΒΕΔ, ΒΞΔ, ΚΞΝ ήμικύκλια· ἐπὶ γὰρ ἴσων εἰσὶ διαμέτρων τῶν ΒΔ, ΚΝ· ίσα ἐστὶ καὶ τὰ ΒΕ, ΒΞ, ΚΞ τεταρτημόρια ἀλλήλοις. ὄσαι άρα εἰσὶν ἐν τῷ ΒΕ τεταρτημορίω πλευραὶ τοῦ πολυγώνου, τοσαῦταί εἰσι καὶ ἐν τοῖς ΒΞ, ΚΞ τεταρτημορίοις ἴσαι ταῖς BK, KΛ, ΛΜ, ΜΕ εὐθείαις. ἐγγεγράφθωσαν καὶ ἔστωσαν αί ΒΟ, ΟΠ, ΠΡ, ΡΞ, ΚΣ, ΣΤ, ΤΥ, ΥΞ, καὶ ἐπεζεύχθωσαν αί ΣΟ, ΤΠ, ΥΡ, καὶ ἀπὸ τῶν Ο, Σ ἐπὶ τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον κάθετοι ἤχθωσαν. πεσοῦνται δὴ ἐπὶ τὰς κοινάς τομάς τῶν ἐπιπέδων τὰς ΒΔ, ΚΝ, ἐπειδήπερ καὶ τὰ τῶν ΒΞΔ, ΚΞΝ ἐπίπεδα ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. πιπτέτωσαν, καὶ ἔστωσαν αἱ ΟΦ, ΣΧ, καὶ ἐπεζεύχθω ἡ ΧΦ. καὶ ἐπεὶ ἐν ἴσοις ἡμικυκλίοις τοῖς BΞΔ, ΚΞΝ ἴσαι ἀπειλημμέναι εἰσὶν αἱ BO, ΚΣ, καὶ κάθετοι ήγμέναι εἰσὶν αἱ Ο $\Phi, \Sigma X,$ ἴση [ἄρα] ἐστὶν ἡ μὲν Ο Φ τῆ $\Sigma X,$ ή δὲ ΒΦ τῆ ΚΧ. ἔστι δὲ καὶ ὅλη ἡ ΒΑ ὅλη τῆ ΚΑ ἴση· καὶ λοιπὴ ἄρα ἡ ΦA λοιπῆ τῆ XA ἐστιν ἴση· ἔστιν ἄρα ὡς ἡ $B\Phi$ πρὸς τὴν ΦΑ, οὕτως ἡ ΚΧ πρὸς τὴν ΧΑ· παράλληλος ἄρα έστιν ή ΧΦ τῆ ΚΒ. και ἐπει ἑκατέρα τῶν ΟΦ, ΣΧ ὀρθή έστι πρός τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, παράλληλος ἄρα ἐστὶν ἡ Ο Φ τῆ ΣX. ἐδείχθη δὲ αὐτῆ καὶ ἴση· καὶ αἱ X $\Phi,$ ΣΟ άρα ἴσαι εἰσὶ καὶ παράλληλοι. καὶ ἐπεὶ παράλληλός ἐστιν ή ΧΦ τῆ ΣΟ, ἀλλὰ ἡ ΧΦ τῆ ΚΒ ἐστι παράλληλος, καὶ ή ΣΟ ἄρα τῆ KB ἐστι παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΒΟ, ΚΣ· τὸ ΚΒΟΣ ἄρα τετράπλευρον ἐν ἑνί έστιν ἐπιπέδω, ἐπειδήπερ, ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, καὶ ἐφ᾽ ἑκατέρας αὐτῶν ληφϑῇ τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις. διὰ τὰ αὐτὰ δὴ καὶ ἑκάτερον τῶν ΣΟΠΤ, ΤΠΡΥ τετραπλεύρων ἐν ἑνί ἐστιν ἐπιπέδω. ἔστι δὲ καὶ τὸ ΥΡΞ τρίγωνον ἐν ἑνὶ ἑπιπέδω. ἐὰν δὴ νοήσωμεν ἀπὸ

circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let BCDE be the circle in the greater sphere, and FGH the circle in the lesser sphere. And let two diameters of them have been drawn at right-angles to one another, (namely), BD and CE. And there being two circles about the same center—(namely), BCDE and FGH—let an equilateral and even-sided polygon have been inscribed in the greater circle, *BCDE*, not touching the lesser circle, FGH [Prop. 12.16], of which let the sides in the quadrant BE be BK, KL, LM, and ME. And, KA being joined, let it have been drawn across to N. And let AOhave been set up at point A, at right-angles to the plane of circle *BCDE*. And let it meet the surface of the (greater) sphere at O. And let planes have been produced through AO and each of BD and KN. So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let BOD and KON be semi-circles on the diameters BD and KN (respectively). And since OAis at right-angles to the plane of circle BCDE, all of the planes through OA are thus also at right-angles to the plane of circle BCDE [Prop. 11.18]. And, hence, the semi-circles BOD and KON are also at right-angles to the plane of circle BCDE. And since semi-circles BED, BOD, and KON are equal—for (they are) on the equal diameters BD and KN [Def. 3.1]—the quadrants BE, BO, and KO are also equal to one another. Thus, as many sides of the polygon as are in quadrant BE, so many are also in quadrants BO and KO equal to the straight-lines BK, KL, LM, and ME. Let them have been inscribed, and let them be BP, PQ, QR, RO, KS, ST, TU, and UO. And let SP, TQ, and UR have been joined. And let perpendiculars have been drawn from Pand S to the plane of circle BCDE [Prop. 11.11]. So, they will fall on the common sections of the planes BDand KN (with BCDE), inasmuch as the planes of BODand KON are also at right-angles to the plane of circle BCDE [Def. 11.4]. Let them have fallen, and let them be PV and SW. And let WV have been joined. And since BP and KS are equal (circumferences) having been cut off in the equal semi-circles *BOD* and *KON* [Def. 3.28], and PV and SW are perpendiculars having been drawn (from them), PV is [thus] equal to SW, and BV to KW[Props. 3.27, 1.26]. And the whole of BA is also equal to the whole of KA. And, thus, as BV is to VA, so KW(is) to WA. WV is thus parallel to KB [Prop. 6.2]. And

τῶν Ο, Σ, Π, Τ, Ρ, Υ σημείων ἐπὶ τὸ Α ἐπιζευγνυμένας εὐθείας, συσταθήσεταί τι σχῆμα στερεὸν πολύεδρον ματαξὺ τῶν ΒΞ, ΚΞ περιφερειῶν ἐκ πυραμίδων συγκείμενον, ῶν βάσεις μὲν τὰ ΚΒΟΣ, ΣΟΠΤ, ΤΠΡΥ τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον, κορυφὴ δὲ τὸ Α σημεῖον. ἐὰν δὲ καὶ ἐπὶ ἑκάστης τῶν ΚΛ, ΛΜ, ΜΕ πλευρῶν καθάπερ ἐπὶ τῆς ΒΚ τὰ αὐτὰ κατασκευάσωμεν καὶ ἔτι τῶν λοιπῶν τριῶν τεταρτημορίων, συσταθήσεταί τι σχῆμα πολύεδρον ἐγγεγραμμένον εἰς τὴν σφαῖραν πυραμίσι περιεχόμενον, ῶν βάσιες [μὲν] τὰ εἰρημένα τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον καὶ τὰ ὁμοταγῆ αὐτοῖς, κορυφὴ δὲ τὸ Α σημεῖον.

Λέγω ὅτι τὸ εἰρημένον πολύεδρον οὐκ ἐφάψεται τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν, ἐφ' ῆς ἐστιν ὁ ΖΗΘ κύκλος.

ΨΑχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον κάθετος ἡ ΑΨ καὶ συμβαλλέτω τῷ ἐπιπέδω κατὰ τὸ Ψ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ $\Psi \mathrm{B}, \Psi \mathrm{K}.$ καὶ ἐπεὶ ἡ ΑΨ ὀρθή ἐστι πρὸς τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ τοῦ τετραπλεύρου ἐπιπέδω ὀρθή ἐστιν. ἡ ΑΨ ἄρα ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΨ, ΨΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῇ AK, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς AB τῷ ἀπὸ τῆς ΑΚ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΒ· ὀρθή γὰρ ἡ πρὸς τῷ Ψ· τῷ δὲ ἀπὸ τῆς ΑΚ ἴσα τὰ άπὸ τῶν ΑΨ, ΨΚ. τὰ ἄρα ἀπὸ τῶν ΑΨ, ΨΒ ἴσα ἐστὶ τοῖς ἀπὸ τῶν Α Ψ , Ψ K. κοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς Α Ψ · λοιπὸν ἄρα τὸ ἀπὸ τῆς ${
m B}\Psi$ λοιπῷ τῷ ἀπὸ τῆς $\Psi{
m K}$ ἴσον ἐστίν· ἴση ἄρα ή $\mathrm{B}\Psi$ τῆ $\Psi\mathrm{K}$. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ἀπὸ τοῦ Ψ ἐπὶ τὰ \mathbf{O}, Σ ἐπιζευγνύμεναι εὐθεῖαι ἴσαι εἰσὶν ἑκατέρα τῶν BΨ, ΨΚ. ὁ ἄρα κέντρω τῷ Ψ καὶ διαστήματι ἑνὶ τῶν ΨΒ, ΨΚ γραφόμενος κύκλος ἤξει καὶ διὰ τῶν Ο, Σ, καὶ ἔσται έν κύκλω τὸ ΚΒΟΣ τετράπλευρον.

Καὶ ἐπεὶ μείζων ἐστὶν ἡ ΚΒ τῆς ΧΦ, ἴση δὲ ἡ ΧΦ τῆ ΣΟ, μείζων ἄρα ἡ ΚΒ τῆς ΣΟ. ἴση δὲ ἡ ΚΒ ἑκατέρα τῶν ΚΣ, ΒΟ· καὶ ἑκατέρα ἄρα τῶν ΚΣ, ΒΟ τῆς ΣΟ μείζων έστίν. καὶ ἐπεὶ ἐν κύκλῷ τετράπλευρόν ἐστι τὸ ΚΒΟΣ, καὶ ἴσαι αἱ KB, BO, KΣ, καὶ ἐλάττων ἡ ΟΣ, καὶ ἐκ τοῦ κέντρου τοῦ κύκλου ἐστὶν ἡ ΒΨ, τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μειζόν ἐστιν ἢ διπλάσιον. ἤχθω ἀπὸ τοῦ Κ ἐπὶ τὴν ΒΦ κάθετος ή ΚΩ. καὶ ἐπεὶ ἡ ΒΔ τῆς ΔΩ ἐλάττων ἐστὶν ἢ διπλῆ, καί ἐστιν ὡς ἡ $B\Delta$ πρὸς τὴν $\Delta\Omega$, οὕτως τὸ ὑπὸ τῶν ΔB , $B\Omega$ πρὸς τὸ ὑπὸ [τῶν] $\Delta \Omega$, ΩB , ἀναγραφομένου ἀπὸ τῆς Β Ω τετραγώνου καὶ συμπληρουμένου τοῦ ἐπὶ τῆς $\Omega\Delta$ παραλληλογράμμου καὶ τὸ ὑπὸ ΔΒ, ΒΩ ἄρα τοῦ ὑπὸ ΔΩ, ΩΒ ἕλαττόν ἐστιν ἢ διπλάσιον. καί ἐστι τῆς ΚΔ ἐπιζευγνυμένης τὸ μὲν ὑπὸ ΔΒ, ΒΩ ἴσον τῷ ἀπὸ τῆς ΒΚ, τὸ δὲ ὑπὸ τῶν $\Delta\Omega,\,\Omega\mathrm{B}$ ἴσον τῷ ἀπὸ τῆς $\mathrm{K}\Omega^{\cdot}$ τὸ ẳρα ἀπὸ τῆς $\mathrm{K}\mathrm{B}$ τοῦ ἀπὸ τῆς ΚΩ ἔλασσόν ἐστιν ἢ διπλάσιον. ἀλλὰ τὸ ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον· μεῖζον άρα τὸ ἀπὸ τῆς ΚΩ τοῦ ἀπὸ τῆς ΒΨ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΑ τῆ ΚΑ, ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΚ. καί

since PV and SW are each at right-angles to the plane of circle BCDE, PV is thus parallel to SW [Prop. 11.6]. And it was also shown (to be) equal to it. And, thus, WV and SP are equal and parallel [Prop. 1.33]. And since WV is parallel to SP, but WV is parallel to KB, SP is thus also parallel to KB [Prop. 11.1]. And BPand KS join them. Thus, the quadrilateral KBPS is in one plane, inasmuch as if there are two parallel straightlines, and a random point is taken on each of them, then the straight-line joining the points is in the same plane as the parallel (straight-lines) [Prop. 11.7]. So, for the same (reasons), each of the quadrilaterals SPQT and TQRU is also in one plane. And triangle URO is also in one plane [Prop. 11.2]. So, if we conceive straightlines joining points P, S, Q, T, R, and U to A then some solid polyhedral figure will have been constructed between the circumferences BO and KO, being composed of pyramids whose bases (are) the quadrilaterals KBPS, SPQT, TQRU, and the triangle URO, and apex the point A. And if we also make the same construction on each of the sides KL, LM, and ME, just as on BK, and, further, (repeat the construction) in the remaining three quadrants, then some polyhedral figure which has been inscribed in the sphere will have been constructed, being contained by pyramids whose bases (are) the aforementioned quadrilaterals, and triangle URO, and the (quadrilaterals and triangles) similarly arranged to them, and apex the point A.

So, I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle FGH is (situated).

Let the perpendicular (straight-line) AX have been drawn from point A to the plane KBPS, and let it meet the plane at point X [Prop. 11.11]. And let XB and XK have been joined. And since AX is at right-angles to the plane of quadrilateral KBPS, it is thus also at rightangles to all of the straight-lines joined to it which are also in the plane of the quadrilateral [Def. 11.3]. Thus, AX is at right-angles to each of BX and XK. And since AB is equal to AK, the (square) on AB is also equal to the (square) on AK. And the (sum of the squares) on AXand XB is equal to the (square) on AB. For the angle at X (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on AX and XK is equal to the (square) on AK [Prop. 1.47]. Thus, the (sum of the squares) on AX and XB is equal to the (sum of the squares) on AX and XK. Let the (square) on AX have been subtracted from both. Thus, the remaining (square) on BX is equal to the remaining (square) on XK. Thus, BX (is) equal to XK. So, similarly, we can show that the straight-lines joined from X to P and S are equal to each of BX and XK.

έστι τῷ μὲν ἀπὸ τῆς ΒΑ ἴσα τὰ ἀπὸ τῶν ΒΨ, ΨΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα τὰ ἀπὸ τῶν ΚΩ, ΩΑ· τὰ ἄρα ἀπὸ τῶν ΒΨ, ΨΑ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΚΩ, ΩΑ, ῶν τὸ ἀπὸ τῆς ΚΩ μεῖζον τοῦ ἀπὸ τῆς ΒΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΩΑ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΨΑ. μείζων ἄρα ἡ ΑΨ τῆς ΑΩ· πολλῷ ἄρα ἡ ΑΨ μείζων ἐστὶ τῆς ΑΗ. καί ἐστιν ἡ μὲν ΑΨ ἐπὶ μίαν τοῦ πολυέδρου βάσιν, ἡ δὲ ΑΗ ἐπὶ τὴν τῆς ἐλάσσονος σφαίρας ἐπιφάνειαν· ὥστε τὸ πολύεδρον οὐ ψαύσει τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Δύο άρα σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγέγραπται μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν· ὅπερ ἔδει ποιῆσαι. Thus, a circle drawn (in the plane of the quadrilateral) with center X, and radius one of XB or XK, will also pass through P and S, and the quadrilateral KBPS will be inside the circle.

And since KB is greater than WV, and WV (is) equal to SP, KB (is) thus greater than SP. And KB (is) equal to each of KS and BP. Thus, KS and BP are each greater than SP. And since quadrilateral KBPS is in a circle, and KB, BP, and KS are equal (to one another), and PS (is) less (than them), and BX is the radius of the circle, the (square) on KB is thus greater than double the (square) on BX^{\dagger} . Let the perpendicular *KY* have been drawn from *K* to BV.[‡] And since *BD* is less than double DY, and as BD is to DY, so the (rectangle contained) by DB and BY (is) to the (rectangle contained) by DY and YB—a square being described on BY, and a (rectangular) parallelogram (with short side equal to BY) completed on YD—the (rectangle contained) by DB and BY is thus also less than double the (rectangle contained) by DY and YB. And, KD being joined, the (rectangle contained) by DB and BY is equal to the (square) on BK, and the (rectangle contained) by DY and YB equal to the (square) on KY [Props. 3.31, 6.8 corr.]. Thus, the (square) on KB is less than double the (square) on KY. But, the (square) on KB is greater than double the (square) on BX. Thus, the (square) on KY (is) greater than the (square) on BX. And since BA is equal to KA, the (square) on BA is equal to the (square) on AK. And the (sum of the squares) on BXand XA is equal to the (square) on BA, and the (sum of the squares) on KY and YA (is) equal to the (square) on KA [Prop. 1.47]. Thus, the (sum of the squares) on BXand XA is equal to the (sum of the squares) on KY and YA, of which the (square) on KY (is) greater than the (square) on BX. Thus, the remaining (square) on YAis less than the (square) on XA. Thus, AX (is) greater than AY. Thus, AX is much greater than AG.[§] And AX is (a perpendicular) on one of the bases of the polyhedron, and AG (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do.

[†] Since KB, BP, and KS are greater than the sides of an inscribed square, which are each of length $\sqrt{2}BX$.

 $^{^\}ddagger$ Note that points Y and V are actually identical.

[§] This conclusion depends on the fact that the chord of the polygon in proposition 12.16 does not touch the inner circle.

Ἐἀν δὲ καὶ εἰς ἑτάραν σφαῖραν τῷ ἐν τῇ ΒΓΔΕ σφαίρα στερεῷ πολυέδρω ὄμοιον στερεὸν πολύεδρον ἐγγραφῆ, τὸ έν τῆ ΒΓΔΕ σφαίρα στερεὸν πολύεδρον πρὸς τὸ ἐν τῆ ἑτέρα σφαίρα στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει, ήπερ ή τῆς ΒΓΔΕ σφαίρας διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον. διαιρεθέντων γάρ τῶν στερεῶν εἰς τὰς ὁμοιοπληθεῖς καὶ ὁμοιοταγεῖς πυραμίδας ἔσονται αἱ πυραμίδες ὄμοιαι. αἱ δὲ ὄμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγω είσι τῶν ὑμολόγων πλευρῶν. ἡ ἄρα πυραμίς, ής βάσις μέν ἐστι τὸ ΚΒΟΣ τετράπλευρον, κορυφή δὲ τὸ Α σημεῖον, πρὸς τὴν ἐν τῆ ἑτέρα σφαίρα ὑμοιοταγῆ πυραμίδα τριπλασίονα λόγον ἔχει, ἤπερ ἡ ὑμόλογος πλευρὰ πρός την δμόλογον πλευράν, τουτέστιν ήπερ ή AB έχ τοῦ κέντρου τῆς σφαίρας τῆς περὶ κέντρον τὸ Α πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. ὁμοίως καὶ ἑκάστη πυραμὶς τῶν ἐν τῇ περὶ κέντρον τὸ Α σφαίρα πρὸς ἑκάστην ὑμοταγῆ πυραμίδα τῶν ἐν τῆ ἑτέρα σφαίρα τριπλασίονα λόγον ἕξει, ήπερ ή AB πρός την έκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. καὶ ὡς ἕν τῶν ἡγουμένων πρὸς ἕν τῶν ἑπομένων, οὕτως άπαντα τὰ ἡγούμενα πρὸς ἄπαντα τὰ ἑπόμενα· ὥστε ὅλον τὸ ἐν τῆ περὶ κέντρον τὸ Α σφαίρα στερεὸν πολύεδρον πρὸς ὅλον τὸ ἐν τῆ ἑτέρα [σφαίρα] στερεὸν πολύεδρον τριπλασίονα λόγον ἕξει, ἤπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας, τουτέστιν ἤπερ ἡ ΒΔ διάμετρος πρὸς την της έτέρας σφαίρας διάμετρον. ὅπερ ἔδει δεῖξαι.

ιη'**.**

Αἱ σφαῖραι πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἰδίων διαμέτρων.



Corollary

And, also, if a similar polyhedral solid to that in sphere BCDE is inscribed in another sphere then the polyhedral solid in sphere BCDE has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere *BCDE* has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral KBPS, and apex the point A, will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius AB of the sphere about center A to the radius of the other sphere. And, similarly, each pyramid in the sphere about center A will have to each similarly situated pyramid in the other sphere the cubed ratio that AB (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center A will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) AB (has) to the radius of the other sphere. That is to say, that diameter BD (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.

Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.



Νενοήσθωσαν σφαῖραι αἱ ΑΒΓ, ΔΕΖ, διάμετροι δὲ αὐτῶν αἱ ΒΓ, ΕΖ· λέγω, ὅτι ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰ γὰρ μὴ ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ, ἕξει ἄρα ἡ ΑΒΓ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ή πρός μείζονα ήπερ ή ΒΓ πρός την ΕΖ. έχέτω πρότερον πρὸς ἐλάσσονα τὴν ΗΘΚ, καὶ νενοήσθω ἡ ΔΕΖ τῆ ΗΘΚ περὶ τὸ αὐτὸ κέντρον, καὶ ἐγγεγράφθω εἰς τὴν μείζονα σφαῖραν τὴν ΔΕΖ στερεὸν πολύεδρον μὴ ψαῦον τῆς ἐλάσσονος σφαίρας τῆς ΗΘΚ κατὰ τὴν ἐπιφάνειαν, έγγεγράφθω δὲ καὶ εἰς τὴν ΑΒΓ σφαῖραν τῷ ἐν τῇ ΔΕΖ σφαίρα στερεῷ πολυέδρῳ ὄμοιον στερεὸν πολύεδρον. τὸ άρα ἐν τῆ ΑΒΓ στερεὸν πολύεδρον πρὸς τὸ ἐν τῆ ΔΕΖ στερεόν πολύεδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἔχει δὲ καὶ ἡ ΑΒΓ σφαῖρα πρὸς τὴν ΗΘΚ σφαῖραν τριπλασίονα λόγον ήπερ ή ΒΓ πρός την ΕΖ· ἔστιν ἄρα ὡς ή ΑΒΓ σφαῖρα πρός την ΗΘΚ σφαῖραν, οὕτως τὸ ἐν τῆ ΑΒΓ σφαίρα στερεὸν πολύεδρον πρὸς τὸ ἐν τῆ ΔΕΖ σφαίρα στερεόν πολύεδρον· ἐναλλὰξ [ἄρα] ὡς ἡ ΑΒΓ σφαῖρα πρός τὸ ἐν αὐτῆ πολύεδρον, οὕτως ἡ ΗΘΚ σφαῖρα πρὸς τὸ ἐν τῆ ΔΕΖ σφαίρα στερεόν πολύεδρον. μείζων δὲ ἡ ΑΒΓ σφαῖρα τοῦ ἐν αὐτῇ πολυέδρου· μείζων ἄρα καὶ ἡ ΗΘΚ σφαῖρα τοῦ ἐν τῆ ΔΕΖ σφαίρα πολυέδρου. ἀλλὰ καὶ ἐλάττων έμπεριέχεται γὰρ ὑπ' αὐτοῦ. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς έλάσσονα τῆς $\Delta \mathrm{EZ}$ σφαίρας τριπλασίονα λόγον ἕχει ἤπερ ἡ ΒΓ διάμετρος πρὸς τὴν ΕΖ. ὑμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ΔΕΖ σφαΐρα πρὸς ἐλάσσονα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ.

Λέγω δή, ὅτι οὐδὲ ἡ ABΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔEZ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ BΓ πρὸς τὴν EZ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μείζονα τὴν ΛΜΝ· ἀνάπαλιν ἄρα ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ διάμετρος πρὸς τὴν ΒΓ διάμετρον. ὡς δὲ ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν, οὕτως ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας, ἐπειδήπερ μείζων ἐστὶν ἡ ΛΜΝ τῆς ΔΕΖ, ὡς ἔμπροσθεν ἑδείχθη. καὶ ἡ ΔΕΖ ἄρα σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ· ὅπερ ἀδύνατον ἑδείχθη. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἐλάσσονα. ἡ ἄρα ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ὅπερ ἕδει δεῖξαι. Let the spheres ABC and DEF have been conceived, and (let) their diameters (be) BC and EF (respectively). I say that sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF.

For if sphere ABC does not have to sphere DEF the cubed ratio that BC (has) to EF then sphere ABC will have to some (sphere) either less than, or greater than, sphere DEF the cubed ratio that BC (has) to EF. Let it, first of all, have (such a ratio) to a lesser (sphere), GHK. And let DEF have been conceived about the same center as GHK. And let a polyhedral solid have been inscribed in the greater sphere *DEF*, not touching the lesser sphere GHK on its surface [Prop. 12.17]. And let a polyhedral solid, similar to the polyhedral solid in sphere *DEF*, have also been inscribed in sphere *ABC*. Thus, the polyhedral solid in sphere ABC has to the polyhedral solid in sphere DEF the cubed ratio that BC(has) to EF [Prop. 12.17 corr.]. And sphere ABC also has to sphere GHK the cubed ratio that BC (has) to EF. Thus, as sphere ABC is to sphere GHK, so the polyhedral solid in sphere ABC (is) to the polyhedral solid is sphere DEF. [Thus], alternately, as sphere ABC (is) to the polygon within it, so sphere GHK (is) to the polyhedral solid within sphere *DEF* [Prop. 5.16]. And sphere ABC (is) greater than the polyhedron within it. Thus, sphere GHK (is) also greater than the polyhedron within sphere DEF [Prop. 5.14]. But, (it is) also less. For it is encompassed by it. Thus, sphere ABC does not have to (a sphere) less than sphere DEF the cubed ratio that diameter BC (has) to EF. So, similarly, we can show that sphere *DEF* does not have to (a sphere) less than sphere ABC the cubed ratio that EF (has) to BC either.

So, I say that sphere ABC does not have to some (sphere) greater than sphere DEF the cubed ratio that BC (has) to EF either.

For, if possible, let it have (the cubed ratio) to a greater (sphere), LMN. Thus, inversely, sphere LMN(has) to sphere ABC the cubed ratio that diameter EF (has) to diameter BC [Prop. 5.7 corr.]. And as sphere LMN (is) to sphere ABC, so sphere DEF(is) to some (sphere) less than sphere ABC, inasmuch as LMN is greater than DEF, as was shown before [Prop. 12.2 lem.]. And, thus, sphere *DEF* has to some (sphere) less than sphere ABC the cubed ratio that EF(has) to BC. The very thing was shown (to be) impossible. Thus, sphere *ABC* does not have to some (sphere) greater than sphere DEF the cubed ratio that BC (has) to EF. And it was shown that neither (does it have such a ratio) to a lesser (sphere). Thus, sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF. (Which is) the very thing it was required to show.
ELEMENTS BOOK 13

The Platonic Solids †

[†]The five regular solids—the cube, tetrahedron (*i.e.*, pyramid), octahedron, icosahedron, and dodecahedron—were problably discovered by the school of Pythagoras. They are generally termed "Platonic" solids because they feature prominently in Plato's famous dialogue *Timaeus*. Many of the theorems contained in this book—particularly those which pertain to the last two solids—are ascribed to Theaetetus of Athens.

α΄.

Έὰν εὐθεῖα γραμμὴ ἄχρον καὶ μέσον λόγον τμηθῆ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου.



Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄχρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ ἐκβεβλήσθω ἐπ' εὐθείας τῆ ΓΑ εὐθεῖα ἡ ΑΔ, καὶ κείσθω τῆς ΑΒ ἡμίσεια ἡ ΑΔ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΓΔ τοῦ ἀπὸ τῆς ΔΑ.

Άναγεγράφθωσαν γὰρ ἀπὸ τῶν ΑΒ, ΔΓ τετράγωνα τὰ ΑΕ, ΔΖ, καὶ καταγεγράφθω ἐν τῷ ΔΖ τὸ σχῆμα, καὶ διήχθω ή ΖΓ ἐπὶ τὸ Η. καὶ ἐπεὶ ή ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατά τὸ Γ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΖΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΖΘ. καὶ ἐπεὶ διπλη ἐστιν ἡ ΒΑ της ΑΔ, ἴση δὲ ἡ μὲν ΒΑ τῃ ΚΑ, ἡ δὲ ΑΔ τῆ ΑΘ, διπλῆ ἄρα καὶ ἡ ΚΑ τῆς ΑΘ. ὡς δὲ ἡ ΚΑ πρὸς τὴν ΑΘ, οὕτως τὸ ΓΚ πρὸς τὸ ΓΘ· διπλάσιον ἄρα τὸ ΓΚ τοῦ ΓΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΓ διπλάσια τοῦ ΓΘ. ἴσον ἄρα τὸ ΚΓ τοῖς ΛΘ, ΘΓ. ἐδείχθη δὲ καὶ τὸ ΓΕ τῷ ΘΖ ἴσον. όλον ἄρα τὸ ΑΕ τετράγωνον ἴσον ἐστὶ τῷ ΜΝΞ γνώμονι. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, τετραπλάσιόν ἐστι τὸ άπὸ τῆς BA τοῦ ἀπὸ τῆς A Δ , τουτέστι τὸ AE τοῦ $\Delta\Theta$. ίσον δὲ τὸ ΑΕ τῷ MNΞ γνώμονι· καὶ ὁ MNΞ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΑΟ· ὅλον ἄρα τὸ ΔΖ πενταπλάσιόν έστι τοῦ AO. καί ἐστι τὸ μὲν ΔZ τὸ ἀπὸ τῆς $\Delta \Gamma$, τὸ δὲ AO τὸ ἀπὸ τῆς ΔA · τὸ ẳρα ἀπὸ τῆς ΓΔ πενταπλάσι
όν ἐστι τοῦ άπὸ τῆς ΔΑ.

Ἐἀν ἄρα εὐθεῖα ἄχρον καὶ μέσον λόγον τμηθῆ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου· ὅπερ ἔδει δεῖξαι.

Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.



For let the straight-line AB have been cut in extreme and mean ratio at point C, and let AC be the greater piece. And let the straight-line AD have been produced in a straight-line with CA. And let AD be made (equal to) half of AB. I say that the (square) on CD is five times the (square) on DA.

For let the squares AE and DF have been described on AB and DC (respectively). And let the figure in DFhave been drawn. And let FC have been drawn across to G. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and FH the (square) on AC. Thus, CE (is) equal to FH. And since BA is double AD, and BA (is) equal to KA, and AD to AH, KA (is) thus also double AH. And as KA (is) to AH, so CK (is) to CH [Prop. 6.1]. Thus, CK (is) double CH. And *LH* plus *HC* is also double *CH* [Prop. 1.43]. Thus, KC (is) equal to LH plus HC. And CE was also shown (to be) equal to HF. Thus, the whole square AE is equal to the gnomon MNO. And since BA is double AD, the (square) on BA is four times the (square) on AD—that is to say, AE (is four times) DH. And AE (is) equal to gnomon MNO. And, thus, gnomon MNO is also four times AP. Thus, the whole of DF is five times AP. And DF is the (square) on DC, and AP the (square) on DA. Thus, the (square) on CD is five times the (square) on DA.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of β΄.

Έαν εύθεῖα γραμμή τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄχρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα τὸ λοιπὸν μέρος ἑστὶ τῆς ἑξ ἀρχῆς εὐθείας.



Εὐθεῖα γὰρ γραμμὴ ἡ AB τμήματος ἑαυτῆς τοῦ AΓ πενταπλάσιον δυνάσθω, τῆς δὲ AΓ διπλῆ ἔστω ἡ ΓΔ. λέγω, ὅτι τῆς ΓΔ ἄχρον καὶ μέσον λόγον τεμνομένος τὸ μεῖζον τμῆμά ἐστιν ἡ ΓΒ.

Άναγεγράφθω γὰρ ἀφ᾽ ἑκατέρας τῶν ΑΒ, ΓΔ τετράγωνα τὰ ΑΖ, ΓΗ, καὶ καταγεγράφθω ἐν τῷ ΑΖ τὸ σχῆμα, καὶ διήχθω ή ΒΕ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπό τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΓ, πενταπλάσιόν ἐστι τὸ ΑΖ τοῦ ΑΘ. τετραπλάσιος ἄρα ὁ ΜΝΞ γνώμων τοῦ ΑΘ. καὶ ἐπεὶ διπλῆ έστιν ή $\Delta\Gamma$ τῆς ΓΑ, τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ $\Delta\Gamma$ τοῦ άπὸ ΓΑ, τουτέστι τὸ ΓΗ τοῦ ΑΘ. ἐδείχθη δὲ καὶ ὁ ΜΝΞ γνώμων τετραπλάσιος τοῦ ΑΘ· ἴσος ἄρα ὁ ΜΝΞ γνώμων τῷ ΓΗ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΔΓ τῆς ΓΑ, ἴση δὲ ἡ μὲν ΔΓ τῆ ΓΚ, ἡ δὲ ΑΓ τῆ ΓΘ, [διπλῆ ἄρα καὶ ἡ ΚΓ τῆς ΓΘ], διπλάσιον ἄρα καὶ τὸ ΚΒ τοῦ ΒΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΒ τοῦ ΘΒ διπλάσια ἴσον ἄρα τὸ ΚΒ τοῖς ΛΘ, ΘΒ. ἐδείχθη δὲ καὶ ὅλος ὁ ΜΝΞ γνώμων ὅλῳ τῷ ΓΗ ἴσος· καὶ λοιπὸν ἄρα τὸ ΘΖ τῷ BH ἐστιν ἴσον. καί ἐστι τὸ μὲν BH τὸ ὑπὸ τῶν ΓΔΒ· ἴση γὰρ
 ή ΓΔ τỹ ΔΗ· τὸ δὲ ΘΖ τὸ ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΓΔΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ. ἔστιν ἄρα ώς ή ΔΓ πρός την ΓΒ, οὕτως ή ΓΒ πρός την ΒΔ. μείζων δὲ ἡ ΔΓ τῆς ΓΒ· μείζων ἄρα καὶ ἡ ΓΒ τῆς ΒΔ. τῆς ΓΔ άρα εύθείας άκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΓΒ.

Έλν ἄρα εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄχρον χαὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα τὸ λοιπὸν μέρος the whole, is five times the square on the half. (Which is) the very thing it was required to show.

Proposition 2

If the square on a straight-line is five times the (square) on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.



For let the square on the straight-line AB be five times the (square) on the piece of it, AC. And let CD be double AC. I say that if CD is cut in extreme and mean ratio then the greater piece is CB.

For let the squares AF and CG have been described on each of AB and CD (respectively). And let the figure in AF have been drawn. And let BE have been drawn across. And since the (square) on BA is five times the (square) on AC, AF is five times AH. Thus, gnomon MNO (is) four times AH. And since DC is double CA, the (square) on DC is thus four times the (square) on CA—that is to say, CG (is four times) AH. And the gnomon MNO was also shown (to be) four times AH. Thus, gnomon MNO (is) equal to CG. And since DC is double CA, and DC (is) equal to CK, and AC to CH, [KC (is) thus also double CH], (and) KB (is) also double BH [Prop. 6.1]. And LH plus HB is also double HB[Prop. 1.43]. Thus, KB (is) equal to LH plus HB. And the whole gnomon MNO was also shown (to be) equal to the whole of CG. Thus, the remainder HF is also equal to (the remainder) BG. And BG is the (rectangle contained) by CDB. For CD (is) equal to DG. And HF(is) the square on CB. Thus, the (rectangle contained) by CDB is equal to the (square) on CB. Thus, as DCis to CB, so CB (is) to BD [Prop. 6.17]. And DC (is) greater than CB (see lemma). Thus, CB (is) also greater than BD [Prop. 5.14]. Thus, if the straight-line CD is cut έστι τῆς ἐξ ἀρχῆς εὐθείας. ὅπερ ἔδει δεῖξαι.

Λῆμμα.

Ότι δὲ ἡ διπλῆ τῆς ΑΓ μείζων ἐστὶ τῆς ΒΓ, οὕτως δειχτέον.

Εἰ γὰρ μή, ἔστω, εἰ δυνατόν, ἡ ΒΓ διπλῆ τῆς ΓΑ. τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΑ· πενταπλάσια ἄρα τὰ ἀπὸ τῶν ΒΓ, ΓΑ τοῦ ἀπὸ τῆς ΓΑ. ὑπόκειται δὲ καὶ τὸ ἀπὸ τῆς ΒΑ πενταπλάσιον τοῦ ἀπὸ τῆς ΓΑ· τὸ ἄρα ἀπὸ τῆς ΒΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΓ, ΓΑ· ὅπερ ἀδύνατον. οὐx ἄρα ἡ ΓΒ διπλασία ἐστὶ τῆς ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ἐλάττων τῆς ΓΒ διπλασίων ἐστὶ τῆς ΓΑ· πολλῷ γὰρ [μεῖζον] τὸ ἀτοπον.

Ή ἄρα τῆς ΑΓ διπλῆ μείζων ἐστὶ τῆς ΓΒ· ὅπερ ἔδει δε
ῖζαι.

in extreme and mean ratio then the greater piece is *CB*.

Thus, if the square on a straight-line is five times the (square) on a piece of itself, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line. (Which is) the very thing it was required to show.

Lemma

And it can be shown that double AC (*i.e.*, DC) is greater than BC, as follows.

For if (double AC is) not (greater than BC), if possible, let BC be double CA. Thus, the (square) on BC (is) four times the (square) on CA. Thus, the (sum of) the (squares) on BC and CA (is) five times the (square) on CA. And the (square) on BA was assumed (to be) five times the (square) on CA. Thus, the (square) on BA is equal to the (sum of) the (squares) on BC and CA. The very thing (is) impossible [Prop. 2.4]. Thus, CB is not double AC. So, similarly, we can show that a (straight-line) less than CB is not double AC either. For (in this case) the absurdity is much [greater].

Thus, double AC is greater than CB. (Which is) the very thing it was required to show.

Proposition 3

Έαν εύθεῖα γραμμή ἄχρον καὶ μέσον λόγον τμηθῆ, τὸ ἔλασσον τμῆμα προσλαβὸν τὴν ἡμίσειαν τοῦ μείζονος τμήματος πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος τετραγώνου.



Εὐθεῖα γάρ τις ἡ ΑΒ ἄχρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Δ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΔ τοῦ ἀπὸ τῆς ΔΓ.

Άναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΕ, καὶ DC.

If a straight-line is cut in extreme and mean ratio then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.



For let some straight-line AB have been cut in extreme and mean ratio at point C. And let AC be the greater piece. And let AC have been cut in half at D. I say that the (square) on BD is five times the (square) on DC.

καταγεγράφθω διπλοῦν τὸ σχῆμα. ἐπεὶ διπλῆ ἐστιν ἡ ΑΓ τῆς $\Delta\Gamma$, τετραπλάσιον ἄρα τὸ ἀπὸ τῆς $A\Gamma$ τοῦ ἀπὸ τῆς $\Delta\Gamma$, τουτέστι τὸ ΡΣ τοῦ ΖΗ. χαὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒΓ ἴσον έστι τῷ ἀπὸ τῆς ΑΓ, καί ἐστι τὸ ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ άρα ΓΕ ίσον ἐστὶ τῷ ΡΣ. τετραπλάσιον δὲ τὸ ΡΣ τοῦ ZH· τετραπλάσιον ἄρα καὶ τὸ ΓΕ τοῦ ΖΗ. πάλιν ἐπεὶ ἴση ἐστὶν ή ΑΔ τῆ ΔΓ, ἴση ἐστὶ καὶ ή ΘΚ τῆ ΚΖ. ὥστε καὶ τὸ ΗΖ τετράγωνον ἴσον ἐστὶ τῷ ΘΛ τετραγώνω. ἴση ἄρα ἡ ΗΚ τῆ ΚΛ, τουτέστιν ἡ ΜΝ τῆ ΝΕ· ὥστε καὶ τὸ ΜΖ τῷ ΖΕ έστιν ίσον. άλλὰ τὸ ΜΖ τῷ ΓΗ ἐστιν ίσον καὶ τὸ ΓΗ ἄρα τῷ ΖΕ ἐστιν ἴσον.
 χοινὸν προσχείσθω τὸ ΓΝ· ὁ ἄρα ΞΟΠ γνώμων ἴσος ἐστὶ τῷ ΓΕ. ἀλλὰ τὸ ΓΕ τετραπλάσιον ἐδείχϑῃ τοῦ ΗΖ· καὶ ὁ ΞΟΠ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΖΗ τετραγώνου. ὁ ΞΟΠ ἄρα γνώμων καὶ τὸ ΖΗ τετράγωνον πενταπλάσιός έστι τοῦ ΖΗ. ἀλλὰ ὁ ΞΟΠ γνώμων καὶ τὸ ΖΗ τετράγωνόν ἐστι τὸ ΔΝ. καί ἐστι τὸ μὲν ΔΝ τὸ ἀπὸ τῆς ΔB , τὸ δὲ HZ τὸ ἀπὸ τῆς $\Delta \Gamma$. τὸ ẳρα ἀπὸ τῆς ΔB πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

For let the square AE have been described on AB. And let the figure have been drawn double. Since AC is double DC, the (square) on AC (is) thus four times the (square) on DC—that is to say, RS (is four times) FG. And since the (rectangle contained) by ABC is equal to the (square) on AC [Def. 6.3, Prop. 6.17], and CE is the (rectangle contained) by ABC, CE is thus equal to RS. And RS (is) four times FG. Thus, CE (is) also four times FG. Again, since AD is equal to DC, HK is also equal to KF. Hence, square GF is also equal to square HL. Thus, GK (is) equal to KL—that is to say, MN to NE. Hence, MF is also equal to FE. But, MF is equal to CG. Thus, CG is also equal to FE. Let CN have been added to both. Thus, gnomon OPQ is equal to CE. But, CE was shown (to be) equal to four times GF. Thus, gnomon OPQ is also four times square FG. Thus, gnomon OPQplus square FG is five times FG. But, gnomon OPQ plus square FG is (square) DN. And DN is the (square) on DB, and GF the (square) on DC. Thus, the (square) on DB is five times the (square) on DC. (Which is) the very thing it was required to show.

Proposition 4

If a straight-line is cut in extreme and mean ratio then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.



Let AB be a straight-line, and let it have been cut in extreme and mean ratio at C, and let AC be the greater piece. I say that the (sum of the squares) on AB and BC is three times the (square) on CA.

For let the square ADEB have been described on AB, and let the (remainder of the) figure have been drawn. Therefore, since AB has been cut in extreme and mean ratio at C, and AC is the greater piece, the (rectangle

δ'.

Έὰν εὐθεῖα γραμμὴ ἄχρον καὶ μέσον λόγον τμηθῆ, τὸ ἀπὸ τῆς ὅλης καὶ τοῦ ἐλάσσονος τμήματος, τὰ συναμφότερα τετράγωνα, τριπλάσιά ἐστι τοῦ ἀπὸ τοῦ μείζονος τμήματος τετραγώνου.



Έστω εὐθεῖα ἡ AB, καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μεῖζον τμῆμα τὸ AΓ· λέγω, ὅτι τὰ ἀπὸ τῶν AB, BΓ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ.

Άναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ ΑΔΕΒ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οῦν ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ AΓ, τὸ ἄρα ὑπὸ τῶν ABΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς AΓ. καί ἐστι τὸ μὲν ὑπὸ τῶν ABΓ τὸ AK, τὸ δὲ ἀπὸ τῆς AΓ τὸ ΘΗ· ίσον ἄρα ἐστὶ τὸ ΑΚ τῷ ΘΗ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΖ τῷ ΖΕ, κοινὸν προσκείσθω τὸ ΓΚ· ὅλον ἄρα τὸ ΑΚ ὅλω τῷ ΓΕ ἐστιν ἴσον· τὰ ἄρα ΑΚ, ΓΕ τοῦ ΑΚ ἐστι διπλάσια. ἀλλὰ τὰ ΑΚ, ΓΕ ὁ ΛΜΝ γνώμων ἐστὶ καὶ τὸ ΓΚ τετράγωνον· ὁ ἄρα ΛΜΝ γνώμων καὶ τὸ ΓΚ τετράγωνον διπλάσιά ἐστι τοῦ ΑΚ. ἀλλὰ μὴν καὶ τὸ ΑΚ τῷ ΘΗ ἐδείχθη ἴσον· ὁ ἄρα ΛΜΝ γνώμων καὶ [τὸ ΓΚ τετράγωνον διπλάσιά ἐστι τοῦ ΘΗ· ὥστε ὁ ΛΜΝ γνώμων καὶ] τὰ ΓΚ, ΘΗ τετράγωνα τριπλάσιά ἐστι τοῦ ΘΗ τετραγώνου. καί ἐστιν ὁ [μὲν] ΛΜΝ γνώμων καὶ τὰ ΓΚ, ΘΗ τετράγωνα ὅλον τὸ ΑΕ καὶ τὸ ΓΚ, ἅπερ ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα, τὸ δὲ ΗΘ τὸ ἀπὸ τῆς ΑΓ τετράγωνον. τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου· ὅπερ ἔδει δεῖξαι.

ε΄.

Έὰν εὐθεῖα γραμμὴ ἄχρον καὶ μέσον λόγον τμηθῆ, καὶ προστεθῆ αὐτῆ ἴση τῷ μείζονι τμήματι, ἡ ὅλη εὐθεῖα ἄχρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ἐξ ἀρχῆς εὐθεῖα.



Εύθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄχρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα ἡ ΑΓ, καὶ τῆ ΑΓ ἴση [κείσθω] ἡ ΑΔ. λέγω, ὅτι ἡ ΔΒ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Α, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ἐξ ἀρχῆς εὐθεῖα ἡ ΑΒ.

Άναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ AE, καὶ καταγεγράφθω τὸ σχῆμα. ἑπεὶ ἡ AB ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ ABΓ ἴσον ἑστὶ τῷ ἀπὸ AΓ. καί ἐστι τὸ μὲν ὑπὸ ABΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς AΓ τὸ ΓΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΘΓ. ἀλλὰ τῷ μὲν ΓΕ ἴσον ἑστὶ τὸ ΘΕ, τῷ δὲ ΘΓ ἴσον τὸ Δ Θ· καὶ τὸ Δ Θ ἄρα ἴσον ἑστὶ τῷ ΘΕ [κοινὸν προσκείσθω τὸ ΘΒ]. ὅλον ἄρα τὸ Δ K ὅλῳ τῷ AE ἑστιν ἴσον. καί ἐστι τὸ μὲν Δ K τὸ ὑπὸ τῶν BΔ, Δ A· ἴση

contained) by ABC is thus equal to the (square) on AC[Def. 6.3, Prop. 6.17]. And AK is the (rectangle contained) by ABC, and HG the (square) on AC. Thus, AK is equal to HG. And since AF is equal to FE[Prop. 1.43], let CK have been added to both. Thus, the whole of AK is equal to the whole of CE. Thus, AKplus CE is double AK. But, AK plus CE is the gnomon LMN plus the square CK. Thus, gnomon LMN plus square CK is double AK. But, indeed, AK was also shown (to be) equal to HG. Thus, gnomon LMN plus [square CK is double HG. Hence, gnomon LMN plus] the squares CK and HG is three times the square HG. And gnomon LMN plus the squares CK and HG is the whole of AE plus CK—which are the squares on ABand BC (respectively)—and GH (is) the square on AC. Thus, the (sum of the) squares on AB and BC is three times the square on AC. (Which is) the very thing it was required to show.

Proposition 5

If a straight-line is cut in extreme and mean ratio, and a (straight-line) equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.



For let the straight-line AB have been cut in extreme and mean ratio at point C. And let AC be the greater piece. And let AD be [made] equal to AC. I say that the straight-line DB has been cut in extreme and mean ratio at A, and that the original straight-line AB is the greater piece.

For let the square AE have been described on AB, and let the (remainder of the) figure have been drawn. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and CH the (square) on AC. But, HE is equal to CE [Prop. 1.43], and DH equal γὰρ ἡ $A\Delta$ τῆ $\Delta\Lambda$ · τὸ δὲ AE τὸ ἀπὸ τῆς AB· τὸ ἄρα ὑπὸ τῶν $B\Delta A$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB. ἔστιν ἄρα ὡς ἡ ΔB πρὸς τὴν BA, οὕτως ἡ BA πρὸς τὴν $A\Delta$. μείζων δὲ ἡ ΔB τῆς BA· μείζων ἄρα καὶ ἡ BA τῆς $A\Delta$.

Ή ἄρα ΔΒ ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Α, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΒ· ὅπερ ἔδει δεῖξαι.

ኖ'.

Έὰν εὐθεῖα ἑητη ἄχρον καὶ μέσον λόγον τμηθῆ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.



Έστω εὐθεῖα ἑητὴ ἡ ΑΒ καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μεῖζον τμῆμα ἡ ΑΓ· λέγω, ὅτι ἑκατέρα τῶν ΑΓ, ΓΒ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΒΑ, καὶ κείσθω τῆς ΒΑ ἡμίσεια ή ΑΔ. ἐπεὶ οὖν εὐθεῖα ἡ ΑΒ τέτμηται ἄχρον χαὶ μέσον λόγον κατὰ τὸ Γ, καὶ τῷ μείζονι τμήματι τῷ ΑΓ πρόσκειται ή $A\Delta$ ήμίσεια οὕσα τῆς AB, τὸ ẳρα ἀπὸ $\Gamma\Delta$ τοῦ ἀπὸ ΔA πενταπλάσιόν ἐστιν. τὸ ἄρα ἀπὸ ΓΔ πρὸς τὸ ἀπὸ ΔΑ λόγον ἕχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· σύμμετρον ἄρα τὸ ἀπὸ Γ Δ τῷ ἀπὸ ΔA . ἑητὸν δὲ τὸ ἀπὸ ΔA · ἑητὴ γάρ [ἐστιν] ἡ ΔA ήμίσεια οὕσα τῆς ΑΒ ἑητῆς οὔσης· ἑητὸν ἄρα καὶ τὸ ἀπὸ ΓΔ· ἡητὴ ἄρα ἐστὶ καὶ ἡ ΓΔ. καὶ ἐπεὶ τὸ ἀπὸ ΓΔ πρὸς τὸ ἀπὸ ΔΑ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀσύμμετρος ἄρα μήκει ἡ Γ Δ τῆ ΔA^{\cdot} αἱ ΓΔ, ΔΑ ἄρα ἑηταί εἰσι δυνάμει μόνον σύμμετροι ἀποτομή ἄρα ἐστὶν ἡ ΑΓ. πάλιν, ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΓ, τὸ ἄρα ὑπὸ ΑΒ, ΒΓ τῷ ἀπὸ ΑΓ ἴσον ἐστίν. τὸ ἄρα ἀπὸ τῆς ΑΓ ἀποτομῆς παρὰ τὴν ΑΒ ἑητὴν παραβληθὲν πλάτος ποιεῖ τὴν ΒΓ. τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ἑητὴν παραβαλλόμενον πλάτος ποιεῖ άποτομὴν πρώτην· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ ΓΒ. ἐδείχϑη δὲ καὶ ἡ ΓΑ ἀποτομή.

Έλν ἄρα εὐθεῖα ἑητὴ ἄχρον καὶ μέσον λόγον τμηθῆ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή ὅπερ ἔδει δεῖξαι. to HC. Thus, DH is also equal to HE. [Let HB have been added to both.] Thus, the whole of DK is equal to the whole of AE. And DK is the (rectangle contained) by BD and DA. For AD (is) equal to DL. And AE (is) the (square) on AB. Thus, the (rectangle contained) by BDA is equal to the (square) on AB. Thus, as DB (is) to BA, so BA (is) to AD [Prop. 6.17]. And DB (is) greater than BA. Thus, BA (is) also greater than AD[Prop. 5.14].

Thus, DB has been cut in extreme and mean ratio at A, and the greater piece is AB. (Which is) the very thing it was required to show.

Proposition 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straightline) called an apotome.



Let AB be a rational straight-line cut in extreme and mean ratio at C, and let AC be the greater piece. I say that AC and CB is each that irrational (straight-line) called an apotome.

For let BA have been produced, and let AD be made (equal) to half of BA. Therefore, since the straightline AB has been cut in extreme and mean ratio at C, and AD, which is half of AB, has been added to the greater piece AC, the (square) on CD is thus five times the (square) on DA [Prop. 13.1]. Thus, the (square) on CD has to the (square) on DA the ratio which a number (has) to a number. The (square) on CD (is) thus commensurable with the (square) on DA [Prop. 10.6]. And the (square) on DA (is) rational. For DA [is] rational, being half of AB, which is rational. Thus, the (square) on CD (is) also rational [Def. 10.4]. Thus, CD is also rational. And since the (square) on CD does not have to the (square) on DA the ratio which a square number (has) to a square number, CD (is) thus incommensurable in length with DA [Prop. 10.9]. Thus, CD and DA are rational (straight-lines which are) commensurable in square only. Thus, AC is an apotome [Prop. 10.73]. Again, since AB has been cut in extreme and mean ratio, and AC is the greater piece, the (rectangle contained) by AB and BC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. Thus, the (square) on the apotome AC, applied to the rational (straight-line) AB, makes BC as width. And the (square) on an apotome, applied to a rational (straight-line), makes a first apotome as width [Prop. 10.97]. Thus, CB is a first apotome. And CA was also shown (to be) an apotome.

۲'.

Έὰν πενταγώνου ἰσοπλεύρου αἰ τρεῖς γωνίαι ἤτοι αἱ κατὰ τὸ ἑξῆς ἢ αἱ μὴ κατὰ τὸ ἑξῆς ἴσαι ῶσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον.



Πενταγώνου γὰρ ἰσοπλεύρον τοῦ ΑΒΓΔΕ αἱ τρεῖς γωνίαι πρότερον αἱ κατὰ τὸ ἑξῆς αἱ πρὸς τοῖς Α, Β, Γ ἴσαι ἀλλήλαις ἔστωσαν· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθωσαν γὰρ αί ΑΓ, ΒΕ, ΖΔ. καὶ ἐπεὶ δύο αί ΓΒ, ΒΑ δυσὶ ταῖς ΒΑ, ΑΕ ἴσαι ἐισὶν ἑκατέρα ἑκατέρα, καὶ γωνία ή ὑπὸ ΓΒΑ γωνία τῆ ὑπὸ ΒΑΕ ἐστιν ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῆ ΒΕ ἐστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΕ τριγώνω ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι έσονται, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΒΓΑ τῆ ὑπὸ ΒΕΑ, ἡ δὲ ὑπὸ ΑΒΕ τῆ ὑπὸ ΓΑΒ· ὥστε καὶ πλευρὰ ή AZ πλευρᾶ τῆ BZ ἐστιν ἴση. ἐδείχθη δὲ καὶ ὅλη ἡ AΓ όλη τη ΒΕ ίση και λοιπή άρα ή ΖΓ λοιπη τη ΖΕ έστιν ίση. έστι δὲ καὶ ἡ ΓΔ τῆ ΔΕ ἴση. δύο δὴ αἱ ΖΓ, ΓΔ δυσὶ ταῖς ZE, EΔ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ZΔ· γωνία ἄρα ή ὑπὸ ΖΓΔ γωνία τῆ ὑπὸ ΖΕΔ ἐστιν ἴση. ἐδείχθη δὲ καὶ ή ὑπὸ ΒΓΑ τῆ ὑπὸ ΑΕΒ ἴση· καὶ ὅλη ἄρα ἡ ὑπὸ ΒΓΔ ὅλῃ τῆ ὑπὸ ΑΕΔ ἴση. ἀλλ' ἡ ὑπὸ ΒΓΔ ἴση ὑπόκειται ταῖς πρὸς τοῖς Α, Β γωνίαις· καὶ ἡ ὑπὸ ΑΕΔ ἄρα ταῖς πρὸς τοῖς Α, Β γωνίαις ἴση ἐστίν. ὑμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ Γ ΔE γωνία ἴση ἐστὶ ταῖς πρὸς τοῖς Α, Β, Γ γωνίαις ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον.

Άλλὰ δὴ μὴ ἔστωσαν ἴσαι αἱ κατὰ τὸ ἑξῆς γωνίαι, ἀλλ² ἔστωσαν ἴσαι αἱ πρὸς τοῖς Α, Γ, Δ σημείοις· λέγω, ὅτι καὶ οὕτως ἰσογώνιόν ἐστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθω γὰρ ἡ ΒΔ. καὶ ἐπεὶ δύο αἱ ΒΑ, ΑΕ δυσὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῆ ΒΔ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῷ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. Thus, if a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

Proposition 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.



For let three angles of the equilateral pentagon ABCDE—first of all, the consecutive (angles) at A, B, and C—be equal to one another. I say that pentagon ABCDE is equiangular.

For let AC, BE, and FD have been joined. And since the two (straight-lines) CB and BA are equal to the two (straight-lines) BA and AE, respectively, and angle CBAis equal to angle BAE, base AC is thus equal to base BE, and triangle ABC equal to triangle ABE, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4], (that is), BCA (equal) to *BEA*, and *ABE* to *CAB*. And hence side *AF* is also equal to side BF [Prop. 1.6]. And the whole of ACwas also shown (to be) equal to the whole of BE. Thus, the remainder FC is also equal to the remainder FE. And *CD* is also equal to *DE*. So, the two (straight-lines) FC and CD are equal to the two (straight-lines) FE and ED (respectively). And FD is their common base. Thus, angle FCD is equal to angle FED [Prop. 1.8]. And BCAwas also shown (to be) equal to AEB. And thus the whole of BCD (is) equal to the whole of AED. But, (angle) BCD was assumed (to be) equal to the angles at A and B. Thus, (angle) AED is also equal to the angles at A and B. So, similarly, we can show that angle CDEis also equal to the angles at A, B, C. Thus, pentagon ABCDE is equiangular.

And so let consecutive angles not be equal, but let the (angles) at points A, C, and D be equal. I say that pentagon ABCDE is also equiangular in this case.

For let BD have been joined. And since the two

ίση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῆ ὑπὸ ΓΔΒ. ἔστι δὲ καὶ ἡ ὑπὸ ΒΕΔ γωνία τῆ ὑπὸ ΒΔΕ ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΒΕ πλευρῷ τῆ ΒΔ ἐστιν ἴση. καὶ ὅλη ἄρα ἡ ὑπὸ ΑΕΔ γωνία ὅλη τῆ ὑπὸ ΓΔΕ ἐστιν ἴση. ἀλλὰ ἡ ὑπὸ ΓΔΕ ταῖς πρὸς τοῖς Α, Γ γωνίαις ὑπόκειται ἴση[.] καὶ ἡ ὑπὸ ΑΕΔ ἄρα γωνία ταῖς πρὸς τοῖς Α, Γ ἴση ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΒΓ ἴση ἐστὶ ταῖς πρὸς τοῖς Α, Γ, Δ γωνίαις. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον[.] ὅπερ ἔδει δεῖξαι.

η'.

Έὰν πενταγώνου ἰσοπλεύρου καὶ ἰσογωνίου τὰς κατὰ τὸ ἑξῆς δύο γωνίας ὑποτείνωσιν εὐθεῖαι, ἄκρον καὶ μέσον λόγον τέμνουσιν ἀλλήλας, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρῷ.



Πενταγώνου γὰρ ἰσοπλεύρον καὶ ἰσογωνίου τοῦ ΑΒΓΔΕ δύο γωνίας τὰς κατὰ τὸ ἑξῆς τὰς πρὸς τοῖς Α, Β ὑποτεινέτωσαν εὐθεῖαι αἱ ΑΓ, ΒΕ τέμνουσαι ἀλλήλας κατὰ τὸ Θ σημεὶον· λέγω, ὅτι ἑκατέρα αὐτῶν ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ σημεῖον, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρῷ.

Περιγεγράφθω γὰρ περὶ τὸ ΑΒΓΔΕ πεντάγωνον κύκλος ὁ ΑΒΓΔΕ. καὶ ἐπεὶ ὁύο εὐθεῖαι αἱ ΕΑ, ΑΒ ὁυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῆ ΑΓ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΒΓ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρα, ὑφʾ ἀς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῆ ὑπὸ ΑΒΕ· ἱπλῆ ἄρα ἡ ὑπὸ ΑΘΕ τῆς ὑπὸ ΒΑΘ. ἔστι δὲ καὶ ἡ ὑπὸ ΕΑΓ τῆς ὑπὸ ΒΑΓ διπλῆ, ἐπειδήπερ καὶ περιφέρεια ἡ ΕΔΓ περιφερείας τῆς ΓΒ ἐστι διπλῆ· ἴση ἄρα ἡ ὑπὸ ΘΑΕ γωνία τῆ ὑπὸ ΑΘΕ· ὥστε καὶ ἡ ΘΕ εὐθεῖα τῆ ΕΑ, τουτέστι τῆ ΑΒ (straight-lines) BA and AE are equal to the (straightlines) BC and CD, and they contain equal angles, base BE is thus equal to base BD, and triangle ABE is equal to triangle BCD, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle AEB is equal to (angle) CDB. And angle BED is also equal to (angle) BDE, since side BE is also equal to side BD [Prop. 1.5]. Thus, the whole angle AED is also equal to the whole (angle) CDE. But, (angle) CDE was assumed (to be) equal to the angles at A and C. Thus, angle AED is also equal to the (angles) at A and C. So, for the same (reasons), (angle) ABC is also equal to the angles at A, C, and D. Thus, pentagon ABCDE is equiangular. (Which is) the very thing it was required to show.

Proposition 8

If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.



For let the two straight-lines, AC and BE, cutting one another at point H, have subtended two consecutive angles, at A and B (respectively), of the equilateral and equiangular pentagon ABCDE. I say that each of them has been cut in extreme and mean ratio at point H, and that their greater pieces are equal to the sides of the pentagon.

For let the circle ABCDE have been circumscribed about pentagon ABCDE [Prop. 4.14]. And since the two straight-lines EA and AB are equal to the two (straightlines) AB and BC (respectively), and they contain equal angles, the base BE is thus equal to the base AC, and triangle ABE is equal to triangle ABC, and the remaining angles will be equal to the remaining angles, respectively, which the equal sides subtend [Prop. 1.4]. Thus, angle BAC is equal to (angle) ABE. Thus, (angle) AHE (is) double (angle) BAH [Prop. 1.32]. And EAC is also douέστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ BA εὐθεῖα τῆ AE, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ABE τῆ ὑπὸ AEB. ἀλλὰ ἡ ὑπὸ ABE τῆ ὑπὸ BAΘ ἐδείχθη ἴση· καὶ ἡ ὑπὸ BEA ἄρα τῆ ὑπὸ BAΘ ἐστιν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ABE καὶ τοῦ ABΘ ἐστιν ἡ ὑπὸ ABE· λοιπὴ ἄρα ἡ ὑπὸ BAE γωνία λοιπῆ τῆ ὑπὸ AΘB ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ABE τρίγωνον τῷ ABΘ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ EB πρὸς τὴν BA, οὕτως ἡ AB πρὸς τὴν BΘ. ἴση δὲ ἡ BA τῆ EΘ· ὡς ἄρα ἡ BE πρὸς τὴν EΘ, οὕτως ἡ EΘ πρὸς τὴν ΘB. μείζων δὲ ἡ BE τῆς EΘ· μείζων ἄρα καὶ ἡ EΘ τῆς ΘB. ἡ BE ἄρα ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ, καὶ τὸ μεῖζον τμῆμα τὸ ΘΕ ἴσον ἐστὶ τῆ τοῦ πενταγώνου πλευρặ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ AΓ ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ, καὶ τὸ μεῖζον αὐτῆς τμῆμα ἡ ΓΘ ἴσον ἐστὶ τῆ τοῦ πενταγώνου πλευρῷ· ὅπερ ἕδει δεῖξαι.

θ'.

Έὰν ἡ τοῦ ἑξαγώνου πλευρὰ καὶ ἡ τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων συντεθῶσιν, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖξον αὐτῆς τμῆμά ἐστιν ἡ τοῦ ἑξαγώνου πλευρά.



ble BAC, inasmuch as circumference EDC is also double circumference CB [Props. 3.28, 6.33]. Thus, angle HAE (is) equal to (angle) AHE. Hence, straight-line HE is also equal to (straight-line) EA-that is to say, to (straight-line) AB [Prop. 1.6]. And since straight-line BA is equal to AE, angle ABE is also equal to AEB[Prop. 1.5]. But, *ABE* was shown (to be) equal to *BAH*. Thus, BEA is also equal to BAH. And (angle) ABE is common to the two triangles ABE and ABH. Thus, the remaining angle *BAE* is equal to the remaining (angle) AHB [Prop. 1.32]. Thus, triangle ABE is equiangular to triangle ABH. Thus, proportionally, as EB is to BA, so AB (is) to BH [Prop. 6.4]. And BA (is) equal to EH. Thus, as BE (is) to EH, so EH (is) to HB. And BE(is) greater than EH. EH (is) thus also greater than HB [Prop. 5.14]. Thus, BE has been cut in extreme and mean ratio at H, and the greater piece HE is equal to the side of the pentagon. So, similarly, we can show that AC has also been cut in extreme and mean ratio at H, and that its greater piece CH is equal to the side of the pentagon. (Which is) the very thing it was required to show.

Proposition 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together then the whole straight-line has been cut in extreme and mean ratio (at the junction point), and its greater piece is the side of the hexagon.[†]



Έστω χύχλος ὁ ΑΒΓ, καὶ τῶν εἰς τὸν ΑΒΓ χύχλον ἐγγραφομένων σχημάτων, δεκαγώνου μὲν ἔστω πλευρὰ ἡ ΒΓ, ἑξαγώνου δὲ ἡ ΓΔ, καὶ ἔστωσαν ἐπ' εὐθείας· λέγω, ὅτι ἡ ὅλη εὐθεῖα ἡ ΒΔ ἄχρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΓΔ.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΒ, ΕΓ, ΕΔ, καὶ διήχθω ἡ ΒΕ ἐπὶ τὸ

Let ABC be a circle. And of the figures inscribed in circle ABC, let BC be the side of a decagon, and CD (the side) of a hexagon. And let them be (laid down) straight-on (to one another). I say that the whole straight-line BD has been cut in extreme and mean ratio (at C), and that CD is its greater piece.

For let the center of the circle, point E, have been

Α. ἐπεὶ δεχαγώνου ἰσοπλεύρον πλευρά ἐστιν ἡ ΒΓ, πενταπλασίων ἄρα ή ΑΓΒ περιφέρεια τῆς ΒΓ περιφερείας· τετραπλασίων ἄρα ή ΑΓ περιφέρεια τῆς ΓΒ. ὡς δὲ ή ΑΓ περιφέρεια πρός την ΓΒ, οὕτως ή ὑπὸ ΑΕΓ γωνία πρός την ύπὸ ΓΕΒ· τετραπλασίων ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΓΕΒ. καὶ έπεὶ ἴση ἡ ὑπὸ ΕΒΓ γωνία τῆ ὑπὸ ΕΓΒ, ἡ ἄρα ὑπὸ ΑΕΓ γωνία διπλασία έστι τῆς ὑπὸ ΕΓΒ. καὶ ἐπεὶ ἴση ἐστιν ἡ ΕΓ εὐθεῖα τ
ῆ ΓΔ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῆ τοῦ ἑξαγώνου πλευρα τοῦ εἰς τὸν ΑΒΓ κύκλον [ἐγγραφομένου]· ἴση ἐστὶ καὶ ἡ ὑπὸ ΓΕΔ γωνία τῆ ὑπὸ ΓΔΕ γωνία. διπλασία ἄρα ή ὑπὸ ΕΓΒ γωνία τῆς ὑπὸ ΕΔΓ. ἀλλὰ τῆς ὑπὸ ΕΓΒ διπλασία ἐδείχθη ἡ ὑπὸ ΑΕΓ· τετραπλασία ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΕΔΓ. ἐδείχϑη δὲ καὶ τῆς ὑπὸ ΒΕΓ τετραπλασία ή ὑπὸ ΑΕΓ· ἴση ἄρα ή ὑπὸ ΕΔΓ τῃ ὑπὸ ΒΕΓ. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τ
ε ${\rm BE}\Gamma$ καὶ τοῦ ${\rm BE}\Delta,$ ἡ ὑπὸ ${\rm EB}\Delta$ γωνία· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΕΔ τῆ ὑπὸ ΕΓΒ ἐστιν ἴση· ίσογώνιον ἄρα ἐστὶ τὸ ΕΒΔ τρίγωνον τῷ ΕΒΓ τριγώνω. άνάλογον ἄρα ἐστίν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΕΒ πρὸς τὴν BΓ. ἴση δὲ ἡ EB τ
ỹ ΓΔ. ἔστιν ἄρα ὡς ἡ BΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΒ. μείζων δὲ ἡ ΒΔ τῆς ΔΓ· μείζων ἄρα καὶ ἡ ΔΓ τῆς ΓΒ. ἡ ΒΔ ἄρα εὐθεῖα ἄκρον και μέσον λόγον τέτμηται [κατὰ τὸ Γ], και τὸ μεῖζον τμῆμα αὐτῆς ἐστιν
ἡ $\Delta\Gamma^{.}$ ὅπερ ἔδει δεῖξαι.

found [Prop. 3.1], and let EB, EC, and ED have been joined, and let BE have been drawn across to A. Since BC is a side on an equilateral decagon, circumference ACB (is) thus five times circumference BC. Thus, circumference AC (is) four times CB. And as circumference AC (is) to CB, so angle AEC (is) to CEB [Prop. 6.33]. Thus, (angle) AEC (is) four times CEB. And since angle EBC (is) equal to ECB [Prop. 1.5], angle AEC is thus double ECB [Prop. 1.32]. And since straight-line EC is equal to CD—for each of them is equal to the side of the hexagon [inscribed] in circle ABC [Prop. 4.15 corr.] angle CED is also equal to angle CDE [Prop. 1.5]. Thus, angle ECB (is) double EDC [Prop. 1.32]. But, AECwas shown (to be) double ECB. Thus, AEC (is) four times EDC. And AEC was also shown (to be) four times BEC. Thus, EDC (is) equal to BEC. And angle EBD(is) common to the two triangles *BEC* and *BED*. Thus, the remaining (angle) *BED* is equal to the (remaining angle) ECB [Prop. 1.32]. Thus, triangle EBD is equiangular to triangle EBC. Thus, proportionally, as DB is to BE, so EB (is) to BC [Prop. 6.4]. And EB (is) equal to CD. Thus, as BD is to DC, so DC (is) to CB. And BD (is) greater than DC. Thus, DC (is) also greater than CB [Prop. 5.14]. Thus, the straight-line BD has been cut in extreme and mean ratio [at C], and DC is its greater piece. (Which is), the very thing it was required to show.

[†] If the circle is of unit radius then the side of the hexagon is 1, whereas the side of the decagon is $(1/2)(\sqrt{5}-1)$.

ι.

Έὰν εἰς κύκλον πεντάγωνον ἰσόπλευρον ἐγγραφῆ, ἡ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων.



Proposition 10

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.[†]



Έστω κύκλος ὁ ΑΒΓΔΕ, καὶ εἰς τὸ ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓΔΕ. λέγω, ὄτι ἡ τοῦ ΑΒΓΔΕ πενταγώνου πλευρὰ δύναται τήν τε τοῦ έξαγώνου καὶ τὴν τοῦ δεκαγώνου πλευρὰν τῶν εἰς τὸν ΑΒΓΔΕ κύκλον ἐγγραφομένων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ζ σημείον, καὶ ἐπιζευχθεῖσα ἡ ΑΖ διήχθω ἐπὶ τὸ Η σημεῖον, καὶ ἐπεζεύχθω ή ZB, καὶ ἀπὸ τοῦ Z ἐπὶ τὴν AB κάθετος ἤχθω ἡ ZΘ, καὶ διήχθω ἐπὶ τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΚΒ, καὶ πάλιν άπὸ τοῦ Z ἐπὶ τὴν AK κάθετος ἤχθω ἡ ZΛ, καὶ διήχθω ἐπὶ τὸ Μ, καὶ ἐπεζεύχθω ἡ ΚΝ.

Ἐπεὶ ἴση ἐστὶν ἡ ΑΒΓΗ περιφέρεια τῆ ΑΕΔΗ περιφερεία, ῶν ἡ ΑΒΓ τῆ ΑΕΔ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΓΗ περιφέρεια λοιπῆ τῆ ΗΔ ἐστιν ἴση. πενταγώνου δὲ ἡ ΓΔ· δεκαγώνου ἄρα ή ΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ή ΖΑ τῆ ΖΒ, καὶ κάθετος ή ΖΘ, ἴση ἄρα καὶ ή ὑπὸ ΑΖΚ γωνία τῆ ὑπὸ ΚΖΒ. ώστε καὶ περιφέρεια ἡ ΑΚ τῆ ΚΒ ἐστιν ἴση· διπλῆ ἄρα ἡ ΑΒ περιφέρεια τῆς ΒΚ περιφερείας· δεκαγώνου ἄρα πλευρά έστιν ή ΑΚ εύθεῖα. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΑΚ τῆς ΚΜ ἐστι διπλῆ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ περιφέρεια τῆς ΒΚ περιφερείας, ἴση δὲ ἡ ΓΔ περιφέρεια τῆ ΑΒ περιφερεία, διπλῆ ἄρα καὶ ἡ Γ Δ περιφέρεια τῆς BK περιφερείας. ἔστι δὲ ἡ Γ Δ περιφέρεια καὶ τῆς ΓΗ διπλῆ· ἴση ἄρα ἡ ΓΗ περιφέρεια τῆ ΒΚ περιφερεία. ἀλλὰ ἡ ΒΚ τῆς ΚΜ ἐστι διπλῆ, ἐπεὶ καὶ ή ΚΑ· καὶ ή ΓΗ ἄρα τῆς ΚΜ ἐστι διπλῆ. ἀλλὰ μὴν καὶ ἡ ΓΒ περιφέρεια τῆς ΒΚ περιφερείας ἐστὶ διπλῆ· ἴση γὰρ ἡ ΓΒ περιφέρεια τῆ ΒΑ. καὶ ὅλη ἄρα ἡ ΗΒ περιφέρεια τῆς BM ἐστι διπλῆ· ὥστε καὶ γωνία ἡ ὑπὸ HZB γωνίας τῆς ὑπὸ BZM [ἐστι] διπλῆ. ἔστι δὲ ἡ ὑπὸ HZB καὶ τῆς ὑπὸ ZAB διπλῆ· ἴση γὰρ ἡ ὑπὸ ΖΑΒ τῆ ὑπὸ ΑΒΖ. καὶ ἡ ὑπὸ ΒΖΝ ἄρα τῆ ὑπὸ ΖΑΒ ἐστιν ἴση. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ABZ καὶ τοῦ BZN, ἡ ὑπὸ ABZ γωνία· λοιπὴ ἄρα ἡ ὑπὸ AZB λοιπῆ τῆ ὑπὸ BNZ ἐστιν ἴση· ἴσογώνιον ἄρα ἐστὶ τὸ ABZ τρίγωνον τῷ BZN τριγώνω. ἀνάλογον ἄρα ἐστὶν ὡς ἡ AB εὐθεῖα πρὸς τὴν BZ, οὕτως ἡ ZB πρὸς τὴν BN· τὸ ẳρα ύπὸ τῶν ABN ἴσον ἐστὶ τῷ ἀπὸ BZ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΛ τῆ ΛΚ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΛΝ, βάσις ẳρα ἡ ΚΝ βάσει τῆ ΑΝ ἐστιν ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ΛΚΝ γωνία τῆ ύπὸ ΛΑΝ ἐστιν ἴση. ἀλλὰ ἡ ὑπὸ ΛΑΝ τῆ ὑπὸ KBN ἐστιν ἴση· καὶ ἡ ὑπὸ ΛΚΝ ἄρα τῆ ὑπὸ ΚΒΝ ἐστιν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΚΒ καὶ τοῦ ΑΚΝ ἡ πρὸς τῷ A. λοιπὴ ἄρα ἡ ὑπὸ AKB λοιπῆ τῆ ὑπὸ KNA ἐστιν ἴση· ίσογώνιον ἄρα ἐστὶ τὸ ΚΒΑ τρίγωνον τῷ ΚΝΑ τριγώνω. άνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ εὐθεῖα πρὸς τὴν ΑΚ, οὕτως ή ΚΑ πρὸς τὴν ΑΝ· τὸ ἄρα ὑπὸ τῶν ΒΑΝ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΚ. ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ΑΒΝ ἴσον τῷ ἀπὸ τῆς BZ· τὸ ἄρα ὑπὸ τῶν ABN μετὰ τοῦ ὑπὸ BAN, ὅπερ έστι τὸ ἀπὸ τῆς ΒΑ, ἴσον ἐστι τῷ ἀπὸ τῆς ΒΖ μετὰ τοῦ ἀπὸ τῆς ΑΚ. καί ἐστιν ἡ μὲν ΒΑ πενταγώνου πλευρά, ἡ δὲ ΒΖ έξαγώνου, ή δὲ ΑΚ δεκαγώνου.

Let *ABCDE* be a circle. And let the equilateral pentagon ABCDE have been inscribed in circle ABCDE. I say that the square on the side of pentagon ABCDE is the (sum of the squares) on the sides of the hexagon and of the decagon inscribed in circle ABCDE.

For let the center of the circle, point F, have been found [Prop. 3.1]. And, AF being joined, let it have been drawn across to point G. And let FB have been joined. And let FH have been drawn from F perpendicular to AB. And let it have been drawn across to K. And let AKand KB have been joined. And, again, let FL have been drawn from F perpendicular to AK. And let it have been drawn across to M. And let KN have been joined.

Since circumference *ABCG* is equal to circumference AEDG, of which ABC is equal to AED, the remaining circumference CG is thus equal to the remaining (circumference) GD. And CD (is the side) of the pentagon. CG (is) thus (the side) of the decagon. And since FA is equal to FB, and FH is perpendicular (to AB), angle AFK (is) thus also equal to KFB [Props. 1.5, 1.26]. Hence, circumference AK is also equal to KB[Prop. 3.26]. Thus, circumference AB (is) double circumference BK. Thus, straight-line AK is the side of the decagon. So, for the same (reasons, circumference) AK is also double KM. And since circumference ABis double circumference BK, and circumference CD (is) equal to circumference AB, circumference CD (is) thus also double circumference BK. And circumference CDis also double CG. Thus, circumference CG (is) equal to circumference BK. But, BK is double KM, since KA (is) also (double KM). Thus, (circumference) CGis also double KM. But, indeed, circumference CB is also double circumference BK. For circumference CB(is) equal to BA. Thus, the whole circumference GBis also double BM. Hence, angle GFB [is] also double angle BFM [Prop. 6.33]. And GFB (is) also double FAB. For FAB (is) equal to ABF. Thus, BFNis also equal to FAB. And angle ABF (is) common to the two triangles ABF and BFN. Thus, the remaining (angle) AFB is equal to the remaining (angle) BNF[Prop. 1.32]. Thus, triangle ABF is equiangular to triangle BFN. Thus, proportionally, as straight-line AB (is) to BF, so FB (is) to BN [Prop. 6.4]. Thus, the (rectangle contained) by ABN is equal to the (square) on BF[Prop. 6.17]. Again, since AL is equal to LK, and LNis common and at right-angles (to KA), base KN is thus equal to base AN [Prop. 1.4]. And, thus, angle LKNis equal to angle LAN. But, LAN is equal to KBN[Props. 3.29, 1.5]. Thus, LKN is also equal to KBN. And the (angle) at A (is) common to the two triangles H ἄρα τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ AKB and AKN. Thus, the remaining (angle) AKB is

έξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι. equal to the remaining (angle) KNA [Prop. 1.32]. Thus, triangle KBA is equiangular to triangle KNA. Thus, proportionally, as straight-line BA is to AK, so KA (is) to AN [Prop. 6.4]. Thus, the (rectangle contained) by BAN is equal to the (square) on AK [Prop. 6.17]. And the (rectangle contained) by ABN was also shown (to be) equal to the (square) on BF. Thus, the (rectangle contained) by ABN which is the (square) on BA [Prop. 2.2], is equal to the (square) on BF (the side) of the hexagon [Prop. 4.15 corr.], and AK (the side) of the decagon.

Thus, the square on the side of the pentagon (inscribed in a circle) is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.

[†] If the circle is of unit radius then the side of the pentagon is $(1/2) \sqrt{10 - 2\sqrt{5}}$.

ıα'.

Έὰν εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῆ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.



Εἰς γὰρ κύκλον τὸν ΑΒΓΔΕ ἑητὴν ἔχοντα τὴν δίαμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓΔΕ· λέγω, ὅτι ἡ τοῦ [ΑΒΓΔΕ] πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ζ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ καὶ διήχθωσαν ἐπὶ τὰ Η, Θ σημεῖα, καὶ ἐπεζεύχθω ἡ ΑΓ, καὶ κείσθω τῆς ΑΖ τέταρτον μέρος ἡ ΖΚ. ἑητὴ δὲ ἡ ΑΖ· ἑητὴ ἄρα καὶ ἡ ΖΚ. ἔστι δὲ καὶ ἡ ΒΖ ἑητή· ὅλη ἄρα ἡ ΒΚ ἑητή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓΗ περιφέρεια τῆ ΑΔΗ περιφερεία, ῶν ἡ ΑΒΓ τῆ ΑΕΔ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΓΗ λοιπῆ τῆ ΗΔ ἐστιν ἴση. καὶ ἐὰν ἐπιζεύξωμεν τὴν ΑΔ, συνάγονται ὀρθαὶ αί

Proposition 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter then the side of the pentagon is that irrational (straight-line) called minor.



For let the equilateral pentagon ABCDE have been inscribed in the circle ABCDE which has a rational diameter. I say that the side of pentagon [ABCDE] is that irrational (straight-line) called minor.

For let the center of the circle, point F, have been found [Prop. 3.1]. And let AF and FB have been joined. And let them have been drawn across to points G and H(respectively). And let AC have been joined. And let FKmade (equal) to the fourth part of AF. And AF (is) rational. FK (is) thus also rational. And BF is also rational. Thus, the whole of BK is rational. And since circumference ACG is equal to circumference ADG, of which

πρὸς τῷ Λ γωνίαι, καὶ διπλῆ ἡ ΓΔ τῆς ΓΛ. διὰ τὰ αὐτὰ δη και αί προς τῷ Μ ὀρθαί εἰσιν, και διπλη ή ΑΓ της ΓΜ. έπει οῦν ἴση ἐστιν ἡ ὑπὸ ΑΛΓ γωνία τῃ ὑπὸ AMZ, κοινὴ δὲ τῶν δύο τριγώνων τοῦ τε ΑΓΛ καὶ τοῦ ΑΜΖ ἡ ὑπὸ ΛΑΓ, λοιπή ἄρα ή ὑπὸ ΑΓΛ λοιπῆ τῆ ὑπὸ ΜΖΑ ἐστιν ἴση· ίσογώνιον ἄρα ἐστὶ τὸ ΑΓΛ τρίγωνον τῷ ΑΜΖ τριγώνω άνάλογον ἄρα ἐστὶν ὡς ἡ ΛΓ πρὸς ΓΑ, οὕτως ἡ ΜΖ πρὸς ΖΑ· καὶ τῶν ἡγουμένων τὰ διπλάσια· ὡς ἄρα ἡ τῆς ΛΓ διπλη πρός την ΓΑ, ούτως ή της ΜΖ διπλη πρός την ΖΑ. ώς δὲ ἡ τῆς ΜΖ διπλῆ πρὸς τὴν ΖΑ, οὕτως ἡ ΜΖ πρὸς τὴν ήμίσειαν τῆς ΖΑ· καὶ ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ΓΑ, οὕτως ἡ MZ πρὸς τὴν ἡμίσειαν τῆς ΖΑ· καὶ τῶν ἑπομένων τὰ ἡμίσεα· ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῆ πρὸς τὴν ἡμίσειαν τῆς ΓΑ, οὕτως ἡ ΜΖ πρὸς τὸ τέτατρον τῆς ΖΑ. καί ἐστι τῆς μὲν ΑΓ διπλῆ ἡ ΔΓ, τῆς δὲ ΓΑ ἡμίσεια ἡ ΓΜ, τῆς δὲ ΖΑ τέτατρον μέρος ή ΖΚ· ἔστιν ἄρα ὡς ή ΔΓ πρὸς τὴν ΓΜ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΚ. συνθέντι καὶ ὡς συναμφότερος ή ΔΓΜ πρός την ΓΜ, οὕτως ή ΜΚ πρός ΚΖ· καὶ ὡς ἄρα τὸ άπὸ συναμφοτέρου τῆς ΔΓΜ πρὸς τὸ ἀπὸ ΓΜ, οὕτως τὸ ἀπὸ ΜΚ πρὸς τὸ ἀπὸ ΚΖ. καὶ ἐπεὶ τῆς ὑπὸ δύο πλευρὰς τοῦ πενταγώνου ὑποτεινούσης, οἶον τῆς ΑΓ, ἄχρον χαὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα ἴσον ἐστὶ τῆ τοῦ πενταγώνου πλευρα, τουτέστι τῆ ΔΓ, τὸ δὲ μεῖζον τμῆμα προσλαβόν την ήμίσειαν της όλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τῆς ὅλης, καί ἐστιν ὅλης τῆς ΑΓ ἡμίσεια ή ΓΜ, τὸ ẳρα ἀπὸ τῆς ΔΓΜ ὡς μιᾶς πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΜ. ὡς δὲ τὸ ἀπὸ τῆς $\Delta \Gamma M$ ὡς μιᾶς πρὸς τὸ ἀπὸ τῆς ΓΜ, οὕτως ἐδείχθη τὸ ἀπὸ τῆς ΜΚ πρὸς τὸ ἀπὸ τῆς ΚΖ· πενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΜΚ τοῦ ἀπὸ τῆς ΚΖ. ἑητὸν δὲ τὸ ἀπὸ τῆς ΚΖ· ἑητὴ γὰρ ἡ διάμετρος· ἑητὸν άρα καὶ τὸ ἀπὸ τῆς ΜΚ· ἑητὴ ἄρα ἐστὶν ἡ ΜΚ [δυνάμει μόνον]. καὶ ἐπεὶ τετραπλασία ἐστὶν ἡ ΒΖ τῆς ΖΚ, πενταπλασία ἄρα ἐστὶν ἡ ΒΚ τῆς ΚΖ· εἰκοσιπενταπλάσιον ἄρα τὸ άπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΖ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΜΚ τοῦ ἀπὸ τῆς ΚΖ· πενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ· τὸ ἄρα ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ ΚΜ λόγον ούκ έχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΚ τῆ ΚΜ μήκει. καί ἐστι ρητή έκατέρα αὐτῶν. aι BK, KM ἄρα ρηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ ἀπὸ ἑητῆς ἑητὴ ἀφαιρεθῆ δυνάμει μόνον σύμμετρος ούσα τῆ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν ἀποτομή άποτομή ἄρα ἐστὶν ή MB, προσαρμόζουσα δὲ αὐτῆ ή ΜΚ. λέγω δή, ὅτι καὶ τετάρτη. ῷ δὴ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς Ν· ή ΒΚ ἄρα τῆς ΚΜ μεῖζον δύναται τῆ Ν. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ KZ τῆ ZB, καὶ συνθέντι σύμμετρός ἐστι ἡ KB τῆ ZB. ἀλλὰ ἡ BZ τῆ BΘ σύμμετρός ἐστιν· καὶ ἡ BK ἄρα τῆ ΒΘ σύμμετρός έστιν. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ, τὸ ἄρα ἀπὸ τῆς ΒΚ πρὸς τὸ άπὸ τῆς ΚΜ λόγον ἔχει, ὃν ε πρὸς ἕν. ἀναστρέψαντι ἄρα τὸ ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ τῆς Ν λόγον ἔχει, ὃν Ξ πρὸς

ABC is equal to AED, the remainder CG is thus equal to the remainder GD. And if we join AD then the angles at L are inferred (to be) right-angles, and CD (is inferred to be) double CL [Prop. 1.4]. So, for the same (reasons), the (angles) at M are also right-angles, and AC (is) double CM. Therefore, since angle ALC (is) equal to AMF, and (angle) LAC (is) common to the two triangles ACLand AMF, the remaining (angle) ACL is thus equal to the remaining (angle) MFA [Prop. 1.32]. Thus, triangle ACL is equiangular to triangle AMF. Thus, proportionally, as LC (is) to CA, so MF (is) to FA [Prop. 6.4]. And (we can take) the doubles of the leading (magnitudes). Thus, as double LC (is) to CA, so double MF (is) to FA. And as double MF (is) to FA, so MF (is) to half of FA. And, thus, as double LC (is) to CA, so MF (is) to half of FA. And (we can take) the halves of the following (magnitudes). Thus, as double LC (is) to half of CA, so MF (is) to the fourth of FA. And DC is double LC, and CM half of CA, and FK the fourth part of FA. Thus, as DC is to CM, so MF (is) to FK. Via composition, as the sum of DCM (i.e., DC and CM) (is) to CM, so MK(is) to KF [Prop. 5.18]. And, thus, as the (square) on the sum of DCM (is) to the (square) on CM, so the (square) on MK (is) to the (square) on KF. And since the greater piece of a (straight-line) subtending two sides of a pentagon, such as AC, (which is) cut in extreme and mean ratio is equal to the side of the pentagon [Prop. 13.8]that is to say, to DC—-and the square on the greater piece added to half of the whole is five times the (square) on half of the whole [Prop. 13.1], and CM (is) half of the whole, AC, thus the (square) on DCM, (taken) as one, is five times the (square) on CM. And the (square) on DCM, (taken) as one, (is) to the (square) on CM, so the (square) on MK was shown (to be) to the (square) on KF. Thus, the (square) on MK (is) five times the (square) on KF. And the square on KF (is) rational. For the diameter (is) rational. Thus, the (square) on MK (is) also rational. Thus, MK is rational [in square only]. And since BF is four times FK, BK is thus five times KF. Thus, the (square) on BK (is) twenty-five times the (square) on KF. And the (square) on MK(is) five times the square on KF. Thus, the (square) on BK (is) five times the (square) on KM. Thus, the (square) on BK does not have to the (square) on KMthe ratio which a square number (has) to a square number. Thus, BK is incommensurable in length with KM[Prop. 10.9]. And each of them is a rational (straightline). Thus, BK and KM are rational (straight-lines which are) commensurable in square only. And if from a rational (straight-line) a rational (straight-line) is subtracted, which is commensurable in square only with the δ, οὐχ ὃν τετράγωνος πρὸς τετράγωνον· ἀσύμμετρος ἄρα ἐστὶν ἡ BK τῆ N· ἡ BK ἄρα τῆς KM μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ. ἐπεὶ οῦν ὅλη ἡ BK τῆς προσαρμοζούσης τῆς KM μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῆ, καὶ ὅλη ἡ BK σύμμετρός ἐστι τῆ ἐκκειμένῃ ῥητῆ τῆ BΘ, ἀποτομὴ ἄρα τετάρτῃ ἐστὶν ἡ MB. τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένῃ αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ ἐλάττων. δύναται δὲ τὸ ὑπὸ τῶν ΘBM ἡ AB διὰ τὸ ἐπιζευγνυμένῃς τῆς ΑΘ ἰσογώνιον γίνεσθαι τὸ ABΘ τρίγωνον τῷ ABM τριγώνῳ καὶ εἶναι ὡς τὴν ΘB πρὸς τὴν BA, οὕτως τὴν AB πρὸς τὴν BM.

Ή ἄρα AB τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων· ὅπερ ἔδει δεῖξαι. whole, then the remainder is that irrational (straight-line called) an apotome [Prop. 10.73]. Thus, MB is an apotome, and MK its attachment. So, I say that (it is) also a fourth (apotome). So, let the (square) on N be (made) equal to that (magnitude) by which the (square) on BKis greater than the (square) on KM. Thus, the square on BK is greater than the (square) on KM by the (square) on N. And since KF is commensurable (in length) with FB then, via composition, KB is also commensurable (in length) with FB [Prop. 10.15]. But, BF is commensurable (in length) with BH. Thus, BK is also commensurable (in length) with BH [Prop. 10.12]. And since the (square) on BK is five times the (square) on KM. the (square) on BK thus has to the (square) on KM the ratio which 5 (has) to one. Thus, via conversion, the (square) on BK has to the (square) on N the ratio which 5 (has) to 4 [Prop. 5.19 corr.], which is not (that) of a square (number) to a square (number). BK is thus incommensurable (in length) with N [Prop. 10.9]. Thus, the square on BK is greater than the (square) on KMby the (square) on (some straight-line which is) incommensurable (in length) with (BK). Therefore, since the square on the whole, BK, is greater than the (square) on the attachment, KM, by the (square) on (some straightline which is) incommensurable (in length) with (BK), and the whole, BK, is commensurable (in length) with the (previously) laid down rational (straight-line) BH, *MB* is thus a fourth apotome [Def. 10.14]. And the rectangle contained by a rational (straight-line) and a fourth apotome is irrational, and its square-root is that irrational (straight-line) called minor [Prop. 10.94]. And the square on AB is the rectangle contained by HBM, on account of joining AH, (so that) triangle ABH becomes equiangular with triangle ABM [Prop. 6.8], and (proportionally) as HB is to BA, so AB (is) to BM.

Thus, the side AB of the pentagon is that irrational (straight-line) called minor.[†] (Which is) the very thing it was required to show.

[†] If the circle has unit radius then the side of the pentagon is $(1/2)\sqrt{10-2\sqrt{5}}$. However, this length can be written in the "minor" form (see Prop. 10.94) $(\rho/\sqrt{2})\sqrt{1+k/\sqrt{1+k^2}} - (\rho/\sqrt{2})\sqrt{1-k/\sqrt{1+k^2}}$, with $\rho = \sqrt{5/2}$ and k = 2.

ιβ΄.

Έλν εἰς κύκλον τρίγωνον ἰσόπλευρον ἐγγραφῆ, ἡ τοῦ τριγώνου πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου.

Έστω κύκλος ὁ ABΓ, καὶ εἰς αὐτὸν τρίγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ABΓ· λέγω, ὅτι τοῦ ABΓ τριγώνου μία πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ ABΓ κύκλου.

Proposition 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the (square) on the radius of the circle.

Let there be a circle ABC, and let the equilateral triangle ABC have been inscribed in it [Prop. 4.2]. I say that the square on one side of triangle ABC is three times the (square) on the radius of circle ABC.



Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου τὸ Δ, καὶ ἐπιζευχθεῖσα ἡ ΑΔ διήχθω ἐπὶ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΒΕ. Καὶ ἐπεὶ ἰσόπλευρόν ἐστι τὸ ΑΒΓ τρίγωνον, ἡ ΒΕΓ ἄρα περιφέρεια τρίτον μέρος ἐστὶ τῆς τοῦ ΑΒΓ κύκλου περιφερείας. ἡ ἄρα ΒΕ περιφέρεια ἕκτον ἐστὶ μέρος τῆς τοῦ κύκλου περιφερείας· ἑξαγώνου ἄρα ἐστὶν ἡ ΒΕ εὐθεῖα· ἴση ἄρα ἐστὶ τῆ ἐκ τοῦ κέντρου τῆ ΔΕ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΕ τῆς ΔΕ, τετραπλάσιον ἐστι τὸ ἀπὸ τῆς ΑΕ τοῦ ἀπὸ τῆς ΕΔ, τουτέστι τοῦ ἀπὸ τῆς ΒΕ. ἴσον δὲ τὸ ἀπὸ τῆς ΑΕ τοῖς ἀπὸ τῶν AB, BE· τὰ ἄρα ἀπὸ τῶν AB, BE τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς BE. διελόντι ἄρα τὸ ἀπὸ τῆς AB τριπλάσιόν ἐστι τοῦ ἀπὸ BE. ἴση δὲ ἡ BE τῆ ΔΕ· τὸ ἄρα ἀπὸ τῆς AB τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΕ.

Ή ἄρα τοῦ τριγώνου πλευρὰ δυνάμει τριπλασία ἐστὶ τῆς ἐκ τοῦ κέντρου [τοῦ κύκλου]· ὅπερ ἔδει δεῖξαι.

ιγ'.

Πυραμίδα συστήσασθαι καὶ σφαίρα περιλαβεῖν τῆ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος.



For let the center, D, of circle ABC have been found [Prop. 3.1]. And AD (being) joined, let it have been drawn across to E. And let BE have been joined.

And since triangle ABC is equilateral, circumference BEC is thus the third part of the circumference of circle ABC. Thus, circumference BE is the sixth part of the circumference of the circle. Thus, straight-line BE is (the side) of a hexagon. Thus, it is equal to the radius DE [Prop. 4.15 corr.]. And since AE is double DE, the (square) on AE is four times the (square) on ED—that is to say, of the (square) on BE. And the (square) on AE (is) equal to the (sum of the squares) on AB and BE [Props. 3.31, 1.47]. Thus, the (sum of the squares) on AB and BE is four times the (square) on BE. Thus, via separation, the (square) on AB is three times the (square) on BE. And BE (square) on BE. Thus, the (square) on BE. Thus, the (square) on BE. Thus, the (square) on BE is three times the (square) on AB is three times the (square) on AB is three times the (square) on BE. Thus, the (square) on BE.

Thus, the square on the side of the triangle is three times the (square) on the radius [of the circle]. (Which is) the very thing it was required to show.

Proposition 13

To construct a (regular) pyramid (*i.e.*, a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας δίαμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε διπλασίαν εἶναι τὴν ΑΓ τῆς ΓΒ· καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ σημείου τῆ ΑΒ πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΑ· καὶ ἐκκείσθω κύκλος ὁ ΕΖΗ ἴσην έχων τὴν ἐκ τοῦ κέντρου τῃ ΔΓ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗ χύχλον τρίγωνον ἰσόπλευρον τὸ ΕΖΗ· καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Θ σημεῖον, καὶ ἐπεζεύχθωσαν αί ΕΘ, ΘΖ, ΘΗ· και άνεστάτω ἀπὸ τοῦ Θ σημείου τῷ τοῦ ΕΖΗ χύκλου ἐπιπέδω πρὸς ὀρθὰς ἡ ΘΚ, καὶ ἀφηρήσθω ἀπὸ τῆς ΘΚ τῆ ΑΓ εὐθεία ἴση ἡ ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΚΕ, ΚΖ, ΚΗ. καὶ ἐπεὶ ἡ ΚΘ ὀρθή ἐστι πρὸς τὸ τοῦ ΕΖΗ κύκλου έπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ τοῦ ΕΖΗ κύκλου ἐπιπέδω ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ αὐτῆς ἑκάστη τῶν ΘΕ, ΘΖ, ΘΗ· ἡ ΘΚ ἄρα πρὸς ἑκάστη τῶν ΘΕ, ΘΖ, ΘΗ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση έστιν ή μέν ΑΓ τῆ ΘΚ, ή δὲ ΓΔ τῆ ΘΕ, και ὀρθὰς γωνίας περιέχουσιν, βάσις ἄρα ή ΔΑ βάσει τῆ ΚΕ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ΚΖ, ΚΗ τῆ ΔΑ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΚΕ, ΚΖ, ΚΗ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ διπλῆ έστιν ή ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ή ΑΒ τῆς ΒΓ. ὡς δὲ ή ΑΒ πρὸς τὴν BΓ, οὕτως τὸ ἀπὸ τῆς AΔ πρὸς τὸ ἀπὸ τῆς ΔΓ, ώς ἑξῆς δειχθήσεται. τριπλάσιον ἄρα τὸ ἀπὸ τῆς $A\Delta$ τοῦ ἀπὸ τῆς
 $\Delta \Gamma.$ ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΖΕ τοῦ ἀπὸ τῆς ΕΘ τριπλάσιον, καί ἐστιν ἴση ἡ $\Delta\Gamma$ τῃ ΕΘ· ἴση ἄρα καὶ ἡ ΔA τῆ ΕΖ. ἀλλὰ ἡ ΔΑ ἑκάστη τῶν ΚΕ, ΚΖ, ΚΗ ἐδείχθη ἴση· καὶ ἑκάστη ἄρα τῶν ΕΖ, ΖΗ, ΗΕ ἑκάστῃ τῶν ΚΕ, ΚΖ, ΚΗ έστιν ἴση· ἰσόπλευρα ἄρα ἐστὶ τὰ τέσσαρα τρίγωνα τὰ ΕΖΗ, ΚΕΖ, ΚΖΗ, ΚΕΗ. πυραμίς ἄρα συνέσταται έκ τεσσάρων τριγώνων ἰσοπλέυρων, ῆς βάσις μέν ἐστι τὸ ΕΖΗ τρίγωνον,



Let the diameter AB of the given sphere be laid out, and let it have been cut at point C such that AC is double CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let CD have been drawn from point Cat right-angles to AB. And let DA have been joined. And let the circle EFG be laid down having a radius equal to DC, and let the equilateral triangle EFG have been inscribed in circle EFG [Prop. 4.2]. And let the center of the circle, point H, have been found [Prop. 3.1]. And let EH, HF, and HG have been joined. And let HKhave been set up, at point H, at right-angles to the plane of circle EFG [Prop. 11.12]. And let HK, equal to the straight-line AC, have been cut off from HK. And let KE, KF, and KG have been joined. And since KH is at right-angles to the plane of circle EFG, it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle *EFG* [Def. 11.3]. And HE, HF, and HG each join it. Thus, HK is at right-angles to each of HE, HF, and HG. And since AC is equal to HK, and CD to HE, and they contain right-angles, the base DA is thus equal to the base KE[Prop. 1.4]. So, for the same (reasons), KF and KG is each equal to DA. Thus, the three (straight-lines) KE, KF, and KG are equal to one another. And since AC is double CB, AB (is) thus triple BC. And as AB (is) to BC, so the (square) on AD (is) to the (square) on DC, as will be shown later [see lemma]. Thus, the (square) on AD (is) three times the (square) on DC. And the (square) on FE is also three times the (square) on EH[Prop. 13.12], and DC is equal to EH. Thus, DA (is)

κορυφή δὲ τὸ Κ σημεῖον.

Δεῖ δὴ αὐτὴν καὶ σφαίρα περιλαβεῖν τῆ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκβεβλήσθω γὰρ ἐπ' εὐθείας τῆ ΚΘ εὐθεῖα ἡ ΘΛ, καὶ κείσθω τῆ ΓΒ ἴση ἡ ΘΛ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΓ πρὸς τὴν $\Gamma\Delta$, οὕτως ή $\Gamma\Delta$ πρὸς τὴν Γ B, ἴση δὲ ή μὲν ΑΓ τῆ KΘ, ή δὲ $\Gamma\Delta$ τῆ ΘΕ, ἡ δὲ Γ Β τῆ ΘΛ, ἔστιν ἄρα ὡς ἡ KΘ πρὸς τὴν ΘΕ, οὕτως
ἡ ΕΘ πρὸς τὴν ΘΛ· τὸ ἄρα ὑπὸ τῶν ΚΘ, ΘΛ ἴσον έστι τῷ ἀπὸ τῆς ΕΘ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΚΘΕ, ΕΘΛ γωνιῶν· τὸ ἄρα ἐπὶ τῆς ΚΛ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ E [ἐπειδήπερ ἐὰν ἐπιζεύξωμεν τὴν $\mathrm{E}\Lambda,$ ὀρθὴ γίνεται ή ὑπὸ ΛΕΚ γωνία διὰ τὸ ἰσογώνιον γίνεσθαι τὸ ΕΛΚ τρίγωνον ἑχατέρω τῶν ΕΛΘ, ΕΘΚ τριγώνων]. ἐἀν δή μενούσης τῆς ΚΛ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῆ, ὄθεν ἤρξατο φέρεσθαι, ἤξει καὶ διὰ τῶν Ζ, Η σημείων ἐπιζευγνυμένων τῶν ΖΛ, ΛΗ καὶ ὀρθῶν όμοίως γινομένων τῶν πρὸς τοῖς Ζ, Η γωνιῶν καὶ ἔσται ή πυραμίς σφαίρα περιειλημμένη τῆ δοθείσῆ. ἡ γὰρ ΚΛ τῆς σφαίρας διάμετρος ἴση ἐστὶ τῆ τῆς δοθείσης σφαίρας διαμετρώ τη AB, ἐπειδήπερ τη μέν AΓ ἴση κεῖται ἡ KΘ, τη δὲ ΓΒ ἡ ΘΛ.

Λέγω δή, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Έπεὶ γὰρ διπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ· ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΓ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΔ [ἐπειδήπερ ἐπιζευγνμένης τῆς ΔΒ ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ διὰ τὴν ὑμοιότητα τῶν ΔΑΒ, ΔΑΓ τριγώνων, καὶ εἶναι ὡς τὴν πρώτην πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς τρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας]. ἡμιόλιον ἄρα καὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ. καί ἐστιν ἡ μὲν ΒΑ ἡ τῆς δοθείσης σφαίρας διάμετρος, ἡ δὲ ΑΔ ἴση τῆ πλευρῷ τῆς πυραμίδος.

Ή ἄρα τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος· ὅπερ ἔδει δεῖξαι.

also equal to EF. But, DA was shown (to be) equal to each of KE, KF, and KG. Thus, EF, FG, and GE are equal to KE, KF, and KG, respectively. Thus, the four triangles EFG, KEF, KFG, and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG, and apex the point K, has been constructed from four equilateral triangles.

So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For let the straight-line HL have been produced in a straight-line with KH, and let HL be made equal to CB. And since as AC (is) to CD, so CD (is) to CB[Prop. 6.8 corr.], and AC (is) equal to KH, and CD to HE, and CB to HL, thus as KH is to HE, so EH (is) to HL. Thus, the (rectangle contained) by KH and HLis equal to the (square) on EH [Prop. 6.17]. And each of the angles *KHE* and *EHL* is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle *ELK* becoming equiangular to each of the triangles ELH and EHK[Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G, (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, KL, of the sphere is equal to the diameter, AB, of the given sphere inasmuch as KH was made equal to AC, and HL to CB.

So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB, AB is thus triple BC. Thus, via conversion, BA is one and a half times AC. And as BA (is) to AC, so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BAis to AD, so DA (is) to AC, on account of the similarity of triangles DAB and DAC. And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD. And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of the pyramid (*i.e.*, tetrahedron) is $\sqrt{8/3}$.



Λῆμμα.

Δει
 κτέον, ὅτι ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α
Δ πρὸς τὸ ἀπὸ τῆς ΔΓ.

Ἐκκείσϑω γὰρ ἡ τοῦ ἡμικυκλίου καταγραφή, καὶ
 ἐπεζεύχθω ἡ ΔB , καὶ ἀναγεγράφθω ἀπὸ τῆς
 $A\Gamma$ τετράγωνον τὸ ΕΓ, καὶ συμπεπληρώσθω τὸ ΖΒ παραλληλόγραμμον.
 ἐπεὶ οὖν διὰ τὸ ἰσογώνιον εἶναι τὸ ΔAB τρίγωνον τ
ῷ $\Delta A\Gamma$ τριγώνω ἐστίν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ, τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΕΒ πρός τὸ ΒΖ, καί ἐστι τὸ μὲν ΕΒ τὸ ὑπὸ τῶν ΒΑ, ΑΓ· ἴση γὰρ ἡ ΕΑ τῆ ΑΓ· τὸ δὲ ΒΖ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, ὡς ἄρα ἡ ΑΒ πρός τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΓ πρὸ τὸ ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΓ ἴσον τῷ ἀπὸ τῆς $A\Delta$, τὸ δὲ ὑπὸ τῶν $A\Gamma B$ ἴσον τῷ ἀπὸ τῆς $\Delta \Gamma$ · ἡ γὰρ ΔΓ κάθετος τῶν τῆς βάσεως τμημάτων τῶν ΑΓ, ΓΒ μέση άνάλογόν ἐστι διὰ τὸ ὀρθὴν εἶναι τὴν ὑπὸ ΑΔΒ. ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς $\Delta \Gamma$ · ὅπερ ἔδει δε
ῖξαι.

ιδ'.

Όκτάεδρον συστήσασθαι καὶ σφαίρα περιλαβεῖν, ἢ καὶ τὰ πρότερα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασία ἐστὶ τῆς πλευρᾶς τοῦ ὀκταέδρου.

Έκκείσθω ή τῆς δοθείσης σφαίρας διάμετρος ή AB, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AΔB, καὶ ἦχθω ἀπὸ τοῦ Γ τῆ AB πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔB, καὶ ἐκκείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον ἑκάστην τῶν πλευρῶν τῆ ΔB, καὶ



It must be shown that as AB is to BC, so the (square) on AD (is) to the (square) on DC.

For, let the figure of the semi-circle have been set out, and let DB have been joined. And let the square EC have been described on AC. And let the parallelogram FB have been completed. Therefore, since, on account of triangle DAB being equiangular to triangle DAC [Props. 6.8, 6.4], (proportionally) as BA is to AD, so DA (is) to AC, the (rectangle contained) by BA and AC is thus equal to the (square) on AD [Prop. 6.17]. And since as AB is to BC, so EB (is) to BF [Prop. 6.1]. And EB is the (rectangle contained) by BA and AC—for *EA* (is) equal to *AC*. And *BF* the (rectangle contained) by AC and CB. Thus, as AB (is) to BC, so the (rectangle contained) by BA and AC (is) to the (rectangle contained) by AC and CB. And the (rectangle contained) by BA and AC is equal to the (square) on AD, and the (rectangle contained) by ACB (is) equal to the (square) on DC. For the perpendicular DC is the mean proportional to the pieces of the base, AC and CB, on account of ADB being a right-angle [Prop. 6.8 corr.]. Thus, as AB (is) to BC, so the (square) on AD (is) to the (square) on DC. (Which is) the very thing it was required to show.

Proposition 14

To construct an octahedron, and to enclose (it) in a (given) sphere, like in the preceding (proposition), and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

Let the diameter AB of the given sphere be laid out, and let it have been cut in half at C. And let the semicircle ADB have been drawn on AB. And let CD be drawn from C at right-angles to AB. And let DB have ἐπεζεύχθωσαν αἱ ΘΖ, ΕΗ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖα ἡ ΚΛ καὶ διήχθω ἐπὶ τὰ ἔτερα μέρη τοῦ ἐπιπέδου ὡς ἡ ΚΜ, καὶ ἀφηρήσθω ἀφ᾽ ἑκατέρας τῶν ΚΛ, ΚΜ μιῷ τῶν ΕΚ, ΖΚ, ΗΚ, ΘΚ ἴση ἑκατέρα τῶν ΚΛ, ΚΜ, καὶ ἐπεζεύχθωσαν αἱ ΛΕ, ΛΖ, ΛΗ, ΛΘ, ΜΕ, ΜΖ, ΜΗ, ΜΘ.



Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΕ τῆ ΚΘ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΕΚΘ γωνία, τὸ ἄρα ἀπὸ τῆς ΘΕ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΛΚ τῆ ΚΕ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΛΚΕ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΛ διπλάσιον ἐστι τοῦ ἀπὸ ΕΚ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΘΕ διπλάσιον τοῦ ἀπὸ τῆς ΕΚ· τὸ ἄρα ἀπὸ τῆς ΛΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ· ἴση ἄρα ἐστὶν ἡ ΛΕ τῆ ΕΘ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΛΘ τῆ ΘΕ ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΕΘ τρίγωνον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ῶν βάσεις μέν εἰσιν αἱ τοῦ ΕΖΗΘ τετραγώνου πλευραί, κορυφαὶ δὲ τὰ Λ, Μ σημεῖα, ἰσόπλευρόν ἐστιν· ὀκτάεδρον ἄρα συνέσταται ὑπὸ ὀκτὼ τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρα περιλαβεῖν τῆ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς.

Έπει γὰρ ai τρεῖς ai ΛK, KM, KE ἴσaι ἀλλήλαις εἰσίν, τὸ ἄρα ἐπὶ τῆς ΛΜ γραφόμενον ἡμικύκλιον ῆξει καὶ διὰ τοῦ Ε. καὶ διὰ τὰ αὐτά, ἐὰν μενούσης τῆς ΛΜ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῆ, ὅθεν ῆρξατο φέρεσθαι, ῆξει καὶ διὰ τῶν Ζ, Η, Θ σημείων, καὶ ἔσται σφαίρα περιειλημμένον τὸ ὀκτάεδρον. λέγω δή, ὅτι καὶ τῆ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΛΚ τῆ KM, κοινὴ δὲ ἡ KE, been joined. And let the square EFGH, having each of its sides equal to DB, be laid out. And let HF and EGhave been joined. And let the straight-line KL have been set up, at point K, at right-angles to the plane of square EFGH [Prop. 11.12]. And let it have been drawn across on the other side of the plane, like KM. And let KL and KM, equal to one of EK, FK, GK, and HK, have been cut off from KL and KM, respectively. And let LE, LF, LG, LH, ME, MF, MG, and MH have been joined.



And since KE is equal to KH, and angle EKH is a right-angle, the (square) on the HE is thus double the (square) on EK [Prop. 1.47]. Again, since LK is equal to KE, and angle LKE is a right-angle, the (square) on EL is thus double the (square) on EK [Prop. 1.47]. And the (square) on HE was also shown (to be) double the (square) on EK. Thus, the (square) on LE is equal to the (square) on EH. Thus, LE is equal to EH. So, for the same (reasons), LH is also equal to HE. Triangle LEH is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square EFGH, and apexes the points L and M, are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) LK, KM, and KE are equal to one another, the semi-circle drawn on LM will thus also pass through E. And, for the same (reasons), if LM remains (fixed), and the semi-circle is car-

καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΛΕ βάσει τῆ EM ἐστιν ἴση. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΕΜ γωνία ἐν ἡμικυκλίῳ γάρ· τὸ ἄρα ἀπὸ τῆς ΛΜ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΛΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΒ, διπλασία ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΛΜ διπλάσιον τοῦ ἀπὸ τῆς ΛΕ. καί ἐστιν ἴσον τὸ ἀπὸ τῆς ΔΒ τῷ ἀπὸ τῆς ΑΕ· ἴση γὰρ κεῖται ἡ ΕΘ τῆ ΔΒ. ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΛΜ· ἴση ἄρα ἡ ΑΒ τῆ ΛΜ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΛΜ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Περιείληπται ἄρα τὸ ὀκτάεδρον τῆ δοθείσῃ σφαίρα. καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς. ὅπερ ἔδει δεῖξαι.

[†] If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

ιε'**.**

Κύβον συστήσασθαι καὶ σφαίρα περιλαβεῖν, ἤ καὶ τὴν πυραμίδα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς.

Έχχείσθω ή τῆς δοθείσης σφαίρας διάμετρος ή AB καὶ τετμήσθω κατὰ τὸ Γ ὥστε διπλῆν εἶναι τὴν AΓ τῆς ΓB, καὶ γεγράφθω ἐπὶ τῆς AB ἡμιχύχλιον τὸ AΔB, καὶ ἀπὸ τοῦ Γ τῆ AB πρὸς ὀρθὰς ἤχθω ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔB, καὶ ἐκχείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον τὴν πλευρὰν τῆ ΔB, καὶ ἀπὸ τῶν Ε, Ζ, Η, Θ τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἀφηρήσθω ἀπὸ ἑκάστης τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΛΜ, MN, ΝΚ· κύβος ἄρα συνέσταται ὁ ΖΝ ὑπὸ ἑξ τετραγώνων ἴσων περιεχόμενος.

Δεῖ δὴ αὐτὸν καὶ σφαίρα περιλαβεῖν τῆ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασία ἐστὶ τῆς πλευρᾶς τοῦ κύβου. ried around, and again established at the same (position) from which it began to be moved, then it will also pass through points F, G, and H, and the octahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since LK is equal to KM, and KE (is) common, and they contain right-angles, the base LE is thus equal to the base EM[Prop. 1.4]. And since angle *LEM* is a right-angle—for (it is) in a semi-circle [Prop. 3.31]—the (square) on LM is thus double the (square) on LE [Prop. 1.47]. Again, since AC is equal to CB, AB is double BC. And as AB(is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BD. And the (square) on LM was also shown (to be) double the (square) on LE. And the (square) on DB is equal to the (square) on LE. For EHwas made equal to DB. Thus, the (square) on AB (is) also equal to the (square) on LM. Thus, AB (is) equal to *LM*. And *AB* is the diameter of the given sphere. Thus, *LM* is equal to the diameter of the given sphere.

Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron.[†] (Which is) the very thing it was required to show.

Proposition 15

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is double CB. And let the semi-circle ADB have been drawn on AB. And let CD have been drawn from C at rightangles to AB. And let DB have been joined. And let the square EFGH, having (its) side equal to DB, be laid out. And let EK, FL, GM, and HN have been drawn from (points) E, F, G, and H, (respectively), at right-angles to the plane of square EFGH. And let EK, FL, GM, and HN, equal to one of EF, FG, GH, and HE, have been cut off from EK, FL, GM, and HN, respectively. And let KL, LM, MN, and NK have been joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.



Έπεζεύχθωσαν γὰρ αἱ ΚΗ, ΕΗ. καὶ ἐπεὶ ὀρθή ἐστιν ή ὑπὸ ΚΕΗ γωνία διὰ τὸ καὶ τὴν ΚΕ ὀρθὴν εἶναι πρὸς τὸ ΕΗ ἐπίπεδον δηλαδή καὶ πρὸς τὴν ΕΗ εὐθεῖαν, τὸ ἄρα έπι τῆς ΚΗ γραφόμενον ἡμικύκλιον ἤξει και διὰ τοῦ Ε σημείου. πάλιν, ἐπεὶ ἡ ΗΖ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΖΛ, ΖΕ, καὶ πρὸς τὸ ΖΚ ἄρα ἐπίπεδον ὀρθή ἐστιν ἡ ΗΖ· ὥστε καὶ ἐὰν ἐπιζεύξωμεν τὴν ΖΚ, ἡ ΗΖ ὀρθἡ ἔσται καὶ πρὸς τὴν ΖΚ· καὶ δὶα τοῦτο πάλιν τὸ ἐπὶ τῆς ΗΚ γραφόμενον ήμικύκλιον ήξει καὶ διὰ τοῦ Ζ. ὁμοίως καὶ δὶα τῶν λοιπῶν τοῦ χύβου σημείων ἥξει. ἐὰν δὴ μενούσης τῆς ΚΗ περιενεχθέν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῆ, ὅθεν ήρξατο φέρεσθαι, ἕσται σφαίρα περιειλημμένος ὁ κύβος. λέγω δή, ὅτι καὶ τῆ δοθείση. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΗΖ τῆ ΖΕ, καί ἐστιν ὀρθή ἡ πρὸς τῷ Ζ γωνία, τὸ ẳρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῆ ΕΚ· τὸ ἄρα άπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ· ὥστε τὰ ἀπὸ τῶν ΗΕ, ΕΚ, τουτέστι τὸ ἀπὸ τῆς ΗΚ, τριπλάσιόν ἐστι τοῦ άπὸ τῆς ΕΚ. καὶ ἐπεὶ τριπλασίων ἐστὶν ἡ ΑΒ τῆς ΒΓ, ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ, τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. έδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΗΚ τοῦ ἀπὸ τῆς ΚΕ τριπλάσιον. καὶ κεῖται ἴση ἡ KE τ
ỹ $\Delta B^{.}$ ἴση ἄρα καὶ ἡ KH τỹ AB. καί έστιν ή AB τῆς δοθείσης σφαίρας διάμετρος καὶ ή KH ẳρα ίση ἐστὶ τῆ τῆς δοθείσης σφαίρας διαμέτρω.

Τῆ δοθείση ἄρα σφαίρα περιείληπται ὁ κύβος· καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς· ὅπερ ἔδει δεῖξαι.



ELEMENTS BOOK 13

For let KG and EG have been joined. And since angle KEG is a right-angle—on account of KE also being at right-angles to the plane EG, and manifestly also to the straight-line EG [Def. 11.3]—the semi-circle drawn on KG will thus also pass through point E. Again, since GF is at right-angles to each of FL and FE, GF is thus also at right-angles to the plane FK [Prop. 11.4]. Hence, if we also join FK then GF will also be at right-angles to FK. And, again, on account of this, the semi-circle drawn on GK will also pass through point F. Similarly, it will also pass through the remaining (angular) points of the cube. So, if KG remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then the cube will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since GF is equal to FE, and the angle at F is a right-angle, the (square) on EG is thus double the (square) on EF [Prop. 1.47]. And EF (is) equal to EK. Thus, the (square) on EGis double the (square) on EK. Hence, the (sum of the squares) on GE and EK—that is to say, the (square) on GK [Prop. 1.47]—is three times the (square) on EK. And since AB is three times BC, and as AB (is) to BC, so the (square) on AB (is) to the (square) on BD[Prop. 6.8, Def. 5.9], the (square) on AB (is) thus three times the (square) on BD. And the (square) on GK was also shown (to be) three times the (square) on KE. And KE was made equal to DB. Thus, KG (is) also equal to AB. And AB is the radius of the given sphere. Thus, KG is also equal to the diameter of the given sphere.

Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on [†] If the radius of the sphere is unity then the side of the cube is $\sqrt{4/3}$.

ເຈ່.

Είκοσάεδρον συστήσασθαι καὶ σφαίρα περιλαβεῖν, ἤ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατά τὸ Γ ὥστε τετραπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἦχθω άπὸ τοῦ Γ τῆ ΑΒ πρὸς ορθὰς γωνίας εὐθεῖα γραμμὴ ἡ ΓΔ, καί ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω κύκλος ὁ ΕΖΗΘΚ, οὕ ἡ ἐν τοῦ κέντρου ἴση ἔστω τῃ ΔΒ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘΚ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ίσογώνιον τὸ ΕΖΗΘΚ, καὶ τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, Θ K, KE περιφέρειαι δίχα κατὰ τὸ Λ, Μ, Ν, Ξ, Ο σημεĩα, καὶ έπεζεύχθωσαν αί ΛΜ, ΜΝ, ΝΞ, ΞΟ, ΟΛ, ΕΟ. ἴσόπλευρον άρα ἐστὶ καὶ τὸ ΛΜΝΞΟ πεντάγωνον, καὶ δεκαγώνου ἡ ΕΟ εύθεῖα. καὶ ἀνεστάτωσαν ἄπὸ τῶν Ε, Ζ, Η, Θ, Κ σημείων τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς γωνίας εὐθεῖαι αί ΕΠ, ΖΡ, ΗΣ, ΘΤ, ΚΥ ἴσαι οὖσαι τῆ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ χύχλου, καὶ ἐπεζεύχθωσαν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ, $\Upsilon\Pi$, ΠΛ, ΛΡ, ΡΜ, ΜΣ, ΣΝ, ΝΤ, ΤΞ, ΞΥ, ΥΟ, ΟΠ.

Καὶ ἐπεὶ ἑκατέρα τῶν ΕΠ, ΚΥ τῷ αὐτῷ ἐπιπέδῳ πρὸς όρθάς ἐστιν, παράλληλος ἄρα ἐστιν ἡ ΕΠ τῆ ΚΥ. ἔστι δὲ αὐτῆ καὶ ἴση· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπιζευγνύουσαι ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι ἴσαι τε καὶ παράλληλοί είσιν. ή ΠΥ άρα τῆ ΕΚ ἴση τε καὶ παράλληλός ἐστιν. πενταγώνου δὲ ἰσοπλεύρου ἡ ΕΚ· πενταγώνου ἄρα ἰσοπλεύρου καὶ ἡ ΠΥ τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ΠΡ, ΡΣ, ΣΤ, ΤΥ πενταγώνου ἐστίν ἰσοπλεύρου τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον έγγραφομένου ισόπλευρον άρα το ΠΡΣΤΥ πεντάγωνον. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΕ, δεκαγώνου δὲ ἡ ΕΟ, καί ἐστιν ὀρθή ἡ ὑπὸ ΠΕΟ, πενταγώνου ἄρα ἐστὶν ἡ ΠΟ· ἡ γὰρ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. διὰ τὰ αὐτὰ δὴ χαὶ ἡ ΟΥ πενταγώνου ἐστὶ in circle EFGHK. Pentagon QRSTU (is) thus equilat-

the side of the cube.[†] (Which is) the very thing it was required to show.

Proposition 16

To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straightline) called minor.



Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let the straight-line CD have been drawn from C at right-angles to AB. And let DB have been joined. And let the circle EFGHK be set down, and let its radius be equal to DB. And let the equilateral and equiangular pentagon EFGHK have been inscribed in circle EFGHK [Prop. 4.11]. And let the circumferences EF, FG, GH, HK, and KE have been cut in half at points L, M, N, O, and P (respectively). And let LM, MN, NO, OP, PL, and EP have been joined. Thus, pentagon LMNOP is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ, FR, GS, HT, and KU, which are equal to the radius of circle EFGHK, have been set up at right-angles to the plane of the circle, at points E, F, FG, H, and K (respectively). And let QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, and PQ have been joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK. And EK (is the side) of an equilateral pentagon (inscribed in circle EFGHK). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle EFGHK. So, for the same (reasons), QR, RS, ST, and TU are also the sides of an equilateral pentagon inscribed πλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΟΥ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΛΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν.



eral. And side QE is (the side) of a hexagon (inscribed in circle EFGHK), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QUis also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR, RMS, SNT, and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM, MSN, NTO, and OUP are each also equilateral.



Εἰλήφθω τὸ κέντρον τοῦ ΕΖΗΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἔτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφηρήσθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἐκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεζεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ.

Καὶ ἐπεὶ ἐκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῷ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστιν ἡ ΦΧ τῆ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἔρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ πενταγώνου ἐστίν, ἐπειδήπερ, Let the center, point V, of circle EFGHK have been found [Prop. 3.1]. And let VZ have been set up, at (point) V, at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like VX. And let VW have been cut off (from XZso as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ, QW, UZ, EV, LV, LX, and XM have been joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33].

έὰν ἐπιζεύξωμεν τὰς ΦΚ, ΧΥ, ἴσαι καὶ ἀπεναντίον ἔσονται, καί ἐστιν ἡ ΦΚ ἐκ τοῦ κέντρου οὖσα ἑξαγώνου. έξαγώνου ἄρα καὶ ἡ ΧΥ. δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθὴ ή ὑπὸ ΥΧΩ· πενταγώνου ἄρα ή ΥΩ. ἔστι δὲ καὶ ή ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΥΩ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ῶν βάσεις μέν είσιν αί ΠΡ, ΡΣ, ΣΤ, ΤΥ εύθεῖαι, χορυφή δὲ τὸ Ω σημεῖον, ἰσόπλευρόν ἐστιν. πάλιν, ἐπεὶ ἑξαγώνου μὲν ἡ $\Phi \Lambda$, δεκαγώνου δὲ ή $\Phi \Psi$, καὶ ὀρθή ἐστιν ἡ ὑπὸ $\Lambda \Phi \Psi$ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΛΨ. διὰ τὰ αὐτὰ δὴ ἐὰν έπιζεύξωμεν τὴν $\mathrm{M}\Phi$ οὖσαν ἑξαγώνου, συνάγεται καὶ ἡ $\mathrm{M}\Psi$ πενταγώνου. ἔστι δὲ καὶ ἡ ΛΜ πενταγώνου· ἰσόπλευρον άρα ἐστὶ τὸ ΛΜΨ τρίγωνον. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ῶν βάσεις μέν εἰσιν αἱ ΜΝ, ΝΞ, ΞΟ, ΟΛ, χορυφή δὲ τὸ Ψ σημείον, ἰσόπλευρόν έστιν. συνέσταται άρα είκοσάεδρον ύπὸ εἴκοσι τριγώνων ίσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρα περιλαβεῖν τῆ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένῃ ἐλάσσων.

Ἐπεὶ γὰρ ἑξαγώνου ἐστὶν ἡ ΦΧ, δεκαγώνου δὲ ἡ ΧΩ, ἡ $\Phi\Omega$ ἄρα ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ X, καὶ τὸ με
ίζον αὐτῆς τμῆμά ἐστιν ἡ $\Phi X^{.}$ ἔστιν ẳρα ὡς ἡ
 $\Omega \Phi$ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ. ἴση δὲ ἡ μὲν ΦΧ τῆ ΦE , ή δè XΩ τῆ $\Phi \Psi$ · ἔστιν ἄρα ὡς ή ΩΦ πρὸς τὴν ΦE , οὕτως ἡ ΕΦ πρὸς τὴν ΦΨ. καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΩΦΕ, ΕΦΨ γωνίαι· ἐὰν ἄρα ἐπιζεύξωμεν τὴν ΕΩ εὐθεὶαν, ὀρθὴ ἔσται ἡ ὑπὸ $\Psi E \Omega$ γωνία διὰ τὴν ὑμοιότητα τῶν $\Psi E \Omega$, $\Phi E \Omega$ τριγώνων. διὰ τὰ αὐτὰ δὴ ἐπεί ἐστιν ὡς ἡ ΩΦ πρὸς τὴν ΦX , οὕτως ή ΦX πρὸς τὴν $X\Omega$, ἴση δὲ ή μὲν $\Omega \Phi$ τῆ ΨX , ή δὲ ΦΧ τῆ ΧΠ, ἔστιν ἄρα ὡς ἡ ΨΧ πρὸς τὴν ΧΠ, οὕτως ή ΠΧ πρός τὴν ΧΩ. καὶ διὰ τοῦτο πάλιν ἐὰν ἐπιζεύξωμεν τὴν ΠΨ, ὀρθὴ ἔσται ἡ πρὸς τῷ Π γωνία τὸ ẳρα ἐπὶ τῆς $\Psi\Omega$ γραφόμενον ήμικύκλιον ήξει καὶ δὶα τοῦ Π. καὶ ἐἀν μενούσης τῆς $\Psi\Omega$ περιενεχθέν τὸ ἡμιχύχλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῆ, ὅθεν ἤρξατο φέρεσθαι, ἤξει καὶ διὰ τοῦ Π καὶ τῶν λοιπῶν σημείων τοῦ εἰκοσαέδρου, καὶ ἔσται σφαίρα περιειλημμένον τὸ εἰχοσάεδρον. λέγω δή, ὅτι καὶ τῆ δοθείση. τετμήσθω γὰρ ἡ ΦΧ δίχα κατὰ τὸ α. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΦΩ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ ἕλασσον αὐτῆς τμῆμά ἐστιν ἡ ΩΧ, ἡ ἄρα ΩΧ προσλαβοῦσα τὴν ἡμίσειαν τοῦ μείζονος τμήματος τὴν Χα πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ωα τοῦ ἀπὸ τῆς αΧ. καί ἐστι τῆς μὲν Ωα διπλῆ ἡ ΩΨ, τὴς δὲ αΧ διπλῆ ή $\Phi \mathrm{X} \cdot$ πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς $\Omega \Psi$ τοῦ ἀπὸ τῆς ΧΦ. καὶ ἐπεὶ τετραπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, πενταπλῆ ἄρα έστιν ή ΑΒ τῆς ΒΓ. ὡς δὲ ή ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς B
Δ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς $\Omega \Psi$ πενταπλάσιον τοῦ ἀπὸ τῆς ΦX . καί ἐστιν ἴση ἡ ΔB τῆ

And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK, being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QUis also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR, RS, ST, and TU, and apexes the point Z, are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a rightangle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV, which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN, NO, OP, and PL, and apexes the point X, are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ(the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W, and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW, so VW (is) to WZ. And VW (is) equal to VE, and WZ to VX. Thus, as ZV is to VE, so EV (is) to VX. And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZthen angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ. [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW, so VW(is) to WZ, and ZV (is) equal to XW, and VW to WQ, thus as XW is to WQ, so QW (is) to WZ. And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZremains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) Q, and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been enΦΧ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῆ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ἴση ἄρα καὶ ἡ ΑΒ τῆ ΨΩ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΨΩ ἄρα ἴση ἐστὶ τῆ τῆς δοθείσης σφαίρας διαμέτρω· τῆ ἄρα δοθείσῃ σφαίρα περιείληπται τὸ εἰκοσάεδρον.

Λέγω δή, ὅτι ἡ τοῦ εἰχοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἐπεὶ γὰρ ῥητή ἐστιν ἡ τῆς σφαίρας διάμετρος, καί ἐστι δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, ῥητὴ ἄρα ἐστὶ καὶ ἡ ἑκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου. ὥστε καὶ ἡ διάμετρος αὐτοῦ ἑητή ἐστιν. ἐὰν δὲ εἰς κύκλον ἑητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῃ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἡ δὲ τοῦ ΕΖΗΘΚ πενταγώνου πλευρὰ ἦ τοῦ εἰκοσαέδρου ἐστίν. ἡ ἄρα τοῦ είκοσαέδρου πλευρὰ ἄλογός ἑστιν ἡ καλουμένη ἐλάττων.

Πόρισμα.

Έκ δὴ τούτου φανερόν, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὖ τὸ εἰκοσάεδρον ἀναγέγραπται, καὶ ὅτι ἡ τῆς σφαίρας διάμετρος σύγκειται ἕκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι. closed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW have been cut in half at a. And since the straight-line VZ has been cut in extreme and mean ratio at W, and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa, is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW. And ZX is double Za, and VW double aW. Thus, the (square) on ZX is five times the (square) on WV. And since AC is four times CB, AB is thus five times BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD. And the (square) on ZX was also shown (to be) five times the (square) on VW. And DB is equal to VW. For each of them is equal to the radius of circle EFGHK. Thus, AB (is) also equal to XZ. And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle EFGHK, the radius of circle EFGHK is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon EFGHK is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.[†]

[†] If the radius of the sphere is unity then the radius of the circle is $2/\sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2/\sqrt{5}$, $1 - 1/\sqrt{5}$, and $(1/\sqrt{5})\sqrt{10 - 2\sqrt{5}}$, respectively.

ιζ΄.

Δωδεκάεδρον συστήσασθαι καὶ σφαίρα περιλαβεῖν, ἢ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straightline) called an apotome.



Έκκείσθωσαν τοῦ προειρημένου κύβου δύο ἐπίπεδα πρὸς ὀρθὰς ἀλλήλοις τὰ ABΓΔ, ΓΒΕΖ, καὶ τετμήσθω ἑκάστη τῶν AB, BΓ, ΓΔ, ΔΑ, ΕΖ, ΕΒ, ΖΓ πλευρῶν δίχα κατὰ τὰ H, Θ, K, Λ, M, N, Ξ, καὶ ἐπεζεύχθωσαν αἱ HK, ΘΛ, ΜΘ, ΝΞ, καὶ τετηήσθω ἑκάστη τῶν NO, ΟΞ, ΘΠ ἄκρον καὶ μέσον λόγον κατὰ τὰ P, Σ, Τ σημεῖα, καὶ ἔστω αὐτῶν μείζονα τμήματα τὰ PO, ΟΣ, ΤΠ, καὶ ἀνεστάτωσαν ἀπὸ τῶν P, Σ, Τ σημείων τοῖς τοῦ κύβου ἐπιπέδοις πρὸς ὀρθὰς ἐπὶ τὰ ἐκτὸς μέρη τοῦ κύβου αἱ PY, ΣΦ, ΤΧ, καὶ κείσθωσαν ἴσαι ταῖς PO, ΟΣ, ΤΠ, καὶ ἐπεζεύχθωσαν αἱ ΥB, BX, ΧΓ, ΓΦ, ΦΥ.

Λέγω, ὅτι τὸ ΥΒΧΓΦ πεντάγωνον ἰσόπλευρόν τε καὶ ἐν ένὶ ἐπιπέδω καὶ ἔτι ἰσογώνιόν ἐστιν. ἐπεζεύχθωσαν γὰρ αἱ PB, ΣB, ΦB. καὶ ἐπεὶ εὐθεῖα ἡ NO ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΟ, τὰ ἄρα ἀπὸ τῶν ΟΝ, NP τριπλάσιά ἐστι τοῦ ἀπὸ τῆς PO. ἴση δὲ ἡ μέν ΟΝ τῆ ΝΒ, ἡ δὲ ΟΡ τῆ ΡΥ· τὰ ἄρα ἀπὸ τῶν ΒΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΝ, ΝΡ τὸ ἀπὸ τῆς BP ἐστιν ἴσον· τὸ ἄρα ἀπὸ τῆς BP τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΡΥ· ὥστε τὰ ἀπὸ τῶν ΒΡ, ΡΥ τετραπλάσιά έστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΡ, ΡΥ ἴσον ἐστι τὸ ἀπὸ τῆς ΒΥ· τὸ ἄρα ἄπὸ τῆς ΒΥ τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΥΡ· διπλῆ ἄρα ἐστὶν ἡ ΒΥ τῆς ΡΥ. ἔστι δὲ καὶ ἡ ΦΥ τῆς ΥΡ διπλη, ἐπειδήπερ καὶ ἡ ΣΡ τῆς ΟΡ, τουτέστι τῆς ΡΥ, έστι διπλῆ· ἴση ἄρα ἡ ΒΥ τῆ ΥΦ. ὑμοίως δὴ δειχθήσεται, ὄτι καὶ ἑκάστη τῶν BX, XΓ, ΓΦ ἑκατέρ
α τῶν BΥ, ΥΦ έστιν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. λέγω δή, ὅτι καὶ ἐν ἑνί ἐστιν ἐπιπέδω. ἤχθω γὰρ ἀπὸ τοῦ Ο έκατέρα τῶν ΡΥ, ΣΦ παράλληλος ἐπὶ τὰ ἐκτὸς τοῦ κύβου μέρη ή ΟΨ, καὶ ἐπεζεύχθωσαν αἱ ΨΘ, ΘΧ· λέγω, ὅτι ἡ $\Psi\Theta X$ εὐθεῖά ἐστιν. ἐπεὶ γὰρ ή $\Theta \Pi$ ἄχρον χαὶ μέσον λόγον τέτμηται κατά τὸ Τ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΠΤ, έστιν ἄρα ώς ή ΘΠ πρός την ΠΤ, οὕτως ή ΠΤ πρός την



Let two planes of the aforementioned cube [Prop. 13.15], *ABCD* and *CBEF*, (which are) at right-angles to one another, be laid out. And let the sides *AB*, *BC*, *CD*, *DA*, *EF*, *EB*, and *FC* have each been cut in half at points *G*, *H*, *K*, *L*, *M*, *N*, and *O* (respectively). And let *GK*, *HL*, *MH*, and *NO* have been joined. And let *NP*, *PO*, and *HQ* have each been cut in extreme and mean ratio at points *R*, *S*, and *T* (respectively). And let their greater pieces be *RP*, *PS*, and *TQ* (respectively). And let *RU*, *SV*, and *TW* have been set up on the exterior side of the cube, at points *R*, *S*, and *T* (respectively), at right-angles to the planes of the cube. And let them be made equal to *RP*, *PS*, and *TQ*. And let *UB*, *BW*, *WC*, *CV*, and *VU* have been joined.

I say that the pentagon UBWCV is equilateral, and in one plane, and, further, equiangular. For let RB, SB, and VB have been joined. And since the straight-line NPhas been cut in extreme and mean ratio at R, and RP is the greater piece, the (sum of the squares) on PN and *NR* is thus three times the (square) on *RP* [Prop. 13.4]. And PN (is) equal to NB, and PR to RU. Thus, the (sum of the squares) on BN and NR is three times the (square) on RU. And the (square) on BR is equal to the (sum of the squares) on BN and NR [Prop. 1.47]. Thus, the (square) on BR is three times the (square) on RU. Hence, the (sum of the squares) on BR and RUis four times the (square) on RU. And the (square) on BU is equal to the (sum of the squares) on BR and RU[Prop. 1.47]. Thus, the (square) on BU is four times the (square) on UR. Thus, BU is double RU. And VU is also double UR, inasmuch as SR is also double PR—that is to say, RU. Thus, BU (is) equal to UV. So, similarly, it can be shown that each of BW, WC, CV is equal to each ΤΘ. ἴση δὲ ἡ μὲν ΘΠ τῆ ΘΟ, ἡ δὲ ΠΤ ἑκατέρα τῶν ΤΧ, ΟΨ· ἔστιν ἄρα ὡς ἡ ΘΟ πρὸς τὴν ΟΨ, οὕτως ἡ ΧΤ πρὸς τὴν ΤΘ. καί ἐστι παράλληλος ἡ μὲν ΘΟ τῆ ΤΧ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἡ δὲ ΤΘ τῆ ΟΨ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΖ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο τρίγωνα συντεθῆ κατὰ μίαν γωνίαν, ὡς τὰ ΨΟΘ, ΘΤΧ, τὰς δύο πλευρὰς ταῖς δυνὶν ἀνάλογον ἔχοντα, ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἰ λοιπαὶ εὐθεῖαι ἐπ' εὐθείας ἔσονται· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΨΘ τῆ ΘΧ. πᾶσα δὲ εὐθεῖα ἐν ἑνί ἐστιν ἐπιπέδῳ· ἐν ἑνὶ ἄρα ἐπιπέδῳ ἐστὶ τὸ ΥΒΧΓΦ πεντάγωνον.

Λέγω δή, ὅτι καὶ ἰσογώνιόν ἐστιν.

Έπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΝΟ ẳϰρον καὶ μέσον λόγον τέτμηται κατά τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΡ [ἔστιν ἄρα ὡς συναμφότερος ἡ ΝΟ, ΟΡ πρὸς τὴν ΟΝ, οὕτως ἡ NO πρὸς τὴν OP], ἴση δὲ ἡ OP τῃ OΣ [ἔστιν ἄρα ὡς ἡ ΣN πρὸς τὴν ΝΟ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΣ], ἡ ΝΣ ἄρα ἄχρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον τμῆμά έστιν ή ΝΟ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ άπὸ τῆς NO. ἴση δὲ ἡ μὲν NO τỹ NB, ἡ δὲ ΟΣ τỹ Σ
Φ· τὰ άρα ἀπὸ τῶν ΝΣ, ΣΦ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς NB· ὥστε τὰ ἀπὸ τῶν
 $\Phi\Sigma,$ ΣN, NB τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ. τοῖς δὲ ἀπὸ τῶν ΣΝ, ΝΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΣB · τὰ ἄρα ἀπὸ τῶν $B\Sigma$, $\Sigma \Phi$, τουτέστι τὸ ἀπὸ τῆς $B\Phi$ [ὀρθή γὰρ ἡ ὑπὸ ΦΣΒ γωνία], τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΒ· διπλῆ ἄρα ἐστὶν ἡ ΦΒ τῆς ΒΝ. ἔστι δὲ καὶ ἡ ΒΓ τῆς ΒΝ διπλῆ· ἴση ἄρα ἐστὶν ἡ ΒΦ τῆ ΒΓ. καὶ ἐπεὶ δύο αἱ $B\Upsilon$, $\Upsilon\Phi$ δυσὶ ταῖς BX, $X\Gamma$ ἴσαι εἰσίν, καὶ βάσις ἡ $B\Phi$ βάσει τῆ ΒΓ ἴση, γωνία ἄρα ἡ ὑπὸ ΒΥΦ γωνία τῆ ὑπὸ ΒΧΓ ἐστιν ίση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΥΦΓ γωνία ἴση ἐστὶ τῆ ὑπὸ ΒΧΓ· αἱ ἄρα ὑπὸ ΒΧΓ, ΒΥΦ, ΥΦΓ τρεῖς γωνίαι ἴσαι ἀλλήλαις εἰσίν. ἐὰν δὲ πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ἴσαι ἀλλήλαις ῶσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον. ίσογώνιον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. ἐδείχθη δὲ καὶ ίσόπλευρον· τὸ ἄρα ΒΥΦΓΧ πεντάγωνον ἰσόπλευρόν ἐστι καὶ ἰσογώνιον, καί ἐστιν ἐπὶ μιᾶς τοῦ κύβου πλευρᾶς τῆς ΒΓ. ἐὰν ἄρα ἐφ' ἑκάστης τῶν τοῦ κύβου δώδεκα πλευρῶν τὰ αὐτὰ κατασκευάσωμεν, συσταθήσεταί τι σχῆμα στερεὸν ύπὸ δώδεκα πενταγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων περιεγόμενον, δ καλεῖται δωδεκάεδρον.

Δεῖ δὴ αὐτὸ καὶ σφαίρα περιλαβεῖν τῆ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένῃ ἀποτομή.

Έκβεβλήσθω γὰρ ἡ ΨΟ, καὶ ἔστω ἡ ΨΩ· συμβάλλει ἄρα ἡ ΟΩ τῆ τοῦ κύβου διαμέτρω, καὶ δίχα τέμνουσιν ἀλλήλας· τοῦτο γὰρ δέδεικται ἐν τῷ παρατελεύτω θεωρήματι τοῦ ἑνδεκάτου βιβλίου. τεμνέτωσαν κατὰ τὸ Ω· τὸ Ω ἄρα κέντρον ἐστὶ τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον, καὶ ἡ ΩΟ ἡμίσεια τῆς πλευρᾶς τοῦ κύβου. ἐπεζεύχθω δὴ ἡ ΥΩ. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΝΣ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΝΟ,

of BU and UV. Thus, pentagon BUVCW is equilateral. So, I say that it is also in one plane. For let PX have been drawn from P, parallel to each of RU and SV, on the exterior side of the cube. And let XH and HW have been joined. I say that XHW is a straight-line. For since HQ has been cut in extreme and mean ratio at T, and QT is its greater piece, thus as HQ is to QT, so QT (is) to TH. And HQ (is) equal to HP, and QT to each of TW and PX. Thus, as HP is to PX, so WT (is) to TH. And HP is parallel to TW. For of each of them is at right-angles to the plane BD [Prop. 11.6]. And TH(is parallel) to PX. For each of them is at right-angles to the plane BF [Prop. 11.6]. And if two triangles, like XPH and HTW, having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, XH is straight-on to HW. And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon UBWCV is in one plane.

So, I say that it is also equiangular.

For since the straight-line NP has been cut in extreme and mean ratio at R, and PR is the greater piece [thus as the sum of NP and PR is to PN, so NP (is) to PR, and PR (is) equal to PS [thus as SN is to NP, so NP (is) to PS], NS has thus also been cut in extreme and mean ratio at P, and NP is the greater piece [Prop. 13.5]. Thus, the (sum of the squares) on NS and SP is three times the (square) on NP [Prop. 13.4]. And NP (is) equal to NB, and PS to SV. Thus, the (sum of the) squares on NS and SV is three times the (square) on NB. Hence, the (sum of the squares) on VS, SN, and *NB* is four times the (square) on *NB*. And the (square) on SB is equal to the (sum of the squares) on SN and *NB* [Prop. 1.47]. Thus, the (sum of the squares) on *BS* and SV—that is to say, the (square) on BV [for angle VSB (is) a right-angle]—is four times the (square) on NB [Def. 11.3, Prop. 1.47]. Thus, VB is double BN. And BC (is) also double BN. Thus, BV is equal to BC. And since the two (straight-lines) BU and UV are equal to the two (straight-lines) BW and WC (respectively), and the base BV (is) equal to the base BC, angle BUVis thus equal to angle BWC [Prop. 1.8]. So, similarly, we can show that angle UVC is equal to angle BWC. Thus, the three angles BWC, BUV, and UVC are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon BUVCW is equiangular. And it was also shown (to be) equilateral. Thus, pentagon BUVCW is equilateral and equiangular, and it is on one of the sides, BC, of the cube. Thus, if we make the

τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΣ τ
ỹ $\Psi\Omega$, ἐπειδήπερ καὶ ἡ μὲν ΝΟ τῆ ΟΩ έστιν ἴση, ἡ δ
ὲ ΨO τῆ ΟΣ. ἀλλὰ μὴν καὶ ἡ ΟΣ τῆ ΨΥ, ἐπεὶ καὶ τῆ ΡΟ· τὰ ἄρα ἀπὸ τῶν ΩΨ, ΨΥ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς NO. τοῖς δὲ ἀπὸ τῶν $\Omega\Psi$, $\Psi\Upsilon$ ἴσον ἐστὶ τὸ ἀπὸ τῆς $\Upsilon\Omega$ · τὸ ἄρα ἀπὸ τῆς ΥΩ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἔστι δὲ καὶ ἡ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον δυνάμει τριπλασίων τῆς ἡμισείας τῆς τοῦ κύβου πλευρᾶς· προδέδεικται γὰρ κύβον συστήσασθαι καὶ σφαίρα περιλαβεῖν καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων έστι τῆς πλευρᾶς τοῦ χύβου. εἰ δὲ ὅλη τῆς ὅλης, καὶ [ἡ] ἡμίσεια τῆς ἡμισείας· καί ἐστιν ἡ ΝΟ ἡμίσεια τῆς τοῦ κύβου πλευρας. ή άρα ΥΩ ίση ἐστὶ τῇ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον. καί ἐστι τὸ Ω κέντρον τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον· τὸ Υ ἄρα σημεῖον πρὸς τῆ ἐπιφανεία ἐστι τῆς σφαίρας. ὑμοίως δή δείξομεν, ότι και έκάστη τῶν λοιπῶν γωνιῶν τοῦ δωδεκαέδρου πρός τῆ ἐπιφανεία ἐστὶ τῆς σφαίρας· περιείληπται άρα τὸ δωδεχαέδρον τῆ δοθείση σφαίρα.

Λέγω δή, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ τῆς ΝΟ ἄχρον χαὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ὁ PO, τῆς δὲ ΟΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ἡ $O\Sigma$, ὅλης ἄρα τῆς $N\Xi$ άκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΣ. [οἶον ἐπεί ἐστιν ὡς ἡ ΝΟ πρὸς τὴν ΟΡ, ἡ ΟΡ πρὸς τὴν ΡΝ, καὶ τὰ διπλάσια· τὰ γὰρ μέρη τοῖς ἰσάκις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον· ὡς ἄρα ἡ ΝΞ πρὸς τὴν $P\Sigma$, οὕτως ἡ ΡΣ πρός συναμφότερον τὴν ΝΡ, ΣΞ. μείζων δὲ ἡ ΝΞ τῆς $P\Sigma$ · μείζων ἄρα καὶ
ἡ $P\Sigma$ συναμφοτέρου τῆς NP, $\Sigma\Xi$ · ἡ NΞ ἄρα ἄχρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ή PΣ.] ἴση δὲ ή PΣ τῃ ΥΦ· τῆς ἄρα NΞ ἄχρον και μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΥΦ. καὶ ἐπεὶ ῥητή ἐστιν τῆς σφαίρας διάμετρος καί ἐστι δυνάμει τριπλασίων τῆς τοῦ χύβου πλευρᾶς, ῥητὴ ἄρα ἐστὶν ἡ ΝΞ πλευρὰ οὕσα τοῦ κύβου. ἐὰν δὲ ἑητὴ γραμμὴ ἄκρον καὶ μέσον λόγον τμηθή, έκάτερον τῶν τμημάτων ἄλογός ἐστιν ἀποτομή.

Ή ΥΦ ἄρα πλευρὰ οῦσα τοῦ δωδεκαέδρου ἄλογός ἐστιν ἀποτομή.

same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will have been constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let *XP* have been produced, and let (the produced straight-line) be XZ. Thus, PZ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at Z. Thus, Z is the center of the sphere enclosing the cube, and ZP(is) half the side of the cube. So, let UZ have been joined. And since the straight-line NS has been cut in extreme and mean ratio at P, and its greater piece is NP, the (sum of the squares) on NS and SP is thus three times the (square) on NP [Prop. 13.4]. And NS (is) equal to XZ, inasmuch as NP is also equal to PZ, and XP to PS. But, indeed, PS (is) also (equal) to XU, since (it is) also (equal) to RP. Thus, the (sum of the squares) on ZX and XU is three times the (square) on NP. And the (square) on UZ is equal to the (sum of the squares) on ZX and XU [Prop. 1.47]. Thus, the (square) on UZis three times the (square) on NP. And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And NP is half of the side of the cube. Thus, UZ is equal to the radius of the sphere enclosing the cube. And Z is the center of the sphere enclosing the cube. Thus, point U is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since RP is the greater piece of NP, which has been cut in extreme and mean ratio, and PS is the greater piece of PO, which has been cut in extreme and mean ratio, RS is thus the greater piece of the whole of NO, which has been cut in extreme and mean ratio. [Thus, since as NP is to PR, (so) PR (is) to RN, and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding

order) [Prop. 5.15]. Thus, as NO (is) to RS, so RS (is) to the sum of NR and SO. And NO (is) greater than RS. Thus, RS (is) also greater than the sum of NR and SO [Prop. 5.14]. Thus, NO has been cut in extreme and mean ratio, and RS is its greater piece.] And RS (is) equal to UV. Thus, UV is the greater piece of NO, which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, NO, which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

Thus, UV, which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

Corollary

So, (it is) clear, from this, that the side of the dodecahedron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio.[†] (Which is) the very thing it was required to show.

[†] If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4/3}$, and the side of the dodecahedron is $(1/3)(\sqrt{15}-\sqrt{3})$.

*ι*η'.

Πόρισμα.

ἄχρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν

ή τοῦ δωδεχαέδρου πλευρά. ὅπερ ἔδει δεῖξαι.

Έκ δή τούτου φανερόν, ὅτι τῆς τοῦ κύβου πλευρᾶς

Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρῖναι πρὸς ἀλλήλας.



Έχχείσθω ή τῆς δοθείσης σφαίρας διάμετρος ή AB, καὶ τετμήσθω κατὰ τὸ Γ ὥστε ἴσην είναι τὴν AΓ τῆ ΓΒ, κατὰ δὲ τὸ Δ ὥστε διπλασίονα είναι τὴν AΔ τῆς ΔB, καὶ γεγράφθω ἐπὶ τῆς AB ἡμιχύχλιον τὸ AEB, καὶ ἀπὸ τῶν Γ, Δ τῆ AB πρὸς ὀρθὰς ἦχθωσαν αἱ ΓΕ, ΔΖ, καὶ ἐπεζεύχθωσαν αἱ AΖ, ZB, EB. καὶ ἐπεὶ διπλῆ ἐστιν ἡ AΔ τῆς ΔB, τριπλῆ ἄρα ἐστὶν ἡ AB τῆς BΔ. ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ BA τῆς AΔ. ὡς δὲ ἡ BA πρὸς τὴν AΔ, οὕτως τὸ ἀπὸ τῆς BA

Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another.^{\dagger}



Let the diameter, AB, of the given sphere be laid out. And let it have been cut at C, such that AC is equal to CB, and at D, such that AD is double DB. And let the semi-circle AEB have been drawn on AB. And let CE and DF have been drawn from C and D (respectively), at right-angles to AB. And let AF, FB, and EB have been joined. And since AD is double DB, AB is thus triple BD. Thus, via conversion, BA is one and a half πρὸς τὸ ἀπὸ τῆς ΑΖ· ἰσογώνιον γάρ ἐστι τὸ ΑΖΒ τρίγωνον τῷ ΑΖΔ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία τῆς πλευρᾶς τῆς πυραμίδος. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΑΖ ἄρα ἴση ἐστὶ τῆ πλευρᾶ τῆς πυραμίδος.

Πάλιν, ἐπεὶ διπλασίων ἐστὶν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· τριπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΒΖ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῆ ΓΒ, διπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΕ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων τῆς τοῦ ὀκταέδρου πλευρᾶς. καὶ ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΒΕ ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

"Ηχθω δὴ ἀπὸ τοῦ Α σημείου τῆ ΑΒ εὐθεία πρὸς ὀρθὰς ή ΑΗ, καὶ κείσθω ἡ ΑΗ ἴση τῆ ΑΒ, καὶ ἐπεζεύχθω ἡ ΗΓ, καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ διπλη έστιν ή ΗΑ της ΑΓ· ἴση γὰρ ή ΗΑ τη ΑΒ· ὡς δὲ ή ΗΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΓ, διπλῆ ẳρα καὶ ἡ ΘΚ τῆς ΚΓ. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘΚ τοῦ ἀπὸ τῆς ΚΓ· τὰ ἄρα ἀπὸ τῶν ΘΚ, ΚΓ, ὅπερ ἐστὶ τὸ άπὸ τῆς ΘΓ, πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΚΓ. ἴση δὲ ή ΘΓ τῆ ΓΒ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΓ τοῦ άπὸ τῆς ΓΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ τῆς ΓΒ, ῶν ἡ $A\Delta$ τῆς ΔB ἐστι διπλῆ, λοιπὴ ἄρα ἡ $B\Delta$ λοιπῆς τῆς $\Delta \Gamma$ έστι διπλῆ. τριπλῆ ἄρα ἡ ΒΓ τῆς ΓΔ· ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΔ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΓΚ τοῦ άπὸ τῆς ΓΔ. μείζων ἄρα ἐστὶν ἡ ΓΚ τῆς ΓΔ. κείσθω τῆ ΓΚ ἴση ἡ ΓΛ, καὶ ἀπὸ τοῦ Λ τῆ AB πρὸς ὀρθὰς ἤχθω ἡ ΛΜ, καὶ ἐπεζεύχθω ἡ MB. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ, καί ἐστι τῆς μὲν ΒΓ διπλῆ ή AB, τῆς δὲ ΓΚ διπλῆ ή ΚΛ, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΚΛ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὕ τὸ εἰκοσάεδρον ἀναγέγραπται. καί ἐστιν ἡ ΑΒ ή τῆς σφαίρας διάμετρος ή ΚΛ ἄρα ἐκ τοῦ κέντρου έστι τοῦ κύκλου, ἀφ' οὕ τὸ εἰκοσάεδρον ἀναγέγραπται· ή ΚΛ ἄρα ἑξαγώνου ἐστὶ πλευρὰ τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ έξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον έγγραφομένων, καί έστιν ή μέν AB ή τῆς σφαίρας διάμετρος, ή δὲ ΚΛ ἑξαγώνου πλευρά, καὶ ἴση ή ΑΚ τῆ ΑΒ, ἑκατέρα ἄρα τῶν ΑΚ, ΑΒ δεκαγώνου ἐστὶ πλευρὰ τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ' οῦ τὸ εἰκοσάεδρον άναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ ΛΒ, ἑξαγώνου

times AD. And as BA (is) to AD, so the (square) on BA (is) to the (square) on AF [Def. 5.9]. For triangle AFB is equiangular to triangle AFD [Prop. 6.8]. Thus, the (square) on BA is one and a half times the (square) on AF. And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And AB is the diameter of the sphere. Thus, AF is equal to the side of the pyramid.

Again, since AD is double DB, AB is thus triple BD. And as AB (is) to BD, so the (square) on AB (is) to the (square) on BF [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is three times the (square) on BF. And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And ABis the diameter of the sphere. Thus, BF is the side of the cube.

And since AC is equal to CB, AB is thus double BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BE [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BE. And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And AB is the diameter of the given sphere. Thus, BE is the side of the octagon.

So let AG have been drawn from point A at rightangles to the straight-line AB. And let AG be made equal to AB. And let GC have been joined. And let HK have been drawn from H, perpendicular to AB. And since GAis double AC. For GA (is) equal to AB. And as GA (is) to AC, so HK (is) to KC [Prop. 6.4]. HK (is) thus also double KC. Thus, the (square) on HK is four times the (square) on KC. Thus, the (sum of the squares) on HKand KC, which is the (square) on HC [Prop. 1.47], is five times the (square) on KC. And HC (is) equal to CB. Thus, the (square) on BC (is) five times the (square) on CK. And since AB is double CB, of which AD is double DB, the remainder BD is thus double the remainder DC. BC (is) thus triple CD. The (square) on BC (is) thus nine times the (square) on CD. And the (square) on BC(is) five times the (square) on CK. Thus, the (square) on CK (is) greater than the (square) on CD. CK is thus greater than CD. Let CL be made equal to CK. And let *LM* have been drawn from *L* at right-angles to *AB*. And let MB have been joined. And since the (square) on BC is five times the (square) on CK, and AB is double BC, and KL double CK, the (square) on AB is thus five times the (square) on KL. And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And AB is the diameter of the sphere. Thus, KL is the radius of the circle from δὲ ἡ ΜΛ· ἴση γάρ ἐστι τῆ ΚΛ, ἐπεὶ καὶ τῆ ΘΚ· ἴσον γὰρ ἀπέχουσιν ἀπὸ τοῦ κέντρου· καί ἐστιν ἑκατέρα τῶν ΘΚ, ΚΛ διπλασίων τῆς ΚΓ· πενταγώνου ἄρα ἐστὶν ἡ MB. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ MB.

Καὶ ἐπεὶ ἡ ΖΒ κύβου ἐστὶ πλευρά, τετμήσθω ἄχρον καὶ μέσον λόγον κατὰ τὸ Ν, καὶ ἔστω μεῖζον τμῆμα τὸ NB[.] ἡ NB ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν AZ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς BE δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ZB δυνάμει τριπλασίων, οἴων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριτος, τῆς δὲ τοῦ κύβου δυνάμει διπλῆ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ὑπλο, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ὑπλῆ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου, δυνάμει ἡμιολία. αἰ μὲν οῦν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσιν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἤ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὕτε πρὸς ἀλλήλας οὕτε πρὸς τὰς προειρημένας εἰσιν ἐν λόγοις ῥητοῖς· ἄλογοι γάρ εἰσιν, ἡ μὲν ἐλάττων, ἡ δὲ ἀποτομή.

Ότι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ MB τῆς τοῦ δωδεκαέδρου τῆς NB, δείξομεν οὕτως.

Ἐπεὶ γὰρ ἰσογώνιόν ἐστι τὸ ΖΔΒ τρίγωνον τῷ ΖΑΒ τριγώνω, ἀνάλογόν ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΖ, οὕτως ή BZ πρός την BA. και έπει τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, έστιν ώς ή πρώτη πρός την τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· ἔστιν ἄρα ὡς ἡ ΔB πρὸς τὴν BA, οὕτως τὸ ἀπὸ τῆς ΔΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· ἀνάπαλιν ἄρα ώς ή AB πρός την BΔ, οὕτως τὸ ἀπὸ τῆς ZB πρὸς τὸ ἀπὸ τῆς ΒΔ. τριπλῆ δὲ ἡ ΑΒ τῆς ΒΔ· τριπλάσιον ἄρα τὸ ἀπὸ τῆς ZB τοῦ ἀπὸ τῆς BΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς AΔ τοῦ ἀπὸ τῆς $\Delta {
m B}$ τετραπλάσιον· διπλῆ γὰρ ἡ ${
m A}\Delta$ τῆς $\Delta {
m B}$ · μεῖζον ἄρα τὸ ἀπὸ τῆς $m A\Delta$ τοῦ ἀπὸ τῆς $m ZB^{.}$ μείζων ἄρα ἡ $m A\Delta$ τῆς ΖΒ· πολλῷ ἄρα ἡ ΑΛ τῆς ΖΒ μείζων ἐστίν. καὶ τῆς μὲν ΑΛ ἄχρον χαὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά έστιν ή ΚΛ, ἐπειδήπερ ή μὲν ΛΚ ἑξαγώνου ἐστίν, ή δὲ ΚΑ δεκαγώνου. τῆς δὲ ΖΒ ἄχρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ NB· μείζων ἄρα ἡ KΛ τῆς NB. ἴση δὲ ἡ ΚΛ τῆ ΛΜ· μείζων ἄρα ἡ ΛΜ τῆς NB [τῆς δὲ ΛΜ μείζων ἐστὶν ἡ MB]. πολλῷ ἄρα ἡ MB πλευρὰ οὖσα τοῦ εἰχοσαέδρου μείζων ἐστὶ τῆς ΝΒ πλευρᾶς οὔσης τοῦ δωδεχαέδρου. ὅπερ ἔδει δεῖξαι.

which the icosahedron has been described. Thus, KL is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and AB is the diameter of the sphere, and KL the side of the hexagon, and AK (is) equal to LB, thus AK and LB are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since LB is (the side) of the decagon. And ML (is the side) of the hexagon—for (it is) equal to KL, since (it is) also (equal) to HK, for they are equally far from the center. And HK and KL are each double *KC. MB* is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, MB is (the side) of the icosahedron.

And since FB is the side of the cube, let it have been cut in extreme and mean ratio at N, and let NB be the greater piece. Thus, NB is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, AF, of the pyramid, and twice the square on (the side), BE, of the octagon, and three times the square on (the side), FB, of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures-I mean, of the pyramid, and of the octahedron, and of the cubeare in rational ratios to one another. And (the sides of) the remaining two (figures)-I mean, of the icosahedron, and of the dodecahedron-are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straightlines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].

(And), we can show that the side, MB, of the icosahedron is greater that the (side), NB, or the dodecahedron, as follows.

For, since triangle FDB is equiangular to triangle FAB [Prop. 6.8], proportionally, as DB is to BF, so BF (is) to BA [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third,

so the (square) on the first (is) to the (square) on the second [Def. 5.9, Prop. 6.20 corr.]. Thus, as DB is to BA, so the (square) on DB (is) to the (square) on BF. Thus, inversely, as AB (is) to BD, so the (square) on FB (is) to the (square) on BD. And AB (is) triple BD. Thus, the (square) on FB (is) three times the (square) on *BD*. And the (square) on *AD* is also four times the (square) on DB. For AD (is) double DB. Thus, the (square) on AD (is) greater than the (square) on FB. Thus, AD (is) greater than FB. Thus, AL is much greater than FB. And KL is the greater piece of AL, which is cut in extreme and mean ratio—inasmuch as LK is (the side) of the hexagon, and KA (the side) of the decagon [Prop. 13.9]. And NB is the greater piece of FB, which is cut in extreme and mean ratio. Thus, KL (is) greater than NB. And KL (is) equal to LM. Thus, LM (is) greater than NB [and MB is greater than LM]. Thus, MB, which is (the side) of the icosahedron, is much greater than NB, which is (the side) of the dodecahedron. (Which is) the very thing it was required to show.

† If the radius of the given sphere is unity then the sides of the pyramid (*i.e.*, tetrahedron), octahedron, cube, icosahedron, and dodecahedron, respectively, satisfy the following inequality: $\sqrt{8/3} > \sqrt{2} > \sqrt{4/3} > (1/\sqrt{5}) \sqrt{10 - 2\sqrt{5}} > (1/3) (\sqrt{15} - \sqrt{3})$.

Λέγω δή, ὅτι παρὰ τὰ εἰρημένα πέντε σχήματα οὐ συσταθήσεται ἔτερον σχῆμα περιεχόμενον ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων ἴσων ἀλλήλοις.

Ύπὸ μὲν γὰρ δύο τριγώνων ἢ ὅλως ἐπιπέδων στερεὰ γωνία ού συνίσταται. ὑπὸ δὲ τριῶν τριγώνων ἡ τῆς πυραμίδος, ὑπὸ δὲ τεσσάρων ἡ τοῦ ὀχταέδρου, ὑπὸ δὲ πέντε ή τοῦ εἰχοσαέδρου. ὑπὸ δὲ ἕξ τριγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων πρὸς ἑνὶ σημείω συνισταμένων οὐκ ἔσται στερεὰ γωνία· οὕσης γὰρ τῆς τοῦ ἰσοπλεύρου τριγώνου γωνίας διμοίρου ὀρθῆς ἔσονται αἱ ἕξ τέσσαρσιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον. ἄπασα γὰρ στερεὰ γωνία ὑπὸ ἐλασσόνων ἢ τεσσάρων ὀρθῶν περέχεται. διὰ τὰ αὐτὰ δὴ οὐδὲ ὑπὸ πλειόνων η εξ γωνιῶν ἐπιπέδων στερεὰ γωνία συνίσταται. ύπὸ δὲ τετραγώνων τριῶν ἡ τοῦ χύβου γωνία περιέχεται· ύπὸ δὲ τεσσάρων ἀδύνατον. ἔσονται γὰρ πάλιν τέσσαρες όρθαί. ὑπὸ δὲ πενταγώνων ἰσοπλεύρων καὶ ἰσογωνίων, ὑπὸ μέν τριῶν ή τοῦ δωδεχαέδρου. ὑπὸ δὲ τεσσάρων ἀδύνατον. ούσης γὰρ τῆς τοῦ πενταγώνου ἰσοπλεύρου γωνίας ὀρθῆς καὶ πέμπτου, ἔσονται αἱ τέσσαρες γωνίαι τεσσάρων ὀρθῶν μείζους. ὅπερ ἀδύνατον. οὐδὲ μὴν ὑπὸ πολυγώνων ἑτέρων σχημάτων περισχεθήσεται στερεά γωνία διά τὸ αὐτὸ ἄτοπον.

Οὐκ ἄρα παρὰ τὰ εἰρημένα πέντε σχήματα ἕτερον σχῆμα στερεὸν συσταθήσεται ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον. ὅπερ ἔδει δεῖξαι. So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to twothirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four

(equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of rightangle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.



Lemma

It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a rightangle, as follows.

For let ABCDE be an equilateral and equiangular pentagon, and let the circle ABCDE have been circumscribed about it [Prop. 4.14]. And let its center, F, have been found [Prop. 3.1]. And let FA, FB, FC, FD, and FE have been joined. Thus, they cut the angles of the pentagon in half at (points) A, B, C, D, and E[Prop. 1.4]. And since the five angles at F are equal (in sum) to four right-angles, and are also equal (to one another), (any) one of them, like AFB, is thus one less a fifth of a right-angle. Thus, the (sum of the) remaining (angles in triangle ABF), FAB and ABF, is one plus a fifth of a right-angle [Prop. 1.32]. And FAB (is) equal to FBC. Thus, the whole angle, ABC, of the pentagon is also one and one-fifth of a right-angle. (Which is) the very thing it was required to show.



Λῆμμα.

Ότι δὲ ἡ τοῦ ἰσοπλεύρου καὶ ἰσογωνίου πενταγώνου γωνία ὀρθή ἐστι καὶ πέμπτου, οὕτω δεικτέον.

Έστω γὰρ πεντάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ABΓΔΕ, καὶ περιγεγράφθω περὶ αὐτὸ κύκλος ὁ ABΓΔΕ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἰ ZA, ZB, ZΓ, ZΔ, ZE. δίχα ἄρα τέμνουσι τὰς πρὸς τοῖς Α, B, Γ, Δ, Ε τοῦ πενταγώνου γωνίας. καὶ ἐπεὶ αἱ πρὸς τῷ Ζ πέντε γωνίαι τέσσαρσιν ὀρθαῖς ἴσαι εἰσὶ καί εἰσιν ἴσαι, μία ἄρα αὐτῶν, ὡς ἡ ὑπὸ AZB, μιᾶς ὀρθῆς ἐστι παρὰ πέμπτου. λοιπαὶ ἄρα αἱ ὑπὸ ZAB, ABZ μιᾶς εἰσιν ὀρθῆς καὶ πέμπτου. ἴση δὲ ἡ ὑπὸ ZAB τῆ ὑπὸ ZBΓ· καὶ ὅλη ἄρα ἡ ὑπὸ ABΓ τοῦ πενταγώνου γωνία μιᾶς ἐστιν ὀρθῆς καὶ πέμπτου· ὅπερ ἔδει δεῖξαι.

GREEK-ENGLISH LEXICON

ABBREVIATIONS: act - active; adj - adjective; adv - adverb; conj - conjunction; fut - future; gen - genitive; imperat - imperative; impf - imperfect; ind - indeclinable; indic - indicative; intr - intransitive; mid - middle; neut - neuter; no - noun; par - particle; part - participle; pass - passive; perf - perfect; pre - preposition; pres - present; pro - pronoun; sg - singular; tr - transitive; vb verb. άγω, άξω, ήγαγον, -ήχα, ήγμαι, ήχθην : vb, lead, draw (a line). gin. $å\delta$ ύνατος -ον : *adj*, impossible. άεί : adv, always, for ever. αίρέω, αἰρήσω, ε[ί]λον, ἤρηκα, ἤρημαι, ἡρέθην : vb, grasp. άιτέω, αἰτήσω, ἤτησα, ἤτηκα, ἤτημαι, ἤτήθη : vb, postulate. αίτημα - ατος, τό : no, postulate. ἀχόλουθος -ον : adj, analogous, consequent on, in conformity with. α̈́κρος -α -ον : *adj*, outermost, end, extreme. άλλά : conj, but, otherwise. άλογος -ον : *adj*, irrational. άμβλυγώνιος - ον : adj, obtuse-angled; τὸ ἀμβλυγώνιον, no, obtuse angle. $d\mu$ βλύς -εῖα -ύ : *adj*, obtuse. άμφότερος -α -ον : pro, both. άναγράφω : vb, describe (a figure); see γράφω. άναλογία, $\dot{\eta}$: no, proportion, (geometric) progression. dv dλoγoς -ov : adj, proportional.ἀν άπαλιν : adv, inverse(ly).αναπληρό ω : vb, fill up. άναστρέφω : vb, turn upside down, convert (ratio); see στρέφω. άναστροφή, $\dot{\eta}$: *no*, turning upside down, conversion (of ratio). άνθυφαιρέω : vb, take away in turn; see αἰρέω. ἀνίστημι : vb, set up; see ἴστημι. ανισος -ον : adj, unequal, uneven. άντιπάσχω : vb, be reciprocally proportional; see πάσχω. άξων - ονος, b : vb, axis. ăπαξ : *adv*, once. απας, απασα, απαν : adj, quite all, the whole. απειρος -ον : adj, infinite. accept. ἀπεναντίον : ind, opposite. ἀπόδειξις -εως, ή : no, proof. άποκαθίστημι : vb, re-establish, restore; see ίστημι. $\dot{\alpha}$ πολαμβάνω : vb, take from, subtract from, cut off from; see λαμβάνω. άποτέμνω : vb, cut off, subtend. άπότμημα -ατος, τό : no, piece cut off, segment. άποτομή, ή : vb, piece cut off, apotome.

άπώτερος -α -ον : adj, further off. άρα : par, thus, as it seems (inferential). ἀριθμός, δ : no, number. άρτιάχις : adv, an even number of times. αρτιόπλευρος - ov : *adj*, having a even number of sides. άρχω, άρξω, ήρξα, ήρχα, ήργμαι, ήρχθην : vb, rule; mid., beἀσύμμετρος -ον : adj, incommensurable. ἀσύμπτωτος -ον : adj, not touching, not meeting. $\overset{\circ}{\alpha}$ ρτιος -α -ον : *adj*, even, perfect. ἄτμητος -ον : adj, uncut. $\dot{\alpha}$ τόπος -ον : *adj*, absurd, paradoxical. αὐτόθεν : adv, immediately, obviously. άφαίρεω : vb, take from, subtract from, cut off from; see αἰρέω. ἁφή, ή : no, point of contact. βάθος -εος, τό : no, depth, height. βαίνω, -βήσομαι, -έβην, βέβηχα, —, — : vb, walk; perf, stand (of angle). βάλλω, βαλῶ, ἔβαλον, βέβληκα, βέβλημαι, ἐβλήθην : vb, throw. βάσις -εως, $\dot{\eta}$: no, base (of a triangle). γάρ : *conj*, for (explanatory). γί[γ]νομαι, γενήσομαι, έγενόμην, γέγονα, γεγένημαι, — : vb, happen, become. γνώμων -ονος, ή : no, gnomon. γραμμή, ή : no, line. γράφω, γράψω, ἕγρα[ψ/φ]α, γέγραφα, γέγραμμαι, ἐραψάμην : vb, draw (a figure). γωνία, $\dot{\eta}$: no, angle. δεĩ : vb, be necessary; δεĩ, it is necessary; ἔδει, it was necassary; δέον, being necessary. δείχνυμι, δείξω, έδειξα, δέδειχα, δέδειγμαι, έδείχθην : vb, show, demonstrate. δ ειχτέον : *ind*, one must show. δεκαγώνος -ον : adj, ten-sided; τὸ δεκαγώνον, no, decagon. δέχομαι, δέξομαι, έδεξάμην, —, δέδεγμαι, έδέχθην : vb, receive, $\delta \eta$: *conj*, so (explanatory). δ ηλαδή : *ind*, quite clear, manifest. δ ῆλος -η -ον : *adj*, clear. δηλονότι : adv, manifestly. διάγω : vb, carry over, draw through, draw across; see åγω. δ ιαγώνιος -ον : *adj*, diagonal. διαλείπω : vb. leave an interval between. διάμετρος -ον : adj, diametrical; ή διάμετρος, no, diameter, diagonal.

διαίρεσις -εως, η : *no*, division, separation.
$\Sigma TOIXEI\Omega N$

διαιρέω : vb , divide (in two); διαρεθέντος -η -ον, <i>adj</i> , separated	ενπεριέχ ω : vb, encompass; see ἕχ ω .
(rano); see $\alpha i \rho \epsilon \omega$.	ἐνπίπτω : see ἐμπίπτω.
$\delta \iota \alpha \sigma \tau \eta \mu \alpha - \alpha \tau \sigma \zeta, \tau \sigma : no, radius.$	ἐντός : <i>pre</i> + <i>gen</i> , inside, interior, within, internal.
οιαφερω : vb, differ; see φερω.	ἑξάγωνος -ον : adj, hexagonal; τὸ ἑξάγωνον, no, hexagon.
οιοωμι, οωσω, εοωχα, οεοωχα, οεοομαι, εσούην : vb, give.	ἑξαπλάσιος -α -ον : adj, sixfold.
διμοίρος -ον : adj, two-thirds.	ἑξ $\tilde{\eta}$ ς : <i>adv</i> , in order, successively, consecutively.
διπλασιάζω : vb, double.	ἕξω ϑ εν : <i>adv</i> , outside, extrinsic.
διπλάσιος -α - ον : adj , double, twofold.	ἐπάνω : <i>adv</i> , above.
διπλασίων -ον : adj, double, twofold.	ἐπαφή, ἡ : no, point of contact.
διπλοῦς -ῆ -οῦν : <i>adj</i> , double.	ἐπεί : conj, since (causal).
δίς : adv, twice.	ἐπειδήπερ : <i>ind</i> , inasmuch as, seeing that.
δίχα : <i>adv</i> , in two, in half.	έπιζεύγνῦμι, ἐπιζεύξω, ἐπέζευξα, —, ἐπέζευγμαι, ἐπέζεύχϑην :
διχορομία, ή : no, point of bisection.	vb, join (by a line).
δυάς -άδος, ή : <i>no</i> , the number two, dyad.	ἐπιλογίζομαι : vb, conclude.
δύναμα: : vb, be able, be capable, generate, square, be when	ἐπινοέω : <i>vb</i> , think of, contrive.
squared; δυναμένη, ή, no, square-root (of area)— <i>i.e.</i> , strai- ght-line whose square is equal to a given area	ἐπιπέδος -ον : adj, level, flat, plane; τὸ ἐπιπέδον, no, plane.
δ ώναμε -εως $\dot{\sigma}$: no power (usually 2nd power when used in	ἐπισκέπτομαι : vb, investigate.
mathematical sence, hence), square.	ἐπίσχεψις -εως, ή : no, inspection, investigation.
δυνατός -ή -όν : adj, possible.	έπιτάσσω : vb, put upon, enjoin; τὸ ἐπιταχθέν, no, the (thing)
δωδεχάεδρος -ον : adj, twelve-sided.	$\dot{z}_{\pi}(z_{0}, z_{0}, z_{0})$: <i>adi</i> one and a third times
ἑαυτοῦ -ῆς -οῦ : adj, of him/her/it/self, his/her/its/own.	à recherce à : no surface
ἐγγίων -ον : adj, nearer, nearest.	žeour v h follow
ἐγγράφω : vb, inscribe; see γράφω.	
είδος -εος, τό : <i>no</i> , figure, form, shape.	ερχομαι, ελευσομαι, ηλυον, εληλυύα, —, — : vo, come, go.
εἰχοσάεδρος -ον : adj, twenty-sided.	ε σ χ α τ ο ζ - η - ο ν : <i>adj</i> , outermost, uttermost, last.
εἴρω/λέγω, ἐρῶ/ερέω, εἴπον, εἴρηχα, εἴρημαι, ἐρρήθην : νb,	έτερόμηχης -ες : <i>adj</i> , oblong; τὸ ἐτερόμηχες, <i>no</i> , rectangle.
say, speak; <i>per pass part</i> , ειρημένος -η -ον, <i>adj</i> , said, afore- mentioned.	
εἴτεεἴτε : ind, either or.	similar part for di rectilinear: The similar part of recti-
ἕκαστος -η -ον :pro, each, every one.	linear figure.
έχατέρος -α -ον : pro, each (of two).	εὐθύς - εῖα - ύ : adj, straight; ἡ εὐθεῖα, no, straight-line; ἐπ'
ἐϰβάλλω : vb, produce (a line); see βάλλω.	εύθεῖας, in a straight-line, straight-on.
ἐκθέω : vb, set out.	εύρίσχω, εύρήσχω, ηὕρον, εὕρεχα, εὕρημαι, εὑρέθην : vb, find.
ἔκκειμαι : νb, be set out, be taken; see κεῖμαι.	ἐφάπτω : vb, bind to; mid, touch; ἡ ἑφαπτομένη, no, tangent;
ἐχτίθημι : vb, set out; see τίθημι.	See άπτω.
ἐκτός : pre + gen, outside, external.	eφαρμοςω, εφαρμοσω, εφηρμοσα, εφημοσμαί, εφημοσμαι στομαί, εφημοσμαί, εφη
ἐλάσσων/ἐλάττων -ον : adj, less, lesser.	έφεξης : adv , in order, adjacent.
ἐλάχιστος -η -ον : adj, least.	έφίστημι : vb. set. stand. place upon: see ιστημι.
ἐλλείπ ω : <i>vb</i> , be less than, fall short of.	$ \tilde{\epsilon}$ χω, $\tilde{\epsilon}$ ζω, έσχον, έσχηχα, -έσχημαι, — : vb , have.
έμπίπτω : vb, meet (of lines), fall on; see πίπτω.	δ_{λ} $\delta_{$
ἕμπροσθεν : <i>adv</i> , in front.	δh : ind already now
ἐναλλάξ : <i>adv</i> , alternate(ly).	$\tilde{\gamma}_{0}$, $\tilde{\gamma}$
έναρμόζω : vb, insert; perf indic pass 3rd sg, ἐνήρμοσται.	$f_{\mu\nu}(x), f_{\mu\nu}(x), \dots, f_{\mu}(x), \dots, f_{\mu\nu}(x)$ in the semi-circle
ἐνδέχομαι : <i>vb</i> , admit, allow.	$\frac{1}{2}$ $\frac{1}$
ἕνεχεν : ind, on account of, for the sake of.	times.
ἐνναπλάσιος -α -ον : adj, nine-fold, nine-times.	ἥμισυς -εια -υ : <i>adj</i> , half.
ἕννοια, ἡ : <i>no</i> , notion.	

ΣΤΟΙΧΕΙΩΝ

ήτοι ή : par, surely, either or; in fact, either or.	λαμβάνω, λήψομαι, ἕλαβον, εἴληφα εἴλημμαι, ἐλήφθην :vb,
ϑ έσις -εως, ή : no, placing, setting, position.	take.
θεωρημα -ατος, τό : no, theorem.	λεγω : vb, say; pres pass part, λεγομενος -η -ον, adj, so-called; see \tilde{e}_{LOG} .
ἴδιος -α -ον : <i>adj</i> , one's own.	λείπω, λείψω, έλιπον, λέλοιπα, λέλειμμαι, έλείωθην : vb. leave.
ίσάχις : <i>adv</i> , the same number of times; ἰσάχις πολλαπλάσια, the same multiples, equal multiples.	leave behind. λημιάτιον, τό : no. diminutive of λήμια.
dσογώνιος -ον : <i>adj</i> , equiangular.	$\lambda \tilde{\eta}_{\mu}$ $\mu \alpha \tau \sigma \sigma$
$d\sigma$ όπλευρος -ον : <i>adj</i> , equilateral.	$\lambda \tilde{n} \mu c_{-s} c_{0} c_{1} \tilde{n} c_{1} c_{0} c_{0} c_{1} c_{0} c$
$d = \frac{1}{2}$ do πληθής -ές : <i>adj</i> , equal in number.	$\lambda \dot{\eta} \gamma \zeta = \delta \zeta \gamma \eta + \kappa \delta \gamma$, taking, exterming.
ίσος -η -ον : adi, equal; έξ ίσου, equally, evenly.	λ_{0} τος, σ · no, rado, proportion, argument.
iσοσχελής -ές : <i>adj</i> , isosceles.	(av)δα (av)
čστημι, στήσω, ἔστησα, —, —, ἐστάθην : vb tr, stand (some- thing).	μέγεθος -εος, τό : <i>no</i> , magnitude, size.
เสาะการ เกิด เลือก เล	μείζων - ov : <i>adj</i> , greater.
up (oneself); Note: perfect <i>I have stood up</i> can be taken	μένω, μενῶ, ἔμεινα, μεμένηκα , —, — : νb, stay, remain.
to mean present I am standing.	μέρος -ους, τό : <i>no</i> , part, direction, side.
ἰσοϋψής -ές : adj, of equal height.	μέσος -η -ον : <i>adj</i> , middle, mean, medial; ἐχ δύο μέσων, bime-
καθάπερ : ind, according as, just as.	ulai.
κάθετος -ον : adj, perpendicular.	ustration : adv between
xαϑόλου : <i>adv</i> , on the whole, in general.	$u_{a} = u_{a} = u_{a}$, between:
xαλέω : vb , call.	ustości : vh measure
κάχεινος = χαὶ ἐχεῖνος .	μ ϵ ρ_{0} μ ϵ r_{0} measure
$x \ddot{\alpha} v = x \alpha \dot{\alpha} v$: <i>ind</i> , even if, and if.	under (under under (under (u
καταγραφή, η : no, diagram, figure.	undérors : adu never
καταγράφω : <i>vb</i> , describe/draw, inscribe (a figure); see γράφω.	$\mu \eta \delta \tau \sigma \sigma c - \alpha - \alpha v$; never.
κατακολουθέω : vb, follow after.	μ $\bar{\mu}$
χαταλείπω : vb, leave behind; see λείπω; τὰ χαταλειπόμενα, no, remainder.	$\mu \eta \chi \delta \zeta = \delta \xi$, to z ho, length, $\mu \eta \chi : par$, truely, indeed.
κατάλληλος -ον : <i>adj</i> , in succession, in corresponding order.	μονάς -άδος, ή : <i>no</i> , unit, unity.
καταμετρέω : vb, measure (exactly).	μοναχός -ή -όν : adj , unique.
καταντάω : <i>vb</i> , come to, arrive at.	μοναχῶς : adv , uniquely.
κατασκευάζω : vb, furnish, construct.	μόνος -η -ον : adj , alone.
κεῖμαι, κεῖσομαι, —, —, —, — : νb, have been placed, lie, be made; see τίθημι.	νοέω, —, νόησα, νενόηκα, νενόημαι, ἐνοήθην : νb, apprehend, conceive.
χέντρον, τό : no, center.	οἴος -α -ον : <i>pre</i> , such as, of what sort.
κλά ω : <i>vb</i> , break off, inflect.	όκτάεδρος -ον : adj, eight-sided.
κλίνω, κλίνω, ἕκλινα, κέκλικα, κέκλιμαι, ἐκλίθην : vb, lean, in- cline.	ὄλος -η -ον : <i>adj</i> , whole.
xλίσις -εως, $\dot{\eta}$: <i>no</i> , inclination, bending.	
χοῖλος -η -ον : adj , hollow, concave.	$o\mu o \iota o \varsigma - \alpha - o v : a a j, similar.$
κορυφή, ή : <i>no</i> , top, summit, apex; κατὰ κορυφήν, vertically opposite (of angles).	δμοιοπληθής -ές : adj, similar in number. δμοιοταγής -ές : adj, similarly arranged.
κρίνω, κρινῶ, ἕκρῖνα, κέκρικα, κέκριμαι, ἐκρίθην : vb, judge.	δμοιότης -ητος, ή <i>: no</i> similarity.
χύβος, ὁ : <i>no</i> , cube.	δμοίως : adv, similarly.
χύχλος, δ : <i>no</i> , circle.	δμόλογος -ον : adj, corresponding, homologous.
χύλινδρος, δ : <i>no</i> , cylinder.	δμοταγής -ές : <i>adj</i> , ranged in the same row or line.
χυρτός -ή -όν : adj, convex.	δμώνυμος -ον : <i>adj</i> , having the same name.
χῶνος, ὁ : <i>no</i> , cone.	ὄνομα -ατος, τό : no, name; ἐκ δύο ὀνομάτων, binomial.

ΣΤΟΙΧΕΙΩΝ

όξυγώνιος -ον : <i>adj</i> , acute-angled; τὸ ὀξυγώνιον, <i>no</i> , acute an- gle.	πάσχω, πείσομαι, ἕπαθον, πέπονθα, —, — : vb, suffer. πεντάγωνος -ον : adj, pentagonal; τὸ πεντάγωνον, no, pen-
όξύς -εῖα -ύ : <i>adj</i> , acute.	tagon.
ὑποιοσοῦν = ὑποῖος -α -ον + οῦν : adj , of whatever kind, any kind whatsoever.	πενταπλάσιος -α -ον : <i>adj</i> , five-fold, five-times. πεντεχαιδεχάγωνον, τό : <i>no</i> . fifteen-sided figure.
ὑπόσος -η -ον : <i>pro</i> , as many, as many as.	πεπερασμένος -η -ογ : <i>adi</i> , finite, limited: see περαίνω.
ὑποσοσδηποτοῦν = ὑπόσος -η -ον + δή + ποτέ + οὖν : adj , of whatever number, any number whatsoever.	περαίνω, περανῶ, ἐπέρανα, —, πεπεράσμαι, ἐπερανάνθην : vb, bring to end, finish, complete; pass, be finite.
ὑποσοσοῦν = ὑπόσος -η -ον + οὕν : <i>adj</i> , of whatever number, any number whatsoever.	πέρας - ατος, τό : <i>no</i> , end, extremity.
δπότερος -α -ον : pro, either (of two), which (of two).	reputow, -, -, -, -, -, - . <i>VD</i> , bring to an end.
όρϑογώνιον, τό : no, rectangle, right-angle.	$\pi \epsilon_{p_1} \gamma_{p_4} \phi_{\phi}$: ν_{p_5} chromosons surround contain comprise: see $\delta_{\nu_{1}}$
ὀρθός -ή -όν : adj, straight, right-angled, perpendicular; πρὸς ὀρθὰς γωνίας, at right-angles.	περιλαμβάνω : vb, enclose; see λαμβάνω.
ὄρος, δ : <i>no</i> , boundary, definition, term (of a ratio).	περισσάχις : adv , an odd number of times.
όσαδηποτοῦν = ὄσα + δή + ποτέ + οὖν : ind, any number	περισσός -ή -όν : <i>adj</i> , odd.
whatsoever.	περιφέρεια, ή : no, circumference.
δσάχις : ind, as many times as, as often as.	περιφέρω : vb, carry round; see φέρω.
όσαπλάσιος -ον : pro, as many times as.	πηλικότης -ητος, ή : <i>no</i> , magnitude, size.
όσος -η -ον : pro, as many as.	πίπτω, πεσοῦμαι, ἔπεσον, πέπτωκα , —, — : <i>vb</i> , fall.
ὄσπερ, ήπερ, ὅπερ : pro, the very man who, the very thing which	πλάτος -εος, τό : <i>no</i> , breadth, width.
όστις, ὅτις, ὅτι : <i>pro</i> , anyone who, anything which.	πλείων - ον : <i>adj</i> , more, several.
$\delta_{T}(x)$: <i>ady</i> when whenever	πλευρά, ἡ : no, side.
δ_{τ_1} δ_{τ_2} δ_{τ_3} δ_{τ_4} δ_{τ_5} δ_{τ	πλῆθος -εος, τό : <i>no</i> , great number, multitude, number.
$o\dot{v}\delta\epsilon(c, o\dot{v}\delta\epsilon u(\alpha, o\dot{v}\delta\epsilon \chi; pro, not one nothing)$	πλήν : adv & prep + gen, more than.
abbits a comparation of the context of the contex	ποιός -ά -όν : <i>adj</i> , of a certain nature, kind, quality, type.
ούθέτερος · see ούδέτερος	πολλαπλασιάζω : vb, multiply.
$a^{i}a^{j}a^{j}$	πολλαπλασιασμός, ὁ : no, multiplication.
a_{i} b_{i} c_{i} a_{i} b_{i} b_{i	πολλαπλάσιον, τό : <i>no</i> , multiple.
o_{true} : adv thusly in this case	πολύεδρος -ον : adj, polyhedral; τό πολύεδρον, no, polyhedron.
$\pi \omega \omega : adv$ back again	πολύγωνος -ον : adj, polygonal; τό πολύγωνον, no, polygon.
TANTO : utv, back, again.	πολύπλευρος -ον : <i>adj</i> , multilateral.
$\pi \alpha \gamma \alpha \gamma \alpha \gamma $	πόρισμα -ατος, τό : no, corollary.
$\pi u \rho u^{2}$. $p \rho \rho^{2} + u c c$, parallel to:	ποτέ : <i>ind</i> , at some time.
παραβαλλώ · νο, apply (a lighter), see βαλλώ.	πρίσμα -ατος, τό : no, prism.
$\pi \alpha \beta \alpha \beta \beta \beta \beta \eta$, $\eta = n \sigma$, application.	προβαίνω : vb, step forward, advance.
$\pi \alpha \rho \alpha \lambda \epsilon \mu \alpha$.	προδείχνυμι : vb, show previously; see δείχνυμι.
παραλλασσώ, παραλλαζώ, —, παρηλλαχα, —, — : νο, miss, ran awry.	προεκτίθημι : vb, set forth beforehand; see τίθημι.
παραλληλεπίπεδος, -ον : <i>adj</i> , with parallel surfaces; τὸ παραλ- ληλεπίπεδον, <i>no</i> , parallelepiped.	προερέω : vb, say beforehand; perf pass part, προειρημένος -η -ον, adj, aforementioned; see εἴρω.
παραλληλόγραμμος -ον : <i>adj</i> , bounded by parallel lines; τὸ πα- ραλληλόγραμμον, <i>no</i> , parallelogram.	προσαναπληρόω : vb, fill up, complete. προσαναγράφω : vb, complete (tracing of); see γράφω.
παράλληλος -ον : <i>adj</i> , parallel; τὸ παράλληλον, <i>no</i> , parallel, parallel-line.	προσαρμόζω : vb, fit to, attach to. προσεχβάλλω : vb, produce (a line): see ἐχβάλλω.
- παραπλήρωμα - ατος, τό : <i>no</i> , complement (of a parallelogram).	προσευρίσχω : vb , find besides, find: see εύρίσχω.
παρατέλυετος -ον : adj, penultimate.	προσλαμβάμω : vb , add.
παρέχ : prep + gen, except.	πρόχειμαι : vb, set before, prescribe: see χεῖμαι.
παρεμπίπτω : vb, insert; see πίπτω.	πρόσχειμαι : vb, be laid on, have been added to: see χεῖμαι.

ΣΤΟΙΧΕΙΩΝ

προσπίπτω : <i>vb</i> , fall on, fall toward, meet; see πίπτω.	τάσσω, τάξω, ἕταξα, τέταχα, τέταγμαι, ἐτάχθην : vb, arrange,
προτασις -εως, ή : no, proposition.	draw up.
προστάσσω : vb, prescribe, enjoin; τὸ τροσταχθέν, no, the thing prescribed; see τάσσω.	τέλειος -α -ον : adj, perfect. τέμνω, τεμνῶ, ἔτεμον, -τέτμηκα, τέτμημαι, ἐτμήθην : vb, cut;
προστίθημι : vb, add; see τίθημι.	pres/jul indic dci 3rd sg, teµet.
πρότερος -α -ον : <i>adj</i> , first (comparative), before, former.	$tetap in \mu opiov, to : no, quadrant.$
προτίθημι : vb , assign; see τίθημι.	terpayout = adv four times
προγωρέω : vb, go/come forward, advance.	$\tau = \tau \alpha \pi \lambda \dot{\alpha} \alpha \sigma \alpha \cdot \alpha \cdot$
πρῶτος -α -ον : adi. first. prime.	τετράπλευρος - ον : adj , quadrilateral.
πυραμίς -(δρς. \dot{n} : <i>no</i> . pyramid.	τετραπλόος - n - ον : adj , fourfold.
$\delta n \tau \delta c \cdot n \cdot \delta v : adi expressible rational$	τίθημι, θήσω, ἔθηχα, τέθηχα, χεῖμαι, ἐτέθην : vb, place, put.
$δ_{0}$ βουβοειδής - ής : <i>adj</i> , expressione, rational.	τμήμα -ατος, τό : no, part cut off, piece, segment.
$\dot{\rho}_{0}$	τοίνυν : par, accordingly.
poppoç, o no, monibus.	τοιοῦτος -αύτη -οῦτο : pro, such as this.
σ_{flat}	τομεύς -έως, δ : no, sector (of circle).
$\sigma \kappa \alpha \kappa \eta \nu \circ \varsigma - \eta - \circ \nu : aaj, scalene.$	τομή, ή : no, cutting, stump, piece.
στερεός -ά -όν : adj , solid; τὸ στερεόν, no, solid, solid body.	τόπος, ὁ : <i>no</i> , place, space.
στοιχεῖον, τό : no, element.	τοσαυτάκις : adv, so many times.
στρέφω, -στρέψω, ἕστρεψα, —, ἐσταμμαι, ἐστάφην :vb, turn.	τοσαυταπλάσιος -α -ον : pro, so many times.
σύγχειμαι : vb , lie together, be the sum of, be composed;	τοσοῦτος -αύτη -οῦτο :pro, so many.
συγκείμενος -η -ον, adj, composed (ratio), compounded;	τουτέστι = τοῦτ' ἔστι : <i>par</i> , that is to say.
$\sigma(x) = x + x + x + x + x + x + x + x + x + x$	τραπέζιον, τό : no, trapezium.
augustic : vb, compare, see April.	τρίγωνος -ον : adj, triangular; τὸ τρίγωνον, no, triangle.
will (1) with threw together most see (1) w	τριπλάσιος -α -ον : <i>adj</i> , triple, threefold.
σ_{0} μ_{0} μ_{0	τριπλευρος -ον : aaj , trilateral.
συμμετρος -ον : <i>aaj</i> , commensurable.	$\tau \rho_{i} \pi \Lambda - o \circ \varsigma - \eta - o \vee : a a j, triple.$
σύμπας -αντος, ο : no, sum, whole.	τροπος, o : <i>no</i> , way.
συμπίπτω : <i>vb</i> , meet together (of lines); see πίπτω.	vb, hit, happen to be at (a place).
συμπληρόω : <i>vb</i> , complete (a figure), fill in.	ὑπάρχω : vb, begin, be, exist; see ἄρχω.
συνάγω : vb, conclude, infer; see ἄγω.	ύπεξαίρεσις -εως, ή : no, removal.
συναμφότεροι -α - α : adj, both together; ὁ συναμφότερος, no, sum (of two things).	ὑπερβάλλω : vb , overshoot, exceed; see βάλλω.
συναποδείκνυμι : no, demonstrate together; see δείκνυμι.	$b_{\text{TEO}}(x_0)$: vb exceed: see δx_0
συναφή, $\dot{\eta}$: no, point of junction.	$\dot{\nu}_{\pi \delta} \eta_{\epsilon \sigma \kappa} - \epsilon \omega_{\epsilon}$, $\dot{n} : n_{\epsilon}$, hypothesis.
σύνδυο, οί, αί, τά : no, two together, in pairs.	bπόχειμα : vb. underlie. be assumed (as hypothesis): see χεῖμα.
συνεχής -ές : adj , continuous; κατὰ τὸ συνεχές, continuously.	ὑπολείπω : vb , leave remaining.
σύνθεσις -εως, \hbar : no, putting together, composition.	ύποτείνω, ύποτενῶ, ὑπέτεινα, ὑποτέταχα, ὑποτέταμαι, ὑπετάθην
σύνθετος -ον : adi , composite.	: <i>vb</i> , subtend.
συ[ν]ίστημ: : vb, construct (a figure), set up together; perf im- perat pass 3rd sg. συχεστάτω: see ίστημι.	ὕψος -εος, τό <i>: no</i> , height. φανερός -ά -όν <i>: adj</i> , visible, manifest.
συντίθημι : <i>vb</i> , put together, add together, compound (ratio);	φημὶ, φήσω, ἔφην, —, —, — : νb, say; ἔφαμεν, we said. φέρω, οἴσω, ἦνεγχον, ἐνήνοχα, ἐνήνεγμαι, ἦνέχθην : νb, carry.
σχέσις -εως, $\dot{\eta}$: <i>no</i> , state, condition.	χώριον, τό : no, place, spot, area, figure.
σχῆμα -ατος, τό : <i>no,</i> figure.	χ ωρις : pre + gen, apart from.
σφαῖρα -ας, ἡ : no, sphere.	$\psi_{\alpha,0}$. <i>vo</i> , touch.
τάξις -εως, ή : <i>no</i> , arrangement, order.	$ω_{\zeta}$ · <i>par</i> , as, inc, for instance. $\dot{ω}_{\zeta}$ ετυγεν : <i>par</i> , at random
ταράσσω, ταράξω, —, —, τετάραγμαι, ἐταράχθην : vb. stir. trou-	$\dot{\omega}$ σαύτως : <i>adv.</i> in the same manner just so.
ble, disturbe; τεταραγμένος -η -ον, <i>adj</i> , disturbed, per- turbed.	ὥστε : <i>conj</i> , so that (causal), hence.