# EUCLID'S ELEMENTS OF GEOMETRY 

The Greek text of J.L. Heiberg (1883-1885)<br>from Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883-1885<br>edited, and provided with a modern English translation, by Richard Fitzpatrick

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## Introduction

Euclid's Elements is by far the most famous mathematical work of classical antiquity, and also has the distinction of being the world's oldest continuously used mathematical textbook. Little is known about the author, beyond the fact that he lived in Alexandria around 300 BCE. The main subjects of the work are geometry, proportion, and number theory.

Most of the theorems appearing in the Elements were not discovered by Euclid himself, but were the work of earlier Greek mathematicians such as Pythagoras (and his school), Hippocrates of Chios, Theaetetus of Athens, and Eudoxus of Cnidos. However, Euclid is generally credited with arranging these theorems in a logical manner, so as to demonstrate (admittedly, not always with the rigour demanded by modern mathematics) that they necessarily follow from five simple axioms. Euclid is also credited with devising a number of particularly ingenious proofs of previously discovered theorems: e.g., Theorem 48 in Book 1.

The geometrical constructions employed in the Elements are restricted to those which can be achieved using a straight-rule and a compass. Furthermore, empirical proofs by means of measurement are strictly forbidden: i.e., any comparison of two magnitudes is restricted to saying that the magnitudes are either equal, or that one is greater than the other.

The Elements consists of thirteen books. Book 1 outlines the fundamental propositions of plane geometry, including the three cases in which triangles are congruent, various theorems involving parallel lines, the theorem regarding the sum of the angles in a triangle, and the Pythagorean theorem. Book 2 is commonly said to deal with "geometric algebra", since most of the theorems contained within it have simple algebraic interpretations. Book 3 investigates circles and their properties, and includes theorems on tangents and inscribed angles. Book 4 is concerned with regular polygons inscribed in, and circumscribed around, circles. Book 5 develops the arithmetic theory of proportion. Book 6 applies the theory of proportion to plane geometry, and contains theorems on similar figures. Book 7 deals with elementary number theory: e.g., prime numbers, greatest common denominators, etc. Book 8 is concerned with geometric series. Book 9 contains various applications of results in the previous two books, and includes theorems on the infinitude of prime numbers, as well as the sum of a geometric series. Book 10 attempts to classify incommensurable (i.e., irrational) magnitudes using the so-called "method of exhaustion", an ancient precursor to integration. Book 11 deals with the fundamental propositions of three-dimensional geometry. Book 12 calculates the relative volumes of cones, pyramids, cylinders, and spheres using the method of exhaustion. Finally, Book 13 investigates the five so-called Platonic solids.

This edition of Euclid's Elements presents the definitive Greek text-i.e., that edited by J.L. Heiberg (1883-1885)-accompanied by a modern English translation, as well as a Greek-English lexicon. Neither the spurious books 14 and 15, nor the extensive scholia which have been added to the Elements over the centuries, are included. The aim of the translation is to make the mathematical argument as clear and unambiguous as possible, whilst still adhering closely to the meaning of the original Greek. Text within square parenthesis (in both Greek and English) indicates material identified by Heiberg as being later interpolations to the original text (some particularly obvious or unhelpful interpolations have been omitted altogether). Text within round parenthesis (in English) indicates material which is implied, but not actually present, in the Greek text.

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## ELEMENTS BOOK 1

## Fundamentals of Plane Geometry Involving Straight-Lines

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## Definitions

1．A point is that of which there is no part．
2．And a line is a length without breadth．
3．And the extremities of a line are points．
4．A straight－line is（any）one which lies evenly with points on itself．

5．And a surface is that which has length and breadth only．

6．And the extremities of a surface are lines．
7．A plane surface is（any）one which lies evenly with the straight－lines on itself．

8．And a plane angle is the inclination of the lines to one another，when two lines in a plane meet one another， and are not lying in a straight－line．

9．And when the lines containing the angle are straight then the angle is called rectilinear．

10．And when a straight－line stood upon（another） straight－line makes adjacent angles（which are）equal to one another，each of the equal angles is a right－angle，and the former straight－line is called a perpendicular to that upon which it stands．

11．An obtuse angle is one greater than a right－angle．
12．And an acute angle（is）one less than a right－angle．
13．A boundary is that which is the extremity of some－ thing．

14．A figure is that which is contained by some bound－ ary or boundaries．

15．A circle is a plane figure contained by a single line ［which is called a circumference］，（such that）all of the straight－lines radiating towards［the circumference］from one point amongst those lying inside the figure are equal to one another．

16．And the point is called the center of the circle．
17．And a diameter of the circle is any straight－line， being drawn through the center，and terminated in each direction by the circumference of the circle．（And）any such（straight－line）also cuts the circle in half．${ }^{\dagger}$

18．And a semi－circle is the figure contained by the diameter and the circumference cuts off by it．And the center of the semi－circle is the same（point）as（the center of）the circle．

19．Rectilinear figures are those（figures）contained by straight－lines：trilateral figures being those contained by three straight－lines，quadrilateral by four，and multi－ lateral by more than four．

20．And of the trilateral figures：an equilateral trian－ gle is that having three equal sides，an isosceles（triangle） that having only two equal sides，and a scalene（triangle） that having three unequal sides．
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21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acuteangled (triangle) that having three acute angles.
22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.
23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).
${ }^{\dagger}$ This should really be counted as a postulate, rather than as part of a definition.

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## Postulates

1. Let it have been postulated ${ }^{\dagger}$ to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side). $\ddagger$
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## Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal. ${ }^{\dagger}$
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

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## Proposition 1

To construct an equilateral triangle on a given finite straight-line.


Let $A B$ be the given finite straight-line.
So it is required to construct an equilateral triangle on the straight-line $A B$.

Let the circle $B C D$ with center $A$ and radius $A B$ have been drawn [Post. 3], and again let the circle $A C E$ with center $B$ and radius $B A$ have been drawn [Post. 3]. And let the straight-lines $C A$ and $C B$ have been joined from the point $C$, where the circles cut one another, ${ }^{\dagger}$ to the points $A$ and $B$ (respectively) [Post. 1].

And since the point $A$ is the center of the circle $C D B$, $A C$ is equal to $A B$ [Def. 1.15]. Again, since the point $B$ is the center of the circle $C A E, B C$ is equal to $B A$ [Def. 1.15]. But $C A$ was also shown (to be) equal to $A B$. Thus, $C A$ and $C B$ are each equal to $A B$. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, $C A$ is also equal to $C B$. Thus, the three (straightlines) $C A, A B$, and $B C$ are equal to one another.

Thus, the triangle $A B C$ is equilateral, and has been constructed on the given finite straight-line $A B$. (Which is) the very thing it was required to do.
${ }^{\dagger}$ The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

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## Proposition $2^{\dagger}$

To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let $A$ be the given point, and $B C$ the given straightline. So it is required to place a straight-line at point $A$ equal to the given straight-line $B C$.

For let the straight-line $A B$ have been joined from point $A$ to point $B$ [Post. 1], and let the equilateral triangle $D A B$ have been been constructed upon it [Prop. 1.1].














And let the straight-lines $A E$ and $B F$ have been produced in a straight-line with $D A$ and $D B$ (respectively) [Post. 2]. And let the circle $C G H$ with center $B$ and radius $B C$ have been drawn [Post. 3], and again let the circle $G K L$ with center $D$ and radius $D G$ have been drawn [Post. 3].


Therefore, since the point $B$ is the center of (the circle) $C G H, B C$ is equal to $B G$ [Def. 1.15]. Again, since the point $D$ is the center of the circle $G K L, D L$ is equal to $D G$ [Def. 1.15]. And within these, $D A$ is equal to $D B$. Thus, the remainder $A L$ is equal to the remainder $B G$ [C.N. 3]. But $B C$ was also shown (to be) equal to $B G$. Thus, $A L$ and $B C$ are each equal to $B G$. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, $A L$ is also equal to $B C$.

Thus, the straight-line $A L$, equal to the given straightline $B C$, has been placed at the given point $A$. (Which is) the very thing it was required to do.
${ }^{\dagger}$ This proposition admits of a number of different cases, depending on the relative positions of the point $A$ and the line $B C$. In such situations, Euclid invariably only considers one particular case-usually, the most difficult-and leaves the remaining cases as exercises for the reader.

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## Proposition 3

For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

Let $A B$ and $C$ be the two given unequal straight-lines, of which let the greater be $A B$. So it is required to cut off a straight-line equal to the lesser $C$ from the greater $A B$.

Let the line $A D$, equal to the straight-line $C$, have been placed at point $A$ [Prop. 1.2]. And let the circle $D E F$ have been drawn with center $A$ and radius $A D$ [Post. 3].




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And since point $A$ is the center of circle $D E F, A E$ is equal to $A D$ [Def. 1.15]. But, $C$ is also equal to $A D$. Thus, $A E$ and $C$ are each equal to $A D$. So $A E$ is also equal to $C$ [C.N. 1].


Thus, for two given unequal straight-lines, $A B$ and $C$, the (straight-line) $A E$, equal to the lesser $C$, has been cut off from the greater $A B$. (Which is) the very thing it was required to do.

## Proposition 4

If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.


Let $A B C$ and $D E F$ be two triangles having the two sides $A B$ and $A C$ equal to the two sides $D E$ and $D F$, respectively. (That is) $A B$ to $D E$, and $A C$ to $D F$. And (let) the angle $B A C$ (be) equal to the angle $E D F$. I say that the base $B C$ is also equal to the base $E F$, and triangle $A B C$ will be equal to triangle $D E F$, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) $A B C$ to $D E F$, and $A C B$ to $D F E$.

For if triangle $A B C$ is applied to triangle $D E F,{ }^{\dagger}$ the point $A$ being placed on the point $D$, and the straight-line













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$A B$ on $D E$, then the point $B$ will also coincide with $E$, on account of $A B$ being equal to $D E$. So (because of) $A B$ coinciding with $D E$, the straight-line $A C$ will also coincide with $D F$, on account of the angle $B A C$ being equal to $E D F$. So the point $C$ will also coincide with the point $F$, again on account of $A C$ being equal to $D F$. But, point $B$ certainly also coincided with point $E$, so that the base $B C$ will coincide with the base $E F$. For if $B$ coincides with $E$, and $C$ with $F$, and the base $B C$ does not coincide with $E F$, then two straight-lines will encompass an area. The very thing is impossible [Post. 1]. $\ddagger$ Thus, the base $B C$ will coincide with $E F$, and will be equal to it [C.N. 4]. So the whole triangle $A B C$ will coincide with the whole triangle $D E F$, and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. (That is) $A B C$ to $D E F$, and $A C B$ to $D F E$ [C.N. 4].

Thus, if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-line equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (Which is) the very thing it was required to show.
$\dagger$ The application of one figure to another should be counted as an additional postulate.
$\ddagger$ Since Post. 1 implicitly assumes that the straight-line joining two given points is unique.

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## Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.


Let $A B C$ be an isosceles triangle having the side $A B$ equal to the side $A C$, and let the straight-lines $B D$ and $C E$ have been produced in a straight-line with $A B$ and $A C$ (respectively) [Post. 2]. I say that the angle $A B C$ is equal to $A C B$, and (angle) $C B D$ to $B C E$.

For let the point $F$ have been taken at random on $B D$, and let $A G$ have been cut off from the greater $A E$, equal
















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to the lesser $A F$ [Prop. 1.3]. Also, let the straight-lines $F C$ and $G B$ have been joined [Post. 1].

In fact, since $A F$ is equal to $A G$, and $A B$ to $A C$, the two (straight-lines) $F A, A C$ are equal to the two (straight-lines) $G A, A B$, respectively. They also encompass a common angle, $F A G$. Thus, the base $F C$ is equal to the base $G B$, and the triangle $A F C$ will be equal to the triangle $A G B$, and the remaining angles subtendend by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) $A C F$ to $A B G$, and $A F C$ to $A G B$. And since the whole of $A F$ is equal to the whole of $A G$, within which $A B$ is equal to $A C$, the remainder $B F$ is thus equal to the remainder $C G$ [C.N. 3]. But $F C$ was also shown (to be) equal to $G B$. So the two (straightlines) $B F, F C$ are equal to the two (straight-lines) $C G$, $G B$, respectively, and the angle $B F C$ (is) equal to the angle $C G B$, and the base $B C$ is common to them. Thus, the triangle $B F C$ will be equal to the triangle $C G B$, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, $F B C$ is equal to $G C B$, and $B C F$ to $C B G$. Therefore, since the whole angle $A B G$ was shown (to be) equal to the whole angle $A C F$, within which $C B G$ is equal to $B C F$, the remainder $A B C$ is thus equal to the remainder $A C B$ [C.N. 3]. And they are at the base of triangle $A B C$. And $F B C$ was also shown (to be) equal to $G C B$. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

## Proposition 6

If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.


Let $A B C$ be a triangle having the angle $A B C$ equal to the angle $A C B$. I say that side $A B$ is also equal to side $A C$.













For if $A B$ is unequal to $A C$ then one of them is greater. Let $A B$ be greater. And let $D B$, equal to the lesser $A C$, have been cut off from the greater $A B$ [Prop. 1.3]. And let $D C$ have been joined [Post. 1].

Therefore, since $D B$ is equal to $A C$, and $B C$ (is) common, the two sides $D B, B C$ are equal to the two sides $A C, C B$, respectively, and the angle $D B C$ is equal to the angle $A C B$. Thus, the base $D C$ is equal to the base $A B$, and the triangle $D B C$ will be equal to the triangle $A C B$ [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, $A B$ is not unequal to $A C$. Thus, (it is) equal. ${ }^{\dagger}$

Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show.


#### Abstract

${ }^{\dagger}$ Here, use is made of the previously unmentioned common notion that if two quantities are not unequal then they must be equal. Later on, use is made of the closely related common notion that if two quantities are not greater than or less than one another, respectively, then they must be equal to one another.


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## Proposition 7

On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines.


For, if possible, let the two straight-lines $A C, C B$, equal to two other straight-lines $A D, D B$, respectively, have been constructed on the same straight-line $A B$, meeting at different points, $C$ and $D$, on the same side (of $A B$ ), and having the same ends (on $A B$ ). So $C A$ is equal to $D A$, having the same end $A$ as it, and $C B$ is equal to $D B$, having the same end $B$ as it. And let $C D$ have been joined [Post. 1].

Therefore, since $A C$ is equal to $A D$, the angle $A C D$ is also equal to angle $A D C$ [Prop. 1.5]. Thus, $A D C$ (is) greater than $D C B$ [C.N. 5]. Thus, $C D B$ is much greater than $D C B$ [C.N. 5]. Again, since $C B$ is equal to $D B$, the angle $C D B$ is also equal to angle $D C B$ [Prop. 1.5]. But it was shown that the former (angle) is also much greater




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(than the latter). The very thing is impossible.
Thus, on the same straight-line, two other straightlines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

## Proposition 8

If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines.


Let $A B C$ and $D E F$ be two triangles having the two sides $A B$ and $A C$ equal to the two sides $D E$ and $D F$, respectively. (That is) $A B$ to $D E$, and $A C$ to $D F$. Let them also have the base $B C$ equal to the base $E F$. I say that the angle $B A C$ is also equal to the angle $E D F$.

For if triangle $A B C$ is applied to triangle $D E F$, the point $B$ being placed on point $E$, and the straight-line $B C$ on $E F$, then point $C$ will also coincide with $F$, on account of $B C$ being equal to $E F$. So (because of) $B C$ coinciding with $E F$, (the sides) $B A$ and $C A$ will also coincide with $E D$ and $D F$ (respectively). For if base $B C$ coincides with base $E F$, but the sides $A B$ and $A C$ do not coincide with $E D$ and $D F$ (respectively), but miss like $E G$ and $G F$ (in the above figure), then we will have constructed upon the same straight-line, two other straightlines equal, respectively, to two (given) straight-lines, and (meeting) at a different point on the same side (of the straight-line), but having the same ends. But (such straight-lines) cannot be constructed [Prop. 1.7]. Thus, the base $B C$ being applied to the base $E F$, the sides $B A$ and $A C$ cannot not coincide with $E D$ and $D F$ (respectively). Thus, they will coincide. So the angle $B A C$ will also coincide with angle $E D F$, and will be equal to it [C.N. 4].

Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base,

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then they will also have equal the angles encompassed by the equal straight-lines. (Which is) the very thing it was required to show.

## Proposition 9

To cut a given rectilinear angle in half.


Let $B A C$ be the given rectilinear angle. So it is required to cut it in half.

Let the point $D$ have been taken at random on $A B$, and let $A E$, equal to $A D$, have been cut off from $A C$ [Prop. 1.3], and let $D E$ have been joined. And let the equilateral triangle $D E F$ have been constructed upon $D E$ [Prop. 1.1], and let $A F$ have been joined. I say that the angle $B A C$ has been cut in half by the straight-line $A F$.

For since $A D$ is equal to $A E$, and $A F$ is common, the two (straight-lines) $D A, A F$ are equal to the two (straight-lines) $E A, A F$, respectively. And the base $D F$ is equal to the base $E F$. Thus, angle $D A F$ is equal to angle $E A F$ [Prop. 1.8].

Thus, the given rectilinear angle $B A C$ has been cut in half by the straight-line $A F$. (Which is) the very thing it was required to do.

## Proposition 10

To cut a given finite straight-line in half.
Let $A B$ be the given finite straight-line. So it is required to cut the finite straight-line $A B$ in half.

Let the equilateral triangle $A B C$ have been constructed upon ( $A B$ ) [Prop. 1.1], and let the angle $A C B$ have been cut in half by the straight-line $C D$ [Prop. 1.9]. I say that the straight-line $A B$ has been cut in half at point $D$.

For since $A C$ is equal to $C B$, and $C D$ (is) common,


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$i \alpha^{\prime}$.


















the two (straight-lines) $A C, C D$ are equal to the two (straight-lines) $B C, C D$, respectively. And the angle $A C D$ is equal to the angle $B C D$. Thus, the base $A D$ is equal to the base $B D$ [Prop. 1.4].


Thus, the given finite straight-line $A B$ has been cut in half at (point) $D$. (Which is) the very thing it was required to do.

## Proposition 11

To draw a straight-line at right-angles to a given straight-line from a given point on it.


Let $A B$ be the given straight-line, and $C$ the given point on it. So it is required to draw a straight-line from the point $C$ at right-angles to the straight-line $A B$.

Let the point $D$ be have been taken at random on $A C$, and let $C E$ be made equal to $C D$ [Prop. 1.3], and let the equilateral triangle $F D E$ have been constructed on $D E$ [Prop. 1.1], and let $F C$ have been joined. I say that the straight-line $F C$ has been drawn at right-angles to the given straight-line $A B$ from the given point $C$ on it.

For since $D C$ is equal to $C E$, and $C F$ is common, the two (straight-lines) $D C, C F$ are equal to the two (straight-lines), $E C, C F$, respectively. And the base $D F$ is equal to the base $F E$. Thus, the angle $D C F$ is equal to the angle $E C F$ [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line




## ${ }^{\prime} \beta^{\prime}$.


 ад $\gamma \alpha \gamma \varepsilon i ̃$.










 $\eta_{\eta} \varkappa$ बа। $\dot{\eta} \Gamma \Theta$.
'Enei $\gamma \dot{\alpha} \mathrm{p}$ 亿̋бn









makes the adjacent angles equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, each of the (angles) $D C F$ and $F C E$ is a right-angle.

Thus, the straight-line $C F$ has been drawn at rightangles to the given straight-line $A B$ from the given point $C$ on it. (Which is) the very thing it was required to do.

## Proposition 12

To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.


Let $A B$ be the given infinite straight-line and $C$ the given point, which is not on $(A B)$. So it is required to draw a straight-line perpendicular to the given infinite straight-line $A B$ from the given point $C$, which is not on (AB).

For let point $D$ have been taken at random on the other side (to $C$ ) of the straight-line $A B$, and let the circle $E F G$ have been drawn with center $C$ and radius $C D$ [Post. 3], and let the straight-line $E G$ have been cut in half at (point) $H$ [Prop. 1.10], and let the straightlines $C G, C H$, and $C E$ have been joined. I say that the (straight-line) $C H$ has been drawn perpendicular to the given infinite straight-line $A B$ from the given point $C$, which is not on $(A B)$.

For since $G H$ is equal to $H E$, and $H C$ (is) common, the two (straight-lines) $G H, H C$ are equal to the two (straight-lines) $E H, H C$, respectively, and the base $C G$ is equal to the base $C E$. Thus, the angle $C H G$ is equal to the angle $E H C$ [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straightline is called a perpendicular to that upon which it stands [Def. 1.10].

Thus, the (straight-line) CH has been drawn perpendicular to the given infinite straight-line $A B$ from the

$$
' \gamma^{\prime} \text {. }
$$




 $\gamma \omega v i ́ \alpha \varsigma ~ \pi o เ \varepsilon i ́ t \omega ~ \tau \alpha ̀ \varsigma ~ i ́ \pi o ̀ ~ Г В А, ~ А В ~ A \cdot ~ \lambda e ̀ \gamma \omega, ~ o ̈ \tau \iota ~ \alpha i ~ u ́ \pi o ̀ ~ Г В А, ~$ $\mathrm{AB} \Delta$ rcuvíal ${ }^{2}$ tol















given point $C$, which is not on $(A B)$. (Which is) the very thing it was required to do.

## Proposition 13

If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two rightangles, or (angles whose sum is) equal to two rightangles.


For let some straight-line $A B$ stood on the straightline $C D$ make the angles $C B A$ and $A B D$. I say that the angles $C B A$ and $A B D$ are certainly either two rightangles, or (have a sum) equal to two right-angles.

In fact, if $C B A$ is equal to $A B D$ then they are two right-angles [Def. 1.10]. But, if not, let $B E$ have been drawn from the point $B$ at right-angles to [the straightline] $C D$ [Prop. 1.11]. Thus, $C B E$ and $E B D$ are two right-angles. And since $C B E$ is equal to the two (angles) $C B A$ and $A B E$, let $E B D$ have been added to both. Thus, the (sum of the angles) $C B E$ and $E B D$ is equal to the (sum of the) three (angles) $C B A, A B E$, and $E B D$ [C.N. 2]. Again, since $D B A$ is equal to the two (angles) $D B E$ and $E B A$, let $A B C$ have been added to both. Thus, the (sum of the angles) $D B A$ and $A B C$ is equal to the (sum of the) three (angles) $D B E, E B A$, and $A B C$ [C.N. 2]. But (the sum of) $C B E$ and $E B D$ was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) $C B E$ and $E B D$ is also equal to (the sum of) $D B A$ and $A B C$. But, (the sum of) $C B E$ and $E B D$ is two right-angles. Thus, (the sum of) $A B D$ and $A B C$ is also equal to two right-angles.

Thus, if a straight-line stood on a(nother) straightline makes angles, it will certainly either make two rightangles, or (angles whose sum is) equal to two rightangles. (Which is) the very thing it was required to show.

## $1 \delta^{\prime}$.



 દย่งยโั๙ા.





 غ̀ $\pi^{\prime}$ عưvzías $\dot{\eta}$ BE.













## เモ'.




## Proposition 14

If two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another.


For let two straight-lines $B C$ and $B D$, not lying on the same side, make adjacent angles $A B C$ and $A B D$ (whose sum is) equal to two right-angles with some straight-line $A B$, at the point $B$ on it. I say that $B D$ is straight-on with respect to $C B$.

For if $B D$ is not straight-on to $B C$ then let $B E$ be straight-on to $C B$.

Therefore, since the straight-line $A B$ stands on the straight-line $C B E$, the (sum of the) angles $A B C$ and $A B E$ is thus equal to two right-angles [Prop. 1.13]. But (the sum of) $A B C$ and $A B D$ is also equal to two rightangles. Thus, (the sum of angles) $C B A$ and $A B E$ is equal to (the sum of angles) $C B A$ and $A B D$ [C.N. 1]. Let (angle) $C B A$ have been subtracted from both. Thus, the remainder $A B E$ is equal to the remainder $A B D$ [C.N. 3], the lesser to the greater. The very thing is impossible. Thus, $B E$ is not straight-on with respect to $C B$. Similarly, we can show that neither (is) any other (straightline) than $B D$. Thus, $C B$ is straight-on with respect to $B D$.

Thus, if two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two rightangles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another. (Which is) the very thing it was required to show.

## Proposition 15

If two straight-lines cut one another then they make the vertically opposite angles equal to one another.

 únò $\Delta \mathrm{EB}, \dot{\eta}$ đ̀̀ ú ù̀ $\Gamma \mathrm{EB}$ rñ újò $\mathrm{AE} \Delta$.

 $\gamma \omega v i ́ \alpha s ~ \pi o เ o u ̃ \sigma \alpha ~ \tau \grave{\alpha} \varsigma ~ i ́ \pi o ̀ ~ \Gamma E A, ~ A E \Delta, ~ \alpha i ~ \alpha ̋ p \alpha ~ ن ́ \pi o ̀ ~ Г Е А, ~ А Е \Delta ~$











## $\varepsilon^{\prime}$.


 $\mu \varepsilon i \zeta \omega \nu$ ह̇бтiv.


 ن́лò ГВА, ВАГ ү $\omega \nu \iota \widetilde{\omega} \nu$.









For let the two straight-lines $A B$ and $C D$ cut one another at the point $E$. I say that angle $A E C$ is equal to (angle) $D E B$, and (angle) $C E B$ to (angle) $A E D$.


For since the straight-line $A E$ stands on the straightline $C D$, making the angles $C E A$ and $A E D$, the (sum of the) angles $C E A$ and $A E D$ is thus equal to two rightangles [Prop. 1.13]. Again, since the straight-line $D E$ stands on the straight-line $A B$, making the angles $A E D$ and $D E B$, the (sum of the) angles $A E D$ and $D E B$ is thus equal to two right-angles [Prop. 1.13]. But (the sum of) $C E A$ and $A E D$ was also shown (to be) equal to two right-angles. Thus, (the sum of) $C E A$ and $A E D$ is equal to (the sum of) $A E D$ and $D E B$ [C.N. 1]. Let $A E D$ have been subtracted from both. Thus, the remainder $C E A$ is equal to the remainder $B E D$ [C.N. 3]. Similarly, it can be shown that $C E B$ and $D E A$ are also equal.

Thus, if two straight-lines cut one another then they make the vertically opposite angles equal to one another. (Which is) the very thing it was required to show.

## Proposition 16

For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let $A B C$ be a triangle, and let one of its sides $B C$ have been produced to $D$. I say that the external angle $A C D$ is greater than each of the internal and opposite angles, $C B A$ and $B A C$.

Let the (straight-line) $A C$ have been cut in half at (point) $E$ [Prop. 1.10]. And $B E$ being joined, let it have been produced in a straight-line to (point) $F .^{\dagger}$ And let $E F$ be made equal to $B E$ [Prop. 1.3], and let $F C$ have been joined, and let $A C$ have been drawn through to (point) $G$.

Therefore, since $A E$ is equal to $E C$, and $B E$ to $E F$, the two (straight-lines) $A E, E B$ are equal to the two










(straight-lines) $C E, E F$, respectively. Also, angle $A E B$ is equal to angle $F E C$, for (they are) vertically opposite [Prop. 1.15]. Thus, the base $A B$ is equal to the base $F C$, and the triangle $A B E$ is equal to the triangle $F E C$, and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 1.4]. Thus, $B A E$ is equal to $E C F$. But $E C D$ is greater than $E C F$. Thus, $A C D$ is greater than $B A E$. Similarly, by having cut $B C$ in half, it can be shown (that) $B C G$-that is to say, $A C D$-(is) also greater than $A B C$.


Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. (Which is) the very thing it was required to show.
${ }^{\dagger}$ The implicit assumption that the point $F$ lies in the interior of the angle $A B C$ should be counted as an additional postulate.
し'.
 $\varepsilon i \sigma \iota ~ \pi \alpha ́ \nu \tau \tilde{n} \mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu o ́ \mu \varepsilon v \alpha \iota$.


 $\beta \alpha \nu o ́ \mu \varepsilon v \alpha l$.

## Proposition 17

For any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles.


Let $A B C$ be a triangle. I say that (the sum of) two angles of triangle $A B C$ taken together in any (possible way) is less than two right-angles.











$$
i n^{\prime} \text {. }
$$

 ínoteível.




 $\dot{\eta} \mathrm{A} \Delta$, x $\alpha \grave{\imath}$ ह̇ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \dot{\eta} \mathrm{~B} \Delta$.








## เv'.

 $\pi \lambda \varepsilon \cup p \alpha ̀$ Úлотєíveเ.
 $\gamma \omega v i ́ \alpha \nu \tau \eta ̃ s ~ i ́ \pi o ̀ ~ В Г А \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~ o ̈ \tau \iota ~ x \alpha i ~ \pi \lambda \varepsilon u p \alpha ̀ ~ \dot{\eta} А \Gamma \pi \lambda \varepsilon u p a ̃ \varsigma$ $\tau \tilde{\eta} s \mathrm{AB} \mu \varepsilon$ í $\omega \nu$ ह̇のтiv.

For let $B C$ have been produced to $D$.
And since the angle $A C D$ is external to triangle $A B C$, it is greater than the internal and opposite angle $A B C$ [Prop. 1.16]. Let $A C B$ have been added to both. Thus, the (sum of the angles) $A C D$ and $A C B$ is greater than the (sum of the angles) $A B C$ and $B C A$. But, (the sum of) $A C D$ and $A C B$ is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) $A B C$ and $B C A$ is less than two rightangles. Similarly, we can show that (the sum of) $B A C$ and $A C B$ is also less than two right-angles, and further (that the sum of) $C A B$ and $A B C$ (is less than two rightangles).

Thus, for any triangle, (the sum of) two angles taken together in any (possible way) is less than two rightangles. (Which is) the very thing it was required to show.

## Proposition 18

In any triangle, the greater side subtends the greater angle.


For let $A B C$ be a triangle having side $A C$ greater than $A B$. I say that angle $A B C$ is also greater than $B C A$.

For since $A C$ is greater than $A B$, let $A D$ be made equal to $A B$ [Prop. 1.3], and let $B D$ have been joined.

And since angle $A D B$ is external to triangle $B C D$, it is greater than the internal and opposite (angle) $D C B$ [Prop. 1.16]. But $A D B$ (is) equal to $A B D$, since side $A B$ is also equal to side $A D$ [Prop. 1.5]. Thus, $A B D$ is also greater than $A C B$. Thus, $A B C$ is much greater than $A C B$.

Thus, in any triangle, the greater side subtends the greater angle. (Which is) the very thing it was required to show.

## Proposition 19

In any triangle, the greater angle is subtended by the greater side.

Let $A B C$ be a triangle having the angle $A B C$ greater than $B C A$. I say that side $A C$ is also greater than side $A B$.











## $x^{\prime}$.

 عiซ॰ $\pi \alpha ́ \nu \tau \eta \mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu o ́ \mu \varepsilon \nu \alpha \iota$.



 $\tau \tilde{\eta} \varsigma \mathrm{A} \Gamma$, $\alpha i \quad \delta \grave{̀} \mathrm{~B} \mathrm{\Gamma}, \Gamma \mathrm{~A} \tau \tilde{\eta} \varsigma \mathrm{AB}$.

For if not, $A C$ is certainly either equal to, or less than, $A B$. In fact, $A C$ is not equal to $A B$. For then angle $A B C$ would also have been equal to $A C B$ [Prop. 1.5]. But it is not. Thus, $A C$ is not equal to $A B$. Neither, indeed, is $A C$ less than $A B$. For then angle $A B C$ would also have been less than $A C B$ [Prop. 1.18]. But it is not. Thus, $A C$ is not less than $A B$. But it was shown that $(A C)$ is not equal (to $A B$ ) either. Thus, $A C$ is greater than $A B$.


Thus, in any triangle, the greater angle is subtended by the greater side. (Which is) the very thing it was required to show.

## Proposition 20

In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side).


For let $A B C$ be a triangle. I say that in triangle $A B C$ (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (So), (the sum of) $B A$ and $A C$ (is greater) than $B C$, (the sum of) $A B$
 そ̋ŋ $\dot{\eta} \mathrm{A} \Delta$, x $\alpha$ è $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \dot{\eta} \Delta \Gamma$.






 $\alpha i ~ \delta e ̀ ~ В \Gamma, ~ Г А ~ \tau \tilde{ŋ} ऽ ~ А В . ~$


$\chi \alpha^{\prime}$.


 ठغ̀ $\gamma \omega v i ́ \alpha \nu \pi \varepsilon \rho เ \varepsilon ́ \xi o u \sigma レ$.


 $\alpha i \mathrm{~B} \Delta, \Delta \Gamma \cdot \lambda \varepsilon ́ \gamma \omega$, ötı $\alpha i \mathrm{~B} \Delta, \Delta \Gamma \tau \widetilde{\omega} \nu \lambda o \iota \pi \widetilde{\omega} \nu$ тои̃ тpıүต́vou


 $\alpha i$ ठúo $\pi \lambda \varepsilon \cup p \alpha i ̀ ~ \tau \tilde{\eta} \varsigma \lambda o เ \pi n ̃ s ~ \mu \varepsilon i \zeta o v e ́ s ~ \varepsilon i \sigma เ \nu, ~ \tau o u ̃ ~ A B E ~ \alpha " p \alpha ~$











and $B C$ than $A C$, and (the sum of) $B C$ and $C A$ than $A B$.

For let $B A$ have been drawn through to point $D$, and let $A D$ be made equal to $C A$ [Prop. 1.3], and let $D C$ have been joined.

Therefore, since $D A$ is equal to $A C$, the angle $A D C$ is also equal to $A C D$ [Prop. 1.5]. Thus, $B C D$ is greater than $A D C$. And since $D C B$ is a triangle having the angle $B C D$ greater than $B D C$, and the greater angle subtends the greater side [Prop. 1.19], $D B$ is thus greater than $B C$. But $D A$ is equal to $A C$. Thus, (the sum of) $B A$ and $A C$ is greater than $B C$. Similarly, we can show that (the sum of) $A B$ and $B C$ is also greater than $C A$, and (the sum of) $B C$ and $C A$ than $A B$.

Thus, in any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (Which is) the very thing it was required to show.

## Proposition 21

If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.


For let the two internal straight-lines $B D$ and $D C$ have been constructed on one of the sides $B C$ of the triangle $A B C$, from its ends $B$ and $C$ (respectively). I say that $B D$ and $D C$ are less than the (sum of the) two remaining sides of the triangle $B A$ and $A C$, but encompass an angle $B D C$ greater than $B A C$.

For let $B D$ have been drawn through to $E$. And since in any triangle (the sum of any) two sides is greater than the remaining (side) [Prop. 1.20], in triangle $A B E$ the (sum of the) two sides $A B$ and $A E$ is thus greater than $B E$. Let $E C$ have been added to both. Thus, (the sum of) $B A$ and $A C$ is greater than (the sum of) $B E$ and $E C$. Again, since in triangle $C E D$ the (sum of the) two sides $C E$ and $E D$ is greater than $C D$, let $D B$ have been added to both. Thus, (the sum of) $C E$ and $E B$ is greater than (the sum of) $C D$ and $D B$. But, (the sum of) $B A$ and $A C$ was shown (to be) greater than (the sum of) $B E$ and $E C$. Thus, (the sum of) $B A$ and $A C$ is much greater than

 тñs únò ВАГ.





## $\chi \beta^{\prime}$.


 $\mu \varepsilon i \zeta o v \alpha s ~ \varepsilon i ̃ \alpha \alpha \iota ~ \pi \alpha ́ \nu \tau \eta ~ \mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu o \mu \varepsilon ́ v \alpha s ~[\delta ı \alpha ̀ ~ \tau o ̀ ~ \chi \alpha i ̀ ~ \pi \alpha \nu \tau o ̀ s ~$
 $\mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu o \mu \varepsilon ́ v \alpha c]$.











 KZH.


(the sum of) $B D$ and $D C$.
Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle $C D E$ the external angle $B D C$ is thus greater than $C E D$. Accordingly, for the same (reason), the external angle $C E B$ of the triangle $A B E$ is also greater than $B A C$. But, $B D C$ was shown (to be) greater than $C E B$. Thus, $B D C$ is much greater than $B A C$.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

## Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].


Let $A, B$, and $C$ be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) $A$ and $B$ (is greater) than $C$, (the sum of) $A$ and $C$ than $B$, and also (the sum of) $B$ and $C$ than $A$. So it is required to construct a triangle from (straight-lines) equal to $A, B$, and $C$.

Let some straight-line $D E$ be set out, terminated at $D$, and infinite in the direction of $E$. And let $D F$ made equal to $A$, and $F G$ equal to $B$, and $G H$ equal to $C$ [Prop. 1.3]. And let the circle $D K L$ have been drawn with center $F$ and radius $F D$. Again, let the circle $K L H$ have been drawn with center $G$ and radius $G H$. And let $K F$ and $K G$ have been joined. I say that the triangle $K F G$ has
 દ̇бтi тoũ $\Lambda \mathrm{K} \Theta$ xúx




 $\sigma \cup \nu \varepsilon ́ \sigma \tau \alpha \tau \alpha l$ tò $\mathrm{KZH} \cdot$ öл $\tau \rho$ है $\delta \varepsilon \iota ~ \pi o เ ท ̃ \sigma \alpha l . ~$

$$
x \gamma^{\prime}
$$


 $\sigma \cup \sigma \tau \eta \dot{\sigma} \alpha \sigma \vartheta \alpha$.









 ย̌兀ぃ $\tau \grave{\eta} \nu \Delta \mathrm{E} \tau \tilde{n} \mathrm{ZH}$.





 $\pi o เ \eta ̃ \sigma \alpha$.
been constructed from three straight-lines equal to $A, B$, and $C$.

For since point $F$ is the center of the circle $D K L, F D$ is equal to $F K$. But, $F D$ is equal to $A$. Thus, $K F$ is also equal to $A$. Again, since point $G$ is the center of the circle $L K H, G H$ is equal to $G K$. But, $G H$ is equal to $C$. Thus, $K G$ is also equal to $C$. And $F G$ is also equal to $B$. Thus, the three straight-lines $K F, F G$, and $G K$ are equal to $A$, $B$, and $C$ (respectively).

Thus, the triangle $K F G$ has been constructed from the three straight-lines $K F, F G$, and $G K$, which are equal to the three given straight-lines $A, B$, and $C$ (respectively). (Which is) the very thing it was required to do.

## Proposition 23

To construct a rectilinear angle equal to a given rectilinear angle at a (given) point on a given straight-line.


Let $A B$ be the given straight-line, $A$ the (given) point on it, and $D C E$ the given rectilinear angle. So it is required to construct a rectilinear angle equal to the given rectilinear angle $D C E$ at the (given) point $A$ on the given straight-line $A B$.

Let the points $D$ and $E$ have been taken at random on each of the (straight-lines) $C D$ and $C E$ (respectively), and let $D E$ have been joined. And let the triangle $A F G$ have been constructed from three straight-lines which are equal to $C D, D E$, and $C E$, such that $C D$ is equal to $A F$, $C E$ to $A G$, and further $D E$ to $F G$ [Prop. 1.22].

Therefore, since the two (straight-lines) $D C, C E$ are equal to the two (straight-lines) $F A, A G$, respectively, and the base $D E$ is equal to the base $F G$, the angle $D C E$ is thus equal to the angle $F A G$ [Prop. 1.8].

Thus, the rectilinear angle $F A G$, equal to the given rectilinear angle $D C E$, has been constructed at the (given) point $A$ on the given straight-line $A B$. (Which

## $\chi \delta^{\prime}$.














 ZH.









 $\mu \varepsilon i \zeta \omega \nu$ व̈p $\alpha$ каi $\dot{\eta}$ ВГ $\tau \tilde{\eta} \varsigma$ EZ.




is) the very thing it was required to do.

## Proposition 24

If two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).


Let $A B C$ and $D E F$ be two triangles having the two sides $A B$ and $A C$ equal to the two sides $D E$ and $D F$, respectively. (That is), $A B$ (equal) to $D E$, and $A C$ to $D F$. Let them also have the angle at $A$ greater than the angle at $D$. I say that the base $B C$ is also greater than the base $E F$.

For since angle $B A C$ is greater than angle $E D F$, let (angle) $E D G$, equal to angle $B A C$, have been constructed at the point $D$ on the straight-line $D E$ [Prop. 1.23]. And let $D G$ be made equal to either of $A C$ or $D F$ [Prop. 1.3], and let $E G$ and $F G$ have been joined.

Therefore, since $A B$ is equal to $D E$ and $A C$ to $D G$, the two (straight-lines) $B A, A C$ are equal to the two (straight-lines) $E D, D G$, respectively. Also the angle $B A C$ is equal to the angle $E D G$. Thus, the base $B C$ is equal to the base $E G$ [Prop. 1.4]. Again, since $D F$ is equal to $D G$, angle $D G F$ is also equal to angle $D F G$ [Prop. 1.5]. Thus, $D F G$ (is) greater than $E G F$. Thus, $E F G$ is much greater than $E G F$. And since triangle $E F G$ has angle $E F G$ greater than $E G F$, and the greater angle is subtended by the greater side [Prop. 1.19], side $E G$ (is) thus also greater than $E F$. But $E G$ (is) equal to $B C$. Thus, $B C$ (is) also greater than $E F$.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).

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\(\chi \varepsilon^{\prime}\).
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 $\beta \alpha ́ \sigma \iota \varsigma ~ \delta \varepsilon ̀ ~ \dot{\eta}$ ВГ $\beta \dot{\alpha} \sigma \varepsilon \omega \varsigma ~ \tau \tilde{\eta} \varsigma \mathrm{EZ} \mu \varepsilon i \zeta \omega \nu$ है $\sigma \tau \omega \cdot \lambda \varepsilon ́ \gamma \omega$, o้тı x $\alpha$














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\chi G^{\prime}
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 $\gamma \omega v i \alpha$.

(Which is) the very thing it was required to show.

## Proposition 25

If two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter).


Let $A B C$ and $D E F$ be two triangles having the two sides $A B$ and $A C$ equal to the two sides $D E$ and $D F$, respectively (That is), $A B$ (equal) to $D E$, and $A C$ to $D F$. And let the base $B C$ be greater than the base $E F$. I say that angle $B A C$ is also greater than $E D F$.

For if not, $(B A C)$ is certainly either equal to, or less than, $(E D F)$. In fact, $B A C$ is not equal to $E D F$. For then the base $B C$ would also have been equal to the base $E F$ [Prop. 1.4]. But it is not. Thus, angle $B A C$ is not equal to $E D F$. Neither, indeed, is $B A C$ less than $E D F$. For then the base $B C$ would also have been less than the base $E F$ [Prop. 1.24]. But it is not. Thus, angle $B A C$ is not less than $E D F$. But it was shown that ( $B A C$ is) not equal (to $E D F$ ) either. Thus, $B A C$ is greater than $E D F$.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter). (Which is) the very thing it was required to show.

## Proposition 26

If two triangles have two angles equal to two angles, respectively, and one side equal to one side-in fact, either that by the equal angles, or that subtending one of the equal angles-then (the triangles) will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle (equal) to the remaining angle.










Ei $\gamma \dot{\alpha} \rho \alpha^{\alpha} \nu \iota \sigma o ́ s ~ \varepsilon ̇ \sigma \tau \iota \nu \dot{\eta} \mathrm{AB} \tau \tilde{n} \Delta \mathrm{E}, \mu i \alpha \alpha$ 人 $\tau \widetilde{\omega} \nu \mu \varepsilon i \zeta \omega \nu$





 $\Delta \mathrm{EZ}$ трıү⿳㇒⿵⺆⿻日乚㇒兀
























Let $A B C$ and $D E F$ be two triangles having the two angles $A B C$ and $B C A$ equal to the two（angles）$D E F$ and $E F D$ ，respectively．（That is）$A B C$（equal）to $D E F$ ， and $B C A$ to $E F D$ ．And let them also have one side equal to one side．First of all，the（side）by the equal angles． （That is）$B C$（equal）to $E F$ ．I say that they will have the remaining sides equal to the corresponding remain－ ing sides．（That is）$A B$（equal）to $D E$ ，and $A C$ to $D F$ ． And（they will have）the remaining angle（equal）to the remaining angle．（That is）$B A C$（equal）to $E D F$ ．


For if $A B$ is unequal to $D E$ then one of them is greater．Let $A B$ be greater，and let $B G$ be made equal to $D E$［Prop．1．3］，and let $G C$ have been joined．

Therefore，since $B G$ is equal to $D E$ ，and $B C$ to $E F$ ， the two（straight－lines）$G B, B C^{\dagger}$ are equal to the two （straight－lines）$D E, E F$ ，respectively．And angle $G B C$ is equal to angle $D E F$ ．Thus，the base $G C$ is equal to the base $D F$ ，and triangle $G B C$ is equal to triangle $D E F$ ， and the remaining angles subtended by the equal sides will be equal to the（corresponding）remaining angles ［Prop．1．4］．Thus，$G C B$（is equal）to $D F E$ ．But，$D F E$ was assumed（to be）equal to $B C A$ ．Thus，$B C G$ is also equal to $B C A$ ，the lesser to the greater．The very thing （is）impossible．Thus，$A B$ is not unequal to $D E$ ．Thus， （it is）equal．And $B C$ is also equal to $E F$ ．So the two （straight－lines）$A B, B C$ are equal to the two（straight－ lines）$D E, E F$ ，respectively．And angle $A B C$ is equal to angle $D E F$ ．Thus，the base $A C$ is equal to the base $D F$ ， and the remaining angle $B A C$ is equal to the remaining angle $E D F$［Prop．1．4］．

But，again，let the sides subtending the equal angles be equal：for instance，（let）$A B$（be equal）to $D E$ ．Again， I say that the remaining sides will be equal to the remain－ ing sides．（That is）$A C$（equal）to $D F$ ，and $B C$ to $E F$ ． Furthermore，the remaining angle $B A C$ is equal to the remaining angle $E D F$ ．

For if $B C$ is unequal to $E F$ then one of them is greater．If possible，let $B C$ be greater．And let $B H$ be made equal to $E F$［Prop．1．3］，and let $A H$ have been joined．And since $B H$ is equal to $E F$ ，and $A B$ to $D E$ ， the two（straight－lines）$A B, B H$ are equal to the two






 ú $\pi o ̀ ~ B A \Gamma ~ \tau \tilde{n} \lambda o l \pi n ̀ ~ \gamma \omega v i ́ \alpha ~ \tau n ̃ ~ u ́ \pi o ̀ ~ E \Delta Z ~ \breve{\sigma} \eta$.






${ }^{\dagger}$ The Greek text has " $B G, B C$ ", which is obviously a mistake.

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 ยひ่ทยĩal.


 $\pi о เ$ ít $\cdot \lambda \varepsilon ́ \gamma \omega$, ơтı $\pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o ́ s ~ \varepsilon ̇ \sigma \tau \iota \nu ~ \dot{\eta} \mathrm{AB} \tau \tilde{\eta} \Gamma \Delta$.


 HEZ $\dot{\eta}$ ย̇x vavtiov tñ ن́ ùò EZH• öл $\Delta \Gamma$ غ่x $\beta \alpha \lambda \lambda o ́ \mu \varepsilon \nu \alpha l ~ \sigma u \mu \pi \varepsilon \sigma o u ̃ v \tau \alpha l ~ \varepsilon ̇ \pi i ~ \tau \grave{\alpha} \mathrm{~B}, \Delta \mu \varepsilon ́ p \eta$. ó $\mu$ oí $\omega \varsigma$
(straight-lines) $D E, E F$, respectively. And the angles they encompass (are also equal). Thus, the base $A H$ is equal to the base $D F$, and the triangle $A B H$ is equal to the triangle $D E F$, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, angle $B H A$ is equal to $E F D$. But, $E F D$ is equal to $B C A$. So, in triangle $A H C$, the external angle $B H A$ is equal to the internal and opposite angle $B C A$. The very thing (is) impossible [Prop. 1.16]. Thus, $B C$ is not unequal to $E F$. Thus, (it is) equal. And $A B$ is also equal to $D E$. So the two (straight-lines) $A B, B C$ are equal to the two (straightlines) $D E, E F$, respectively. And they encompass equal angles. Thus, the base $A C$ is equal to the base $D F$, and triangle $A B C$ (is) equal to triangle $D E F$, and the remaining angle $B A C$ (is) equal to the remaining angle $E D F$ [Prop. 1.4].

Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side-in fact, either that by the equal angles, or that subtending one of the equal angles-then (the triangles) will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle (equal) to the remaining angle. (Which is) the very thing it was required to show.

## Proposition 27

If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.


For let the straight-line $E F$, falling across the two straight-lines $A B$ and $C D$, make the alternate angles $A E F$ and $E F D$ equal to one another. I say that $A B$ and $C D$ are parallel.

For if not, being produced, $A B$ and $C D$ will certainly meet together: either in the direction of $B$ and $D$, or (in the direction) of $A$ and $C$ [Def. 1.23]. Let them have been produced, and let them meet together in the direction of $B$ and $D$ at (point) $G$. So, for the triangle

 $\dot{\eta} \mathrm{AB} \tau \tilde{\eta} \Gamma \Delta$.








 EZ $\tau \grave{\eta} \nu$ ह̉x

 őtı $\pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota \nu ~ \dot{\eta} \mathrm{AB} \tau \tilde{n} \Gamma \Delta$.


 $\mathrm{AB} \tau \tilde{n} \Gamma \Delta$.




 ह̇ $\sigma \tau i \nu \dot{\eta} \mathrm{AB} \tau \tilde{n} \Gamma \Delta$.


$G E F$, the external angle $A E F$ is equal to the interior and opposite (angle) $E F G$. The very thing is impossible [Prop. 1.16]. Thus, being produced, $A B$ and $C D$ will not meet together in the direction of $B$ and $D$. Similarly, it can be shown that neither (will they meet together) in (the direction of) $A$ and $C$. But (straight-lines) meeting in neither direction are parallel [Def. 1.23]. Thus, $A B$ and $C D$ are parallel.

Thus, if a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

## Proposition 28

If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two rightangles, then the (two) straight-lines will be parallel to one another.


For let $E F$, falling across the two straight-lines $A B$ and $C D$, make the external angle $E G B$ equal to the internal and opposite angle $G H D$, or the (sum of the) internal (angles) on the same side, $B G H$ and $G H D$, equal to two right-angles. I say that $A B$ is parallel to $C D$.

For since (in the first case) $E G B$ is equal to $G H D$, but $E G B$ is equal to $A G H$ [Prop. 1.15], $A G H$ is thus also equal to $G H D$. And they are alternate (angles). Thus, $A B$ is parallel to $C D$ [Prop. 1.27].

Again, since (in the second case, the sum of) $B G H$ and $G H D$ is equal to two right-angles, and (the sum of) $A G H$ and $B G H$ is also equal to two right-angles [Prop. 1.13], (the sum of) $A G H$ and $B G H$ is thus equal to (the sum of) $B G H$ and $G H D$. Let $B G H$ have been subtracted from both. Thus, the remainder $A G H$ is equal to the remainder $G H D$. And they are alternate (angles). Thus, $A B$ is parallel to $C D$ [Prop. 1.27].


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 ै $\sigma \alpha \varsigma$.


 $\mu \varepsilon i \zeta o v e ́ s ~ \varepsilon i \sigma t \nu . ~ \alpha ̀ \lambda \lambda \alpha ̀ ~ \alpha i ~ u ́ \pi o ̀ ~ A H \Theta, ~ B H \Theta ~ \delta u \sigma i ̀ ~ o ̉ p \vartheta \alpha i ̃ s ~ \imath ̋ \sigma \alpha l ~$














Thus, if a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two rightangles, then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

## Proposition 29

A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.


For let the straight-line $E F$ fall across the parallel straight-lines $A B$ and $C D$. I say that it makes the alternate angles, $A G H$ and $G H D$, equal, the external angle $E G B$ equal to the internal and opposite (angle) $G H D$, and the (sum of the) internal (angles) on the same side, $B G H$ and $G H D$, equal to two right-angles.

For if $A G H$ is unequal to $G H D$ then one of them is greater. Let $A G H$ be greater. Let $B G H$ have been added to both. Thus, (the sum of) $A G H$ and $B G H$ is greater than (the sum of) $B G H$ and $G H D$. But, (the sum of) $A G H$ and $B G H$ is equal to two right-angles [Prop 1.13]. Thus, (the sum of) $B G H$ and $G H D$ is [also] less than two right-angles. But (straight-lines) being produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, $A B$ and $C D$, being produced to infinity, will meet together. But they do not meet, on account of them (initially) being assumed parallel (to one another) [Def. 1.23]. Thus, $A G H$ is not unequal to $G H D$. Thus, (it is) equal. But, $A G H$ is equal to $E G B$ [Prop. 1.15]. And $E G B$ is thus also equal to $G H D$. Let $B G H$ be added to both. Thus, (the sum of) $E G B$ and $B G H$ is equal to (the sum of) $B G H$ and $G H D$. But, (the sum of) $E G B$ and $B G H$ is equal to two right-


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"E $\sigma \tau \omega \dot{\varepsilon} x \alpha \tau \varepsilon ́ \rho \alpha \tau \widetilde{\omega} \nu \mathrm{AB}, \Gamma \Delta \tau \tilde{n} \mathrm{EZ} \pi \alpha \rho \alpha \dot{\lambda} \lambda \eta \lambda o \varsigma \cdot \lambda \varepsilon ́ \gamma \omega$, ötı x $\alpha \dot{\eta} \dot{\eta} \mathrm{AB} \tau \tilde{n} \Gamma \Delta$ ह̇ $\sigma \tau \iota \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o \varsigma$.
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 «$p \alpha$ ध̇бтiv $\dot{\eta} \mathrm{AB} \tau \tilde{n} \Gamma \Delta$.



## $\lambda \alpha^{\prime}$.









angles [Prop. 1.13]. Thus, (the sum of) $B G H$ and $G H D$ is also equal to two right-angles.

Thus, a straight-line falling across parallel straightlines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles. (Which is) the very thing it was required to show.

## Proposition 30

(Straight-lines) parallel to the same straight-line are also parallel to one another.


Let each of the (straight-lines) $A B$ and $C D$ be parallel to $E F$. I say that $A B$ is also parallel to $C D$.

For let the straight-line $G K$ fall across $(A B, C D$, and $E F)$.

And since the straight-line $G K$ has fallen across the parallel straight-lines $A B$ and $E F$, (angle) $A G K$ (is) thus equal to $G H F$ [Prop. 1.29]. Again, since the straight-line $G K$ has fallen across the parallel straight-lines $E F$ and $C D$, (angle) $G H F$ is equal to $G K D$ [Prop. 1.29]. But $A G K$ was also shown (to be) equal to $G H F$. Thus, $A G K$ is also equal to $G K D$. And they are alternate (angles). Thus, $A B$ is parallel to $C D$ [Prop. 1.27].
[Thus, (straight-lines) parallel to the same straightline are also parallel to one another.] (Which is) the very thing it was required to show.

## Proposition 31

To draw a straight-line parallel to a given straight-line, through a given point.

Let $A$ be the given point, and $B C$ the given straightline. So it is required to draw a straight-line parallel to the straight-line $B C$, through the point $A$.

Let the point $D$ have been taken a random on $B C$, and let $A D$ have been joined. And let (angle) $D A E$, equal to angle $A D C$, have been constructed on the straight-line






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$D A$ at the point $A$ on it [Prop. 1.23]. And let the straightline $A F$ have been produced in a straight-line with $E A$.


And since the straight-line $A D$, (in) falling across the two straight-lines $B C$ and $E F$, has made the alternate angles $E A D$ and $A D C$ equal to one another, $E A F$ is thus parallel to $B C$ [Prop. 1.27].

Thus, the straight-line $E A F$ has been drawn parallel to the given straight-line $B C$, through the given point $A$. (Which is) the very thing it was required to do.

## Proposition 32

In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.


Let $A B C$ be a triangle, and let one of its sides $B C$ have been produced to $D$. I say that the external angle $A C D$ is equal to the (sum of the) two internal and opposite angles $C A B$ and $A B C$, and the (sum of the) three internal angles of the triangle- $A B C, B C A$, and $C A B-$ is equal to two right-angles.

For let $C E$ have been drawn through point $C$ parallel to the straight-line $A B$ [Prop. 1.31].

And since $A B$ is parallel to $C E$, and $A C$ has fallen across them, the alternate angles $B A C$ and $A C E$ are equal to one another [Prop. 1.29]. Again, since $A B$ is parallel to $C E$, and the straight-line $B D$ has fallen across them, the external angle $E C D$ is equal to the internal and opposite (angle) $A B C$ [Prop. 1.29]. But $A C E$ was also shown (to be) equal to $B A C$. Thus, the whole an-
 трıбì т $\alpha i \varsigma ~ \cup ́ \pi o ̀ ~ А В Г, ~ В Г А, ~ Г А В ~ i ̋ \sigma \alpha ı ~ \varepsilon i \sigma i v . ~ \alpha ̀ \lambda \lambda ’ ~ \alpha i ~ ن ́ \pi o ̀ ~ А Г \Delta, ~$


 $\beta \lambda \eta \vartheta \varepsilon i ́ \sigma \eta s \dot{\eta}$ ह̉xtòs $\gamma \omega v i ́ \alpha ~ \delta u \sigma i ̀ ~ \tau \alpha i ̃ \varsigma ~ ह ̉ v t o ̀ s ~ x \alpha i ̀ ~ \alpha ̉ \pi \varepsilon v \alpha \nu \tau i ́ o v ~$



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gle $A C D$ is equal to the（sum of the）two internal and opposite（angles）$B A C$ and $A B C$ ．

Let $A C B$ have been added to both．Thus，（the sum of）$A C D$ and $A C B$ is equal to the（sum of the）three （angles）$A B C, B C A$ ，and $C A B$ ．But，（the sum of）$A C D$ and $A C B$ is equal to two right－angles［Prop．1．13］．Thus， （the sum of）$A C B, C B A$ ，and $C A B$ is also equal to two right－angles．

Thus，in any triangle，（if）one of the sides（is）pro－ duced（then）the external angle is equal to the（sum of the）two internal and opposite（angles），and the（sum of the）three internal angles of the triangle is equal to two right－angles．（Which is）the very thing it was required to show．

## Proposition 33

Straight－lines joining equal and parallel（straight－ lines）on the same sides are themselves also equal and parallel．


Let $A B$ and $C D$ be equal and parallel（straight－lines）， and let the straight－lines $A C$ and $B D$ join them on the same sides．I say that $A C$ and $B D$ are also equal and parallel．

Let $B C$ have been joined．And since $A B$ is paral－ lel to $C D$ ，and $B C$ has fallen across them，the alter－ nate angles $A B C$ and $B C D$ are equal to one another ［Prop．1．29］．And since $A B$ is equal to $C D$ ，and $B C$ is common，the two（straight－lines）$A B, B C$ are equal to the two（straight－lines）$D C, C B .^{\dagger}$ And the angle $A B C$ is equal to the angle $B C D$ ．Thus，the base $A C$ is equal to the base $B D$ ，and triangle $A B C$ is equal to triangle $D C B^{\ddagger}$ ，and the remaining angles will be equal to the corresponding remaining angles subtended by the equal sides［Prop．1．4］．Thus，angle $A C B$ is equal to $C B D$ ． Also，since the straight－line $B C$ ，（in）falling across the two straight－lines $A C$ and $B D$ ，has made the alternate angles（ $A C B$ and $C B D$ ）equal to one another，$A C$ is thus parallel to $B D$［Prop．1．27］．And（ $A C$ ）was also shown （to be）equal to（ $B D$ ）．

Thus，straight－lines joining equal and parallel（straight－
${ }^{\dagger}$ The Greek text has＂$B C, C D$＂，which is obviously a mistake．
$\ddagger$ The Greek text has＂$D C B$＂，which is obviously a mistake．
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lines）on the same sides are themselves also equal and parallel．（Which is）the very thing it was required to show．

## Proposition 34

In parallelogrammic figures the opposite sides and angles are equal to one another，and a diagonal cuts them in half．


Let $A C D B$ be a parallelogrammic figure，and $B C$ its diagonal．I say that for parallelogram $A C D B$ ，the oppo－ site sides and angles are equal to one another，and the diagonal $B C$ cuts it in half．

For since $A B$ is parallel to $C D$ ，and the straight－line $B C$ has fallen across them，the alternate angles $A B C$ and $B C D$ are equal to one another［Prop．1．29］．Again，since $A C$ is parallel to $B D$ ，and $B C$ has fallen across them， the alternate angles $A C B$ and $C B D$ are equal to one another［Prop．1．29］．So $A B C$ and $B C D$ are two tri－ angles having the two angles $A B C$ and $B C A$ equal to the two（angles）$B C D$ and $C B D$ ，respectively，and one side equal to one side－the（one）by the equal angles and common to them，（namely）BC．Thus，they will also have the remaining sides equal to the corresponding re－ maining（sides），and the remaining angle（equal）to the remaining angle［Prop．1．26］．Thus，side $A B$ is equal to $C D$ ，and $A C$ to $B D$ ．Furthermore，angle $B A C$ is equal to $C D B$ ．And since angle $A B C$ is equal to $B C D$ ，and $C B D$ to $A C B$ ，the whole（angle）$A B D$ is thus equal to the whole（angle）$A C D$ ．And $B A C$ was also shown（to be）equal to $C D B$ ．

Thus，in parallelogrammic figures the opposite sides and angles are equal to one another．

And，I also say that a diagonal cuts them in half．For since $A B$ is equal to $C D$ ，and $B C$（is）common，the two （straight－lines）$A B, B C$ are equal to the two（straight－ lines）$D C, C B^{\dagger}$ ，respectively．And angle $A B C$ is equal to angle $B C D$ ．Thus，the base $A C$（is）also equal to $D B$ ，
${ }^{\dagger}$ The Greek text has " $C D, B C$ ", which is obviously a mistake.
₹ The Greek text has " $A B C D$ ", which is obviously a mistake.
$\lambda \varepsilon^{\prime}$.





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and triangle $A B C$ is equal to triangle $B C D$ [Prop. 1.4].
Thus, the diagonal $B C$ cuts the parallelogram $A C D B^{\ddagger}$ in half. (Which is) the very thing it was required to show.

## Proposition 35

Parallelograms which are on the same base and between the same parallels are equal ${ }^{\dagger}$ to one another.


Let $A B C D$ and $E B C F$ be parallelograms on the same base $B C$, and between the same parallels $A F$ and $B C$. I say that $A B C D$ is equal to parallelogram $E B C F$.

For since $A B C D$ is a parallelogram, $A D$ is equal to $B C$ [Prop. 1.34]. So, for the same (reasons), $E F$ is also equal to $B C$. So $A D$ is also equal to $E F$. And $D E$ is common. Thus, the whole (straight-line) $A E$ is equal to the whole (straight-line) $D F$. And $A B$ is also equal to $D C$. So the two (straight-lines) $E A, A B$ are equal to the two (straight-lines) $F D, D C$, respectively. And angle $F D C$ is equal to angle $E A B$, the external to the internal [Prop. 1.29]. Thus, the base $E B$ is equal to the base $F C$, and triangle $E A B$ will be equal to triangle $D F C$ [Prop. 1.4]. Let $D G E$ have been taken away from both. Thus, the remaining trapezium $A B G D$ is equal to the remaining trapezium $E G C F$. Let triangle $G B C$ have been added to both. Thus, the whole parallelogram $A B C D$ is equal to the whole parallelogram $E B C F$.

Thus, parallelograms which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.
${ }^{\dagger}$ Here, for the first time, "equal" means "equal in area", rather than "congruent".

$$
\lambda \varepsilon^{\prime}
$$







## Proposition 36

Parallelograms which are on equal bases and between the same parallels are equal to one another.

Let $A B C D$ and $E F G H$ be parallelograms which are on the equal bases $B C$ and $F G$, and (are) between the same parallels $A H$ and $B G$. I say that the parallelogram
$\lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu \tau \widetilde{\tau}$ ЕZHЄ.













 ठєi̧̋al.

$$
\lambda \zeta^{\prime} .
$$




${ }^{\prime} E \sigma \tau \omega \tau \rho i ́ \gamma \omega v \alpha \tau \dot{\alpha} \mathrm{AB} \mathrm{\Gamma}, \Delta \mathrm{~B} \Gamma$ ह́лì $\tau \tilde{\eta} \varsigma \alpha \cup ̉ \tau \tilde{\eta} \varsigma \beta \alpha ́ \sigma \varepsilon \omega \varsigma \tau \tilde{\eta} \varsigma$











$A B C D$ is equal to $E F G H$.


For let $B E$ and $C H$ have been joined. And since $B C$ is equal to $F G$, but $F G$ is equal to $E H$ [Prop. 1.34], $B C$ is thus equal to $E H$. And they are also parallel, and $E B$ and $H C$ join them. But (straight-lines) joining equal and parallel (straight-lines) on the same sides are (themselves) equal and parallel [Prop. 1.33] [thus, $E B$ and $H C$ are also equal and parallel]. Thus, $E B C H$ is a parallelogram [Prop. 1.34], and is equal to $A B C D$. For it has the same base, $B C$, as $(A B C D)$, and is between the same parallels, $B C$ and $A H$, as ( $A B C D$ ) [Prop. 1.35]. So, for the same (reasons), $E F G H$ is also equal to the same (parallelogram) $E B C H$ [Prop. 1.34]. So that the parallelogram $A B C D$ is also equal to $E F G H$.

Thus, parallelograms which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

## Proposition 37

Triangles which are on the same base and between the same parallels are equal to one another.


Let $A B C$ and $D B C$ be triangles on the same base $B C$, and between the same parallels $A D$ and $B C$. I say that triangle $A B C$ is equal to triangle $D B C$.

Let $A D$ have been produced in both directions to $E$ and $F$, and let the (straight-line) $B E$ have been drawn through $B$ parallel to $C A$ [Prop. 1.31], and let the (straight-line) $C F$ have been drawn through $C$ parallel to $B D$ [Prop. 1.31]. Thus, $E B C A$ and $D B C F$ are both parallelograms, and are equal. For they are on the same base $B C$, and between the same parallels $B C$ and $E F$ [Prop. 1.35]. And the triangle $A B C$ is half of the parallelogram $E B C A$. For the diagonal $A B$ cuts the latter in




${ }^{\dagger}$ This is an additional common notion.

$$
\lambda \eta^{\prime}
$$












 НВГА $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \alpha \mu о u ~ \eta ̆ \mu ı \sigma u ~ t o ̀ ~ А В Г ~ \tau р i ́ \gamma \omega v o v . ~ \dot{\eta}$ үàp







## $\lambda \vartheta^{\prime}$.





half [Prop. 1.34]. And the triangle $D B C$ (is) half of the parallelogram $D B C F$. For the diagonal $D C$ cuts the latter in half [Prop. 1.34]. [And the halves of equal things are equal to one another. $]^{\dagger}$ Thus, triangle $A B C$ is equal to triangle $D B C$.

Thus, triangles which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

## Proposition 38

Triangles which are on equal bases and between the same parallels are equal to one another.


Let $A B C$ and $D E F$ be triangles on the equal bases $B C$ and $E F$, and between the same parallels $B F$ and $A D$. I say that triangle $A B C$ is equal to triangle $D E F$.

For let $A D$ have been produced in both directions to $G$ and $H$, and let the (straight-line) $B G$ have been drawn through $B$ parallel to $C A$ [Prop. 1.31], and let the (straight-line) $F H$ have been drawn through $F$ parallel to $D E$ [Prop. 1.31]. Thus, $G B C A$ and $D E F H$ are each parallelograms. And $G B C A$ is equal to $D E F H$. For they are on the equal bases $B C$ and $E F$, and between the same parallels $B F$ and $G H$ [Prop. 1.36]. And triangle $A B C$ is half of the parallelogram $G B C A$. For the diagonal $A B$ cuts the latter in half [Prop. 1.34]. And triangle $F E D$ (is) half of parallelogram $D E F H$. For the diagonal $D F$ cuts the latter in half. [And the halves of equal things are equal to one another.] Thus, triangle $A B C$ is equal to triangle $D E F$.

Thus, triangles which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

## Proposition 39

Equal triangles which are on the same base, and on the same side, are also between the same parallels.

Let $A B C$ and $D B C$ be equal triangles which are on the same base $B C$, and on the same side (of it). I say that


 $\mathrm{A} \Delta \tilde{n}_{\mathrm{n}} \mathrm{B} \Gamma$.







 $\pi \alpha \rho \alpha \dot{\alpha} \lambda \eta \lambda о \varsigma$.




$$
\mu^{\prime}
$$

 $\mu \varepsilon ́ p \eta ~ x \alpha i ̀ ~ \varepsilon ้ \nu ~ \tau \alpha i ̃ s ~ \alpha u ̉ \tau \alpha i ̃ s ~ \pi \alpha p \alpha \lambda \lambda \dot{n} \lambda o l s ~ \varepsilon ُ \sigma \tau i ้ \nu . ~$


 $\pi \alpha p \alpha \lambda \lambda$ ŋ́ $\lambda$ ols $\varepsilon$ ध̇ $\sigma \tau$.
 $\mathrm{A} \Delta \tau \tilde{n} \mathrm{BE}$.






they are also between the same parallels.


For let $A D$ have been joined. I say that $A D$ and $B C$ are parallel.

For, if not, let $A E$ have been drawn through point A parallel to the straight-line $B C$ [Prop. 1.31], and let $E C$ have been joined. Thus, triangle $A B C$ is equal to triangle $E B C$. For it is on the same base as it, $B C$, and between the same parallels [Prop. 1.37]. But $A B C$ is equal to $D B C$. Thus, $D B C$ is also equal to $E B C$, the greater to the lesser. The very thing is impossible. Thus, $A E$ is not parallel to $B C$. Similarly, we can show that neither (is) any other (straight-line) than $A D$. Thus, $A D$ is parallel to $B C$.

Thus, equal triangles which are on the same base, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

## Proposition $40^{\dagger}$

Equal triangles which are on equal bases, and on the same side, are also between the same parallels.


Let $A B C$ and $C D E$ be equal triangles on the equal bases $B C$ and $C E$ (respectively), and on the same side (of $B E$ ). I say that they are also between the same parallels.

For let $A D$ have been joined. I say that $A D$ is parallel to $B E$.

For if not, let $A F$ have been drawn through $A$ parallel to $B E$ [Prop. 1.31], and let $F E$ have been joined. Thus, triangle $A B C$ is equal to triangle $F C E$. For they are on equal bases, $B C$ and $C E$, and between the same parallels, $B E$ and $A F$ [Prop. 1.38]. But, triangle $A B C$ is equal

 $\dot{\eta} \mathrm{A} \Delta \alpha \alpha_{\alpha} \alpha \tilde{n} \mathrm{BE}$ धं $\sigma \tau \iota \pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ оऽ.

 ठєi̧̋al.
to [triangle] $D C E$. Thus, [triangle] $D C E$ is also equal to triangle $F C E$, the greater to the lesser. The very thing is impossible. Thus, $A F$ is not parallel to $B E$. Similarly, we can show that neither (is) any other (straight-line) than $A D$. Thus, $A D$ is parallel to $B E$.

Thus, equal triangles which are on equal bases, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This whole proposition is regarded by Heiberg as a relatively early interpolation to the original text.

## $\mu \alpha^{\prime}$.


 тò $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ ó $ү \rho \alpha \mu \mu \circ \nu$ тои̃ трเүढ́vou.



 АВГ $\Delta \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu$ тои̃ ВЕГ трıү'́vou.


 $\dot{\alpha} \lambda \lambda \dot{\alpha}$ тò $\mathrm{AB} \Gamma \Delta \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu \circ \nu \delta \iota \pi \lambda \alpha \dot{\sigma} เ o ́ v$ ह̇ $\sigma \tau \iota$ то̃̃ $\mathrm{AB} \mathrm{\Gamma}$

 бьл入人́бเоv.




$$
\mu \beta^{\prime}
$$







## Proposition 41

If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.


For let parallelogram $A B C D$ have the same base $B C$ as triangle $E B C$, and let it be between the same parallels, $B C$ and $A E$. I say that parallelogram $A B C D$ is double (the area) of triangle $B E C$.

For let $A C$ have been joined. So triangle $A B C$ is equal to triangle $E B C$. For it is on the same base, $B C$, as $(E B C)$, and between the same parallels, $B C$ and $A E$ [Prop. 1.37]. But, parallelogram $A B C D$ is double (the area) of triangle $A B C$. For the diagonal $A C$ cuts the former in half [Prop. 1.34]. So parallelogram $A B C D$ is also double (the area) of triangle $E B C$.

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

## Proposition 42

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let $A B C$ be the given triangle, and $D$ the given rectilinear angle. So it is required to construct a parallelogram equal to triangle $A B C$ in the rectilinear angle $D$.













 бovะíon $\tau \tilde{n} \Delta$.




$$
\mu \gamma^{\prime}
$$

П $\alpha \tau$ тòऽ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \circ \cup \tau \widetilde{\omega} \nu \pi \varepsilon \rho \grave{\tau} \tau \grave{\nu} \nu \delta \iota \alpha ́ \mu \varepsilon \tau \rho o \nu \pi \alpha-$

"Ебтш $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu ~ \tau o ̀ ~ А В Г \Delta, ~ \delta \iota \alpha ́ \mu \varepsilon \tau р о \varsigma ~ \delta e ̀ ~$
 $\tau \dot{\alpha} \mathrm{E} \Theta, \mathrm{ZH}$, $\tau \grave{\alpha}$ $\delta$ è $\lambda \varepsilon \gamma o ́ \mu \varepsilon v \alpha \pi \alpha \rho \alpha \pi \lambda n \rho \omega ́ \mu \alpha \tau \alpha \tau \dot{\alpha} \mathrm{BK}, \mathrm{K} \Delta \cdot$
 $\pi \lambda \eta \rho \dot{\rho} \mu \alpha \tau$.












Let $B C$ have been cut in half at $E$ [Prop. 1.10], and let $A E$ have been joined. And let (angle) $C E F$, equal to angle $D$, have been constructed at the point $E$ on the straight-line $E C$ [Prop. 1.23]. And let $A G$ have been drawn through $A$ parallel to $E C$ [Prop. 1.31], and let $C G$ have been drawn through $C$ parallel to $E F$ [Prop. 1.31]. Thus, $F E C G$ is a parallelogram. And since $B E$ is equal to $E C$, triangle $A B E$ is also equal to triangle $A E C$. For they are on the equal bases, $B E$ and $E C$, and between the same parallels, $B C$ and $A G$ [Prop. 1.38]. Thus, triangle $A B C$ is double (the area) of triangle $A E C$. And parallelogram $F E C G$ is also double (the area) of triangle $A E C$. For it has the same base as ( $A E C$ ), and is between the same parallels as ( $A E C$ ) [Prop. 1.41]. Thus, parallelogram $F E C G$ is equal to triangle $A B C$. ( $F E C G$ ) also has the angle $C E F$ equal to the given (angle) $D$.

Thus, parallelogram $F E C G$, equal to the given triangle $A B C$, has been constructed in the angle $C E F$, which is equal to $D$. (Which is) the very thing it was required to do.

## Proposition 43

For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

Let $A B C D$ be a parallelogram, and $A C$ its diagonal. And let $E H$ and $F G$ be the parallelograms about $A C$, and $B K$ and $K D$ the so-called complements (about $A C$ ). I say that the complement $B K$ is equal to the complement $K D$.

For since $A B C D$ is a parallelogram, and $A C$ its diagonal, triangle $A B C$ is equal to triangle $A C D$ [Prop. 1.34]. Again, since $E H$ is a parallelogram, and $A K$ is its diagonal, triangle $A E K$ is equal to triangle $A H K$ [Prop. 1.34]. So, for the same (reasons), triangle $K F C$ is also equal to (triangle) $K G C$. Therefore, since triangle $A E K$ is equal to triangle $A H K$, and $K F C$ to $K G C$, triangle $A E K$ plus $K G C$ is equal to triangle $A H K$ plus $K F C$. And the whole triangle $A B C$ is also equal to the whole (triangle) $A D C$. Thus, the remaining complement $B K$ is equal to
$\pi \lambda \eta \rho \omega ́ \mu \alpha \tau i ́$ ह̇ $\sigma \tau \iota \nu$ ’ैбov.





$$
\mu \delta^{\prime}
$$


 $\mu \varphi$.

 tò $\Gamma, \dot{\eta} \delta \varepsilon ̀ ~ \delta o \vartheta \varepsilon i ̃ \sigma \alpha ~ \gamma \omega v i ́ \alpha ~ \varepsilon u ̉ \vartheta u ́ \gamma \rho \alpha \mu \mu o \varsigma ~ \dot{\eta} \Delta \cdot \delta \varepsilon 亢 ̃ ~ \delta \grave{\eta} \pi \alpha \rho \alpha ̀$











the remaining complement $K D$.


Thus, for any parallelogramic figure, the complements of the parallelograms about the diagonal are equal to one another. (Which is) the very thing it was required to show.

## Proposition 44

To apply a parallelogram equal to a given triangle to a given straight-line in a given rectilinear angle.


Let $A B$ be the given straight-line, $C$ the given triangle, and $D$ the given rectilinear angle. So it is required to apply a parallelogram equal to the given triangle $C$ to the given straight-line $A B$ in an angle equal to (angle) $D$.

Let the parallelogram $B E F G$, equal to the triangle $C$, have been constructed in the angle $E B G$, which is equal to $D$ [Prop. 1.42]. And let it have been placed so that $B E$ is straight-on to $A B .^{\dagger}$ And let $F G$ have been drawn through to $H$, and let $A H$ have been drawn through A parallel to either of $B G$ or $E F$ [Prop. 1.31], and let $H B$ have been joined. And since the straight-line $H F$ falls across the parallels $A H$ and $E F$, the (sum of the) angles $A H F$ and $H F E$ is thus equal to two right-angles


 $\alpha i \quad \Theta A, H B$ غ̇лi $\tau \dot{\alpha} \Lambda, M$ $\sigma \eta \mu \varepsilon i ̃ \alpha . ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu \alpha \not p \alpha$
 $\pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu \alpha \mu \varepsilon ̀ v ~ \tau \alpha ̀ ~ A H, ~ M E, ~ \tau \grave{\alpha} \delta \varepsilon ̀ ~ \lambda \varepsilon \gamma o ́ \mu \varepsilon v \alpha \alpha \pi \alpha \alpha-$








${ }^{\dagger}$ This can be achieved using Props. 1.3, 1.23, and 1.31.

## $\mu \varepsilon^{\prime}$.





 ү $\omega$ vía $\tau \tilde{n} \mathrm{E}$.
${ }^{3} \mathrm{E} \tau \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \dot{\eta} \Delta \mathrm{~B}$, x $\alpha \grave{\imath} \sigma \cup \nu \varepsilon \sigma \tau \alpha \dot{\tau} \omega \tau \widetilde{\varphi} \mathrm{AB} \Delta \tau \rho \downarrow \gamma \omega \dot{\nu} \omega$




















[Prop. 1.29]. Thus, (the sum of) $B H G$ and $G F E$ is less than two right-angles. And (straight-lines) produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced, $H B$ and $F E$ will meet together. Let them have been produced, and let them meet together at $K$. And let $K L$ have been drawn through point $K$ parallel to either of $E A$ or $F H$ [Prop. 1.31]. And let $H A$ and $G B$ have been produced to points $L$ and $M$ (respectively). Thus, $H L K F$ is a parallelogram, and $H K$ its diagonal. And $A G$ and $M E$ (are) parallelograms, and $L B$ and $B F$ the so-called complements, about $H K$. Thus, $L B$ is equal to $B F$ [Prop. 1.43]. But, $B F$ is equal to triangle $C$. Thus, $L B$ is also equal to $C$. Also, since angle $G B E$ is equal to $A B M$ [Prop. 1.15], but $G B E$ is equal to $D, A B M$ is thus also equal to angle $D$.

Thus, the parallelogram $L B$, equal to the given triangle $C$, has been applied to the given straight-line $A B$ in the angle $A B M$, which is equal to $D$. (Which is) the very thing it was required to do.

## Proposition 45

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Let $A B C D$ be the given rectilinear figure, ${ }^{\dagger}$ and $E$ the given rectilinear angle. So it is required to construct a parallelogram equal to the rectilinear figure $A B C D$ in the given angle $E$.

Let $D B$ have been joined, and let the parallelogram $F H$, equal to the triangle $A B D$, have been constructed in the angle $H K F$, which is equal to $E$ [Prop. 1.42]. And let the parallelogram $G M$, equal to the triangle $D B C$, have been applied to the straight-line $G H$ in the angle $G H M$, which is equal to $E$ [Prop. 1.44]. And since angle $E$ is equal to each of (angles) $H K F$ and $G H M$, (angle) $H K F$ is thus also equal to $G H M$. Let $K H G$ have been added to both. Thus, (the sum of) $F K H$ and $K H G$ is equal to (the sum of) $K H G$ and $G H M$. But, (the sum of) $F K H$ and $K H G$ is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) $K H G$ and $G H M$ is also equal to two right-angles. So two straight-lines, $K H$ and $H M$, not lying on the same side, make adjacent angles with some straight-line $G H$, at the point $H$ on it, (whose sum is) equal to two right-angles. Thus, $K H$ is straight-on to $H M$ [Prop. 1.14]. And since the straightline $H G$ falls across the parallels $K M$ and $F G$, the alternate angles $M H G$ and $H G F$ are equal to one another [Prop. 1.29]. Let $H G L$ have been added to both. Thus, (the sum of) $M H G$ and $H G L$ is equal to (the sum of)




 غ̇бтiv ̂̋ซov.




$H G F$ and $H G L$. But, (the sum of) $M H G$ and $H G L$ is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) $H G F$ and $H G L$ is also equal to two right-angles. Thus, $F G$ is straight-on to $G L$ [Prop. 1.14]. And since $F K$ is equal and parallel to $H G$ [Prop. 1.34], but also $H G$ to $M L$ [Prop. 1.34], $K F$ is thus also equal and parallel to $M L$ [Prop. 1.30]. And the straight-lines $K M$ and $F L$ join them. Thus, $K M$ and $F L$ are equal and parallel as well [Prop. 1.33]. Thus, KFLM is a parallelogram. And since triangle $A B D$ is equal to parallelogram $F H$, and $D B C$ to $G M$, the whole rectilinear figure $A B C D$ is thus equal to the whole parallelogram $K F L M$.


Thus, the parallelogram $K F L M$, equal to the given rectilinear figure $A B C D$, has been constructed in the angle $F K M$, which is equal to the given (angle) $E$. (Which is) the very thing it was required to do.
${ }^{\dagger}$ The proof is only given for a four-sided figure. However, the extension to many-sided figures is trivial.

$$
\mu \xi^{\prime}
$$








 $\tau \tilde{n} \Delta \mathrm{E}, \dot{\eta}$ ठ̀̀ $\mathrm{A} \Delta \tau \tilde{n} \mathrm{BE} . \dot{\alpha} \lambda \lambda \dot{\alpha} \dot{\eta} \mathrm{AB} \tau \tilde{n} \mathrm{~A} \Delta$ ह̇ $\sigma \tau \iota ้$ ้ै $\sigma$.






## Proposition 46

To describe a square on a given straight-line.
Let $A B$ be the given straight-line. So it is required to describe a square on the straight-line $A B$.

Let $A C$ have been drawn at right-angles to the straight-line $A B$ from the point $A$ on it [Prop. 1.11], and let $A D$ have been made equal to $A B$ [Prop. 1.3]. And let $D E$ have been drawn through point $D$ parallel to $A B$ [Prop. 1.31], and let $B E$ have been drawn through point $B$ parallel to $A D$ [Prop. 1.31]. Thus, $A D E B$ is a parallelogram. Therefore, $A B$ is equal to $D E$, and $A D$ to $B E$ [Prop. 1.34]. But, $A B$ is equal to $A D$. Thus, the four (sides) $B A, A D, D E$, and $E B$ are equal to one another. Thus, the parallelogram $A D E B$ is equilateral. So I say that (it is) also right-angled. For since the straight-line








## $\mu \zeta^{\prime}$.



 траүढ́vols.


















$A D$ falls across the parallels $A B$ and $D E$, the (sum of the) angles $B A D$ and $A D E$ is equal to two right-angles [Prop. 1.29]. But $B A D$ (is a) right-angle. Thus, $A D E$ (is) also a right-angle. And for parallelogrammic figures, the opposite sides and angles are equal to one another [Prop. 1.34]. Thus, each of the opposite angles $A B E$ and $B E D$ (are) also right-angles. Thus, $A D E B$ is rightangled. And it was also shown (to be) equilateral.


Thus, $(A D E B)$ is a square [Def. 1.22]. And it is described on the straight-line $A B$. (Which is) the very thing it was required to do.

## Proposition 47

In right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides containing the right-angle.

Let $A B C$ be a right-angled triangle having the angle $B A C$ a right-angle. I say that the square on $B C$ is equal to the (sum of the) squares on $B A$ and $A C$.

For let the square $B D E C$ have been described on $B C$, and (the squares) $G B$ and $H C$ on $A B$ and $A C$ (respectively) [Prop. 1.46]. And let $A L$ have been drawn through point $A$ parallel to either of $B D$ or $C E$ [Prop. 1.31]. And let $A D$ and $F C$ have been joined. And since angles $B A C$ and $B A G$ are each right-angles, then two straight-lines $A C$ and $A G$, not lying on the same side, make the adjacent angles with some straight-line $B A$, at the point $A$ on it, (whose sum is) equal to two right-angles. Thus, $C A$ is straight-on to $A G$ [Prop. 1.14]. So, for the same (reasons), $B A$ is also straight-on to $A H$. And since angle $D B C$ is equal to $F B A$, for (they are) both right-angles, let $A B C$ have been added to both. Thus, the whole (angle) $D B A$ is equal to the whole (angle) $F B C$. And since $D B$ is equal to $B C$, and $F B$ to $B A$, the two (straight-lines) $D B, B A$ are equal to the













 $\pi \lambda \varepsilon \cup р \omega ั ้ ~ \tau \varepsilon \tau р \alpha \gamma(\omega)$ vos.






[^2]two (straight-lines) $C B, B F,{ }^{\dagger}$ respectively. And angle $D B A$ (is) equal to angle $F B C$. Thus, the base $A D$ [is] equal to the base $F C$, and the triangle $A B D$ is equal to the triangle $F B C$ [Prop. 1.4]. And parallelogram $B L$ [is] double (the area) of triangle $A B D$. For they have the same base, $B D$, and are between the same parallels, $B D$ and $A L$ [Prop. 1.41]. And square $G B$ is double (the area) of triangle $F B C$. For again they have the same base, $F B$, and are between the same parallels, $F B$ and $G C$ [Prop. 1.41]. [And the doubles of equal things are equal to one another. $]^{\ddagger}$ Thus, the parallelogram $B L$ is also equal to the square $G B$. So, similarly, $A E$ and $B K$ being joined, the parallelogram $C L$ can be shown (to be) equal to the square $H C$. Thus, the whole square $B D E C$ is equal to the (sum of the) two squares $G B$ and $H C$. And the square $B D E C$ is described on $B C$, and the (squares) $G B$ and $H C$ on $B A$ and $A C$ (respectively). Thus, the square on the side $B C$ is equal to the (sum of the) squares on the sides $B A$ and $A C$.


Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show.

## $\mu \eta^{\prime}$.






 $\tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \circ \nu$ ้̋ $\sigma o \nu$ है $\sigma \tau \omega$ тоїऽ $\alpha \pi o ̀ \tau \widetilde{\omega} \nu \mathrm{BA}, ~ А \Gamma ~ \pi \lambda \varepsilon \cup \rho \widetilde{\omega} \nu \tau \varepsilon-$

















 $\tau \varepsilon \tau \rho \alpha \gamma \omega ́ v o l s, \dot{\eta} \pi \varepsilon \rho เ \varepsilon \chi \circ \mu \varepsilon ́ v \eta ~ \gamma \omega v i ́ \alpha ~ \cup ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu \lambda о \iota \widetilde{\omega} \nu$ тоũ


## Proposition 48

If the square on one of the sides of a triangle is equal to the (sum of the) squares on the two remaining sides of the triangle then the angle contained by the two remaining sides of the triangle is a right-angle.


For let the square on one of the sides, $B C$, of triangle $A B C$ be equal to the (sum of the) squares on the sides $B A$ and $A C$. I say that angle $B A C$ is a right-angle.

For let $A D$ have been drawn from point $A$ at rightangles to the straight-line $A C$ [Prop. 1.11], and let $A D$ have been made equal to $B A$ [Prop. 1.3], and let $D C$ have been joined. Since $D A$ is equal to $A B$, the square on $D A$ is thus also equal to the square on $A B .^{\dagger}$ Let the square on $A C$ have been added to both. Thus, the (sum of the) squares on $D A$ and $A C$ is equal to the (sum of the) squares on $B A$ and $A C$. But, the (square) on $D C$ is equal to the (sum of the squares) on $D A$ and $A C$. For angle $D A C$ is a right-angle [Prop. 1.47]. But, the (square) on $B C$ is equal to (sum of the squares) on $B A$ and $A C$. For (that) was assumed. Thus, the square on $D C$ is equal to the square on $B C$. So side $D C$ is also equal to (side) $B C$. And since $D A$ is equal to $A B$, and $A C$ (is) common, the two (straight-lines) $D A, A C$ are equal to the two (straight-lines) $B A, A C$. And the base $D C$ is equal to the base $B C$. Thus, angle $D A C$ [is] equal to angle $B A C$ [Prop. 1.8]. But $D A C$ is a right-angle. Thus, $B A C$ is also a right-angle.

Thus, if the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining two sides of the triangle then the angle contained by the remaining two sides of the triangle is a right-angle. (Which is) the very thing it was required to show.

[^3]
## ELEMENTS BOOK 2

Fundamentals of Geometric Algebra
${ }^{\circ}$ Opor.

 ยที่ยะハ๘้.
$\beta^{\prime}$. Паvтòऽ ס̀̀ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu о и ~ \chi \omega р i ́ o u ~ \tau \omega ̃ \nu ~ \pi \varepsilon \rho i ̀ ~ \tau \grave{̀ \nu}$



$$
\alpha^{\prime} .
$$









 $\tau \widetilde{\varphi}$

 $\ddot{\eta} \chi \vartheta \omega \dot{\eta} \mathrm{H} \Theta, \delta \iota \dot{\alpha} \delta \dot{\varepsilon} \tau \widetilde{\omega} \nu \Delta, \mathrm{E}, \Gamma \tau \tilde{n} \mathrm{BH} \tau \alpha \rho \dot{\alpha} \lambda \lambda \eta \lambda o \iota ~ \eta ँ \chi \vartheta \omega \sigma \alpha \nu$ ai $\Delta \mathrm{K}, \mathrm{E} \Lambda, \Gamma \Theta$.













## Definitions

1. Any rectangular parallelogram is said to be contained by the two straight-lines containing the rightangle.
2. And in any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a gnomon.

## Proposition $1^{\dagger}$

If there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line).

A


Let $A$ and $B C$ be the two straight-lines, and let $B C$ be cut, at random, at points $D$ and $E$. I say that the rectangle contained by $A$ and $B C$ is equal to the rectangle(s) contained by $A$ and $B D$, by $A$ and $D E$, and, finally, by $A$ and $E C$.

For let $B F$ have been drawn from point $B$, at rightangles to $B C$ [Prop. 1.11], and let $B G$ be made equal to $A$ [Prop. 1.3], and let $G H$ have been drawn through (point) $G$, parallel to $B C$ [Prop. 1.31], and let $D K, E L$, and $C H$ have been drawn through (points) $D, E$, and $C$ (respectively), parallel to $B G$ [Prop. 1.31].

So the (rectangle) $B H$ is equal to the (rectangles) $B K, D L$, and $E H$. And $B H$ is the (rectangle contained) by $A$ and $B C$. For it is contained by $G B$ and $B C$, and $B G$ (is) equal to $A$. And $B K$ (is) the (rectangle contained) by $A$ and $B D$. For it is contained by $G B$ and $B D$, and $B G$ (is) equal to $A$. And $D L$ (is) the (rectangle contained) by $A$ and $D E$. For $D K$, that is to say $B G$ [Prop. 1.34], (is) equal to $A$. Similarly, $E H$ (is) also the (rectangle contained) by $A$ and $E C$. Thus, the (rectangle contained) by $A$ and $B C$ is equal to the (rectangles contained) by $A$

and $B D$, by $A$ and $D E$, and, finally, by $A$ and $E C$.
Thus, if there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity: $a(b+c+d+\cdots)=a b+a c+a d+\cdots$.










「Z.
 тò á $\pi o ̀ ~ \tau n ̃ ऽ ~ A B ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o v, ~ \tau o ̀ ~ \delta \grave{\varepsilon} \mathrm{AZ}$ tò ú $\pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{BA}$,








## Proposition $2^{\dagger}$

If a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole.


For let the straight-line $A B$ have been cut, at random, at point $C$. I say that the rectangle contained by $A B$ and $B C$, plus the rectangle contained by $B A$ and $A C$, is equal to the square on $A B$.

For let the square $A D E B$ have been described on $A B$ [Prop. 1.46], and let $C F$ have been drawn through $C$, parallel to either of $A D$ or $B E$ [Prop. 1.31].

So the (square) $A E$ is equal to the (rectangles) $A F$ and $C E$. And $A E$ is the square on $A B$. And $A F$ (is) the rectangle contained by the (straight-lines) $B A$ and $A C$. For it is contained by $D A$ and $A C$, and $A D$ (is) equal to $A B$. And $C E$ (is) the (rectangle contained) by $A B$ and $B C$. For $B E$ (is) equal to $A B$. Thus, the (rectangle contained) by $B A$ and $A C$, plus the (rectangle contained) by $A B$ and $B C$, is equal to the square on $A B$.

Thus, if a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straightline), and each of the pieces (of the straight-line), is equal to the square on the whole. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity：$a b+a c=a^{2}$ if $a=b+c$ ．

## $\gamma^{\prime}$.







Eủvยĩ $\gamma \dot{\alpha} \rho \dot{\eta} \mathrm{AB} \tau \varepsilon \tau \mu \dot{\gamma} \sigma \vartheta \omega$ ，$\dot{\omega} \varsigma$ है $\tau \cup \chi \varepsilon \nu, x \alpha \tau \grave{\alpha}$ 七ò $\Gamma$ ．

 $\mu \varepsilon \tau \grave{\alpha}$ тои̃ $\alpha \pi o ̀ ~ \tau \tilde{\eta} \varsigma ~ В \Gamma ~ \tau \varepsilon \tau \rho \alpha \gamma \omega ́ v o u . ~$








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 ठєĭそ $\alpha$ ．

## Proposition $3^{\dagger}$

If a straight－line is cut at random then the rectangle contained by the whole（straight－line），and one of the pieces（of the straight－line），is equal to the rectangle con－ tained by（both of）the pieces，and the square on the aforementioned piece．


For let the straight－line $A B$ have been cut，at random， at（point）$C$ ．I say that the rectangle contained by $A B$ and $B C$ is equal to the rectangle contained by $A C$ and $C B$ ，plus the square on $B C$ ．

For let the square $C D E B$ have been described on $C B$ ［Prop．1．46］，and let $E D$ have been drawn through to $F$ ，and let $A F$ have been drawn through $A$ ，parallel to either of $C D$ or $B E$［Prop．1．31］．So the（rectangle）$A E$ is equal to the（rectangle）$A D$ and the（square）$C E$ ．And $A E$ is the rectangle contained by $A B$ and $B C$ ．For it is contained by $A B$ and $B E$ ，and $B E$（is）equal to $B C$ ．And $A D$（is）the（rectangle contained）by $A C$ and $C B$ ．For $D C$（is）equal to $C B$ ．And $D B$（is）the square on $C B$ ． Thus，the rectangle contained by $A B$ and $B C$ is equal to the rectangle contained by $A C$ and $C B$ ，plus the square on $B C$ ．

Thus，if a straight－line is cut at random then the rect－ angle contained by the whole（straight－line），and one of the pieces（of the straight－line），is equal to the rectangle contained by（both of）the pieces，and the square on the aforementioned piece．（Which is）the very thing it was required to show．
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity：$(a+b) a=a b+a^{2}$ ．

$$
\delta^{\prime} .
$$





## Proposition $4^{\dagger}$

If a straight－line is cut at random then the square on the whole（straight－line）is equal to the（sum of the） squares on the pieces（of the straight－line），and twice the
$\gamma \omega v i ́ \varphi$.






 EB $\pi \alpha p \alpha ́ \lambda \lambda \eta \lambda o s ~ \eta ̈ \chi \vartheta \omega \dot{\eta} \Gamma Z$, ठı̀ $\delta \varepsilon ̀ ~ \tau о \tilde{u} H$ ó $\pi о \tau \varepsilon ́ p \alpha ~ \tau \widetilde{\omega} \nu \mathrm{AB}$,


























 òpখorevíc.

rectangle contained by the pieces.


For let the straight-line $A B$ have been cut, at random, at (point) $C$. I say that the square on $A B$ is equal to the (sum of the) squares on $A C$ and $C B$, and twice the rectangle contained by $A C$ and $C B$.

For let the square $A D E B$ have been described on $A B$ [Prop. 1.46], and let $B D$ have been joined, and let $C F$ have been drawn through $C$, parallel to either of $A D$ or $E B$ [Prop. 1.31], and let $H K$ have been drawn through $G$, parallel to either of $A B$ or $D E$ [Prop. 1.31]. And since $C F$ is parallel to $A D$, and $B D$ has fallen across them, the external angle $C G B$ is equal to the internal and opposite (angle) $A D B$ [Prop. 1.29]. But, $A D B$ is equal to $A B D$, since the side $B A$ is also equal to $A D$ [Prop. 1.5]. Thus, angle $C G B$ is also equal to $G B C$. So the side $B C$ is equal to the side $C G$ [Prop. 1.6]. But, $C B$ is equal to $G K$, and $C G$ to $K B$ [Prop. 1.34]. Thus, $G K$ is also equal to $K B$. Thus, $C G K B$ is equilateral. So I say that (it is) also right-angled. For since $C G$ is parallel to $B K$ [and the straight-line $C B$ has fallen across them], the angles $K B C$ and $G C B$ are thus equal to two right-angles [Prop. 1.29]. But $K B C$ (is) a right-angle. Thus, $B C G$ (is) also a rightangle. So the opposite (angles) $C G K$ and $G K B$ are also right-angles [Prop. 1.34]. Thus, $C G K B$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square. And it is on $C B$. So, for the same (reasons), $H F$ is also a square. And it is on $H G$, that is to say [on] $A C$ [Prop. 1.34]. Thus, the squares $H F$ and $K C$ are on $A C$ and $C B$ (respectively). And the (rectangle) $A G$ is equal to the (rectangle) $G E$ [Prop. 1.43]. And $A G$ is the (rectangle contained) by $A C$ and $C B$. For $G C$ (is) equal to $C B$. Thus, $G E$ is also equal to the (rectangle contained) by $A C$ and $C B$. Thus, the (rectangles) $A G$ and $G E$ are equal to twice the (rectangle contained) by $A C$ and $C B$. And $H F$ and $C K$ are the squares on $A C$ and $C B$ (respectively). Thus, the four (figures) $H F, C K$, $A G$, and $G E$ are equal to the (sum of the) squares on



$A C$ and $B C$, and twice the rectangle contained by $A C$ and $C B$. But, the (figures) $H F, C K, A G$, and $G E$ are (equivalent to) the whole of $A D E B$, which is the square on $A B$. Thus, the square on $A B$ is equal to the (sum of the) squares on $A C$ and $C B$, and twice the rectangle contained by $A C$ and $C B$.

Thus, if a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity: $(a+b)^{2}=a^{2}+b^{2}+2 a b$.

$$
\varepsilon^{\prime}
$$


























## Proposition $5^{\ddagger}$

If a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line).


For let any straight-line $A B$ have been cut-equally at $C$, and unequally at $D$. I say that the rectangle contained by $A D$ and $D B$, plus the square on $C D$, is equal to the square on $C B$.

For let the square $C E F B$ have been described on $C B$ [Prop. 1.46], and let $B E$ have been joined, and let $D G$ have been drawn through $D$, parallel to either of $C E$ or $B F$ [Prop. 1.31], and again let $K M$ have been drawn through $H$, parallel to either of $A B$ or $E F$ [Prop. 1.31], and again let $A K$ have been drawn through $A$, parallel to either of $C L$ or $B M$ [Prop. 1.31]. And since the complement $C H$ is equal to the complement $H F$ [Prop. 1.43], let the (square) $D M$ have been added to both. Thus, the whole (rectangle) $C M$ is equal to the whole (rectangle) $D F$. But, (rectangle) $C M$ is equal to (rectangle) $A L$, since $A C$ is also equal to $C B$ [Prop. 1.36]. Thus, (rectangle) $A L$ is also equal to (rectangle) $D F$. Let (rectangle) CH have been added to both. Thus, the whole (rectangle) $A H$ is equal to the gnomon $N O P$. But, $A H$

 A $\Delta, \Delta \mathrm{B} \pi \varepsilon \rho เ \varepsilon \chi o ́ \mu \varepsilon v o v ~ o ̉ p \vartheta o \gamma(\omega) v i o v ~ \mu \varepsilon \tau \alpha ̀ ~ \tau о u ̃ ~ \alpha ̉ \pi o ̀ ~ \tau \tilde{\eta} \varsigma \Gamma \Delta$





is the (rectangle contained) by $A D$ and $D B$. For $D H$ (is) equal to $D B$. Thus, the gnomon $N O P$ is also equal to the (rectangle contained) by $A D$ and $D B$. Let $L G$, which is equal to the (square) on $C D$, have been added to both. Thus, the gnomon $N O P$ and the (square) $L G$ are equal to the rectangle contained by $A D$ and $D B$, and the square on $C D$. But, the gnomon $N O P$ and the (square) $L G$ is (equivalent to) the whole square $C E F B$, which is on $C B$. Thus, the rectangle contained by $A D$ and $D B$, plus the square on $C D$, is equal to the square on $C B$.

Thus, if a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ Note the (presumably mistaken) double use of the label $M$ in the Greek text.
${ }^{\ddagger}$ This proposition is a geometric version of the algebraic identity: $a b+[(a+b) / 2-b]^{2}=[(a+b) / 2]^{2}$.

$$
\varsigma^{\prime}
$$



















## Proposition $6^{\dagger}$

If a straight-line is cut in half, and any straight-line added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added.


For let any straight-line $A B$ have been cut in half at point $C$, and let any straight-line $B D$ have been added to it straight-on. I say that the rectangle contained by $A D$ and $D B$, plus the square on $C B$, is equal to the square on $C D$.

For let the square $C E F D$ have been described on $C D$ [Prop. 1.46], and let $D E$ have been joined, and let $B G$ have been drawn through point $B$, parallel to either of $E C$ or $D F$ [Prop. 1.31], and let $K M$ have been drawn through point $H$, parallel to either of $A B$ or $E F$ [Prop. 1.31], and finally let $A K$ have been drawn











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through $A$ ，parallel to either of $C L$ or $D M$［Prop．1．31］．
Therefore，since $A C$ is equal to $C B$ ，（rectangle）$A L$ is also equal to（rectangle）$C H$［Prop．1．36］．But，（rectan－ gle）$C H$ is equal to（rectangle）$H F$［Prop．1．43］．Thus， （rectangle）$A L$ is also equal to（rectangle）$H F$ ．Let（rect－ angle）$C M$ have been added to both．Thus，the whole （rectangle）$A M$ is equal to the gnomon $N O P$ ．But，$A M$ is the（rectangle contained）by $A D$ and $D B$ ．For $D M$ is equal to $D B$ ．Thus，gnomon $N O P$ is also equal to the ［rectangle contained］by $A D$ and $D B$ ．Let $L G$ ，which is equal to the square on $B C$ ，have been added to both． Thus，the rectangle contained by $A D$ and $D B$ ，plus the square on $C B$ ，is equal to the gnomon $N O P$ and the （square）$L G$ ．But the gnomon $N O P$ and the（square） $L G$ is（equivalent to）the whole square $C E F D$ ，which is on $C D$ ．Thus，the rectangle contained by $A D$ and $D B$ ， plus the square on $C B$ ，is equal to the square on $C D$ ．

Thus，if a straight－line is cut in half，and any straight－ line added to it straight－on，then the rectangle contained by the whole（straight－line）with the（straight－line）hav－ ing being added，and the（straight－line）having being added，plus the square on half（of the original straight－ line），is equal to the square on the sum of half（of the original straight－line）and the（straight－line）having been added．（Which is）the very thing it was required to show．
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity：$(2 a+b) b+a^{2}=(a+b)^{2}$ ．

$$
\zeta^{\prime}
$$




 тєтраү⿳㇒⿵冂一兀．





## Proposition $7{ }^{\dagger}$

If a straight－line is cut at random then the sum of the squares on the whole（straight－line），and one of the pieces（of the straight－line），is equal to twice the rectan－ gle contained by the whole，and the said piece，and the square on the remaining piece．


For let any straight－line $A B$ have been cut，at random， at point $C$ ．I say that the（sum of the）squares on $A B$ and $B C$ is equal to twice the rectangle contained by $A B$ and

 жаı $\varkappa \alpha \tau \alpha \gamma \varepsilon \gamma \rho \alpha ́ \varphi \vartheta \vartheta \omega$ тò $\sigma \chi \tilde{\eta} \mu \alpha$ ．




















$B C$ ，and the square on $C A$ ．
For let the square $A D E B$ have been described on $A B$ ［Prop．1．46］，and let the（rest of）the figure have been drawn．

Therefore，since（rectangle）$A G$ is equal to（rectan－ gle）$G E$［Prop．1．43］，let the（square）$C F$ have been added to both．Thus，the whole（rectangle）$A F$ is equal to the whole（rectangle）$C E$ ．Thus，（rectangle）$A F$ plus （rectangle）$C E$ is double（rectangle）$A F$ ．But，（rectan－ gle）$A F$ plus（rectangle）$C E$ is the gnomon $K L M$ ，and the square $C F$ ．Thus，the gnomon $K L M$ ，and the square $C F$ ，is double the（rectangle）$A F$ ．But double the（rect－ angle）$A F$ is also twice the（rectangle contained）by $A B$ and $B C$ ．For $B F$（is）equal to $B C$ ．Thus，the gnomon $K L M$ ，and the square $C F$ ，are equal to twice the（rect－ angle contained）by $A B$ and $B C$ ．Let $D G$ ，which is the square on $A C$ ，have been added to both．Thus，the gnomon $K L M$ ，and the squares $B G$ and $G D$ ，are equal to twice the rectangle contained by $A B$ and $B C$ ，and the square on $A C$ ．But，the gnomon $K L M$ and the squares $B G$ and $G D$ is（equivalent to）the whole of $A D E B$ and $C F$ ，which are the squares on $A B$ and $B C$（respectively）． Thus，the（sum of the）squares on $A B$ and $B C$ is equal to twice the rectangle contained by $A B$ and $B C$ ，and the square on $A C$ ．

Thus，if a straight－line is cut at random then the sum of the squares on the whole（straight－line），and one of the pieces（of the straight－line），is equal to twice the rect－ angle contained by the whole，and the said piece，and the square on the remaining piece．（Which is）the very thing it was required to show．
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity：$(a+b)^{2}+a^{2}=2(a+b) a+b^{2}$ ．

## $\eta^{\prime}$ ．









 тєтраү⿳㇒⿵冂一叩．


 $\sigma \chi \tilde{\eta} \mu \alpha$ ．

## Proposition $8^{\dagger}$

If a straight－line is cut at random then four times the rectangle contained by the whole（straight－line），and one of the pieces（of the straight－line），plus the square on the remaining piece，is equal to the square described on the whole and the former piece，as on one（complete straight－ line）．

For let any straight－line $A B$ have been cut，at random， at point $C$ ．I say that four times the rectangle contained by $A B$ and $B C$ ，plus the square on $A C$ ，is equal to the square described on $A B$ and $B C$ ，as on one（complete straight－line）．

For let $B D$ have been produced in a straight－line ［with the straight－line $A B$ ］，and let $B D$ be made equal to $C B$［Prop．1．3］，and let the square $A E F D$ have been described on $A D$［Prop．1．46］，and let the（rest of the） figure have been drawn double．







































Therefore, since $C B$ is equal to $B D$, but $C B$ is equal to $G K$ [Prop. 1.34], and $B D$ to $K N$ [Prop. 1.34], $G K$ is thus also equal to $K N$. So, for the same (reasons), $Q R$ is equal to $R P$. And since $B C$ is equal to $B D$, and $G K$ to $K N$, (square) $C K$ is thus also equal to (square) $K D$, and (square) $G R$ to (square) $R N$ [Prop. 1.36]. But, (square) $C K$ is equal to (square) $R N$. For (they are) complements in the parallelogram $C P$ [Prop. 1.43]. Thus, (square) $K D$ is also equal to (square) $G R$. Thus, the four (squares) $D K, C K, G R$, and $R N$ are equal to one another. Thus, the four (taken together) are quadruple (square) $C K$. Again, since $C B$ is equal to $B D$, but $B D$ (is) equal to $B K$-that is to say, $C G$-and $C B$ is equal to $G K$ - that is to say, $G Q-C G$ is thus also equal to $G Q$. And since $C G$ is equal to $G Q$, and $Q R$ to $R P$, (rectangle) $A G$ is also equal to (rectangle) $M Q$, and (rectangle) $Q L$ to (rectangle) $R F$ [Prop. 1.36]. But, (rectangle) $M Q$ is equal to (rectangle) $Q L$. For (they are) complements in the parallelogram $M L$ [Prop. 1.43]. Thus, (rectangle) $A G$ is also equal to (rectangle) $R F$. Thus, the four (rectangles) $A G, M Q, Q L$, and $R F$ are equal to one another. Thus, the four (taken together) are quadruple (rectangle) $A G$. And it was also shown that the four (squares) $C K, K D, G R$, and $R N$ (taken together are) quadruple (square) $C K$. Thus, the eight (figures taken together), which comprise the gnomon $S T U$, are quadruple (rectangle) $A K$. And since $A K$ is the (rectangle contained) by $A B$ and $B D$, for $B K$ (is) equal to $B D$, four times the (rectangle contained) by $A B$ and $B D$ is quadruple (rectangle) $A K$. But the gnomon $S T U$ was also shown (to be equal to) quadruple (rectangle) $A K$. Thus, four times the (rectangle contained) by $A B$ and $B D$ is equal to the gnomon $S T U$. Let $O H$, which is equal to the square on $A C$, have been added to both. Thus, four times the rectangle contained by $A B$ and $B D$, plus the square on $A C$, is equal to the gnomon $S T U$, and the (square) $O H$. But,


the gnomon $S T U$ and the (square) $O H$ is (equivalent to) the whole square $A E F D$, which is on $A D$. Thus, four times the (rectangle contained) by $A B$ and $B D$, plus the (square) on $A C$, is equal to the square on $A D$. And $B D$ (is) equal to $B C$. Thus, four times the rectangle contained by $A B$ and $B C$, plus the square on $A C$, is equal to the (square) on $A D$, that is to say the square described on $A B$ and $B C$, as on one (complete straight-line).

Thus, if a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity: $4(a+b) a+b^{2}=[(a+b)+a]^{2}$.

## $\vartheta^{\prime}$.



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 ЕАГ, АЕГ $\mu \iota \tilde{\alpha}$ ỏpখñ






## Proposition $9^{\dagger}$

If a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces.


For let any straight-line $A B$ have been cut-equally at $C$, and unequally at $D$. I say that the (sum of the) squares on $A D$ and $D B$ is double the (sum of the squares) on $A C$ and $C D$.

For let $C E$ have been drawn from (point) $C$, at rightangles to $A B$ [Prop. 1.11], and let it be made equal to each of $A C$ and $C B$ [Prop. 1.3], and let $E A$ and $E B$ have been joined. And let $D F$ have been drawn through (point) $D$, parallel to $E C$ [Prop. 1.31], and (let) $F G$ (have been drawn) through (point) $F$, (parallel) to $A B$ [Prop. 1.31]. And let $A F$ have been joined. And since $A C$ is equal to $C E$, the angle $E A C$ is also equal to the (angle) $A E C$ [Prop. 1.5]. And since the (angle) at $C$ is a right-angle, the (sum of the) remaining angles (of triangle $A E C$ ), $E A C$ and $A E C$, is thus equal to one right-































angle [Prop. 1.32]. And they are equal. Thus, (angles) $C E A$ and $C A E$ are each half a right-angle. So, for the same (reasons), (angles) $C E B$ and $E B C$ are also each half a right-angle. Thus, the whole (angle) $A E B$ is a right-angle. And since $G E F$ is half a right-angle, and $E G F$ (is) a right-angle-for it is equal to the internal and opposite (angle) $E C B$ [Prop. 1.29]-the remaining (angle) $E F G$ is thus half a right-angle [Prop. 1.32]. Thus, angle $G E F$ [is] equal to $E F G$. So the side $E G$ is also equal to the (side) $G F$ [Prop. 1.6]. Again, since the angle at $B$ is half a right-angle, and (angle) $F D B$ (is) a right-angle-for again it is equal to the internal and opposite (angle) $E C B$ [Prop. 1.29]-the remaining (angle) $B F D$ is half a right-angle [Prop. 1.32]. Thus, the angle at $B$ (is) equal to $D F B$. So the side $F D$ is also equal to the side $D B$ [Prop. 1.6]. And since $A C$ is equal to $C E$, the (square) on $A C$ (is) also equal to the (square) on $C E$. Thus, the (sum of the) squares on $A C$ and $C E$ is double the (square) on $A C$. And the square on $E A$ is equal to the (sum of the) squares on $A C$ and $C E$. For angle $A C E$ (is) a right-angle [Prop. 1.47]. Thus, the (square) on $E A$ is double the (square) on $A C$. Again, since $E G$ is equal to $G F$, the (square) on $E G$ (is) also equal to the (square) on $G F$. Thus, the (sum of the squares) on $E G$ and $G F$ is double the square on $G F$. And the square on $E F$ is equal to the (sum of the) squares on $E G$ and $G F$ [Prop. 1.47]. Thus, the square on $E F$ is double the (square) on $G F$. And $G F$ (is) equal to $C D$ [Prop. 1.34]. Thus, the (square) on $E F$ is double the (square) on $C D$. And the (square) on $E A$ is also double the (square) on $A C$. Thus, the (sum of the) squares on $A E$ and $E F$ is double the (sum of the) squares on $A C$ and $C D$. And the square on $A F$ is equal to the (sum of the squares) on $A E$ and $E F$. For the angle $A E F$ is a right-angle [Prop. 1.47]. Thus, the square on $A F$ is double the (sum of the squares) on $A C$ and $C D$. And the (sum of the squares) on $A D$ and $D F$ (is) equal to the (square) on $A F$. For the angle at $D$ is a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on $A D$ and $D F$ is double the (sum of the) squares on $A C$ and $C D$. And $D F$ (is) equal to $D B$. Thus, the (sum of the) squares on $A D$ and $D B$ is double the (sum of the) squares on $A C$ and $C D$.

Thus, if a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity: $a^{2}+b^{2}=2\left[([a+b] / 2)^{2}+([a+b] / 2-b)^{2}\right]$.

## $i^{\prime}$.




















 $\pi \varepsilon \sigma о \tilde{u} \nu \tau \alpha l$. غ̇x $\beta \varepsilon \beta \lambda \dot{\eta} \sigma \vartheta \omega \sigma \alpha \nu$ x $\alpha \grave{\imath} \sigma \cup \mu \pi \iota \pi \tau \varepsilon ́ \tau \omega \sigma \alpha \nu x \alpha \tau \grave{\alpha}$ 七ò H ,
















## Proposition $10^{\dagger}$

If a straight-line is cut in half, and any straight-line added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line).


For let any straight-line $A B$ have been cut in half at (point) $C$, and let any straight-line $B D$ have been added to it straight-on. I say that the (sum of the) squares on $A D$ and $D B$ is double the (sum of the) squares on $A C$ and $C D$.

For let $C E$ have been drawn from point $C$, at rightangles to $A B$ [Prop. 1.11], and let it be made equal to each of $A C$ and $C B$ [Prop. 1.3], and let $E A$ and $E B$ have been joined. And let $E F$ have been drawn through $E$, parallel to $A D$ [Prop. 1.31], and let $F D$ have been drawn through $D$, parallel to $C E$ [Prop. 1.31]. And since some straight-line $E F$ falls across the parallel straight-lines $E C$ and $F D$, the (internal angles) $C E F$ and $E F D$ are thus equal to two right-angles [Prop. 1.29]. Thus, $F E B$ and $E F D$ are less than two right-angles. And (straight-lines) produced from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced in the direction of $B$ and $D$, the (straight-lines) $E B$ and $F D$ will meet. Let them have been produced, and let them meet together at $G$, and let $A G$ have been joined. And since $A C$ is equal to $C E$, angle $E A C$ is also equal to (angle) $A E C$ [Prop. 1.5]. And the (angle) at $C$ (is) a right-angle. Thus, $E A C$ and $A E C$ [are] each half a right-angle [Prop. 1.32]. So, for the same (reasons), $C E B$ and $E B C$ are also each half a right-angle. Thus, (angle) $A E B$ is a right-angle. And since $E B C$ is half a right-angle, $D B G$ (is) thus also half a rightangle [Prop. 1.15]. And $B D G$ is also a right-angle. For it is equal to $D C E$. For (they are) alternate (angles)


























[Prop. 1.29]. Thus, the remaining (angle) $D G B$ is half a right-angle. Thus, $D G B$ is equal to $D B G$. So side $B D$ is also equal to side $G D$ [Prop. 1.6]. Again, since $E G F$ is half a right-angle, and the (angle) at $F$ (is) a right-angle, for it is equal to the opposite (angle) at $C$ [Prop. 1.34], the remaining (angle) $F E G$ is thus half a right-angle. Thus, angle $E G F$ (is) equal to $F E G$. So the side $G F$ is also equal to the side $E F$ [Prop. 1.6]. And since [ $E C$ is equal to $C A$ ] the square on $E C$ is [also] equal to the square on $C A$. Thus, the (sum of the) squares on $E C$ and $C A$ is double the square on $C A$. And the (square) on $E A$ is equal to the (sum of the squares) on $E C$ and $C A$ [Prop. 1.47]. Thus, the square on $E A$ is double the square on $A C$. Again, since $F G$ is equal to $E F$, the (square) on $F G$ is also equal to the (square) on $F E$. Thus, the (sum of the squares) on $G F$ and $F E$ is double the (square) on $E F$. And the (square) on $E G$ is equal to the (sum of the squares) on $G F$ and $F E$ [Prop. 1.47]. Thus, the (square) on $E G$ is double the (square) on $E F$. And $E F$ (is) equal to $C D$ [Prop. 1.34]. Thus, the square on $E G$ is double the (square) on $C D$. But it was also shown that the (square) on $E A$ (is) double the (square) on $A C$. Thus, the (sum of the) squares on $A E$ and $E G$ is double the (sum of the) squares on $A C$ and $C D$. And the square on $A G$ is equal to the (sum of the) squares on $A E$ and $E G$ [Prop. 1.47]. Thus, the (square) on $A G$ is double the (sum of the squares) on $A C$ and $C D$. And the (sum of the squares) on $A D$ and $D G$ is equal to the (square) on $A G$ [Prop. 1.47]. Thus, the (sum of the) [squares] on $A D$ and $D G$ is double the (sum of the) [squares] on $A C$ and $C D$. And $D G$ (is) equal to $D B$. Thus, the (sum of the) [squares] on $A D$ and $D B$ is double the (sum of the) squares on $A C$ and $C D$.

Thus, if a straight-line is cut in half, and any straightline added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is a geometric version of the algebraic identity: $(2 a+b)^{2}+b^{2}=2\left[a^{2}+(a+b)^{2}\right]$.

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## Proposition $11^{\dagger}$

To cut a given straight-line such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.



 $\tau \mu \dot{\mu} \mu \alpha \tau о \varsigma ~ \tau \varepsilon \tau \rho \alpha \gamma \omega ́ v \omega$.


 EZ, xal̀ $\alpha \nu \alpha \gamma \varepsilon \gamma \rho \alpha ́ \varphi \vartheta \omega$ árò $\tau \tilde{n} \varsigma \mathrm{AZ} \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o \nu$ tò $Z \Theta$, xà̀


















 $\tau \widetilde{\varphi}$ а̉лò $\Theta \mathrm{A} \tau \varepsilon \tau \rho \alpha \gamma \omega ่ \nu \varphi$.





Let $A B$ be the given straight-line. So it is required to cut $A B$ such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straightline), is equal to the square on the remaining piece.

For let the square $A B D C$ have been described on $A B$ [Prop. 1.46], and let $A C$ have been cut in half at point $E$ [Prop. 1.10], and let $B E$ have been joined. And let $C A$ have been drawn through to (point) $F$, and let $E F$ be made equal to $B E$ [Prop. 1.3]. And let the square $F H$ have been described on $A F$ [Prop. 1.46], and let $G H$ have been drawn through to (point) $K$. I say that $A B$ has been cut at $H$ such as to make the rectangle contained by $A B$ and $B H$ equal to the square on $A H$.

For since the straight-line $A C$ has been cut in half at $E$, and $F A$ has been added to it, the rectangle contained by $C F$ and $F A$, plus the square on $A E$, is thus equal to the square on $E F$ [Prop. 2.6]. And $E F$ (is) equal to $E B$. Thus, the (rectangle contained) by $C F$ and $F A$, plus the (square) on $A E$, is equal to the (square) on $E B$. But, the (sum of the squares) on $B A$ and $A E$ is equal to the (square) on $E B$. For the angle at $A$ (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by $C F$ and $F A$, plus the (square) on $A E$, is equal to the (sum of the squares) on $B A$ and $A E$. Let the square on $A E$ have been subtracted from both. Thus, the remaining rectangle contained by $C F$ and $F A$ is equal to the square on $A B$. And $F K$ is the (rectangle contained) by $C F$ and $F A$. For $A F$ (is) equal to $F G$. And $A D$ (is) the (square) on $A B$. Thus, the (rectangle) $F K$ is equal to the (square) $A D$. Let (rectangle) $A K$ have been subtracted from both. Thus, the remaining (square) $F H$ is equal to the (rectangle) $H D$. And $H D$ is the (rectangle contained) by $A B$ and $B H$. For $A B$ (is) equal to $B D$. And $F H$ (is) the (square) on $A H$. Thus, the rectangle contained by $A B$
and $B H$ is equal to the square on $H A$.
Thus, the given straight-line $A B$ has been cut at (point) $H$ such as to make the rectangle contained by $A B$ and $B H$ equal to the square on $H A$. (Which is) the very thing it was required to do.
$\dagger$ This manner of cutting a straight-line—so that the ratio of the whole to the larger piece is equal to the ratio of the larger to the smaller piece-is
sometimes called the "Golden Section".

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 $\gamma \omega \nu i \alpha$.























## Proposition $12^{\dagger}$

In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle.


Let $A B C$ be an obtuse-angled triangle, having the angle $B A C$ obtuse. And let $B D$ be drawn from point $B$, perpendicular to $C A$ produced [Prop. 1.12]. I say that the square on $B C$ is greater than the (sum of the) squares on $B A$ and $A C$, by twice the rectangle contained by $C A$ and $A D$.

For since the straight-line $C D$ has been cut, at random, at point $A$, the (square) on $D C$ is thus equal to the (sum of the) squares on $C A$ and $A D$, and twice the rectangle contained by $C A$ and $A D$ [Prop. 2.4]. Let the (square) on $D B$ have been added to both. Thus, the (sum of the squares) on $C D$ and $D B$ is equal to the (sum of the) squares on $C A, A D$, and $D B$, and twice the [rectangle contained] by $C A$ and $A D$. But, the (square) on $C B$ is equal to the (sum of the squares) on $C D$ and $D B$. For the angle at $D$ (is) a right-angle [Prop. 1.47]. And the (square) on $A B$ (is) equal to the (sum of the squares) on $A D$ and $D B$ [Prop. 1.47]. Thus, the square on $C B$ is equal to the (sum of the) squares on $C A$ and $A B$, and twice the rectangle contained by $C A$ and $A D$. So the square on $C B$ is greater than the (sum of the) squares on




$C A$ and $A B$ by twice the rectangle contained by $C A$ and $A D$.

Thus, in obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to the well-known cosine formula: $B C^{2}=A B^{2}+A C^{2}-2 A B A C \cos B A C$, since $\cos B A C=-A D / A B$.

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 $\tau \widetilde{\omega} \nu \dot{\alpha} \pi \grave{o} \tau \widetilde{\omega} \nu \tau \grave{\eta} \nu$ ỏ $\xi \varepsilon i ̃ \alpha \nu \gamma \omega \nu i ́ \alpha \nu \pi \varepsilon \rho เ \varepsilon \chi о \cup \sigma \widetilde{\omega} \nu \pi \lambda \varepsilon \cup \rho \widetilde{\omega} \nu \tau \varepsilon-$





















## Proposition $13^{\dagger}$

In acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle.


Let $A B C$ be an acute-angled triangle, having the angle at (point) $B$ acute. And let $A D$ have been drawn from point $A$, perpendicular to $B C$ [Prop. 1.12]. I say that the square on $A C$ is less than the (sum of the) squares on $C B$ and $B A$, by twice the rectangle contained by $C B$ and $B D$.

For since the straight-line $C B$ has been cut, at random, at (point) $D$, the (sum of the) squares on $C B$ and $B D$ is thus equal to twice the rectangle contained by $C B$ and $B D$, and the square on $D C$ [Prop. 2.7]. Let the square on $D A$ have been added to both. Thus, the (sum of the) squares on $C B, B D$, and $D A$ is equal to twice the rectangle contained by $C B$ and $B D$, and the (sum of the) squares on $A D$ and $D C$. But, the (square) on $A B$ (is) equal to the (sum of the squares) on $B D$ and $D A$. For the angle at (point) $D$ is a right-angle [Prop. 1.47].
$\tau \widetilde{\omega} \nu \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \Gamma \mathrm{~B}, \mathrm{BA} \tau \varepsilon \tau \rho \alpha \not \omega \dot{\nu \omega \nu} \tau \widetilde{\varphi}$ $\delta i \varsigma ~ ن ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu \Gamma \mathrm{~B}, \mathrm{~B} \Delta$









And the (square) on $A C$ (is) equal to the (sum of the squares) on $A D$ and $D C$ [Prop. 1.47]. Thus, the (sum of the squares) on $C B$ and $B A$ is equal to the (square) on $A C$, and twice the (rectangle contained) by $C B$ and $B D$. So the (square) on $A C$ alone is less than the (sum of the) squares on $C B$ and $B A$ by twice the rectangle contained by $C B$ and $B D$.

Thus, in acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to the well-known cosine formula: $A C^{2}=A B^{2}+B C^{2}-2 A B B C \cos A B C$, since $\cos A B C=B D / A B$.
$\delta^{\prime}$.
 $\vartheta \alpha l$.






 $\mathrm{BE}, \mathrm{E} \Delta \mu \varepsilon i \zeta \omega \nu$ ह̀бтí. हैन $\tau \omega \mu \varepsilon i \zeta \omega \nu \dot{\eta} \mathrm{BE}, x \alpha i$ ह̇x $\beta \varepsilon \beta \lambda \dot{\eta} \sigma \vartheta \omega$ ย̇лì tò Z , x $\alpha i$ x










## Proposition 14

To construct a square equal to a given rectilinear figure.


Let $A$ be the given rectilinear figure. So it is required to construct a square equal to the rectilinear figure $A$.

For let the right-angled parallelogram $B D$, equal to the rectilinear figure $A$, have been constructed [Prop. 1.45]. Therefore, if $B E$ is equal to $E D$ then that (which) was prescribed has taken place. For the square $B D$, equal to the rectilinear figure $A$, has been constructed. And if not, then one of the (straight-lines) $B E$ or $E D$ is greater (than the other). Let $B E$ be greater, and let it have been produced to $F$, and let $E F$ be made equal to $E D$ [Prop. 1.3]. And let $B F$ have been cut in half at (point) $G$ [Prop. 1.10]. And, with center $G$, and radius one of the (straight-lines) $G B$ or $G F$, let the semi-circle $B H F$ have been drawn. And let $D E$ have been produced to $H$, and let $G H$ have been joined.

Therefore, since the straight-line $B F$ has been cutequally at $G$, and unequally at $E$-the rectangle con-










 $\pi o เ ท ̃ \sigma \alpha l$.
tained by $B E$ and $E F$, plus the square on $E G$, is thus equal to the square on $G F$ [Prop. 2.5]. And $G F$ (is) equal to $G H$. Thus, the (rectangle contained) by $B E$ and $E F$, plus the (square) on $G E$, is equal to the (square) on $G H$. And the (sum of the) squares on $H E$ and $E G$ is equal to the (square) on $G H$ [Prop. 1.47]. Thus, the (rectangle contained) by $B E$ and $E F$, plus the (square) on $G E$, is equal to the (sum of the squares) on $H E$ and $E G$. Let the square on $G E$ have been taken from both. Thus, the remaining rectangle contained by $B E$ and $E F$ is equal to the square on $E H$. But, $B D$ is the (rectangle contained) by $B E$ and $E F$. For $E F$ (is) equal to $E D$. Thus, the parallelogram $B D$ is equal to the square on $H E$. And $B D$ (is) equal to the rectilinear figure $A$. Thus, the rectilinear figure $A$ is also equal to the square (which) can be described on $E H$.

Thus, a square-(namely), that (which) can be described on $E H$-has been constructed, equal to the given rectilinear figure $A$. (Which is) the very thing it was required to do.

## ELEMENTS BOOK 3

## Fundamentals of Plane Geometry Involving Circles

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 $\chi \vartheta \varepsilon เ \sigma \widetilde{\omega} \nu$ ยひ่ทยเฮั้．

 $\dot{\eta} \gamma \omega v i \alpha$ ．


 ن $\pi^{\prime} \alpha \cup \cup \tau \widetilde{\omega} \nu \pi \varepsilon \rho ⿺ \varphi \varepsilon \rho \varepsilon i \alpha \varsigma$.



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## Definitions

1．Equal circles are（circles）whose diameters are equal，or whose（distances）from the centers（to the cir－ cumferences）are equal（i．e．，whose radii are equal）．

2．A straight－line said to touch a circle is any（straight－ line）which，meeting the circle and being produced，does not cut the circle．

3．Circles said to touch one another are any（circles） which，meeting one another，do not cut one another．

4．In a circle，straight－lines are said to be equally far from the center when the perpendiculars drawn to them from the center are equal．

5．And（that straight－line）is said to be further（from the center）on which the greater perpendicular falls （from the center）．

6．A segment of a circle is the figure contained by a straight－line and a circumference of a circle．

7．And the angle of a segment is that contained by a straight－line and a circumference of a circle．

8．And the angle in a segment is the angle contained by the joined straight－lines，when any point is taken on the circumference of a segment，and straight－lines are joined from it to the ends of the straight－line which is the base of the segment．

9．And when the straight－lines containing an angle cut off some circumference，the angle is said to stand upon that（circumference）．

10．And a sector of a circle is the figure contained by the straight－lines surrounding an angle，and the circum－ ference cut off by them，when the angle is constructed at the center of a circle．

11．Similar segments of circles are those accepting equal angles，or in which the angles are equal to one an－ other．

## Proposition 1

To find the center of a given circle．
Let $A B C$ be the given circle．So it is required to find the center of circle $A B C$ ．

Let some straight－line $A B$ have been drawn through $(A B C)$ ，at random，and let $(A B)$ have been cut in half at point $D$［Prop．1．9］．And let $D C$ have been drawn from $D$ ，at right－angles to $A B$［Prop．1．11］．And let（ $C D$ ）have been drawn through to $E$ ．And let $C E$ have been cut in half at $F$［Prop．1．9］．I say that（point）$F$ is the center of the［circle］$A B C$ ．

For（if）not then，if possible，let $G$（be the center of the circle），and let $G A, G D$ ，and $G B$ have been joined．And since $A D$ is equal to $D B$ ，and $D G$（is）common，the two









## По́рьгид.




$\dagger$ The Greek text has " $G D, D B$ ", which is obviously a mistake.

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(straight-lines) $A D, D G$ are equal to the two (straightlines) $B D, D G,{ }^{\dagger}$ respectively. And the base $G A$ is equal to the base $G B$. For (they are both) radii. Thus, angle $A D G$ is equal to angle $G D B$ [Prop. 1.8]. And when a straight-line stood upon (another) straight-line make adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, $G D B$ is a right-angle. And $F D B$ is also a right-angle. Thus, $F D B$ (is) equal to $G D B$, the greater to the lesser. The very thing is impossible. Thus, (point) $G$ is not the center of the circle $A B C$. So, similarly, we can show that neither is any other (point) except $F$.


Thus, point $F$ is the center of the [circle] $A B C$.

## Corollary

So, from this, (it is) manifest that if any straight-line in a circle cuts any (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line). - (Which is) the very thing it was required to do.

## Proposition 2

If two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle.

Let $A B C$ be a circle, and let two points $A$ and $B$ have been taken at random on its circumference. I say that the straight-line joining $A$ to $B$ will fall inside the circle.

For (if) not then, if possible, let it fall outside (the circle), like $A E B$ (in the figure). And let the center of the circle $A B C$ have been found [Prop. 3.1], and let it be (at point) $D$. And let $D A$ and $D B$ have been joined, and let $D F E$ have been drawn through.

Therefore, since $D A$ is equal to $D B$, the angle $D A E$
$\pi \lambda \varepsilon \cup \rho \dot{\alpha} \pi \rho о \sigma \varepsilon \chi \beta \dot{\beta} \beta \lambda \eta \tau \alpha l \dot{\eta} \mathrm{AEB}, \mu \varepsilon i \zeta \omega \nu$ ӓ $\rho \alpha \dot{\eta}$ ن́ $\pi o ̀ ~ \Delta E B$













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(is) thus also equal to $D B E$ [Prop. 1.5]. And since in triangle $D A E$ the one side, $A E B$, has been produced, angle $D E B$ (is) thus greater than $D A E$ [Prop. 1.16]. And $D A E$ (is) equal to $D B E$ [Prop. 1.5]. Thus, $D E B$ (is) greater than $D B E$. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, $D B$ (is) greater than $D E$. And $D B$ (is) equal to $D F$. Thus, $D F$ (is) greater than $D E$, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining $A$ to $B$ will not fall outside the circle. So, similarly, we can show that neither (will it fall) on the circumference itself. Thus, (it will fall) inside (the circle).


Thus, if two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle. (Which is) the very thing it was required to show.

## Proposition 3

In a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half.

Let $A B C$ be a circle, and, within it, let some straightline through the center, $C D$, cut in half some straight-line not through the center, $A B$, at the point $F$. I say that $(C D)$ also cuts $(A B)$ at right-angles.

For let the center of the circle $A B C$ have been found [Prop. 3.1], and let it be (at point) $E$, and let $E A$ and $E B$ have been joined.

And since $A F$ is equal to $F B$, and $F E$ (is) common, two (sides of triangle $A F E$ ) [are] equal to two (sides of triangle $B F E$ ). And the base $E A$ (is) equal to the base $E B$. Thus, angle $A F E$ is equal to angle $B F E$ [Prop. 1.8]. And when a straight-line stood upon (another) straightline makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, $A F E$ and $B F E$ are each right-angles. Thus, the





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(straight-line) $C D$, which is through the center and cuts in half the (straight-line) $A B$, which is not through the center, also cuts $(A B)$ at right-angles.


And so let $C D$ cut $A B$ at right-angles. I say that it also cuts $(A B)$ in half. That is to say, that $A F$ is equal to $F B$.

For, with the same construction, since $E A$ is equal to $E B$, angle $E A F$ is also equal to $E B F$ [Prop. 1.5]. And the right-angle $A F E$ is also equal to the right-angle $B F E$. Thus, $E A F$ and $E F B$ are two triangles having two angles equal to two angles, and one side equal to one side-(namely), their common (side) $E F$, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, $A F$ (is) equal to $F B$.

Thus, in a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half. (Which is) the very thing it was required to show.

## Proposition 4

In a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half.

Let $A B C D$ be a circle, and within it, let two straightlines, $A C$ and $B D$, which are not through the center, cut one another at (point) $E$. I say that they do not cut one another in half.

For, if possible, let them cut one another in half, such that $A E$ is equal to $E C$, and $B E$ to $E D$. And let the center of the circle $A B C D$ have been found [Prop. 3.1], and let it be (at point) $F$, and let $F E$ have been joined.

Therefore, since some straight-line through the center, $F E$, cuts in half some straight-line not through the center, $A C$, it also cuts it at right-angles [Prop. 3.3]. Thus, $F E A$ is a right-angle. Again, since some straight-line $F E$
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cuts in half some straight－line $B D$ ，it also cuts it at right－ angles［Prop．3．3］．Thus，$F E B$（is）a right－angle．But $F E A$ was also shown（to be）a right－angle．Thus，$F E A$ （is）equal to $F E B$ ，the lesser to the greater．The very thing is impossible．Thus，$A C$ and $B D$ do not cut one another in half．


Thus，in a circle，if two straight－lines，which are not through the center，cut one another then they do not cut one another in half．（Which is）the very thing it was re－ quired to show．

## Proposition 5

If two circles cut one another then they will not have the same center．


For let the two circles $A B C$ and $C D G$ cut one another at points $B$ and $C$ ．I say that they will not have the same center．

For，if possible，let $E$ be（the common center），and let $E C$ have been joined，and let $E F G$ have been drawn through（the two circles），at random．And since point $E$ is the center of the circle $A B C, E C$ is equal to $E F$ ． Again，since point $E$ is the center of the circle $C D G, E C$ is equal to $E G$ ．But $E C$ was also shown（to be）equal to $E F$ ．Thus，$E F$ is also equal to $E G$ ，the lesser to the greater．The very thing is impossible．Thus，point $E$ is not

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 $\delta เ ท ่ \chi \vartheta \omega, \dot{\omega} \varsigma$ हैт $\tau \chi \varepsilon \nu, \dot{\eta}$ ZEB.









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the (common) center of the circles $A B C$ and $C D G$.
Thus, if two circles cut one another then they will not have the same center. (Which is) the very thing it was required to show.

## Proposition 6

If two circles touch one another then they will not have the same center.


For let the two circles $A B C$ and $C D E$ touch one another at point $C$. I say that they will not have the same center.

For, if possible, let $F$ be (the common center), and let $F C$ have been joined, and let $F E B$ have been drawn through (the two circles), at random.

Therefore, since point $F$ is the center of the circle $A B C, F C$ is equal to $F B$. Again, since point $F$ is the center of the circle $C D E, F C$ is equal to $F E$. But $F C$ was shown (to be) equal to $F B$. Thus, $F E$ is also equal to $F B$, the lesser to the greater. The very thing is impossible. Thus, point $F$ is not the (common) center of the circles $A B C$ and $C D E$.

Thus, if two circles touch one another then they will not have the same center. (Which is) the very thing it was required to show.

## Proposition 7

If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straightlines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straightline) nearer ${ }^{\dagger}$ to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each






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 $\dot{\eta} \mu \varepsilon ̀ \nu \mathrm{ZB} \tau \tilde{\eta} \varsigma \mathrm{Z} \Gamma, \dot{\eta} \delta$ ¿̀ $\mathrm{Z} \Gamma$ 七 $\tilde{\eta} \varsigma \mathrm{ZH}$ ．













（side）of the least（straight－line）．


Let $A B C D$ be a circle，and let $A D$ be its diameter，and let some point $F$ ，which is not the center of the circle， have been taken on $A D$ ．Let $E$ be the center of the circle． And let some straight－lines，$F B, F C$ ，and $F G$ ，radiate from $F$ towards（the circumference of）circle $A B C D$ ．I say that $F A$ is the greatest（straight－line），$F D$ the least， and of the others，$F B$（is）greater than $F C$ ，and $F C$ than $F G$ ．

For let $B E, C E$ ，and $G E$ have been joined．And since for every triangle（any）two sides are greater than the remaining（side）［Prop．1．20］，$E B$ and $E F$ is thus greater than $B F$ ．And $A E$（is）equal to $B E$［thus，$B E$ and $E F$ is equal to $A F]$ ．Thus，$A F$（is）greater than $B F$ ．Again， since $B E$ is equal to $C E$ ，and $F E$（is）common，the two （straight－lines）$B E, E F$ are equal to the two（straight－ lines）$C E, E F$（respectively）．But，angle $B E F$（is）also greater than angle $C E F .{ }^{\ddagger}$ Thus，the base $B F$ is greater than the base $C F$ ．Thus，the base $B F$ is greater than the base $C F$［Prop．1．24］．So，for the same（reasons），$C F$ is also greater than $F G$ ．

Again，since $G F$ and $F E$ are greater than $E G$ ［Prop．1．20］，and $E G$（is）equal to $E D, G F$ and $F E$ are thus greater than $E D$ ．Let $E F$ have been taken from both．Thus，the remainder $G F$ is greater than the re－ mainder $F D$ ．Thus，$F A$（is）the greatest（straight－line）， $F D$ the least，and $F B$（is）greater than $F C$ ，and $F C$ than $F G$ ．

I also say that from point $F$ only two equal（straight－ lines）will radiate towards（the circumference of）circle $A B C D$ ，（one）on each（side）of the least（straight－line） $F D$ ．For let the（angle）$F E H$ ，equal to angle $G E F$ ，have been constructed on the straight－line $E F$ ，at the point $E$ on it［Prop．1．23］，and let $F H$ have been joined．There－ fore，since $G E$ is equal to $E H$ ，and $E F$（is）common，








${ }^{\dagger}$ Presumably, in an angular sense.
$\ddagger$ This is not proved, except by reference to the figure.

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the two (straight-lines) $G E, E F$ are equal to the two (straight-lines) $H E, E F$ (respectively). And angle $G E F$ (is) equal to angle $H E F$. Thus, the base $F G$ is equal to the base $F H$ [Prop. 1.4]. So I say that another (straightline) equal to $F G$ will not radiate towards (the circumference of) the circle from point $F$. For, if possible, let $F K$ (so) radiate. And since $F K$ is equal to $F G$, but $F H$ [is equal] to $F G, F K$ is thus also equal to $F H$, the nearer to the (straight-line) through the center equal to the further away. The very thing (is) impossible. Thus, another (straight-line) equal to $G F$ will not radiate from the point $F$ towards (the circumference of) the circle. Thus, (there is) only one (such straight-line).

Thus, if some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the same point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

## Proposition 8

If some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer ${ }^{\dagger}$ to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straightlines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Let $A B C$ be a circle, and let some point $D$ have been taken outside $A B C$, and from it let some straight-lines, $D A, D E, D F$, and $D C$, have been drawn through (the circle), and let $D A$ be through the center. I say that for the straight-lines radiating towards the concave (part of

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the）circumference，$A E F C$ ，the greatest is the one（pass－ ing）through the center，（namely）$A D$ ，and（that）$D E$（is） greater than $D F$ ，and $D F$ than $D C$ ．For the straight－lines radiating towards the convex（part of the）circumference， $H L K G$ ，the least is the one between the point and the di－ ameter $A G$ ，（namely）$D G$ ，and a（straight－line）nearer to the least（straight－line）$D G$ is always less than one far－ ther away，（so that）$D K$（is less）than $D L$ ，and $D L$ than than $D H$ ．


For let the center of the circle have been found ［Prop．3．1］，and let it be（at point）$M$［Prop．3．1］．And let $M E, M F, M C, M K, M L$ ，and $M H$ have been joined．

And since $A M$ is equal to $E M$ ，let $M D$ have been added to both．Thus，$A D$ is equal to $E M$ and $M D$ ．But， $E M$ and $M D$ is greater than $E D$［Prop．1．20］．Thus， $A D$ is also greater than $E D$ ．Again，since $M E$ is equal to $M F$ ，and $M D$（is）common，the（straight－lines）$E M$ ， $M D$ are thus equal to $F M, M D$ ．And angle $E M D$ is greater than angle $F M D .{ }^{\ddagger}$ Thus，the base $E D$ is greater than the base $F D$［Prop．1．24］．So，similarly，we can show that $F D$ is also greater than $C D$ ．Thus，$A D$（is）the greatest（straight－line），and $D E$（is）greater than $D F$ ， and $D F$ than $D C$ ．

And since $M K$ and $K D$ is greater than $M D$［Prop． 1．20］，and $M G$（is）equal to $M K$ ，the remainder $K D$ is thus greater than the remainder $G D$ ．So $G D$ is less than $K D$ ．And since in triangle $M L D$ ，the two inter－ nal straight－lines $M K$ and $K D$ were constructed on one of the sides，$M D$ ，then $M K$ and $K D$ are thus less than $M L$ and $L D$［Prop．1．21］．And $M K$（is）equal to $M L$ ． Thus，the remainder $D K$ is less than the remainder $D L$ ． So，similarly，we can show that $D L$ is also less than $D H$ ． Thus，$D G$（is）the least（straight－line），and $D K$（is）less than $D L$ ，and $D L$ than $D H$ ．

I also say that only two equal（straight－lines）will radi－








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ate from point $D$ towards (the circumference of) the circle, (one) on each (side) on the least (straight-line), $D G$. Let the angle $D M B$, equal to angle $K M D$, have been constructed on the straight-line $M D$, at the point $M$ on it [Prop. 1.23], and let $D B$ have been joined. And since $M K$ is equal to $M B$, and $M D$ (is) common, the two (straight-lines) $K M, M D$ are equal to the two (straightlines) $B M, M D$, respectively. And angle $K M D$ (is) equal to angle $B M D$. Thus, the base $D K$ is equal to the base $D B$ [Prop. 1.4]. [So] I say that another (straightline) equal to $D K$ will not radiate towards the (circumference of the) circle from point $D$. For, if possible, let (such a straight-line) radiate, and let it be $D N$. Therefore, since $D K$ is equal to $D N$, but $D K$ is equal to $D B$, then $D B$ is thus also equal to $D N$, (so that) a (straightline) nearer to the least (straight-line) $D G$ [is] equal to one further away. The very thing was shown (to be) impossible. Thus, not more than two equal (straight-lines) will radiate towards (the circumference of) circle $A B C$ from point $D$, (one) on each side of the least (straightline) $D G$.

Thus, if some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straightlines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer to the (straightline) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straightlines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

## Proposition 9

If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle.

Let $A B C$ be a circle, and $D$ a point inside it, and let more than two equal straight-lines, $D A, D B$, and $D C$, radiate from $D$ towards (the circumference of) circle $A B C$.




















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I say that point $D$ is the center of circle $A B C$.


For let $A B$ and $B C$ have been joined, and (then) have been cut in half at points $E$ and $F$ (respectively) [Prop. 1.10]. And $E D$ and $F D$ being joined, let them have been drawn through to points $G, K, H$, and $L$.

Therefore, since $A E$ is equal to $E B$, and $E D$ (is) common, the two (straight-lines) $A E, E D$ are equal to the two (straight-lines) $B E, E D$ (respectively). And the base $D A$ (is) equal to the base $D B$. Thus, angle $A E D$ is equal to angle $B E D$ [Prop. 1.8]. Thus, angles $A E D$ and $B E D$ (are) each right-angles [Def. 1.10]. Thus, $G K$ cuts $A B$ in half, and at right-angles. And since, if some straight-line in a circle cuts some (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line) [Prop. 3.1 corr.], the center of the circle is thus on $G K$. So, for the same (reasons), the center of circle $A B C$ is also on $H L$. And the straight-lines $G K$ and $H L$ have no common (point) other than point $D$. Thus, point $D$ is the center of circle $A B C$.

Thus, if some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle. (Which is) the very thing it was required to show.

## Proposition 10

A circle does not cut a(nother) circle at more than two points.

For, if possible, let the circle $A B C$ cut the circle $D E F$ at more than two points, $B, G, F$, and $H$. And $B H$ and $B G$ being joined, let them (then) have been cut in half at points $K$ and $L$ (respectively). And $K C$ and $L M$ being drawn at right-angles to $B H$ and $B G$ from $K$ and $L$ (respectively) [Prop. 1.11], let them (then) have been drawn through to points $A$ and $E$ (respectively).















## $i \alpha^{\prime}$.



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$\Delta$ úo $\gamma \dot{\alpha} \rho$ кúx $\lambda$ ol oi $\mathrm{AB} \mathrm{\Gamma}, \mathrm{~A} \Delta \mathrm{E} \dot{\varepsilon} \varphi \alpha \pi \tau \varepsilon ́ \sigma \vartheta \omega \sigma \alpha \nu \dot{\alpha} \lambda \lambda \dot{\eta} \lambda \omega \nu$



 غ่ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu \alpha i \mathrm{AZ}, \mathrm{AH}$.





 $\pi \varepsilon \sigma \varepsilon \tau ँ \tau \alpha$.


Therefore, since in circle $A B C$ some straight-line $A C$ cuts some (other) straight-line $B H$ in half, and at right-angles, the center of circle $A B C$ is thus on $A C$ [Prop. 3.1 corr.]. Again, since in the same circle $A B C$ some straight-line $N O$ cuts some (other straight-line) $B G$ in half, and at right-angles, the center of circle $A B C$ is thus on $N O$ [Prop. 3.1 corr.]. And it was also shown (to be) on $A C$. And the straight-lines $A C$ and $N O$ meet at no other (point) than $P$. Thus, point $P$ is the center of circle $A B C$. So, similarly, we can show that $P$ is also the center of circle $D E F$. Thus, two circles cutting one another, $A B C$ and $D E F$, have the same center $P$. The very thing is impossible [Prop. 3.5].

Thus, a circle does not cut a(nother) circle at more than two points. (Which is) the very thing it was required to show.

## Proposition 11

If two circles touch one another internally, and their centers are found, then the straight-line joining their centers, being produced, will fall upon the point of union of the circles.

For let two circles, $A B C$ and $A D E$, touch one another internally at point $A$, and let the center $F$ of circle $A B C$ have been found [Prop. 3.1], and (the center) $G$ of (circle) $A D E$ [Prop. 3.1]. I say that the straight-line joining $G$ to $F$, being produced, will fall on $A$.

For (if) not then, if possible, let it fall like $F G H$ (in the figure), and let $A F$ and $A G$ have been joined.

Therefore, since $A G$ and $G F$ is greater than $F A$, that is to say $F H$ [Prop. 1.20], let $F G$ have been taken from both. Thus, the remainder $A G$ is greater than the remainder $G H$. And $A G$ (is) equal to $G D$. Thus, $G D$ is also greater than $G H$, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining $F$ to $G$ will not fall outside (one circle but inside the other). Thus, it will fall upon the point of union (of the circles)






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$\Delta u ́ o ~ \gamma \dot{\alpha} \rho ~ x u ́ x \lambda o l ~ o i ~ A B \Gamma, ~ A ~ \Delta E ~ ह ं \varphi \alpha \pi \tau \varepsilon ́ \sigma \vartheta \omega \sigma \alpha \nu ~ \dot{\alpha} \lambda \lambda \eta \dot{n} \lambda \omega \nu$









at point $A$.


Thus, if two circles touch one another internally, [and their centers are found], then the straight-line joining their centers, [being produced], will fall upon the point of union of the circles. (Which is) the very thing it was required to show.

## Proposition 12

If two circles touch one another externally then the (straight-line) joining their centers will go through the point of union.


For let two circles, $A B C$ and $A D E$, touch one another externally at point $A$, and let the center $F$ of $A B C$ have been found [Prop. 3.1], and (the center) $G$ of $A D E$ [Prop. 3.1]. I say that the straight-line joining $F$ to $G$ will go through the point of union at $A$.

For (if) not then, if possible, let it go like $F C D G$ (in the figure), and let $A F$ and $A G$ have been joined.

Therefore, since point $F$ is the center of circle $A B C$, $F A$ is equal to $F C$. Again, since point $G$ is the center of circle $A D E, G A$ is equal to $G D$. And $F A$ was also shown









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Kúx入os xúx入ou oủx $\dot{\varepsilon} \varphi \alpha ́ \pi \tau \varepsilon \tau \alpha<~ x \alpha \tau \grave{\alpha} \pi \lambda \varepsilon i ́ o v \alpha ~ \sigma \eta \mu \varepsilon i ̃ \alpha ~ \grave{\eta}$



 B．
 §è $\mathrm{EBZ} \Delta$ тò $\Theta$ ．


 $\mu \varepsilon i \zeta \omega \nu \alpha \nsim \alpha \dot{\eta} B H$ т $\tilde{\varsigma} \Theta \Delta \cdot \pi o \lambda \lambda \widetilde{\varphi} \alpha \nsim \alpha \mu \varepsilon i \zeta \omega \nu \dot{\eta} B \Theta \tau \tilde{\eta} \varsigma \Theta \Delta$ ．


 $x \alpha \tau \dot{\alpha} \pi \lambda \varepsilon$ íov $\alpha$ $\sigma \eta \mu \varepsilon i \alpha \alpha$ $\hat{\eta}$ हैv．

Ei үàp ठuvatóv，xúx入os ò АГК xúx入ou тoũ АВГロ
 غ̇лદ $દ \cup ́ \chi \vartheta \omega \dot{\eta} А \Gamma$ ．






（to be）equal to $F C$ ．Thus，the（straight－lines）$F A$ and $A G$ are equal to the（straight－lines）$F C$ and $G D$ ．So the whole of $F G$ is greater than $F A$ and $A G$ ．But，（it is）also less［Prop．1．20］．The very thing is impossible．Thus，the straight－line joining $F$ to $G$ cannot not go through the point of union at $A$ ．Thus，（it will go）through it．

Thus，if two circles touch one another externally then the［straight－line］joining their centers will go through the point of union．（Which is）the very thing it was re－ quired to show．

## Proposition 13

A circle does not touch a（nother）circle at more than one point，whether they touch internally or externally．


For，if possible，let circle $A B D C^{\dagger}$ touch circle $E B F D$－ first of all，internally－at more than one point，$D$ and $B$ ．

And let the center $G$ of circle $A B D C$ have been found ［Prop．3．1］，and（the center）$H$ of $E B F D$［Prop．3．1］．

Thus，the（straight－line）joining $G$ and $H$ will fall on $B$ and $D$［Prop．3．11］．Let it fall like $B G H D$（in the figure）．And since point $G$ is the center of circle $A B D C$ ， $B G$ is equal to $G D$ ．Thus，$B G$（is）greater than $H D$ ． Thus，$B H$（is）much greater than $H D$ ．Again，since point $H$ is the center of circle $E B F D, B H$ is equal to $H D$ ． But it was also shown（to be）much greater than it．The very thing（is）impossible．Thus，a circle does not touch a（nother）circle internally at more than one point．

So，I say that neither（does it touch）externally（at more than one point）．

For，if possible，let circle $A C K$ touch circle $A B D C$ externally at more than one point，$A$ and $C$ ．And let $A C$ have been joined．

Therefore，since two points，$A$ and $C$ ，have been taken at random on the circumference of each of the circles $A B D C$ and $A C K$ ，the straight－line joining the points will fall inside each（circle）［Prop．3．2］．But，it fell inside $A B D C$ ，and outside $A C K$［Def．3．3］．The very thing

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${ }^{\dagger}$ The Greek text has " $A B C D$ ", which is obviously a mistake.

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 غ̇бti x $\alpha i l \dot{\eta} \mathrm{AB} \tau \tilde{n} \Gamma \Delta$.
(is) absurd. Thus, a circle does not touch a(nother) circle externally at more than one point. And it was shown that neither (does it) internally.

Thus, a circle does not touch a(nother) circle at more than one point, whether they touch internally or externally. (Which is) the very thing it was required to show.

## Proposition 14

In a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another.


Let $A B D C^{\dagger}$ be a circle, and let $A B$ and $C D$ be equal straight-lines within it. I say that $A B$ and $C D$ are equally far from the center.

For let the center of circle $A B D C$ have been found [Prop. 3.1], and let it be (at) $E$. And let $E F$ and $E G$ have been drawn from (point) $E$, perpendicular to $A B$ and $C D$ (respectively) [Prop. 1.12]. And let $A E$ and $E C$ have been joined.

Therefore, since some straight-line, $E F$, through the center (of the circle), cuts some (other) straight-line, $A B$, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, $A F$ (is) equal to $F B$. Thus, $A B$ (is) double $A F$. So, for the same (reasons), $C D$ is also double $C G$. And $A B$ is equal to $C D$. Thus, $A F$ (is) also equal to $C G$. And since $A E$ is equal to $E C$, the (square) on $A E$ (is) also equal to the (square) on $E C$. But, the (sum of the squares) on $A F$ and $E F$ (is) equal to the (square) on $A E$. For the angle at $F$ (is) a rightangle [Prop. 1.47]. And the (sum of the squares) on $E G$ and $G C$ (is) equal to the (square) on $E C$. For the angle at $G$ (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on $A F$ and $F E$ is equal to the (sum of the squares) on $C G$ and $G E$, of which the (square) on $A F$ is equal to the (square) on $C G$. For $A F$ is equal to $C G$.








 $\dot{\eta} \Gamma \Delta$ ' 亿ø $\quad$ व̈p $\alpha \dot{\eta}$ AB $\tau \tilde{n} \Gamma \Delta$.



${ }^{\dagger}$ The Greek text has " $A B C D$ ", which is obviously a mistake.

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 $\mu \varepsilon i \zeta \omega \nu$ д̀̀ $\dot{\eta}$ ВГ $\tau \tilde{\eta} \varsigma \mathrm{ZH}$.




 EH.








Thus, the remaining (square) on $F E$ is equal to the (remaining square) on $E G$. Thus, $E F$ (is) equal to $E G$. And straight-lines in a circle are said to be equally far from the center when perpendicular (straight-lines) which are drawn to them from the center are equal [Def. 3.4]. Thus, $A B$ and $C D$ are equally far from the center.

So, let the straight-lines $A B$ and $C D$ be equally far from the center. That is to say, let $E F$ be equal to $E G$. I say that $A B$ is also equal to $C D$.

For, with the same construction, we can, similarly, show that $A B$ is double $A F$, and $C D$ (double) $C G$. And since $A E$ is equal to $C E$, the (square) on $A E$ is equal to the (square) on $C E$. But, the (sum of the squares) on $E F$ and $F A$ is equal to the (square) on $A E$ [Prop. 1.47]. And the (sum of the squares) on $E G$ and $G C$ (is) equal to the (square) on $C E$ [Prop. 1.47]. Thus, the (sum of the squares) on $E F$ and $F A$ is equal to the (sum of the squares) on $E G$ and $G C$, of which the (square) on $E F$ is equal to the (square) on $E G$. For $E F$ (is) equal to $E G$. Thus, the remaining (square) on $A F$ is equal to the (remaining square) on $C G$. Thus, $A F$ (is) equal to $C G$. And $A B$ is double $A F$, and $C D$ double $C G$. Thus, $A B$ (is) equal to $C D$.

Thus, in a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another. (Which is) the very thing it was required to show.

## Proposition 15

In a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away.

Let $A B C D$ be a circle, and let $A D$ be its diameter, and $E$ (its) center. And let $B C$ be nearer to the diameter $A D,{ }^{\dagger}$ and $F G$ further away. I say that $A D$ is the greatest (straight-line), and $B C$ (is) greater than $F G$.

For let $E H$ and $E K$ have been drawn from the center $E$, at right-angles to $B C$ and $F G$ (respectively) [Prop. 1.12]. And since $B C$ is nearer to the center, and $F G$ further away, $E K$ (is) thus greater than $E H$ [Def. 3.5]. Let $E L$ be made equal to $E H$ [Prop. 1.3]. And $L M$ being drawn through $L$, at right-angles to $E K$ [Prop. 1.11], let it have been drawn through to $N$. And let $M E, E N, F E$, and $E G$ have been joined.

And since $E H$ is equal to $E L, B C$ is also equal to $M N$ [Prop. 3.14]. Again, since $A E$ is equal to $E M$, and $E D$ to $E N, A D$ is thus equal to $M E$ and $E N$. But, $M E$ and $E N$ is greater than $M N$ [Prop. 1.20] [also $A D$ is
$\dot{\eta} \mathrm{MN} \beta \alpha \dot{\alpha} \sigma \varepsilon \omega \varsigma \tau \tilde{\eta} \varsigma \mathrm{ZH} \mu \varepsilon i \zeta \omega \nu$ ह̇ $\sigma \tau i \downarrow . \quad \dot{\alpha} \lambda \lambda \dot{\alpha} \dot{\eta} \mathrm{MN} \tau \tilde{n} \mathrm{~B} \Gamma$






greater than $M N]$, and $M N$ (is) equal to $B C$. Thus, $A D$ is greater than $B C$. And since the two (straight-lines) $M E, E N$ are equal to the two (straight-lines) $F E, E G$ (respectively), and angle $M E N$ [is] greater than angle $F E G{ }^{\ddagger}$ the base $M N$ is thus greater than the base $F G$ [Prop. 1.24]. But, $M N$ was shown (to be) equal to $B C$ [(so) $B C$ is also greater than $F G$ ]. Thus, the diameter $A D$ (is) the greatest (straight-line), and $B C$ (is) greater than $F G$.


Thus, in a circle, a diameter (is) the greatest (straightline), and for the others, a (straight-line) nearer to the center is always greater than one further away. (Which is) the very thing it was required to show.
${ }^{\dagger}$ Euclid should have said "to the center", rather than "to the diameter $A D$ ", since $B C, A D$ and $F G$ are not necessarily parallel.
$\ddagger$ This is not proved, except by reference to the figure.

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## Proposition 16

A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).

Let $A B C$ be a circle around the center $D$ and the diameter $A B$. I say that the (straight-line) drawn from $A$, at right-angles to $A B$ [Prop 1.11], from its end, will fall outside the circle.

For (if) not then, if possible, let it fall inside, like $C A$ (in the figure), and let $D C$ have been joined.

Since $D A$ is equal to $D C$, angle $D A C$ is also equal to angle $A C D$ [Prop. 1.5]. And $D A C$ (is) a right-angle. Thus, $A C D$ (is) also a right-angle. So, in triangle $A C D$, the two angles $D A C$ and $A C D$ are equal to two rightangles. The very thing is impossible [Prop. 1.17]. Thus, the (straight-line) drawn from point $A$, at right-angles


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$\Lambda \varepsilon ́ \gamma \omega$, ơть x $\alpha i \dot{\eta} \mu \varepsilon ̀ \nu$ тоũ $\dot{\eta} \mu \iota x \cup x \lambda$ íou $\gamma \omega \nu i ́ \alpha \dot{\eta} \pi \varepsilon p เ \varepsilon \chi о \mu \varepsilon ́ v \eta$





Ei $\gamma \grave{\rho} \rho$ ह̇ $\sigma \tau i ́ t i s ~ \gamma \omega v i ́ \alpha ~ \varepsilon u ̉ \vartheta u ́ \gamma p \alpha \mu \mu o s ~ \mu \varepsilon i \zeta \omega \nu ~ \mu \varepsilon ̀ \nu ~ \tau \eta ̃ s ~$










 $\tau \tilde{\eta} \varsigma ~ Г \Theta А ~ \pi \varepsilon p เ \varphi \varepsilon p \varepsilon i ́ \alpha s ~ x \alpha i ̀ ~ \tau \tilde{\eta} \varsigma ~ A E ~ \varepsilon \cup ̉ \vartheta \varepsilon i ́ \alpha s . ~$
to $B A$, will not fall inside the circle. So, similarly, we can show that neither (will it fall) on the circumference. Thus, (it will fall) outside (the circle).


Let it fall like $A E$ (in the figure). So, I say that another straight-line cannot be inserted into the space between the straight-line $A E$ and the circumference $C H A$.

For, if possible, let it be inserted like $F A$ (in the figure), and let $D G$ have been drawn from point $D$, perpendicular to $F A$ [Prop. 1.12]. And since $A G D$ is a rightangle, and $D A G$ (is) less than a right-angle, $A D$ (is) thus greater than $D G$ [Prop. 1.19]. And $D A$ (is) equal to $D H$. Thus, $D H$ (is) greater than $D G$, the lesser than the greater. The very thing is impossible. Thus, another straight-line cannot be inserted into the space between the straight-line $(A E)$ and the circumference.

And I also say that the semi-circular angle contained by the straight-line $B A$ and the circumference $C H A$ is greater than any acute rectilinear angle whatsoever, and the remaining (angle) contained by the circumference $C H A$ and the straight-line $A E$ is less than any acute rectilinear angle whatsoever.

For if any rectilinear angle is greater than the (angle) contained by the straight-line $B A$ and the circumference $C H A$, or less than the (angle) contained by the circumference $C H A$ and the straight-line $A E$, then a straight-line can be inserted into the space between the circumference $C H A$ and the straight-line $A E$-anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line $B A$ and the circumference $C H A$, or less than the (angle) contained by the circumference $C H A$ and the straightline $A E$. But (such a straight-line) cannot be inserted. Thus, an acute (angle) contained by straight-lines cannot be greater than the angle contained by the straight-line $B A$ and the circumference $C H A$, neither (can it be) less than the (angle) contained by the circumference CHA and the straight-line $A E$.

## Пópıбца.






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 óp७̀̀s ク้ $\chi \vartheta \omega \dot{\eta} \Delta Z$, x $\alpha \grave{l}$ ह่ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ E Z, ~ A B \cdot ~ \lambda \varepsilon ́ ~ \gamma \omega, ~$
 $\dot{\eta} \mathrm{AB}$.














## Corollary

So, from this, (it is) manifest that a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [and that the straight-line touches the circle at a single point, inasmuch as it was also shown that a (straight-line) meeting (the circle) at two (points) falls inside it [Prop. 3.2]]. (Which is) the very thing it was required to show.

## Proposition 17

To draw a straight-line touching a given circle from a given point.


Let $A$ be the given point, and $B C D$ the given circle. So it is required to draw a straight-line touching circle $B C D$ from point $A$.

For let the center $E$ of the circle have been found [Prop. 3.1], and let $A E$ have been joined. And let (the circle) $A F G$ have been drawn with center $E$ and radius $E A$. And let $D F$ have been drawn from from (point) $D$, at right-angles to $E A$ [Prop. 1.11]. And let $E F$ and $A B$ have been joined. I say that the (straight-line) $A B$ has been drawn from point $A$ touching circle $B C D$.

For since $E$ is the center of circles $B C D$ and $A F G$, $E A$ is thus equal to $E F$, and $E D$ to $E B$. So the two (straight-lines) $A E, E B$ are equal to the two (straightlines) $F E, E D$ (respectively). And they contain a common angle at $E$. Thus, the base $D F$ is equal to the base $A B$, and triangle $D E F$ is equal to triangle $E B A$, and the remaining angles (are equal) to the (corresponding) remaining angles [Prop. 1.4]. Thus, (angle) $E D F$ (is) equal to $E B A$. And $E D F$ (is) a right-angle. Thus, $E B A$ (is) also a right-angle. And $E B$ is a radius. And a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [Prop. 3.16 corr.]. Thus, $A B$ touches circle $B C D$.

Thus, the straight-line $A B$ has been drawn touching

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the given circle $B C D$ from the given point $A$. (Which is) the very thing it was required to do.

## Proposition 18

If some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent.


For let some straight-line $D E$ touch the circle $A B C$ at point $C$, and let the center $F$ of circle $A B C$ have been found [Prop. 3.1], and let $F C$ have been joined from $F$ to $C$. I say that $F C$ is perpendicular to $D E$.

For if not, let $F G$ have been drawn from $F$, perpendicular to $D E$ [Prop. 1.12].

Therefore, since angle $F G C$ is a right-angle, (angle) $F C G$ is thus acute [Prop. 1.17]. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, $F C$ (is) greater than $F G$. And $F C$ (is) equal to $F B$. Thus, $F B$ (is) also greater than $F G$, the lesser than the greater. The very thing is impossible. Thus, $F G$ is not perpendicular to $D E$. So, similarly, we can show that neither (is) any other (straight-line) except $F C$. Thus, $F C$ is perpendicular to $D E$.

Thus, if some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent. (Which is) the very thing it was required to show.

## Proposition 19

If some straight-line touches a circle, and a straightline is drawn from the point of contact, at right-[angles] to the tangent, then the center (of the circle) will be on the (straight-line) so drawn.

For let some straight-line $D E$ touch the circle $A B C$ at point $C$. And let $C A$ have been drawn from $C$, at right-

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 $\dot{\eta}$ ú $\pi o ̀ ~ В Е Г ~ o ̈ \lambda \eta s ~ \tau \tilde{\eta} s ~ u ́ \pi o ̀ ~ B A \Gamma ~ \varepsilon ̇ \sigma \tau l ~ \delta \iota \pi \lambda \tilde{\eta}$.
angles to $D E$ [Prop. 1.11]. I say that the center of the circle is on $A C$.


For (if) not, if possible, let $F$ be (the center of the circle), and let $C F$ have been joined.
[Therefore], since some straight-line $D E$ touches the circle $A B C$, and $F C$ has been joined from the center to the point of contact, $F C$ is thus perpendicular to $D E$ [Prop. 3.18]. Thus, $F C E$ is a right-angle. And $A C E$ is also a right-angle. Thus, $F C E$ is equal to $A C E$, the lesser to the greater. The very thing is impossible. Thus, $F$ is not the center of circle $A B C$. So, similarly, we can show that neither is any (point) other (than one) on $A C$.

Thus, if some straight-line touches a circle, and a straightline is drawn from the point of contact, at right-angles to the tangent, then the center (of the circle) will be on the (straight-line) so drawn. (Which is) the very thing it was required to show.

## Proposition 20

In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.

Let $A B C$ be a circle, and let $B E C$ be an angle at its center, and $B A C$ (one) at (its) circumference. And let them have the same circumference base $B C$. I say that angle $B E C$ is double (angle) $B A C$.

For being joined, let $A E$ have been drawn through to $F$.

Therefore, since $E A$ is equal to $E B$, angle $E A B$ (is) also equal to $E B A$ [Prop. 1.5]. Thus, angle $E A B$ and $E B A$ is double (angle) $E A B$. And $B E F$ (is) equal to $E A B$ and $E B A$ [Prop. 1.32]. Thus, $B E F$ is also double $E A B$. So, for the same (reasons), $F E C$ is also double $E A C$. Thus, the whole (angle) $B E C$ is double the whole (angle) $B A C$.


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So let another (straight-line) have been inflected, and let there be another angle, $B D C$. And $D E$ being joined, let it have been produced to $G$. So, similarly, we can show that angle $G E C$ is double $E D C$, of which $G E B$ is double $E D B$. Thus, the remaining (angle) $B E C$ is double the (remaining angle) $B D C$.

Thus, in a circle, the angle at the center is double that at the circumference, when [the angles] have the same circumference base. (Which is) the very thing it was required to show.

## Proposition 21

In a circle, angles in the same segment are equal to one another.


Let $A B C D$ be a circle, and let $B A D$ and $B E D$ be angles in the same segment $B A E D$. I say that angles $B A D$ and $B E D$ are equal to one another.

For let the center of circle $A B C D$ have been found [Prop. 3.1], and let it be (at point) $F$. And let $B F$ and $F D$ have been joined.

And since angle $B F D$ is at the center, and $B A D$ at the circumference, and they have the same circumference base $B C D$, angle $B F D$ is thus double $B A D$ [Prop. 3.20].




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 $\mathrm{AB}, ~ В А Г, ~ А Г В ~ \delta u \sigma i \nu ~ o ̉ p \vartheta \alpha i ̆ s ~ i ̂ \sigma \alpha l ~ \varepsilon i \sigma i ́ v . ~ x \alpha i ̀ ~ \alpha i ~ ن ́ \pi o ̀ ~ А В Г, ~$





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So, for the same (reasons), $B F D$ is also double $B E D$. Thus, $B A D$ (is) equal to $B E D$.

Thus, in a circle, angles in the same segment are equal to one another. (Which is) the very thing it was required to show.

## Proposition 22

For quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles.


Let $A B C D$ be a circle, and let $A B C D$ be a quadrilateral within it. I say that the (sum of the) opposite angles is equal to two right-angles.

Let $A C$ and $B D$ have been joined.
Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles $C A B, A B C$, and $B C A$ of triangle $A B C$ are thus equal to two right-angles. And $C A B$ (is) equal to $B D C$. For they are in the same segment $B A D C$ [Prop. 3.21]. And $A C B$ (is equal) to $A D B$. For they are in the same segment $A D C B$ [Prop. 3.21]. Thus, the whole of $A D C$ is equal to $B A C$ and $A C B$. Let $A B C$ have been added to both. Thus, $A B C, B A C$, and $A C B$ are equal to $A B C$ and $A D C$. But, $A B C, B A C$, and $A C B$ are equal to two right-angles. Thus, $A B C$ and $A D C$ are also equal to two right-angles. Similarly, we can show that angles $B A D$ and $D C B$ are also equal to two right-angles.

Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

## Proposition 23

Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

For, if possible, let the two similar and unequal segments of circles, $A C B$ and $A D B$, have been constructed on the same side of the same straight-line $A B$. And let
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$A C D$ have been drawn through (the segments), and let $C B$ and $D B$ have been joined.


Therefore, since segment $A C B$ is similar to segment $A D B$, and similar segments of circles are those accepting equal angles [Def. 3.11], angle $A C B$ is thus equal to $A D B$, the external to the internal. The very thing is impossible [Prop. 1.16].

Thus, two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

## Proposition 24

Similar segments of circles on equal straight-lines are equal to one another.


For let $A E B$ and $C F D$ be similar segments of circles on the equal straight-lines $A B$ and $C D$ (respectively). I say that segment $A E B$ is equal to segment $C F D$.

For if the segment $A E B$ is applied to the segment $C F D$, and point $A$ is placed on (point) $C$, and the straight-line $A B$ on $C D$, then point $B$ will also coincide with point $D$, on account of $A B$ being equal to $C D$. And if $A B$ coincides with $C D$ then the segment $A E B$ will also coincide with $C F D$. For if the straight-line $A B$ coincides with $C D$, and the segment $A E B$ does not coincide with $C F D$, then it will surely either fall inside it, outside (it), ${ }^{\dagger}$ or it will miss like $C G D$ (in the figure), and a circle (will) cut (another) circle at more than two points. The very
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${ }^{\dagger}$ Both this possibility，and the previous one，are precluded by Prop．3．23．

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thing is impossible［Prop．3．10］．Thus，if the straight－line $A B$ is applied to $C D$ ，the segment $A E B$ cannot not also coincide with $C F D$ ．Thus，it will coincide，and will be equal to it［C．N．4］．

Thus，similar segments of circles on equal straight－ lines are equal to one another．（Which is）the very thing it was required to show．

## Proposition 25

For a given segment of a circle，to complete the circle， the very one of which it is a segment．


Let $A B C$ be the given segment of a circle．So it is re－ quired to complete the circle for segment $A B C$ ，the very one of which it is a segment．

For let $A C$ have been cut in half at（point）$D$ ［Prop．1．10］，and let $D B$ have been drawn from point $D$ ，at right－angles to $A C$［Prop．1．11］．And let $A B$ have been joined．Thus，angle $A B D$ is surely either greater than，equal to，or less than（angle）$B A D$ ．

First of all，let it be greater．And let（angle）$B A E$ ， equal to angle $A B D$ ，have been constructed on the straight－line $B A$ ，at the point $A$ on it［Prop．1．23］．And let $D B$ have been drawn through to $E$ ，and let $E C$ have been joined．Therefore，since angle $A B E$ is equal to $B A E$ ，the straight－line $E B$ is thus also equal to $E A$ ［Prop．1．6］．And since $A D$ is equal to $D C$ ，and $D E$（is） common，the two（straight－lines）$A D, D E$ are equal to the two（straight－lines）$C D, D E$ ，respectively．And angle $A D E$ is equal to angle $C D E$ ．For each（is）a right－angle． Thus，the base $A E$ is equal to the base $C E$［Prop．1．4］． But，$A E$ was shown（to be）equal to $B E$ ．Thus，$B E$ is also equal to $C E$ ．Thus，the three（straight－lines）$A E$ ， $E B$ ，and $E C$ are equal to one another．Thus，if a cir－ cle is drawn with center $E$ ，and radius one of $A E, E B$ ， or $E C$ ，it will also go through the remaining points（of the segment），and the（associated circle）will have been completed［Prop．3．9］．Thus，a circle has been completed from the given segment of a circle．And（it is）clear that the segment $A B C$ is less than a semi－circle，because the center $E$ happens to lie outside it．

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[And], similarly, even if angle $A B D$ is equal to $B A D$, (since) $A D$ becomes equal to each of $B D$ [Prop. 1.6] and $D C$, the three (straight-lines) $D A, D B$, and $D C$ will be equal to one another. And point $D$ will be the center of the completed circle. And $A B C$ will manifestly be a semi-circle.

And if $A B D$ is less than $B A D$, and we construct (angle $B A E$ ), equal to angle $A B D$, on the straight-line $B A$, at the point $A$ on it [Prop. 1.23], then the center will fall on $D B$, inside the segment $A B C$. And segment $A B C$ will manifestly be greater than a semi-circle.

Thus, a circle has been completed from the given segment of a circle. (Which is) the very thing it was required to do.

## Proposition 26

In equal circles, equal angles stand upon equal circumferences whether they are standing at the center or at the circumference.


Let $A B C$ and $D E F$ be equal circles, and within them let $B G C$ and $E H F$ be equal angles at the center, and $B A C$ and $E D F$ (equal angles) at the circumference. I say that circumference $B K C$ is equal to circumference $E L F$.

For let $B C$ and $E F$ have been joined.
And since circles $A B C$ and $D E F$ are equal, their radii are equal. So the two (straight-lines) $B G, G C$ (are) equal to the two (straight-lines) $E H, H F$ (respectively). And the angle at $G$ (is) equal to the angle at $H$. Thus, the base $B C$ is equal to the base $E F$ [Prop. 1.4]. And since the angle at $A$ is equal to the (angle) at $D$, the segment $B A C$ is thus similar to the segment $E D F$ [Def. 3.11]. And they are on equal straight-lines [ $B C$ and $E F$ ]. And similar segments of circles on equal straight-lines are equal to one another [Prop. 3.24]. Thus, segment $B A C$ is equal to (segment) $E D F$. And the whole circle $A B C$ is also equal to the whole circle $D E F$. Thus, the remaining circumference $B K C$ is equal to the (remaining) circumference ELF.

Thus, in equal circles, equal angles stand upon equal circumferences, whether they are standing at the center

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or at the circumference. (Which is) the very thing which it was required to show.

## Proposition 27

In equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference.


For let the angles $B G C$ and $E H F$ at the centers $G$ and $H$, and the (angles) $B A C$ and $E D F$ at the circumferences, stand upon the equal circumferences $B C$ and $E F$, in the equal circles $A B C$ and $D E F$ (respectively). I say that angle $B G C$ is equal to (angle) $E H F$, and $B A C$ is equal to $E D F$.

For if $B G C$ is unequal to $E H F$, one of them is greater. Let $B G C$ be greater, and let the (angle) $B G K$, equal to angle $E H F$, have been constructed on the straight-line $B G$, at the point $G$ on it [Prop. 1.23]. But equal angles (in equal circles) stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference $B K$ (is) equal to circumference $E F$. But, $E F$ is equal to $B C$. Thus, $B K$ is also equal to $B C$, the lesser to the greater. The very thing is impossible. Thus, angle $B G C$ is not unequal to $E H F$. Thus, (it is) equal. And the (angle) at $A$ is half $B G C$, and the (angle) at $D$ half $E H F$ [Prop. 3.20]. Thus, the angle at $A$ (is) also equal to the (angle) at $D$.

Thus, in equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference. (Which is) the very thing it was required to show.

## Proposition 28

In equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser.

Let $A B C$ and $D E F$ be equal circles, and let $A B$ and $D E$ be equal straight-lines in these circles, cutting off the greater circumferences $A C B$ and $D F E$, and the lesser (circumferences) $A G B$ and $D H E$ (respectively). I say that the greater circumference $A C B$ is equal to the greater circumference $D F E$, and the lesser circumfer-

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ence $A G B$ to (the lesser) $D H E$.


For let the centers of the circles, $K$ and $L$, have been found [Prop. 3.1], and let $A K, K B, D L$, and $L E$ have been joined.

And since ( $A B C$ and $D E F$ ) are equal circles, their radii are also equal [Def. 3.1]. So the two (straightlines) $A K, K B$ are equal to the two (straight-lines) $D L$, $L E$ (respectively). And the base $A B$ (is) equal to the base $D E$. Thus, angle $A K B$ is equal to angle $D L E$ [Prop. 1.8]. And equal angles stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference $A G B$ (is) equal to $D H E$. And the whole circle $A B C$ is also equal to the whole circle $D E F$. Thus, the remaining circumference $A C B$ is also equal to the remaining circumference $D F E$.

Thus, in equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser. (Which is) the very thing it was required to show.

## Proposition 29

In equal circles, equal straight-lines subtend equal circumferences.


Let $A B C$ and $D E F$ be equal circles, and within them let the equal circumferences $B G C$ and $E H F$ have been cut off. And let the straight-lines $B C$ and $E F$ have been joined. I say that $B C$ is equal to $E F$.

For let the centers of the circles have been found [Prop. 3.1], and let them be (at) $K$ and $L$. And let $B K$,







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$K C, E L$, and $L F$ have been joined.
And since the circumference $B G C$ is equal to the circumference $E H F$, the angle $B K C$ is also equal to (angle) $E L F$ [Prop. 3.27]. And since the circles $A B C$ and $D E F$ are equal, their radii are also equal [Def. 3.1]. So the two (straight-lines) $B K, K C$ are equal to the two (straight-lines) $E L, L F$ (respectively). And they contain equal angles. Thus, the base $B C$ is equal to the base $E F$ [Prop. 1.4].

Thus, in equal circles, equal straight-lines subtend equal circumferences. (Which is) the very thing it was required to show.

## Proposition 30

To cut a given circumference in half.


Let $A D B$ be the given circumference. So it is required to cut circumference $A D B$ in half.

Let $A B$ have been joined, and let it have been cut in half at (point) $C$ [Prop. 1.10]. And let $C D$ have been drawn from point $C$, at right-angles to $A B$ [Prop. 1.11]. And let $A D$, and $D B$ have been joined.

And since $A C$ is equal to $C B$, and $C D$ (is) common, the two (straight-lines) $A C, C D$ are equal to the two (straight-lines) $B C, C D$ (respectively). And angle $A C D$ (is) equal to angle $B C D$. For (they are) each rightangles. Thus, the base $A D$ is equal to the base $D B$ [Prop. 1.4]. And equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser [Prop. 1.28]. And the circumferences $A D$ and $D B$ are each less than a semicircle. Thus, circumference $A D$ (is) equal to circumference $D B$.

Thus, the given circumference has been cut in half at point $D$. (Which is) the very thing it was required to do.

## Proposition 31

In a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser segment (is) greater than a rightangle. And, further, the angle of a segment greater (than a semi-circle) is greater than a right-angle, and the an-




































gle of a segment less (than a semi-circle) is less than a right-angle.


Let $A B C D$ be a circle, and let $B C$ be its diameter, and $E$ its center. And let $B A, A C, A D$, and $D C$ have been joined. I say that the angle $B A C$ in the semi-circle $B A C$ is a right-angle, and the angle $A B C$ in the segment $A B C$, (which is) greater than a semi-circle, is less than a rightangle, and the angle $A D C$ in the segment $A D C$, (which is) less than a semi-circle, is greater than a right-angle.

Let $A E$ have been joined, and let $B A$ have been drawn through to $F$.

And since $B E$ is equal to $E A$, angle $A B E$ is also equal to $B A E$ [Prop. 1.5]. Again, since $C E$ is equal to $E A, A C E$ is also equal to $C A E$ [Prop. 1.5]. Thus, the whole (angle) $B A C$ is equal to the two (angles) $A B C$ and $A C B$. And $F A C$, (which is) external to triangle $A B C$, is also equal to the two angles $A B C$ and $A C B$ [Prop. 1.32]. Thus, angle $B A C$ (is) also equal to $F A C$. Thus, (they are) each right-angles. [Def. 1.10]. Thus, the angle $B A C$ in the semi-circle $B A C$ is a right-angle.

And since the two angles $A B C$ and $B A C$ of triangle $A B C$ are less than two right-angles [Prop. 1.17], and $B A C$ is a right-angle, angle $A B C$ is thus less than a rightangle. And it is in segment $A B C$, (which is) greater than a semi-circle.

And since $A B C D$ is a quadrilateral within a circle, and for quadrilaterals within circles the (sum of the) opposite angles is equal to two right-angles [Prop. 3.22] [angles $A B C$ and $A D C$ are thus equal to two rightangles], and (angle) $A B C$ is less than a right-angle. The remaining angle $A D C$ is thus greater than a right-angle. And it is in segment $A D C$, (which is) less than a semicircle.

I also say that the angle of the greater segment, (namely) that contained by the circumference $A B C$ and the straight-line $A C$, is greater than a right-angle. And the angle of the lesser segment, (namely) that contained







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by the circumference $A D[C]$ and the straight-line $A C$, is less than a right-angle. And this is immediately apparent. For since the (angle contained by) the two straight-lines $B A$ and $A C$ is a right-angle, the (angle) contained by the circumference $A B C$ and the straight-line $A C$ is thus greater than a right-angle. Again, since the (angle contained by) the straight-lines $A C$ and $A F$ is a right-angle, the (angle) contained by the circumference $A D[C]$ and the straight-line $C A$ is thus less than a right-angle.

Thus, in a circle, the angle in a semi-circle is a rightangle, and that in a greater segment (is) less than a right-angle, and that in a lesser [segment] (is) greater than a right-angle. And, further, the [angle] of a segment greater (than a semi-circle) [is] greater than a rightangle, and the [angle] of a segment less (than a semicircle) is less than a right-angle. (Which is) the very thing it was required to show.

## Proposition 32

If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.


For let some straight-line $E F$ touch the circle $A B C D$ at the point $B$, and let some (other) straight-line $B D$ have been drawn from point $B$ into the circle $A B C D$, cutting it (in two). I say that the angles $B D$ makes with the tangent $E F$ will be equal to the angles in the alternate segments of the circle. That is to say, that angle $F B D$ is equal to the angle constructed in segment $B A D$, and angle $E B D$ is equal to the angle constructed in segment $D C B$.

For let $B A$ have been drawn from $B$, at right-angles to $E F$ [Prop. 1.11]. And let the point $C$ have been taken at random on the circumference $B D$. And let $A D, D C$,



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and $C B$ have been joined.
And since some straight-line $E F$ touches the circle $A B C D$ at point $B$, and $B A$ has been drawn from the point of contact, at right-angles to the tangent, the center of circle $A B C D$ is thus on $B A$ [Prop. 3.19]. Thus, $B A$ is a diameter of circle $A B C D$. Thus, angle $A D B$, being in a semi-circle, is a right-angle [Prop. 3.31]. Thus, the remaining angles (of triangle $A D B$ ) $B A D$ and $A B D$ are equal to one right-angle [Prop. 1.32]. And $A B F$ is also a right-angle. Thus, $A B F$ is equal to $B A D$ and $A B D$. Let $A B D$ have been subtracted from both. Thus, the remaining angle $D B F$ is equal to the angle $B A D$ in the alternate segment of the circle. And since $A B C D$ is a quadrilateral in a circle, (the sum of) its opposite angles is equal to two right-angles [Prop. 3.22]. And $D B F$ and $D B E$ is also equal to two right-angles [Prop. 1.13]. Thus, $D B F$ and $D B E$ is equal to $B A D$ and $B C D$, of which $B A D$ was shown (to be) equal to $D B F$. Thus, the remaining (angle) $D B E$ is equal to the angle $D C B$ in the alternate segment $D C B$ of the circle.

Thus, if some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle. (Which is) the very thing it was required to show.

## Proposition 33

To draw a segment of a circle, accepting an angle equal to a given rectilinear angle, on a given straight-line.


Let $A B$ be the given straight-line, and $C$ the given rectilinear angle. So it is required to draw a segment of a circle, accepting an angle equal to $C$, on the given straight-line $A B$.

So the [angle] $C$ is surely either acute, a right-angle, or obtuse. First of all, let it be acute. And, as in the first diagram (from the left), let (angle) $B A D$, equal to angle $C$, have been constructed on the straight-line $A B$, at the point A (on it) [Prop. 1.23]. Thus, $B A D$ is also acute. Let $A E$ have been drawn, at right-angles to $D A$ [Prop. 1.11].






 AE $\delta \iota \alpha \mu \varepsilon ́ \tau \rho o u ~ \alpha ̇ \pi o ̀ ~ \tau о u ̃ ~ A ~ \tau \tilde{n} A E \pi \rho o ̀ \varsigma ~ o ̉ p \vartheta \alpha ́ s ~ \varepsilon ̇ \sigma \tau \tau \nu ~ \dot{\eta} A \Delta$,







 $\tau \tilde{n} \delta o \vartheta \varepsilon i ́ \sigma \eta \tau \tilde{n} \pi \rho o ̀ s ~ \tau \widetilde{\varphi} \Gamma$.





 ó AEB.









 $\ddot{\eta} \chi \vartheta \omega \dot{\eta} \mathrm{AE}, x \alpha \grave{\imath} \tau \varepsilon \tau \mu \dot{\eta} \sigma \vartheta \omega \pi \dot{\alpha} \lambda \iota \nu \dot{\eta} \mathrm{AB}$ бí $\chi \alpha$ x $\alpha \tau \dot{\alpha}$ 七ò Z , $x \alpha \grave{l}$ $\tau \tilde{\eta} \mathrm{AB} \pi \rho o ̀ s ~ o ̉ \rho \vartheta \grave{\alpha} \varsigma ~ \eta ँ \chi \vartheta \omega \dot{\eta} \mathrm{ZH}$, x $\alpha$ ह̀ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \dot{\eta} \mathrm{HB}$.










 $\pi \rho o ̀ s \tau \widetilde{\varphi} \Gamma$.

 हैסॄเ $\pi \circ เ \tilde{\eta} \sigma \alpha l$.

And let $A B$ have been cut in half at $F$ [Prop. 1.10]. And let $F G$ have been drawn from point $F$, at right-angles to $A B$ [Prop. 1.11]. And let $G B$ have been joined.

And since $A F$ is equal to $F B$, and $F G$ (is) common, the two (straight-lines) $A F, F G$ are equal to the two (straight-lines) $B F, F G$ (respectively). And angle $A F G$ (is) equal to [angle] $B F G$. Thus, the base $A G$ is equal to the base $B G$ [Prop. 1.4]. Thus, the circle drawn with center $G$, and radius $G A$, will also go through $B$ (as well as $A$ ). Let it have been drawn, and let it be (denoted) $A B E$. And let $E B$ have been joined. Therefore, since $A D$ is at the extremity of diameter $A E$, (namely, point) $A$, at right-angles to $A E$, the (straight-line) $A D$ thus touches the circle $A B E$ [Prop. 3.16 corr.]. Therefore, since some straight-line $A D$ touches the circle $A B E$, and some (other) straight-line $A B$ has been drawn across from the point of contact $A$ into circle $A B E$, angle $D A B$ is thus equal to the angle $A E B$ in the alternate segment of the circle [Prop. 3.32]. But, $D A B$ is equal to $C$. Thus, angle $C$ is also equal to $A E B$.

Thus, a segment $A E B$ of a circle, accepting the angle $A E B$ (which is) equal to the given (angle) $C$, has been drawn on the given straight-line $A B$.

And so let $C$ be a right-angle. And let it again be necessary to draw a segment of a circle on $A B$, accepting an angle equal to the right-[angle] $C$. Let the (angle) $B A D$ [again] have been constructed, equal to the rightangle $C$ [Prop. 1.23], as in the second diagram (from the left). And let $A B$ have been cut in half at $F$ [Prop. 1.10]. And let the circle $A E B$ have been drawn with center $F$, and radius either $F A$ or $F B$.

Thus, the straight-line $A D$ touches the circle $A B E$, on account of the angle at $A$ being a right-angle [Prop. 3.16 corr.]. And angle $B A D$ is equal to the angle in segment $A E B$. For (the latter angle), being in a semi-circle, is also a right-angle [Prop. 3.31]. But, $B A D$ is also equal to $C$. Thus, the (angle) in (segment) $A E B$ is also equal to $C$.

Thus, a segment $A E B$ of a circle, accepting an angle equal to $C$, has again been drawn on $A B$.

And so let (angle) $C$ be obtuse. And let (angle) $B A D$, equal to $(C)$, have been constructed on the straight-line $A B$, at the point $A$ (on it) [Prop. 1.23], as in the third diagram (from the left). And let $A E$ have been drawn, at right-angles to $A D$ [Prop. 1.11]. And let $A B$ have again been cut in half at $F$ [Prop. 1.10]. And let $F G$ have been drawn, at right-angles to $A B$ [Prop. 1.10]. And let $G B$ have been joined.

And again, since $A F$ is equal to $F B$, and $F G$ (is) common, the two (straight-lines) $A F, F G$ are equal to the two (straight-lines) $B F, F G$ (respectively). And angle $A F G$ (is) equal to angle $B F G$. Thus, the base $A G$ is

## $\lambda \delta^{\prime}$.

















 $\mu \omega$ т $\tilde{n} \pi \rho o ̀ s ~ \tau \widetilde{\varphi} \Delta^{\prime}$ ö $\pi \varepsilon \rho$ है $\delta \varepsilon \iota ~ \pi o เ \tilde{\eta} \sigma \alpha l$.
equal to the base $B G$ [Prop. 1.4]. Thus, a circle of center $G$, and radius $G A$, being drawn, will also go through $B$ (as well as $A$ ). Let it go like $A E B$ (in the third diagram from the left). And since $A D$ is at right-angles to the diameter $A E$, at its extremity, $A D$ thus touches circle $A E B$ [Prop. 3.16 corr.]. And $A B$ has been drawn across (the circle) from the point of contact $A$. Thus, angle $B A D$ is equal to the angle constructed in the alternate segment $A H B$ of the circle [Prop. 3.32]. But, angle $B A D$ is equal to $C$. Thus, the angle in segment $A H B$ is also equal to $C$.

Thus, a segment $A H B$ of a circle, accepting an angle equal to $C$, has been drawn on the given straight-line $A B$. (Which is) the very thing it was required to do.

## Proposition 34

To cut off a segment, accepting an angle equal to a given rectilinear angle, from a given circle.


Let $A B C$ be the given circle, and $D$ the given rectilinear angle. So it is required to cut off a segment, accepting an angle equal to the given rectilinear angle $D$, from the given circle $A B C$.

Let $E F$ have been drawn touching $A B C$ at point $B .^{\dagger}$ And let (angle) $F B C$, equal to angle $D$, have been constructed on the straight-line $F B$, at the point $B$ on it [Prop. 1.23].

Therefore, since some straight-line $E F$ touches the circle $A B C$, and $B C$ has been drawn across (the circle) from the point of contact $B$, angle $F B C$ is thus equal to the angle constructed in the alternate segment $B A C$ [Prop. 1.32]. But, $F B C$ is equal to $D$. Thus, the (angle) in the segment $B A C$ is also equal to [angle] $D$.

Thus, the segment $B A C$, accepting an angle equal to the given rectilinear angle $D$, has been cut off from the given circle $A B C$. (Which is) the very thing it was required to do.

[^5]point $B$, at right-angles to the aforementioned straight-line [Prop. 1.11].
$$
\lambda \varepsilon^{\prime} .
$$









 $\tau \widetilde{\omega} \nu \mathrm{AE}, \mathrm{E}, \Delta \mathrm{E}, \mathrm{EB}$ каì $\tau o ̀ ~ U ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AE}, \mathrm{E} \Gamma \pi \varepsilon p เ \varepsilon \chi o ́ \mu \varepsilon \nu o \nu$
 ȯpvoravíc.



























## Proposition 35

If two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.


For let the two straight-lines $A C$ and $B D$, in the circle $A B C D$, cut one another at point $E$. I say that the rectangle contained by $A E$ and $E C$ is equal to the rectangle contained by $D E$ and $E B$.

In fact, if $A C$ and $B D$ are through the center (as in the first diagram from the left), so that $E$ is the center of circle $A B C D$, then (it is) clear that, $A E, E C, D E$, and $E B$ being equal, the rectangle contained by $A E$ and $E C$ is also equal to the rectangle contained by $D E$ and $E B$.

So let $A C$ and $D B$ not be though the center (as in the second diagram from the left), and let the center of $A B C D$ have been found [Prop. 3.1], and let it be (at) $F$. And let $F G$ and $F H$ have been drawn from $F$, perpendicular to the straight-lines $A C$ and $D B$ (respectively) [Prop. 1.12]. And let $F B, F C$, and $F E$ have been joined.

And since some straight-line, $G F$, through the center, cuts at right-angles some (other) straight-line, $A C$, not through the center, then it also cuts it in half [Prop. 3.3]. Thus, $A G$ (is) equal to $G C$. Therefore, since the straightline $A C$ is cut equally at $G$, and unequally at $E$, the rectangle contained by $A E$ and $E C$ plus the square on $E G$ is thus equal to the (square) on $G C$ [Prop. 2.5]. Let the (square) on $G F$ have been added [to both]. Thus, the (rectangle contained) by $A E$ and $E C$ plus the (sum of the squares) on $G E$ and $G F$ is equal to the (sum of the squares) on $C G$ and $G F$. But, the (square) on $F E$ is equal to the (sum of the squares) on $E G$ and $G F$ [Prop. 1.47], and the (square) on $F C$ is equal to the (sum of the squares) on $C G$ and $G F$ [Prop. 1.47]. Thus, the (rectangle contained) by $A E$ and $E C$ plus the (square) on $F E$ is equal to the (square) on $F C$. And $F C$ (is) equal to $F B$. Thus, the (rectangle contained) by $A E$ and $E C$ plus the (square) on $F E$ is equal to the (square) on $F B$. So, for the same (reasons), the (rectangle contained) by $D E$ and $E B$ plus the (square) on $F E$ is equal



$$
\lambda \epsilon^{\prime}
$$





 $\pi \tau о \mu \varepsilon ́ v \eta$ т тєтраүढ́vต．





 тєтраү⿳㇒⿵冂人ฺ．








 тoũs $\alpha \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{ZB}, \mathrm{B} \Delta$ ．xoเvòv $\alpha \varphi n p n ́ \sigma \vartheta \omega$ tò $\alpha \pi o ̀ ~ \tau n ̃ s ~ Z B . ~$

to the（square）on $F B$ ．And the（rectangle contained） by $A E$ and $E C$ plus the（square）on $F E$ was also shown （to be）equal to the（square）on $F B$ ．Thus，the（rect－ angle contained）by $A E$ and $E C$ plus the（square）on $F E$ is equal to the（rectangle contained）by $D E$ and $E B$ plus the（square）on $F E$ ．Let the（square）on $F E$ have been taken from both．Thus，the remaining rectangle con－ tained by $A E$ and $E C$ is equal to the rectangle contained by $D E$ and $E B$ ．

Thus，if two straight－lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other． （Which is）the very thing it was required to show．

## Proposition 36

If some point is taken outside a circle，and two straight－lines radiate from it towards the circle，and（one） of them cuts the circle，and the（other）touches（it），then the（rectangle contained）by the whole（straight－line） cutting（the circle），and the（part of it）cut off outside （the circle），between the point and the convex circumfer－ ence，will be equal to the square on the tangent（line）．


For let some point $D$ have been taken outside circle $A B C$ ，and let two straight－lines，$D C[A]$ and $D B$ ，radi－ ate from $D$ towards circle $A B C$ ．And let $D C A$ cut circle $A B C$ ，and let $B D$ touch（it）．I say that the rectangle contained by $A D$ and $D C$ is equal to the square on $D B$ ．
$[D] C A$ is surely either through the center，or not．Let it first of all be through the center，and let $F$ be the cen－ ter of circle $A B C$ ，and let $F B$ have been joined．Thus， （angle）$F B D$ is a right－angle［Prop．3．18］．And since straight－line $A C$ is cut in half at $F$ ，let $C D$ have been added to it．Thus，the（rectangle contained）by $A D$ and $D C$ plus the（square）on $F C$ is equal to the（square）on $F D$［Prop．2．6］．And $F C$（is）equal to $F B$ ．Thus，the （rectangle contained）by $A D$ and $D C$ plus the（square） on $F B$ is equal to the（square）on $F D$ ．And the（square） on $F D$ is equal to the（sum of the squares）on $F B$ and $B D$［Prop．1．47］．Thus，the（rectangle contained）by $A D$

غे $\varphi \alpha \pi \tau 0 \mu \varepsilon ́ v \eta{ }^{2}$.



















 $\tau \widetilde{\varphi} \alpha \dot{\alpha} \pi o ̀ ~ \tau \tilde{\eta} \varsigma \Delta \mathrm{~B}$.







## $\lambda \zeta \zeta^{\prime}$.

 бпuعíou $\pi \rho o ̀ s ~ \tau o ̀ v ~ \chi u ́ x \lambda o v ~ \pi \rho o \sigma \pi i ́ \pi \tau \omega \sigma l ~ \delta u ́ o ~ \varepsilon u ̉ v \varepsilon i ̃ \alpha l, ~ x \alpha i ̀ ~$


and $D C$ plus the (square) on $F B$ is equal to the (sum of the squares) on $F B$ and $B D$. Let the (square) on $F B$ have been subtracted from both. Thus, the remaining (rectangle contained) by $A D$ and $D C$ is equal to the (square) on the tangent $D B$.

And so let $D C A$ not be through the center of circle $A B C$, and let the center $E$ have been found, and let $E F$ have been drawn from $E$, perpendicular to $A C$ [Prop. 1.12]. And let $E B, E C$, and $E D$ have been joined. (Angle) $E B D$ (is) thus a right-angle [Prop. 3.18]. And since some straight-line, $E F$, through the center, cuts some (other) straight-line, $A C$, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, $A F$ is equal to $F C$. And since the straight-line $A C$ is cut in half at point $F$, let $C D$ have been added to it. Thus, the (rectangle contained) by $A D$ and $D C$ plus the (square) on $F C$ is equal to the (square) on $F D$ [Prop. 2.6]. Let the (square) on $F E$ have been added to both. Thus, the (rectangle contained) by $A D$ and $D C$ plus the (sum of the squares) on $C F$ and $F E$ is equal to the (sum of the squares) on $F D$ and $F E$. But the (square) on $E C$ is equal to the (sum of the squares) on $C F$ and $F E$. For [angle] $E F C$ [is] a right-angle [Prop. 1.47]. And the (square) on $E D$ is equal to the (sum of the squares) on $D F$ and $F E$ [Prop. 1.47]. Thus, the (rectangle contained) by $A D$ and $D C$ plus the (square) on $E C$ is equal to the (square) on $E D$. And $E C$ (is) equal to $E B$. Thus, the (rectangle contained) by $A D$ and $D C$ plus the (square) on $E B$ is equal to the (square) on $E D$. And the (sum of the squares) on $E B$ and $B D$ is equal to the (square) on $E D$. For $E B D$ (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by $A D$ and $D C$ plus the (square) on $E B$ is equal to the (sum of the squares) on $E B$ and $B D$. Let the (square) on $E B$ have been subtracted from both. Thus, the remaining (rectangle contained) by $A D$ and $D C$ is equal to the (square) on $B D$.

Thus, if some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and (the other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line). (Which is) the very thing it was required to show.

## Proposition 37

If some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-

 тои̃ жи́клои.



















 ย̇лì $\tau \tilde{\eta} \varsigma ~ А \Gamma ~ \tau u \gamma \chi \alpha ́ v n . ~$
 бпиعiou $\pi \rho o ̀ s ~ t o ̀ v ~ x u ́ x \lambda o v ~ \pi \rho o \sigma \pi i ́ \pi \tau \omega \sigma l ~ \delta u ́ o ~ \varepsilon u ̛ \vartheta \varepsilon i ̃ \alpha l, ~ x \alpha i ̀ ~ \dot{\eta}$





line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle.


For let some point $D$ have been taken outside circle $A B C$, and let two straight-lines, $D C A$ and $D B$, radiate from $D$ towards circle $A B C$, and let $D C A$ cut the circle, and let $D B$ meet (the circle). And let the (rectangle contained) by $A D$ and $D C$ be equal to the (square) on $D B$. I say that $D B$ touches circle $A B C$.

For let $D E$ have been drawn touching $A B C$ [Prop. 3.17], and let the center of the circle $A B C$ have been found, and let it be (at) $F$. And let $F E, F B$, and $F D$ have been joined. (Angle) $F E D$ is thus a right-angle [Prop. 3.18]. And since $D E$ touches circle $A B C$, and $D C A$ cuts (it), the (rectangle contained) by $A D$ and $D C$ is thus equal to the (square) on $D E$ [Prop. 3.36]. And the (rectangle contained) by $A D$ and $D C$ was also equal to the (square) on $D B$. Thus, the (square) on $D E$ is equal to the (square) on $D B$. Thus, $D E$ (is) equal to $D B$. And $F E$ is also equal to $F B$. So the two (straight-lines) $D E$, $E F$ are equal to the two (straight-lines) $D B, B F$ (respectively). And their base, $F D$, is common. Thus, angle $D E F$ is equal to angle $D B F$ [Prop. 1.8]. And $D E F$ (is) a right-angle. Thus, $D B F$ (is) also a right-angle. And $F B$ produced is a diameter, And a (straight-line) drawn at right-angles to a diameter of a circle, at its extremity, touches the circle [Prop. 3.16 corr.]. Thus, $D B$ touches circle $A B C$. Similarly, (the same thing) can be shown, even if the center happens to be on $A C$.

Thus, if some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straightline) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle. (Which is) the very thing it
was required to show.

## ELEMENTS BOOK 4

## Construction of Rectilinear Figures In and Around Circles

"Opor.
















 pıүpáperal, व̈ $\pi \tau \eta \tau \alpha l$.



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\alpha^{\prime} .
$$










## Definitions

1. A rectilinear figure is said to be inscribed in a(nother) rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the (figure) in which it is inscribed.
2. And, similarly, a (rectilinear) figure is said to be circumscribed about a(nother rectilinear) figure when the respective sides of the circumscribed (figure) touch the respective angles of the (figure) about which it is circumscribed.
3. A rectilinear figure is said to be inscribed in a circle when each angle of the inscribed (figure) touches the circumference of the circle.
4. And a rectilinear figure is said to be circumscribed about a circle when each side of the circumscribed (figure) touches the circumference of the circle.
5. And, similarly, a circle is said to be inscribed in a (rectilinear) figure when the circumference of the circle touches each side of the (figure) in which it is inscribed.
6. And a circle is said to be circumscribed about a rectilinear (figure) when the circumference of the circle touches each angle of the (figure) about which it is circumscribed.
7. A straight-line is said to be inserted into a circle when its extemities are on the circumference of the circle.

## Proposition 1

To insert a straight-line equal to a given straight-line into a circle, (the latter straight-line) not being greater than the diameter of the circle.

D


Let $A B C$ be the given circle, and $D$ the given straightline (which is) not greater than the diameter of the circle. So it is required to insert a straight-line, equal to the straight-line $D$, into the circle $A B C$.

Let a diameter $B C$ of circle $A B C$ have been drawn. ${ }^{\dagger}$







 $\tau \tilde{n} \Delta$ 亿ै $\sigma \eta$ ह̇vńp

Therefore, if $B C$ is equal to $D$ then that (which) was prescribed has taken place. For the (straight-line) $B C$, equal to the straight-line $D$, has been inserted into the circle $A B C$. And if $B C$ is greater than $D$ then let $C E$ be made equal to $D$ [Prop. 1.3], and let the circle $E A F$ have been drawn with center $C$ and radius $C E$. And let $C A$ have been joined.

Therefore, since the point $C$ is the center of circle $E A F, C A$ is equal to $C E$. But, $C E$ is equal to $D$. Thus, $D$ is also equal to $C A$.

Thus, $C A$, equal to the given straight-line $D$, has been inserted into the given circle $A B C$. (Which is) the very thing it was required to do.
${ }^{\dagger}$ Presumably, by finding the center of the circle [Prop. 3.1], and then drawing a line through it.

$$
\beta^{\prime} .
$$

 трírตvov ह̇үүpá廿al.





 $\tau \widetilde{\varphi}$ A $\tau \tilde{n}$ ú $\pi o ̀ ~ \Delta E Z ~ \gamma \omega v i ́ \alpha$ '̈ $\sigma \eta \dot{\eta}$ ú $\pi o ̀ ~ \Theta A \Gamma, ~ \pi \rho o ̀ \varsigma ~ \delta e ̀ ~ \tau \tilde{n} ~ A H ~$














## Proposition 2

To inscribe a triangle, equiangular with a given triangle, in a given circle.


Let $A B C$ be the given circle, and $D E F$ the given triangle. So it is required to inscribe a triangle, equiangular with triangle $D E F$, in circle $A B C$.

Let $G H$ have been drawn touching circle $A B C$ at $A .^{\dagger}$ And let (angle) $H A C$, equal to angle $D E F$, have been constructed on the straight-line $A H$ at the point $A$ on it, and (angle) $G A B$, equal to [angle] $D F E$, on the straightline $A G$ at the point $A$ on it [Prop. 1.23]. And let $B C$ have been joined.

Therefore, since some straight-line $A H$ touches the circle $A B C$, and the straight-line $A C$ has been drawn across (the circle) from the point of contact $A$, (angle) $H A C$ is thus equal to the angle $A B C$ in the alternate segment of the circle [Prop. 3.32]. But, $H A C$ is equal to $D E F$. Thus, angle $A B C$ is also equal to $D E F$. So, for the same (reasons), $A C B$ is also equal to $D F E$. Thus, the remaining (angle) $B A C$ is equal to the remaining (angle) $E D F$ [Prop. 1.32]. [Thus, triangle $A B C$ is equiangular with triangle $D E F$, and has been inscribed in circle
${ }^{\dagger}$ See the footnote to Prop. 3.34.
$\gamma^{\prime}$.
 трír $\omega$ vov $\pi \varepsilon \rho เ \gamma \rho \alpha ́ \psi \alpha l . ~$









 xúx $\lambda$ ou ai $\Lambda$ AM, MBN, NГム.

















$A B C]$.
Thus, a triangle, equiangular with the given triangle, has been inscribed in the given circle. (Which is) the very thing it was required to do.

## Proposition 3

To circumscribe a triangle, equiangular with a given triangle, about a given circle.


Let $A B C$ be the given circle, and $D E F$ the given triangle. So it is required to circumscribe a triangle, equiangular with triangle $D E F$, about circle $A B C$.

Let $E F$ have been produced in each direction to points $G$ and $H$. And let the center $K$ of circle $A B C$ have been found [Prop. 3.1]. And let the straight-line $K B$ have been drawn, at random, across ( $A B C$ ). And let (angle) $B K A$, equal to angle $D E G$, have been constructed on the straight-line $K B$ at the point $K$ on it, and (angle) $B K C$, equal to $D F H$ [Prop. 1.23]. And let the (straight-lines) $L A M, M B N$, and $N C L$ have been drawn through the points $A, B$, and $C$ (respectively), touching the circle $A B C .{ }^{\dagger}$

And since $L M, M N$, and $N L$ touch circle $A B C$ at points $A, B$, and $C$ (respectively), and $K A, K B$, and $K C$ are joined from the center $K$ to points $A, B$, and $C$ (respectively), the angles at points $A, B$, and $C$ are thus right-angles [Prop. 3.18]. And since the (sum of the) four angles of quadrilateral $A M B K$ is equal to four rightangles, inasmuch as $A M B K$ (can) also (be) divided into two triangles [Prop. 1.32], and angles $K A M$ and $K B M$ are (both) right-angles, the (sum of the) remaining (angles), $A K B$ and $A M B$, is thus equal to two right-angles. And $D E G$ and $D E F$ is also equal to two right-angles [Prop. 1.13]. Thus, $A K B$ and $A M B$ is equal to $D E G$ and $D E F$, of which $A K B$ is equal to $D E G$. Thus, the remainder $A M B$ is equal to the remainder $D E F$. So, similarly, it can be shown that $L N B$ is also equal to $D F E$. Thus, the remaining (angle) $M L N$ is also equal to the
${ }^{\dagger}$ See the footnote to Prop. 3.34.

## $\delta^{\prime}$.







 عưvยías xávetol ai $\Delta \mathrm{E}, \Delta \mathrm{Z}, \Delta \mathrm{H}$.
















 tò $\mathrm{AB} \mathrm{\Gamma}$ трírcovov. غ̀ $\gamma \gamma \varepsilon \gamma \rho \alpha ́ \varphi \vartheta \vartheta \omega$ ìs ò ZHE.


[remaining] (angle) $E D F$ [Prop. 1.32]. Thus, triangle $L M N$ is equiangular with triangle $D E F$. And it has been drawn around circle $A B C$.

Thus, a triangle, equiangular with the given triangle, has been circumscribed about the given circle. (Which is) the very thing it was required to do.

## Proposition 4

To inscribe a circle in a given triangle.


Let $A B C$ be the given triangle. So it is required to inscribe a circle in triangle $A B C$.

Let the angles $A B C$ and $A C B$ have been cut in half by the straight-lines $B D$ and $C D$ (respectively) [Prop. 1.9], and let them meet one another at point $D$, and let $D E$, $D F$, and $D G$ have been drawn from point $D$, perpendicular to the straight-lines $A B, B C$, and $C A$ (respectively) [Prop. 1.12].

And since angle $A B D$ is equal to $C B D$, and the rightangle $B E D$ is also equal to the right-angle $B F D, E B D$ and $F B D$ are thus two triangles having two angles equal to two angles, and one side equal to one side-the (one) subtending one of the equal angles (which is) common to the (triangles)-(namely), $B D$. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, $D E$ (is) equal to $D F$. So, for the same (reasons), $D G$ is also equal to $D F$. Thus, the three straight-lines $D E, D F$, and $D G$ are equal to one another. Thus, the circle drawn with center $D$, and radius one of $E, F$, or $G,{ }^{\dagger}$ will also go through the remaining points, and will touch the straight-lines $A B, B C$, and $C A$, on account of the angles at $E, F$, and $G$ being right-angles. For if it cuts (one of) them then it will be a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, falling inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center $D$, and radius one of $E, F$,
or $G$, does not cut the straight-lines $A B, B C$, and $C A$. Thus, it will touch them and will be the circle inscribed in triangle $A B C$. Let it have been (so) inscribed, like $F G E$ (in the figure).

Thus, the circle $E F G$ has been inscribed in the given triangle $A B C$. (Which is) the very thing it was required to do.
${ }^{\dagger}$ Here, and in the following propositions, it is understood that the radius is actually one of $D E, D F$, or $D G$.

$$
\varepsilon^{\prime} .
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Proposition 5
To circumscribe a circle about a given triangle.


Let $A B C$ be the given triangle. So it is required to circumscribe a circle about the given triangle $A B C$.

Let the straight-lines $A B$ and $A C$ have been cut in half at points $D$ and $E$ (respectively) [Prop. 1.10]. And let $D F$ and $E F$ have been drawn from points $D$ and $E$, at right-angles to $A B$ and $A C$ (respectively) [Prop. 1.11]. So ( $D F$ and $E F$ ) will surely either meet inside triangle $A B C$, on the straight-line $B C$, or beyond $B C$.

Let them, first of all, meet inside (triangle $A B C$ ) at (point) $F$, and let $F B, F C$, and $F A$ have been joined. And since $A D$ is equal to $D B$, and $D F$ is common and at right-angles, the base $A F$ is thus equal to the base $F B$ [Prop. 1.4]. So, similarly, we can show that $C F$ is also equal to $A F$. So that $F B$ is also equal to $F C$. Thus, the three (straight-lines) $F A, F B$, and $F C$ are equal to one another. Thus, the circle drawn with center $F$, and radius one of $A, B$, or $C$, will also go through the remaining points. And the circle will have been circumscribed about triangle $A B C$. Let it have been (so) circumscribed, like $A B C$ (in the first diagram from the left).

And so, let $D F$ and $E F$ meet on the straight-line $B C$ at (point) $F$, like in the second diagram (from the left). And let $A F$ have been joined. So, similarly, we can show that point $F$ is the center of the circle circumscribed about triangle $A B C$.

And so, let $D F$ and $E F$ meet outside triangle $A B C$, again at (point) $F$, like in the third diagram (from the left). And let $A F, B F$, and $C F$ have been joined. And, again, since $A D$ is equal to $D B$, and $D F$ is common and at right-angles, the base $A F$ is thus equal to the base $B F$ [Prop. 1.4]. So, similarly, we can show that $C F$ is also equal to $A F$. So that $B F$ is also equal to $F C$. Thus,

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 $\Delta \mathrm{A}$.









 тòv АВГД xúx入ov.
 тò $А В Г \Delta \cdot$ ö $\pi \varepsilon \rho$ है $\delta \varepsilon \iota ~ \pi о เ ท ̃ \sigma \alpha \iota . ~$
[again] the circle drawn with center $F$, and radius one of $F A, F B$, and $F C$, will also go through the remaining points. And it will have been circumscribed about triangle $A B C$.

Thus, a circle has been circumscribed about the given triangle. (Which is) the very thing it was required to do.

## Proposition 6

To inscribe a square in a given circle.


Let $A B C D$ be the given circle. So it is required to inscribe a square in circle $A B C D$.

Let two diameters of circle $A B C D, A C$ and $B D$, have been drawn at right-angles to one another. ${ }^{\dagger}$ And let $A B$, $B C, C D$, and $D A$ have been joined.

And since $B E$ is equal to $E D$, for $E$ (is) the center (of the circle), and $E A$ is common and at right-angles, the base $A B$ is thus equal to the base $A D$ [Prop. 1.4]. So, for the same (reasons), each of $B C$ and $C D$ is equal to each of $A B$ and $A D$. Thus, the quadrilateral $A B C D$ is equilateral. So I say that (it is) also right-angled. For since the straight-line $B D$ is a diameter of circle $A B C D$, $B A D$ is thus a semi-circle. Thus, angle $B A D$ (is) a rightangle [Prop. 3.31]. So, for the same (reasons), (angles) $A B C, B C D$, and $C D A$ are also each right-angles. Thus, the quadrilateral $A B C D$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been inscribed in circle $A B C D$.

Thus, the square $A B C D$ has been inscribed in the given circle. (Which is) the very thing it was required to do.

[^6]$$
\zeta^{\prime}
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## Proposition 7

To circumscribe a square about a given circle.




 $\sigma \alpha \nu$ ह́ $\varphi \pi \tau o ́ \mu \varepsilon v \alpha l ~ \tau o u ̃ ~ А В Г \Delta ~ x u ́ x \lambda o u ~ \alpha i ~ Z H, ~ H \Theta, ~ \Theta K, ~ K Z . ~$


 ס̀̀ xai ai $\pi \rho o ̀ s ~ t o i ̃ s ~ B, ~ \Gamma, ~ \Delta ~ \sigma \eta \mu \varepsilon i ́ o l s ~ \gamma \omega v i ́ \alpha l ~ o ̉ p \vartheta \alpha i ́ ~ \varepsilon i \sigma เ v . ~$



 $\tau \widetilde{\omega} \nu \mathrm{HZ}, \Theta \mathrm{K} \tau \tilde{n} \mathrm{BE} \Delta \dot{\varepsilon} \sigma \tau \iota \pi \alpha p \alpha ́ \lambda \lambda \eta \lambda o \varsigma . \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu \alpha$





 үàp $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o ́ v ~ \varepsilon ̇ \sigma \tau l ~ \tau o ̀ ~ H B E A, ~ x \alpha i ́ ~ \varepsilon ̇ \sigma \tau 兀 \nu ~ o ̉ p \vartheta \grave{\eta} \dot{\eta}$


 «$p \alpha$ ह̇ $\sigma \tau i v . ~ x \alpha i ~ \pi \varepsilon \rho เ \gamma \varepsilon ́ \gamma p \alpha \pi \tau \alpha u ~ \pi \varepsilon p i ~ \tau o ̀ v ~ А В Г \Delta ~ x u ́ x \lambda о \nu . ~$



Let $A B C D$ be the given circle. So it is required to circumscribe a square about circle $A B C D$.


Let two diameters of circle $A B C D, A C$ and $B D$, have been drawn at right-angles to one another. ${ }^{\dagger}$ And let $F G$, $G H, H K$, and $K F$ have been drawn through points $A$, $B, C$, and $D$ (respectively), touching circle $A B C D . .^{\ddagger}$

Therefore, since $F G$ touches circle $A B C D$, and $E A$ has been joined from the center $E$ to the point of contact $A$, the angles at $A$ are thus right-angles [Prop. 3.18]. So, for the same (reasons), the angles at points $B, C$, and $D$ are also right-angles. And since angle $A E B$ is a rightangle, and $E B G$ is also a right-angle, $G H$ is thus parallel to $A C$ [Prop. 1.29]. So, for the same (reasons), $A C$ is also parallel to $F K$. So that $G H$ is also parallel to $F K$ [Prop. 1.30]. So, similarly, we can show that $G F$ and $H K$ are each parallel to $B E D$. Thus, $G K, G C, A K, F B$, and $B K$ are (all) parallelograms. Thus, $G F$ is equal to $H K$, and $G H$ to $F K$ [Prop. 1.34]. And since $A C$ is equal to $B D$, but $A C$ (is) also (equal) to each of $G H$ and $F K$, and $B D$ is equal to each of $G F$ and $H K$ [Prop. 1.34] [and each of $G H$ and $F K$ is thus equal to each of $G F$ and $H K$ ], the quadrilateral $F G H K$ is thus equilateral. So I say that (it is) also right-angled. For since $G B E A$ is a parallelogram, and $A E B$ is a right-angle, $A G B$ is thus also a right-angle [Prop. 1.34]. So, similarly, we can show that the angles at $H, K$, and $F$ are also right-angles. Thus, $F G H K$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been circumscribed about circle $A B C D$.

Thus, a square has been circumscribed about the given circle. (Which is) the very thing it was required to do.

[^7]
## $\eta^{\prime}$.








 $\mathrm{AK}, \mathrm{KB}, \mathrm{A} \Theta, \Theta \Delta, \mathrm{AH}, \mathrm{H}, \mathrm{BH}, \mathrm{H} \Delta$, xaì ai $\alpha \pi \varepsilon v \alpha \nu \tau i o v$














 тò $А В Г \Delta$ т $\varepsilon \tau \rho \alpha ́ \gamma \omega v o v$.



## $\vartheta^{\prime}$.





## Proposition 8

To inscribe a circle in a given square.
Let the given square be $A B C D$. So it is required to inscribe a circle in square $A B C D$.


Let $A D$ and $A B$ each have been cut in half at points $E$ and $F$ (respectively) [Prop. 1.10]. And let $E H$ have been drawn through $E$, parallel to either of $A B$ or $C D$, and let $F K$ have been drawn through $F$, parallel to either of $A D$ or $B C$ [Prop. 1.31]. Thus, $A K, K B, A H, H D, A G, G C$, $B G$, and $G D$ are each parallelograms, and their opposite sides [are] manifestly equal [Prop. 1.34]. And since $A D$ is equal to $A B$, and $A E$ is half of $A D$, and $A F$ half of $A B, A E$ (is) thus also equal to $A F$. So that the opposite (sides are) also (equal). Thus, $F G$ (is) also equal to $G E$. So, similarly, we can also show that each of $G H$ and $G K$ is equal to each of $F G$ and $G E$. Thus, the four (straightlines) $G E, G F, G H$, and $G K$ [are] equal to one another. Thus, the circle drawn with center $G$, and radius one of $E, F, H$, or $K$, will also go through the remaining points. And it will touch the straight-lines $A B, B C, C D$, and $D A$, on account of the angles at $E, F, H$, and $K$ being right-angles. For if the circle cuts $A B, B C, C D$, or $D A$, then a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, will fall inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center $G$, and radius one of $E, F, H$, or $K$, does not cut the straight-lines $A B, B C$, $C D$, or $D A$. Thus, it will touch them, and will have been inscribed in the square $A B C D$.

Thus, a circle has been inscribed in the given square. (Which is) the very thing it was required to do.

## Proposition 9

To circumscribe a circle about a given square.
Let $A B C D$ be the given square. So it is required to circumscribe a circle about square $A B C D$.
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 оัтєр ع́ठєเ $\pi о เ ท ̃ \sigma \alpha เ$.

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\therefore \text {. }
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 $\pi \varepsilon$ ì tò АГ $\Delta$ трír $\omega$ vov 犭úx入os ò $А Г \Delta$.
$A C$ and $B D$ being joined, let them cut one another at $E$.


And since $D A$ is equal to $A B$, and $A C$ (is) common, the two (straight-lines) $D A, A C$ are thus equal to the two (straight-lines) $B A, A C$. And the base $D C$ (is) equal to the base $B C$. Thus, angle $D A C$ is equal to angle $B A C$ [Prop. 1.8]. Thus, the angle $D A B$ has been cut in half by $A C$. So, similarly, we can show that $A B C, B C D$, and $C D A$ have each been cut in half by the straight-lines $A C$ and $D B$. And since angle $D A B$ is equal to $A B C$, and $E A B$ is half of $D A B$, and $E B A$ half of $A B C, E A B$ is thus also equal to $E B A$. So that side $E A$ is also equal to $E B$ [Prop. 1.6]. So, similarly, we can show that each of the [straight-lines] $E A$ and $E B$ are also equal to each of $E C$ and $E D$. Thus, the four (straight-lines) $E A, E B$, $E C$, and $E D$ are equal to one another. Thus, the circle drawn with center $E$, and radius one of $A, B, C$, or $D$, will also go through the remaining points, and will have been circumscribed about the square $A B C D$. Let it have been (so) circumscribed, like $A B C D$ (in the figure).

Thus, a circle has been circumscribed about the given square. (Which is) the very thing it was required to do.

## Proposition 10

To construct an isosceles triangle having each of the angles at the base double the remaining (angle).

Let some straight-line $A B$ be taken, and let it have been cut at point $C$ so that the rectangle contained by $A B$ and $B C$ is equal to the square on $C A$ [Prop. 2.11]. And let the circle $B D E$ have been drawn with center $A$, and radius $A B$. And let the straight-line $B D$, equal to the straight-line $A C$, being not greater than the diameter of circle $B D E$, have been inserted into circle $B D E$ [Prop. 4.1]. And let $A D$ and $D C$ have been joined. And let the circle $A C D$ have been circumscribed about triangle $A C D$ [Prop. 4.5].




 $\pi \rho \circ \sigma \pi \varepsilon \pi \tau \omega ́ \omega \alpha \sigma l$ ठט́o $\varepsilon \cup ̉ \vartheta \varepsilon \tau ̃ \alpha l ~ \alpha i ~ B A, ~ B \Delta, ~ x \alpha i ̀ ~ \dot{\eta} \mu \varepsilon ̀ \nu ~ \alpha u ̉ \tau \tilde{\omega} \nu$



















 धं $\sigma \tau \iota \delta \iota \pi \lambda \tilde{\eta}$.



$i \alpha^{\prime}$.



And since the (rectangle contained) by $A B$ and $B C$ is equal to the (square) on $A C$, and $A C$ (is) equal to $B D$, the (rectangle contained) by $A B$ and $B C$ is thus equal to the (square) on $B D$. And since some point $B$ has been taken outside of circle $A C D$, and two straightlines $B A$ and $B D$ have radiated from $B$ towards the circle $A C D$, and (one) of them cuts (the circle), and (the other) meets (the circle), and the (rectangle contained) by $A B$ and $B C$ is equal to the (square) on $B D, B D$ thus touches circle $A C D$ [Prop. 3.37]. Therefore, since $B D$ touches (the circle), and $D C$ has been drawn across (the circle) from the point of contact $D$, the angle $B D C$ is thus equal to the angle $D A C$ in the alternate segment of the circle [Prop. 3.32]. Therefore, since $B D C$ is equal to $D A C$, let $C D A$ have been added to both. Thus, the whole of $B D A$ is equal to the two (angles) $C D A$ and $D A C$. But, the external (angle) $B C D$ is equal to $C D A$ and $D A C$ [Prop. 1.32]. Thus, $B D A$ is also equal to $B C D$. But, $B D A$ is equal to $C B D$, since the side $A D$ is also equal to $A B$ [Prop. 1.5]. So that $D B A$ is also equal to $B C D$. Thus, the three (angles) $B D A, D B A$, and $B C D$ are equal to one another. And since angle $D B C$ is equal to $B C D$, side $B D$ is also equal to side $D C$ [Prop. 1.6]. But, $B D$ was assumed (to be) equal to $C A$. Thus, $C A$ is also equal to $C D$. So that angle $C D A$ is also equal to angle $D A C$ [Prop. 1.5]. Thus, $C D A$ and $D A C$ is double $D A C$. But $B C D$ (is) equal to $C D A$ and $D A C$. Thus, $B C D$ is also double $C A D$. And $B C D$ (is) equal to to each of $B D A$ and $D B A$. Thus, $B D A$ and $D B A$ are each double $D A B$.

Thus, the isosceles triangle $A B D$ has been constructed having each of the angles at the base $B D$ double the remaining (angle). (Which is) the very thing it was required to do.

## Proposition 11

To inscribe an equilateral and equiangular pentagon




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in a given circle.


Let $A B C D E$ be the given circle. So it is required to inscribed an equilateral and equiangular pentagon in circle $A B C D E$.

Let the the isosceles triangle $F G H$ be set up having each of the angles at $G$ and $H$ double the (angle) at $F$ [Prop. 4.10]. And let triangle $A C D$, equiangular to $F G H$, have been inscribed in circle $A B C D E$, such that $C A D$ is equal to the angle at $F$, and the (angles) at $G$ and $H$ (are) equal to $A C D$ and $C D A$, respectively [Prop. 4.2]. Thus, $A C D$ and $C D A$ are each double $C A D$. So let $A C D$ and $C D A$ have been cut in half by the straight-lines $C E$ and $D B$, respectively [Prop. 1.9]. And let $A B, B C, D E$ and $E A$ have been joined.

Therefore, since angles $A C D$ and $C D A$ are each double $C A D$, and are cut in half by the straight-lines $C E$ and $D B$, the five angles $D A C, A C E, E C D, C D B$, and $B D A$ are thus equal to one another. And equal angles stand upon equal circumferences [Prop. 3.26]. Thus, the five circumferences $A B, B C, C D, D E$, and $E A$ are equal to one another [Prop. 3.29]. Thus, the pentagon $A B C D E$ is equilateral. So I say that (it is) also equiangular. For since the circumference $A B$ is equal to the circumference $D E$, let $B C D$ have been added to both. Thus, the whole circumference $A B C D$ is equal to the whole circumference $E D C B$. And the angle $A E D$ stands upon circumference $A B C D$, and angle $B A E$ upon circumference $E D C B$. Thus, angle $B A E$ is also equal to $A E D$ [Prop. 3.27]. So, for the same (reasons), each of the angles $A B C, B C D$, and $C D E$ is also equal to each of $B A E$ and $A E D$. Thus, pentagon $A B C D E$ is equiangular. And it was also shown (to be) equilateral.

Thus, an equilateral and equiangular pentagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

## Proposition 12

To circumscribe an equilateral and equiangular pentagon about a given circle.


 $\pi \varepsilon \rho เ \gamma \rho \alpha ́ \psi \alpha$.

 $\Gamma \Delta, \Delta \mathrm{E}, \mathrm{EA} \pi \varepsilon \rho \iota \varphi \varepsilon \rho \varepsilon i \alpha \varsigma^{\circ}$ каі $\delta \dot{\alpha} \tau \widetilde{\omega} \nu \mathrm{A}, \mathrm{B}, \Gamma, \Delta, \mathrm{E}$

 ย่ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu$ גi $\mathrm{ZB}, \mathrm{ZK}, \mathrm{Z}, \mathrm{Z} \Lambda, \mathrm{Z} \Delta$.
 тò $\Gamma$, 白兀ò ठè toũ Z x






























Let $A B C D E$ be the given circle. So it is required to circumscribe an equilateral and equiangular pentagon about circle $A B C D E$.

Let $A, B, C, D$, and $E$ have been conceived as the angular points of a pentagon having been inscribed (in circle $A B C D E$ ) [Prop. 3.11], such that the circumferences $A B, B C, C D, D E$, and $E A$ are equal. And let $G H, H K$, $K L, L M$, and $M G$ have been drawn through (points) $A$, $B, C, D$, and $E$ (respectively), touching the circle. ${ }^{\dagger}$ And let the center $F$ of the circle $A B C D E$ have been found [Prop. 3.1]. And let $F B, F K, F C, F L$, and $F D$ have been joined.

And since the straight-line $K L$ touches (circle) $A B C D E$ at $C$, and $F C$ has been joined from the center $F$ to the point of contact $C, F C$ is thus perpendicular to $K L$ [Prop. 3.18]. Thus, each of the angles at $C$ is a rightangle. So, for the same (reasons), the angles at $B$ and $D$ are also right-angles. And since angle $F C K$ is a rightangle, the (square) on $F K$ is thus equal to the (sum of the squares) on $F C$ and $C K$ [Prop. 1.47]. So, for the same (reasons), the (square) on $F K$ is also equal to the (sum of the squares) on $F B$ and $B K$. So that the (sum of the squares) on $F C$ and $C K$ is equal to the (sum of the squares) on $F B$ and $B K$, of which the (square) on $F C$ is equal to the (square) on $F B$. Thus, the remaining (square) on $C K$ is equal to the remaining (square) on $B K$. Thus, $B K$ (is) equal to $C K$. And since $F B$ is equal to $F C$, and $F K$ (is) common, the two (straightlines) $B F, F K$ are equal to the two (straight-lines) $C F$, $F K$. And the base $B K$ [is] equal to the base $C K$. Thus, angle $B F K$ is equal to [angle] $K F C$ [Prop. 1.8]. And $B K F$ (is equal) to $F K C$ [Prop. 1.8]. Thus, $B F C$ (is) double $K F C$, and $B K C$ (is double) $F K C$. So, for the same (reasons), $C F D$ is also double $C F L$, and $D L C$ (is also double) $F L C$. And since circumference $B C$ is equal to $C D$, angle $B F C$ is also equal to $C F D$ [Prop. 3.27]. And $B F C$ is double $K F C$, and $D F C$ (is double) $L F C$. Thus, $K F C$ is also equal to $L F C$. And angle $F C K$ is also equal to $F C L$. So, $F K C$ and $F L C$ are two triangles hav-



 $\tau \tilde{\eta} s ~ \delta e ̀ ~ u ́ \pi o ̀ ~ Z \Lambda \Gamma ~ \delta ı \pi \lambda \tilde{\eta} \dot{\eta}$ ú $\pi o ̀ \mathrm{~K} \Lambda \mathrm{M}$, xai $\dot{\eta}$ ú $\pi o ̀ ~ \Theta K \Lambda ~ \alpha ̈ p \alpha ~$








${ }^{\dagger}$ See the footnote to Prop. 3.34.

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 غ̇үүpáчa.




ing two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle to the remaining angle [Prop. 1.26]. Thus, the straight-line $K C$ (is) equal to $C L$, and the angle $F K C$ to $F L C$. And since $K C$ is equal to $C L, K L$ (is) thus double $K C$. So, for the same (reasons), it can be shown that $H K$ (is) also double $B K$. And $B K$ is equal to $K C$. Thus, $H K$ is also equal to $K L$. So, similarly, each of $H G, G M$, and $M L$ can also be shown (to be) equal to each of $H K$ and $K L$. Thus, pentagon $G H K L M$ is equilateral. So I say that (it is) also equiangular. For since angle $F K C$ is equal to $F L C$, and $H K L$ was shown (to be) double $F K C$, and $K L M$ double $F L C, H K L$ is thus also equal to $K L M$. So, similarly, each of $K H G, H G M$, and $G M L$ can also be shown (to be) equal to each of $H K L$ and $K L M$. Thus, the five angles $G H K, H K L, K L M, L M G$, and $M G H$ are equal to one another. Thus, the pentagon $G H K L M$ is equiangular. And it was also shown (to be) equilateral, and has been circumscribed about circle $A B C D E$.
[Thus, an equilateral and equiangular pentagon has been circumscribed about the given circle]. (Which is) the very thing it was required to do.

## Proposition 13

To inscribe a circle in a given pentagon, which is equilateral and equiangular.


Let $A B C D E$ be the given equilateral and equiangular pentagon. So it is required to inscribe a circle in pentagon $A B C D E$.

For let angles $B C D$ and $C D E$ have each been cut in half by each of the straight-lines $C F$ and $D F$ (respectively) [Prop. 1.9]. And from the point $F$, at which the straight-lines $C F$ and $D F$ meet one another, let the





































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straight-lines $F B, F A$, and $F E$ have been joined. And since $B C$ is equal to $C D$, and $C F$ (is) common, the two (straight-lines) $B C, C F$ are equal to the two (straightlines) $D C, C F$. And angle $B C F$ [is] equal to angle $D C F$. Thus, the base $B F$ is equal to the base $D F$, and triangle $B C F$ is equal to triangle $D C F$, and the remaining angles will be equal to the (corresponding) remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle $C B F$ (is) equal to $C D F$. And since $C D E$ is double $C D F$, and $C D E$ (is) equal to $A B C$, and $C D F$ to $C B F, C B A$ is thus also double $C B F$. Thus, angle $A B F$ is equal to $F B C$. Thus, angle $A B C$ has been cut in half by the straight-line $B F$. So, similarly, it can be shown that $B A E$ and $A E D$ have been cut in half by the straight-lines $F A$ and $F E$, respectively. So let $F G, F H$, $F K, F L$, and $F M$ have been drawn from point $F$, perpendicular to the straight-lines $A B, B C, C D, D E$, and $E A$ (respectively) [Prop. 1.12]. And since angle $H C F$ is equal to $K C F$, and the right-angle $F H C$ is also equal to the [right-angle] $F K C, F H C$ and $F K C$ are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) $F C$, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, the perpendicular $F H$ (is) equal to the perpendicular $F K$. So, similarly, it can be shown that $F L, F M$, and $F G$ are each equal to each of $F H$ and $F K$. Thus, the five straight-lines $F G$, $F H, F K, F L$, and $F M$ are equal to one another. Thus, the circle drawn with center $F$, and radius one of $G, H$, $K$, $L$, or $M$, will also go through the remaining points, and will touch the straight-lines $A B, B C, C D, D E$, and $E A$, on account of the angles at points $G, H, K, L$, and $M$ being right-angles. For if it does not touch them, but cuts them, it follows that a (straight-line) drawn at rightangles to the diameter of the circle, from its extremity, falls inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center $F$, and radius one of $G, H, K, L$, or $M$, does not cut the straight-lines $A B, B C, C D, D E$, or $E A$. Thus, it will touch them. Let it have been drawn, like GHKLM (in the figure).

Thus, a circle has been inscribed in the given pentagon which is equilateral and equiangular. (Which is) the very thing it was required to do.

## Proposition 14

To circumscribe a circle about a given pentagon which is equilateral and equiangular.

Let $A B C D E$ be the given pentagon which is equilat-
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 $\chi \alpha \vartheta^{\prime}$ ö $\sigma \cup \mu \beta \alpha ́ \lambda \lambda \lambda o v \sigma レ ~ \alpha i ~ \varepsilon u ̉ \vartheta \varepsilon i ̃ \alpha l, ~ غ ̇ \pi i ̀ ~ \tau \grave{\alpha} \mathrm{~B}, \mathrm{~A}, \mathrm{E} \sigma \eta \mu \varepsilon i ̃ \alpha$

 AE $\Delta \gamma \omega \nu \iota \widetilde{\omega} \nu$ ठí $\chi \alpha \tau \varepsilon ́ \tau \mu \eta \tau \alpha l$ ن́ $\pi o ̀ ~ \varepsilon ́ x \alpha ́ \sigma \tau \eta ร ~ \tau \widetilde{\omega} \nu \mathrm{ZB}, \mathrm{ZA}, \mathrm{ZE}$













## $\iota \varepsilon$＇．

Eis tòv $\delta о \vartheta \varepsilon ́ v \tau \alpha ~ x u ́ x \lambda o v ~ \varepsilon ́ \xi ' \alpha ́ \gamma \omega \nu o v ~ i \sigma o ́ \pi \lambda \varepsilon u p o ́ v ~ \tau \varepsilon ~ x \alpha i ̀ ~$ iooүóviov ย̉ $\gamma \gamma \rho \alpha ́ \psi \alpha$ ．

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eral and equiangular．So it is required to circumscribe a circle about the pentagon $A B C D E$ ．


So let angles $B C D$ and $C D E$ have been cut in half by the（straight－lines）$C F$ and $D F$ ，respectively［Prop．1．9］． And let the straight－lines $F B, F A$ ，and $F E$ have been joined from point $F$ ，at which the straight－lines meet， to the points $B, A$ ，and $E$（respectively）．So，similarly， to the（proposition）before this（one），it can be shown that angles $C B A, B A E$ ，and $A E D$ have also been cut in half by the straight－lines $F B, F A$ ，and $F E$ ，respec－ tively．And since angle $B C D$ is equal to $C D E$ ，and $F C D$ is half of $B C D$ ，and $C D F$ half of $C D E, F C D$ is thus also equal to $F D C$ ．So that side $F C$ is also equal to side $F D$［Prop．1．6］．So，similarly，it can be shown that $F B$ ， $F A$ ，and $F E$ are also each equal to each of $F C$ and $F D$ ． Thus，the five straight－lines $F A, F B, F C, F D$ ，and $F E$ are equal to one another．Thus，the circle drawn with center $F$ ，and radius one of $F A, F B, F C, F D$ ，or $F E$ ， will also go through the remaining points，and will have been circumscribed．Let it have been（so）circumscribed， and let it be $A B C D E$ ．

Thus，a circle has been circumscribed about the given pentagon，which is equilateral and equiangular．（Which is）the very thing it was required to do．

## Proposition 15

To inscribe an equilateral and equiangular hexagon in a given circle．

Let $A B C D E F$ be the given circle．So it is required to inscribe an equilateral and equiangular hexagon in circle $A B C D E F$ ．

Let the diameter $A D$ of circle $A B C D E F$ have been drawn，${ }^{\dagger}$ and let the center $G$ of the circle have been found［Prop．3．1］．And let the circle $E G C H$ have been drawn，with center $D$ ，and radius $D G$ ．And $E G$ and $C G$ being joined，let them have been drawn across（the cir－
































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cle) to points $B$ and $F$ (respectively). And let $A B, B C$, $C D, D E, E F$, and $F A$ have been joined. I say that the hexagon $A B C D E F$ is equilateral and equiangular.


For since point $G$ is the center of circle $A B C D E F$, $G E$ is equal to $G D$. Again, since point $D$ is the center of circle $G C H, D E$ is equal to $D G$. But, $G E$ was shown (to be) equal to $G D$. Thus, $G E$ is also equal to $E D$. Thus, triangle $E G D$ is equilateral. Thus, its three angles $E G D, G D E$, and $D E G$ are also equal to one another, inasmuch as the angles at the base of isosceles triangles are equal to one another [Prop. 1.5]. And the three angles of the triangle are equal to two right-angles [Prop. 1.32]. Thus, angle $E G D$ is one third of two rightangles. So, similarly, $D G C$ can also be shown (to be) one third of two right-angles. And since the straight-line $C G$, standing on $E B$, makes adjacent angles $E G C$ and $C G B$ equal to two right-angles [Prop. 1.13], the remaining angle $C G B$ is thus also one third of two right-angles. Thus, angles $E G D, D G C$, and $C G B$ are equal to one another. And hence the (angles) opposite to them $B G A$, $A G F$, and $F G E$ are also equal [to $E G D, D G C$, and $C G B$ (respectively)] [Prop. 1.15]. Thus, the six angles $E G D, D G C, C G B, B G A, A G F$, and $F G E$ are equal to one another. And equal angles stand on equal circumferences [Prop. 3.26]. Thus, the six circumferences $A B, B C, C D, D E, E F$, and $F A$ are equal to one another. And equal circumferences are subtended by equal straight-lines [Prop. 3.29]. Thus, the six straight-lines ( $A B, B C, C D, D E, E F$, and $F A$ ) are equal to one another. Thus, hexagon $A B C D E F$ is equilateral. So, I say that (it is) also equiangular. For since circumference $F A$ is equal to circumference $E D$, let circumference $A B C D$ have been added to both. Thus, the whole of $F A B C D$ is equal to the whole of $E D C B A$. And angle $F E D$ stands on circumference $F A B C D$, and angle $A F E$ on circumference $E D C B A$. Thus, angle $A F E$ is equal


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${ }^{\dagger}$ See the footnote to Prop. 4.6.

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to $D E F$ [Prop. 3.27]. Similarly, it can also be shown that the remaining angles of hexagon $A B C D E F$ are individually equal to each of the angles $A F E$ and $F E D$. Thus, hexagon $A B C D E F$ is equiangular. And it was also shown (to be) equilateral. And it has been inscribed in circle $A B C D E$.

Thus, an equilateral and equiangular hexagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

## Corollary

So, from this, (it is) manifest that a side of the hexagon is equal to the radius of the circle.

And similarly to a pentagon, if we draw tangents to the circle through the (sixfold) divisions of the (circumference of the) circle, an equilateral and equiangular hexagon can be circumscribed about the circle, analogously to the aforementioned pentagon. And, further, by (means) similar to the aforementioned pentagon, we can inscribe and circumscribe a circle in (and about) a given hexagon. (Which is) the very thing it was required to do.

## Proposition 16

To inscribe an equilateral and equiangular fifteensided figure in a given circle.


Let $A B C D$ be the given circle. So it is required to inscribe an equilateral and equiangular fifteen-sided figure in circle $A B C D$.

Let the side $A C$ of an equilateral triangle inscribed in (the circle) [Prop. 4.2], and (the side) $A B$ of an (inscribed) equilateral pentagon [Prop. 4.11], have been inscribed in circle $A B C D$. Thus, just as the circle $A B C D$ is (made up) of fifteen equal pieces, the circumference $A B C$, being a third of the circle, will be (made up) of five

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such (pieces), and the circumference $A B$, being a fifth of the circle, will be (made up) of three. Thus, the remainder $B C$ (will be made up) of two equal (pieces). Let (circumference) $B C$ have been cut in half at $E$ [Prop. 3.30]. Thus, each of the circumferences $B E$ and $E C$ is one fifteenth of the circle $A B C D E$.

Thus, if, joining $B E$ and $E C$, we continuously insert straight-lines equal to them into circle $A B C D[E]$ [Prop. 4.1], then an equilateral and equiangular fifteensided figure will have been inserted into (the circle). (Which is) the very thing it was required to do.

And similarly to the pentagon, if we draw tangents to the circle through the (fifteenfold) divisions of the (circumference of the) circle, we can circumscribe an equilateral and equiangular fifteen-sided figure about the circle. And, further, through similar proofs to the pentagon, we can also inscribe and circumscribe a circle in (and about) a given fifteen-sided figure. (Which is) the very thing it was required to do.

## ELEMENTS BOOK 5

## Proportion ${ }^{\dagger}$

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 $\lambda เ x o ́ \tau \eta \tau \alpha ́ ~ \pi o l \alpha \sigma \chi \varepsilon ́ \sigma ı s$.
$\delta^{\prime}$. $\Lambda o ́ \gamma o \nu$ है $\chi \varepsilon เ \nu \pi \rho o ̀ s ~ \alpha ̈ \lambda \lambda \eta \lambda \alpha \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta \eta ~ \lambda \varepsilon ́ \gamma \varepsilon \tau \alpha l, ~ \alpha ̈ ~ \delta u ́ v \alpha \tau \alpha l$







 $\chi \alpha \lambda \varepsilon \dot{\sigma} \sigma \vartheta \omega$.
 $\pi \rho \omega ́ \tau o u ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma เ \circ \nu ~ \cup ́ \pi \varepsilon \rho \varepsilon ́ \chi \eta ~ \tau o u ̃ ~ \tau о u ̃ ~ \delta \varepsilon \cup \tau \varepsilon ́ \rho o u ~ \pi o \lambda-~$


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## Definitions

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater. ${ }^{\dagger}$
2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.
3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind. ${ }^{\ddagger}$
4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another. ${ }^{\S}$
5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever. ${ }^{\text {® }}$
6. And let magnitudes having the same ratio be called proportional.*
7. And when for equal multiples (as in Def. 5), the multiple of the first (magnitude) exceeds the multiple of the second, and the multiple of the third (magnitude) does not exceed the multiple of the fourth, then the first (magnitude) is said to have a greater ratio to the second than the third (magnitude has) to the fourth.
8. And a proportion in three terms is the smallest (possible). ${ }^{\$}$
9. And when three magnitudes are proportional, the first is said to have to the third the squared ${ }^{\|}$ratio of that (it has) to the second. ${ }^{\dagger \dagger}$
10. And when four magnitudes are (continuously) proportional, the first is said to have to the fourth the cubed ${ }^{\ddagger \ddagger}$ ratio of that (it has) to the second. ${ }^{\S \S}$ And so on, similarly, in successive order, whatever the (continuous) proportion might be.
11. These magnitudes are said to be corresponding (magnitudes): the leading to the leading (of two ratios), and the following to the following.
12. An alternate ratio is a taking of the (ratio of the) leading (magnitude) to the leading (of two equal ratios), and (setting it equal to) the (ratio of the) following (magnitude) to the following. ${ }^{\top \pi}$
13. An inverse ratio is a taking of the (ratio of the) following (magnitude) as the leading and the leading (magnitude) as the following.**
14. A composition of a ratio is a taking of the (ratio of the) leading plus the following (magnitudes), as one, to the following (magnitude) by itself. ${ }^{\$ \$}$
$\chi \alpha \vartheta \vartheta^{\prime} \dot{\cup} \pi \varepsilon \xi \alpha i \rho \varepsilon \sigma \omega \tau \tau \widetilde{ } \mu \varepsilon \varepsilon^{\prime} \sigma \omega \nu$.






15. A separation of a ratio is a taking of the (ratio of the) excess by which the leading (magnitude) exceeds the following to the following (magnitude) by itself.|"I
16. A conversion of a ratio is a taking of the (ratio of the) leading (magnitude) to the excess by which the leading (magnitude) exceeds the following. ${ }^{\dagger \dagger \dagger}$
17. There being several magnitudes, and other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, a ratio via equality (or ex aequali) occurs when as the first is to the last in the first (set of) magnitudes, so the first (is) to the last in the second (set of) magnitudes. Or alternately, (it is) a taking of the (ratio of the) outer (magnitudes) by the removal of the inner (magnitudes). ${ }^{\ddagger \ddagger \ddagger}$
18. There being three magnitudes, and other (magnitudes) of equal number to them, a perturbed proportion occurs when as the leading is to the following in the first (set of) magnitudes, so the leading (is) to the following in the second (set of) magnitudes, and as the following (is) to some other (i.e., the remaining magnitude) in the first (set of) magnitudes, so some other (is) to the leading in the second (set of) magnitudes. ${ }^{\S \S \S}$
${ }^{\dagger}$ In other words, $\alpha$ is said to be a part of $\beta$ if $\beta=m \alpha$.
$\ddagger$ In modern notation, the ratio of two magnitudes, $\alpha$ and $\beta$, is denoted $\alpha: \beta$.
§ In other words, $\alpha$ has a ratio with respect to $\beta$ if $m \alpha>\beta$ and $n \beta>\alpha$, for some $m$ and $n$.
बIn other words, $\alpha: \beta:: \gamma: \delta$ if and only if $m \alpha>n \beta$ whenever $m \gamma>n \delta$, and $m \alpha=n \beta$ whenever $m \gamma=n \delta$, and $m \alpha<n \beta$ whenever $m \gamma<n \delta$, for all $m$ and $n$. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if $\alpha$, $\beta$, etc., are irrational.
${ }^{*}$ Thus if $\alpha$ and $\beta$ have the same ratio as $\gamma$ and $\delta$ then they are proportional. In modern notation, $\alpha: \beta:: \gamma: \delta$.
$\$$ In modern notation, a proportion in three terms- $\alpha, \beta$, and $\gamma$-is written: $\alpha: \beta:: \beta: \gamma$.
|| Literally, "double".
${ }^{\dagger \dagger}$ In other words, if $\alpha: \beta:: \beta: \gamma$ then $\alpha: \gamma:: \alpha^{2}: \beta^{2}$.
$\ddagger \ddagger$ Literally, "triple".
${ }^{\S}$ In other words, if $\alpha: \beta:: \beta: \gamma:: \gamma: \delta$ then $\alpha: \delta:: \alpha^{3}: \beta^{3}$.
【 $\mathbb{T}$ In other words, if $\alpha: \beta:: \gamma: \delta$ then the alternate ratio corresponds to $\alpha: \gamma:: \beta: \delta$.
${ }^{* *}$ In other words, if $\alpha: \beta$ then the inverse ratio corresponds to $\beta: \alpha$.
$\$ \$$ In other words, if $\alpha: \beta$ then the composed ratio corresponds to $\alpha+\beta: \beta$.
$\|\|\|$ In other words, if $\alpha: \beta$ then the separated ratio corresponds to $\alpha-\beta: \beta$.
$\dagger \dagger \dagger$ In other words, if $\alpha: \beta$ then the converted ratio corresponds to $\alpha: \alpha-\beta$.
$\ddagger \ddagger \ddagger$ In other words, if $\alpha, \beta, \gamma$ are the first set of magnitudes, and $\delta, \epsilon, \zeta$ the second set, and $\alpha: \beta: \gamma:: \delta: \epsilon: \zeta$, then the ratio via equality (or ex aequali) corresponds to $\alpha: \gamma:: \delta: \zeta$.
${ }^{\S} \S \S$ In other words, if $\alpha, \beta, \gamma$ are the first set of magnitudes, and $\delta, \epsilon, \zeta$ the second set, and $\alpha: \beta:: \delta: \epsilon$ as well as $\beta: \gamma:: \zeta: \delta$, then the proportion is said to be perturbed.

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## Proposition $1^{\dagger}$

If there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many
$\pi \alpha ́ \nu \tau \alpha \tau \widetilde{\omega} \nu \pi \alpha ́ \nu \tau \omega \nu$.

"E $\sigma \tau \omega$ ò $\pi о \sigma \alpha o u ̃ \nu ~ \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta ~ \tau \grave{\alpha} \mathrm{AB}, \Gamma \Delta$ ó $\pi о \sigma \omega \nu o u ̃ \nu \mu \varepsilon-$
 $\pi \circ \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma \iota \circ \cdot \lambda \varepsilon ́ \gamma \omega$, ơтı ó $\sigma \alpha \pi \lambda \alpha ́ \sigma เ o ́ v ~ \varepsilon ̇ \sigma \tau l ~ \tau o ̀ ~ A B ~ \tau o u ̃ ~ E, ~$
















times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second).


Let there be any number of magnitudes whatsoever, $A B, C D$, (which are) equal multiples, respectively, of some (other) magnitudes, $E, F$, of equal number (to them). I say that as many times as $A B$ is (divisible) by $E$, so many times will $A B, C D$ also be (divisible) by $E, F$.

For since $A B, C D$ are equal multiples of $E, F$, thus as many magnitudes as (there) are in $A B$ equal to $E$, so many (are there) also in $C D$ equal to $F$. Let $A B$ have been divided into magnitudes $A G, G B$, equal to $E$, and $C D$ into (magnitudes) $C H, H D$, equal to $F$. So, the number of (divisions) $A G, G B$ will be equal to the number of (divisions) $C H, H D$. And since $A G$ is equal to $E$, and $C H$ to $F, A G$ (is) thus equal to $E$, and $A G, C H$ to $E$, $F$. So, for the same (reasons), $G B$ is equal to $E$, and $G B$, $H D$ to $E, F$. Thus, as many (magnitudes) as (there) are in $A B$ equal to $E$, so many (are there) also in $A B, C D$ equal to $E, F$. Thus, as many times as $A B$ is (divisible) by $E$, so many times will $A B, C D$ also be (divisible) by $E, F$.

Thus, if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second). (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads $m \alpha+m \beta+\cdots=m(\alpha+\beta+\cdots)$.

## $\beta^{\prime}$.




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 $\pi \varepsilon ́ \mu \pi \tau о \nu ~ \tau o ̀ ~ B H ~ \delta \varepsilon u \tau \varepsilon ́ p o u ~ \tau o u ̃ ~ \Gamma ~ i ́ \sigma \alpha ́ x ו s ~ \pi о \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma เ o v ~ \chi \alpha i ̀ ~$




## Proposition $2^{\dagger}$

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and the sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively).

For let a first (magnitude) $A B$ and a third $D E$ be equal multiples of a second $C$ and a fourth $F$ (respectively). And let a fifth (magnitude) $B G$ and a sixth $E H$ also be (other) equal multiples of the second $C$ and the fourth $F$ (respectively). I say that the first (magnitude) and the fifth, being added together, (to give) $A G$, and the third (magnitude) and the sixth, (being added together,


## $\mathrm{Z} \longmapsto$















${ }^{\dagger}$ In modern notation, this propostion reads $m \alpha+n \alpha=(m+n) \alpha$.

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to give) $D H$, will also be equal multiples of the second (magnitude) $C$ and the fourth $F$ (respectively).

$\mathrm{F} \longmapsto$
For since $A B$ and $D E$ are equal multiples of $C$ and $F$ (respectively), thus as many (magnitudes) as (there) are in $A B$ equal to $C$, so many (are there) also in $D E$ equal to $F$. And so, for the same (reasons), as many (magnitudes) as (there) are in $B G$ equal to $C$, so many (are there) also in $E H$ equal to $F$. Thus, as many (magnitudes) as (there) are in the whole of $A G$ equal to $C$, so many (are there) also in the whole of $D H$ equal to $F$. Thus, as many times as $A G$ is (divisible) by $C$, so many times will $D H$ also be divisible by $F$. Thus, the first (magnitude) and the fifth, being added together, (to give) $A G$, and the third (magnitude) and the sixth, (being added together, to give) $D H$, will also be equal multiples of the second (magnitude) $C$ and the fourth $F$ (respectively).

Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively). (Which is) the very thing it was required to show.

## Proposition $3^{\dagger}$

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively.

For let a first (magnitude) $A$ and a third $C$ be equal multiples of a second $B$ and a fourth $D$ (respectively), and let the equal multiples $E F$ and $G H$ have been taken of $A$ and $C$ (respectively). I say that $E F$ and $G H$ are equal multiples of $B$ and $D$ (respectively).

For since $E F$ and $G H$ are equal multiples of $A$ and $C$ (respectively), thus as many (magnitudes) as (there) are in $E F$ equal to $A$, so many (are there) also in $G H$
 $\mathrm{H} \Lambda, \Lambda \Theta$. x $\alpha$ દ̇ $\pi \varepsilon i$ ỉ $\sigma \alpha ́ x เ \varsigma ~ \varepsilon ُ \sigma \tau i ̀ ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma เ o v ~ \tau o ̀ ~ A ~ \tau o u ̃ ~ B ~ x \alpha \grave{~}$















${ }^{\dagger}$ In modern notation, this proposition reads $m(n \alpha)=(m n) \alpha$.

$$
\delta^{\prime}
$$

 трítov $\pi \rho o ̀ \varsigma ~ \tau \varepsilon ́ \tau \alpha p \tau о \nu, ~ \chi \alpha i ̀ ~ \tau \alpha ̀ ~ i \sigma \alpha ́ x ı \varsigma ~ \pi о \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma \iota \alpha ~ \tau о u ̃ ~ \tau \varepsilon ~$








equal to $C$. Let $E F$ have been divided into magnitudes $E K, K F$ equal to $A$, and $G H$ into (magnitudes) $G L, L H$ equal to $C$. So, the number of (magnitudes) $E K, K F$ will be equal to the number of (magnitudes) $G L, L H$. And since $A$ and $C$ are equal multiples of $B$ and $D$ (respectively), and $E K$ (is) equal to $A$, and $G L$ to $C, E K$ and $G L$ are thus equal multiples of $B$ and $D$ (respectively). So, for the same (reasons), $K F$ and $L H$ are equal multiples of $B$ and $D$ (respectively). Therefore, since the first (magnitude) $E K$ and the third $G L$ are equal multiples of the second $B$ and the fourth $D$ (respectively), and the fifth (magnitude) $K F$ and the sixth $L H$ are also equal multiples of the second $B$ and the fourth $D$ (respectively), then the first (magnitude) and fifth, being added together, (to give) $E F$, and the third (magnitude) and sixth, (being added together, to give) $G H$, are thus also equal multiples of the second (magnitude) $B$ and the fourth $D$ (respectively) [Prop. 5.2].


Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively. (Which is) the very thing it was required to show.

## Proposition $4^{\dagger}$

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever.

For let a first (magnitude) $A$ have the same ratio to a second $B$ that a third $C$ (has) to a fourth $D$. And let equal multiples $E$ and $F$ have been taken of $A$ and $C$ (respectively), and other random equal multiples $G$ and


 N.










 E трòs tò H , oút $\omega$ s tò Z тpòs tò $\Theta$.

 $\pi \rho \omega ́ \tau о \cup ~ x \alpha i ̀ ~ \tau p i ́ t o u ~ \pi \rho o ̀ \varsigma ~ \tau \alpha ̀ ~ i \sigma \alpha ́ x ı \zeta ~ \pi о \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma เ \alpha ~ \tau о и ̃ ~ \delta \varepsilon \cup \tau \varepsilon ́ p o u ~$


$H$ of $B$ and $D$ (respectively). I say that as $E$ (is) to $G$, so $F$ (is) to $H$.


$$
\mathrm{N} \longmapsto \quad 1
$$

For let equal multiples $K$ and $L$ have been taken of $E$ and $F$ (respectively), and other random equal multiples $M$ and $N$ of $G$ and $H$ (respectively).
[And] since $E$ and $F$ are equal multiples of $A$ and $C$ (respectively), and the equal multiples $K$ and $L$ have been taken of $E$ and $F$ (respectively), $K$ and $L$ are thus equal multiples of $A$ and $C$ (respectively) [Prop. 5.3]. So, for the same (reasons), $M$ and $N$ are equal multiples of $B$ and $D$ (respectively). And since as $A$ is to $B$, so $C$ (is) to $D$, and the equal multiples $K$ and $L$ have been taken of $A$ and $C$ (respectively), and the other random equal multiples $M$ and $N$ of $B$ and $D$ (respectively), then if $K$ exceeds $M$ then $L$ also exceeds $N$, and if ( $K$ is) equal (to $M$ then $L$ is also) equal (to $N$ ), and if ( $K$ is) less (than $M$ then $L$ is also) less (than $N$ ) [Def. 5.5]. And $K$ and $L$ are equal multiples of $E$ and $F$ (respectively), and $M$ and $N$ other random equal multiples of $G$ and $H$ (respectively). Thus, as $E$ (is) to $G$, so $F$ (is) to $H$ [Def. 5.5].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever. (Which is) the very thing it was required to show.

[^9]
## $\varepsilon^{\prime}$.


 है $\sigma \tau \alpha \iota ~ \pi о \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma เ o \nu, \delta \delta \sigma \alpha \pi \lambda \alpha ́ \sigma เ o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ o ̋ \lambda o \nu ~ \tau o u ̃ ~ o ̋ \lambda o u . ~$


 $\lambda \varepsilon ́ \gamma \omega$, őtı xaì $\lambda о \iota \pi o ̀ v ~ t o ̀ ~ E B ~ \lambda o ı \pi o u ̃ ~ \tau o u ̃ ~ Z \Delta ~ i \sigma \sigma \alpha ́ x ı s ~ ह ै \sigma \tau \alpha \iota ~$
















 oัт


${ }^{\dagger}$ In modern notation, this proposition reads $m \alpha-m \beta=m(\alpha-\beta)$.

$$
\tau^{\prime}
$$



 $\lambda \alpha \pi \lambda \alpha ́ \sigma \iota$.
$\Delta$ úo $\gamma \grave{\alpha} \rho \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta$ خ̀̀ $\mathrm{AB}, \Gamma \Delta$ ठúo $\mu \varepsilon \gamma \varepsilon \vartheta \widetilde{\omega} \nu \tau \widetilde{\omega} \nu \mathrm{E}, \mathrm{Z}$

## Proposition $5^{\dagger}$

If a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively).


## $\mathrm{G} \quad \mathrm{C} \quad \mathrm{F} \quad \mathrm{D}$

For let the magnitude $A B$ be the same multiple of the magnitude $C D$ that the (part) taken away $A E$ (is) of the (part) taken away $C F$ (respectively). I say that the remainder $E B$ will also be the same multiple of the remainder $F D$ as that which the whole $A B$ (is) of the whole $C D$ (respectively).

For as many times as $A E$ is (divisible) by $C F$, so many times let $E B$ also have been made (divisible) by $C G$.

And since $A E$ and $E B$ are equal multiples of $C F$ and $G C$ (respectively), $A E$ and $A B$ are thus equal multiples of $C F$ and $G F$ (respectively) [Prop. 5.1]. And $A E$ and $A B$ are assumed (to be) equal multiples of $C F$ and $C D$ (respectively). Thus, $A B$ is an equal multiple of each of $G F$ and $C D$. Thus, $G F$ (is) equal to $C D$. Let $C F$ have been subtracted from both. Thus, the remainder $G C$ is equal to the remainder $F D$. And since $A E$ and $E B$ are equal multiples of $C F$ and $G C$ (respectively), and $G C$ (is) equal to $D F, A E$ and $E B$ are thus equal multiples of $C F$ and $F D$ (respectively). And $A E$ and $A B$ are assumed (to be) equal multiples of $C F$ and $C D$ (respectively). Thus, $E B$ and $A B$ are equal multiples of $F D$ and $C D$ (respectively). Thus, the remainder $E B$ will also be the same multiple of the remainder $F D$ as that which the whole $A B$ (is) of the whole $C D$ (respectively).

Thus, if a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively). (Which is) the very thing it was required to show.

## Proposition $6^{\dagger}$

If two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples


 $\pi о \lambda \lambda \alpha \pi \lambda \alpha \dot{\sigma} \sigma$.




















${ }^{\dagger}$ In modern notation, this proposition reads $m \alpha-n \alpha=(m-n) \alpha$.

$$
\zeta^{\prime}
$$

 $\pi \rho o ̀ s ~ \tau \grave{\alpha}$ ' $\sigma \alpha$.

 тòv aútòv है $\chi \varepsilon \iota ~ \lambda o ́ \gamma o v, ~ x \alpha i ~ \tau o ̀ ~ \Gamma ~ \pi \rho o ̀ \varsigma ~ \varepsilon ̇ x \alpha ́ \tau \varepsilon p o \nu ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}$.
of them (respectively).
For let two magnitudes $A B$ and $C D$ be equal multiples of two magnitudes $E$ and $F$ (respectively). And let the (parts) taken away (from the former) $A G$ and CH be equal multiples of $E$ and $F$ (respectively). I say that the remainders $G B$ and $H D$ are also either equal to $E$ and $F$ (respectively), or (are) equal multiples of them.

$\mathrm{F} \longmapsto$
For let $G B$ be, first of all, equal to $E$. I say that $H D$ is also equal to $F$.

For let $C K$ be made equal to $F$. Since $A G$ and $C H$ are equal multiples of $E$ and $F$ (respectively), and $G B$ (is) equal to $E$, and $K C$ to $F, A B$ and $K H$ are thus equal multiples of $E$ and $F$ (respectively) [Prop. 5.2]. And $A B$ and $C D$ are assumed (to be) equal multiples of $E$ and $F$ (respectively). Thus, $K H$ and $C D$ are equal multiples of $F$ and $F$ (respectively). Therefore, $K H$ and $C D$ are each equal multiples of $F$. Thus, $K H$ is equal to $C D$. Let $C H$ have be taken away from both. Thus, the remainder $K C$ is equal to the remainder $H D$. But, $F$ is equal to $K C$. Thus, $H D$ is also equal to $F$. Hence, if $G B$ is equal to $E$ then $H D$ will also be equal to $F$.

So, similarly, we can show that even if $G B$ is a multiple of $E$ then $H D$ will also be the same multiple of $F$.

Thus, if two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively). (Which is) the very thing it was required to show.

## Proposition 7

Equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Let $A$ and $B$ be equal magnitudes, and $C$ some other random magnitude. I say that $A$ and $B$ each have the








 тpòs tò $\Gamma$, oútcus tò B тpòs tò $\Gamma$.








 $\alpha \cup ̉ \tau o ̀ ~ \pi \rho o ̀ s ~ \tau \alpha ̀ ~ " ~ ' \sigma \alpha . ~$

## По́pıб $\quad$.



same ratio to $C$, and (that) $C$ (has the same ratio) to each of $A$ and $B$.

$\mathrm{B} \longmapsto \mathrm{E} \longmapsto, ~$,
$\mathrm{C} \longmapsto \mathrm{F} \longmapsto \perp$
For let the equal multiples $D$ and $E$ have been taken of $A$ and $B$ (respectively), and the other random multiple $F$ of $C$.

Therefore, since $D$ and $E$ are equal multiples of $A$ and $B$ (respectively), and $A$ (is) equal to $B, D$ (is) thus also equal to $E$. And $F$ (is) different, at random. Thus, if $D$ exceeds $F$ then $E$ also exceeds $F$, and if ( $D$ is) equal (to $F$ then $E$ is also) equal (to $F$ ), and if ( $D$ is) less (than $F$ then $E$ is also) less (than $F$ ). And $D$ and $E$ are equal multiples of $A$ and $B$ (respectively), and $F$ another random multiple of $C$. Thus, as $A$ (is) to $C$, so $B$ (is) to $C$ [Def. 5.5].
[So] I say that $C^{\dagger}$ also has the same ratio to each of $A$ and $B$.

For, similarly, we can show, by the same construction, that $D$ is equal to $E$. And $F$ (has) some other (value). Thus, if $F$ exceeds $D$ then it also exceeds $E$, and if ( $F$ is) equal (to $D$ then it is also) equal (to $E$ ), and if ( $F$ is) less (than $D$ then it is also) less (than $E$ ). And $F$ is a multiple of $C$, and $D$ and $E$ other random equal multiples of $A$ and $B$. Thus, as $C$ (is) to $A$, so $C$ (is) to $B$ [Def. 5.5].

Thus, equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

## Corollary ${ }^{\ddagger}$

So (it is) clear, from this, that if some magnitudes are proportional then they will also be proportional inversely. (Which is) the very thing it was required to show.
${ }^{\dagger}$ The Greek text has " $E$ ", which is obviously a mistake.
${ }^{\ddagger}$ In modern notation, this corollary reads that if $\alpha: \beta:: \gamma: \delta$ then $\beta: \alpha:: \delta: \gamma$.

$$
\eta^{\prime}
$$









## Proposition 8

For unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater.

Let $A B$ and $C$ be unequal magnitudes, and let $A B$ be the greater (of the two), and $D$ another random magnitude. I say that $A B$ has a greater ratio to $D$ than $C$ (has) to $D$, and (that) $D$ has a greater ratio to $C$ than (it has) to $A B$.
































 й $\pi \varepsilon \rho$ tò $\Delta$ т $\rho$ òs tò AB .








$\mathrm{L} \longmapsto$,
$\mathrm{M} \longmapsto$, ,
For since $A B$ is greater than $C$, let $B E$ be made equal to $C$. So, the lesser of $A E$ and $E B$, being multiplied, will sometimes be greater than $D$ [Def. 5.4]. First of all, let $A E$ be less than $E B$, and let $A E$ have been multiplied, and let $F G$ be a multiple of it which (is) greater than $D$. And as many times as $F G$ is (divisible) by $A E$, so many times let $G H$ also have become (divisible) by $E B$, and $K$ by $C$. And let the double multiple $L$ of $D$ have been taken, and the triple multiple $M$, and several more, (each increasing) in order by one, until the (multiple) taken becomes the first multiple of $D$ (which is) greater than $K$. Let it have been taken, and let it also be the quadruple multiple $N$ of $D$-the first (multiple) greater than $K$.

Therefore, since $K$ is less than $N$ first, $K$ is thus not less than $M$. And since $F G$ and $G H$ are equal multiples of $A E$ and $E B$ (respectively), $F G$ and $F H$ are thus equal multiples of $A E$ and $A B$ (respectively) [Prop. 5.1]. And $F G$ and $K$ are equal multiples of $A E$ and $C$ (respectively). Thus, $F H$ and $K$ are equal multiples of $A B$ and $C$ (respectively). Thus, $F H, K$ are equal multiples of $A B, C$. Again, since $G H$ and $K$ are equal multiples of $E B$ and $C$, and $E B$ (is) equal to $C, G H$ (is) thus also equal to $K$. And $K$ is not less than $M$. Thus, $G H$ not less than $M$ either. And $F G$ (is) greater than $D$. Thus, the whole of $F H$ is greater than $D$ and $M$ (added) together. But, $D$ and $M$ (added) together is equal to $N$, inasmuch as $M$ is three times $D$, and $M$ and $D$ (added) together is four times $D$, and $N$ is also four times $D$. Thus, $M$ and $D$ (added) together is equal to $N$. But, $F H$ is greater than $M$ and $D$. Thus, $F H$ exceeds $N$. And $K$ does not exceed $N$. And $F H, K$ are equal multiples of $A B, C$, and $N$ another random multiple of $D$. Thus, $A B$ has a greater ratio to $D$ than $C$ (has) to $D$ [Def. 5.7].

So, I say that $D$ also has a greater ratio to $C$ than $D$ (has) to $A B$.

For, similarly, by the same construction, we can show that $N$ exceeds $K$, and $N$ does not exceed $F H$. And $N$ is a multiple of $D$, and $F H, K$ other random equal multiples of $A B, C$ (respectively). Thus, $D$ has a greater













 סєֹ̋̌al.

## $\vartheta^{\prime}$.


 غ̇бтiv.

'Eरétc $\gamma \grave{\alpha} \rho$ ह́x $\alpha ́ \tau \varepsilon \rho o \nu ~ \tau \widetilde{\omega} \nu ~ A, ~ B ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \Gamma ~ \tau o ̀ v ~ \alpha u ̉ \tau o ̀ v ~$


Eí $\gamma \dot{\alpha} \rho \mu \dot{n}$, oủx $\ddot{\alpha} \nu \dot{\varepsilon} x \dot{\alpha} \tau \varepsilon \rho o \nu \tau \tilde{\omega} \nu \mathrm{~A}, \mathrm{~B} \pi \rho o ̀ s ~ \tau o ̀ ~ \Gamma ~ \tau o ̀ v ~$









$$
\therefore \text {. }
$$



ratio to $C$ than $D$ (has) to $A B$ [Def. 5.5].
And so let $A E$ be greater than $E B$. So, the lesser, $E B$, being multiplied, will sometimes be greater than $D$. Let it have been multiplied, and let $G H$ be a multiple of $E B$ (which is) greater than $D$. And as many times as $G H$ is (divisible) by $E B$, so many times let $F G$ also have become (divisible) by $A E$, and $K$ by $C$. So, similarly (to the above), we can show that $F H$ and $K$ are equal multiples of $A B$ and $C$ (respectively). And, similarly (to the above), let the multiple $N$ of $D$, (which is) the first (multiple) greater than $F G$, have been taken. So, $F G$ is again not less than $M$. And $G H$ (is) greater than $D$. Thus, the whole of $F H$ exceeds $D$ and $M$, that is to say $N$. And $K$ does not exceed $N$, inasmuch as $F G$, which (is) greater than $G H$-that is to say, $K$-also does not exceed $N$. And, following the above (arguments), we (can) complete the proof in the same manner.

Thus, for unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater. (Which is) the very thing it was required to show.

## Proposition 9

(Magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal.


For let $A$ and $B$ each have the same ratio to $C$. I say that $A$ is equal to $B$.

For if not, $A$ and $B$ would not each have the same ratio to $C$ [Prop. 5.8]. But they do. Thus, $A$ is equal to $B$.

So, again, let $C$ have the same ratio to each of $A$ and $B$. I say that $A$ is equal to $B$.

For if not, $C$ would not have the same ratio to each of $A$ and $B$ [Prop. 5.8]. But it does. Thus, $A$ is equal to $B$.

Thus, (magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal. (Which is) the very thing it was required to show.

## Proposition 10

For (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is


## $\mathrm{A} \longmapsto \quad \mathrm{B} \longmapsto$ <br> 

'ExÉtc ràp tò $\mathrm{A} \pi \rho o ̀ s ~ t o ̀ ~ \Gamma ~ \mu \varepsilon i \zeta o v a ~ \lambda o ́ \gamma o v ~ \eta ̈ ँ \pi \varepsilon \rho ~ t o ̀ ~ B ~$















 ह̌̀ $\lambda \alpha \tau \tau 0$ वैp $\alpha$ ह̀бтì tò B тоũ A .




$$
1 \alpha^{\prime}
$$




 ötı ह̇ $\sigma \tau i v$ c̀s tò A tpòs tò B , oütcus tò E tpòs tò Z .

 $\Lambda, \mathrm{M}, \mathrm{N}$.





(the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser.


For let $A$ have a greater ratio to $C$ than $B$ (has) to $C$. I say that $A$ is greater than $B$.

For if not, $A$ is surely either equal to or less than $B$. In fact, $A$ is not equal to $B$. For (then) $A$ and $B$ would each have the same ratio to $C$ [Prop. 5.7]. But they do not. Thus, $A$ is not equal to $B$. Neither, indeed, is $A$ less than $B$. For (then) $A$ would have a lesser ratio to $C$ than $B$ (has) to $C$ [Prop. 5.8]. But it does not. Thus, $A$ is not less than $B$. And it was shown not (to be) equal either. Thus, $A$ is greater than $B$.

So, again, let $C$ have a greater ratio to $B$ than $C$ (has) to $A$. I say that $B$ is less than $A$.

For if not, (it is) surely either equal or greater. In fact, $B$ is not equal to $A$. For (then) $C$ would have the same ratio to each of $A$ and $B$ [Prop. 5.7]. But it does not. Thus, $A$ is not equal to $B$. Neither, indeed, is $B$ greater than $A$. For (then) $C$ would have a lesser ratio to $B$ than (it has) to $A$ [Prop. 5.8]. But it does not. Thus, $B$ is not greater than $A$. And it was shown that (it is) not equal (to $A$ ) either. Thus, $B$ is less than $A$.

Thus, for (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser. (Which is) the very thing it was required to show.

## Proposition $11^{\dagger}$

(Ratios which are) the same with the same ratio are also the same with one another.


For let it be that as $A$ (is) to $B$, so $C$ (is) to $D$, and as $C$ (is) to $D$, so $E$ (is) to $F$. I say that as $A$ is to $B$, so $E$ (is) to $F$.

For let the equal multiples $G, H, K$ have been taken of $A, C, E$ (respectively), and the other random equal multiples $L, M, N$ of $B, D, F$ (respectively).

And since as $A$ is to $B$, so $C$ (is) to $D$, and the equal multiples $G$ and $H$ have been taken of $A$ and $C$ (respectively), and the other random equal multiples $L$ and $M$ of $B$ and $D$ (respectively), thus if $G$ exceeds $L$ then $H$ also exceeds $M$, and if ( $G$ is) equal (to $L$ then $H$ is also)









 тpòs tò B , oưt $\omega \varsigma$ tò E тpòs tò Z .


equal (to $M$ ), and if ( $G$ is) less (than $L$ then $H$ is also) less (than $M$ ) [Def. 5.5]. Again, since as $C$ is to $D$, so $E$ (is) to $F$, and the equal multiples $H$ and $K$ have been taken of $C$ and $E$ (respectively), and the other random equal multiples $M$ and $N$ of $D$ and $F$ (respectively), thus if $H$ exceeds $M$ then $K$ also exceeds $N$, and if ( $H$ is) equal (to $M$ then $K$ is also) equal (to $N$ ), and if ( $H$ is) less (than $M$ then $K$ is also) less (than $N$ ) [Def. 5.5]. But (we saw that) if $H$ was exceeding $M$ then $G$ was also exceeding $L$, and if ( $H$ was) equal (to $M$ then $G$ was also) equal (to $L$ ), and if ( $H$ was) less (than $M$ then $G$ was also) less (than $L$ ). And, hence, if $G$ exceeds $L$ then $K$ also exceeds $N$, and if ( $G$ is) equal (to $L$ then $K$ is also) equal (to $N$ ), and if ( $G$ is) less (than $L$ then $K$ is also) less (than $N$ ). And $G$ and $K$ are equal multiples of $A$ and $E$ (respectively), and $L$ and $N$ other random equal multiples of $B$ and $F$ (respectively). Thus, as $A$ is to $B$, so $E$ (is) to $F$ [Def. 5.5].

Thus, (ratios which are) the same with the same ratio are also the same with one another. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ and $\gamma: \delta:: \epsilon: \zeta$ then $\alpha: \beta:: \epsilon: \zeta$.

## $\beta^{\prime}$.


 $\dot{\eta} \gamma o u ́ \mu \varepsilon \nu \alpha \pi \rho o ̀ s ~ \alpha ั \pi \alpha \nu \tau \alpha ~ \tau \grave{\alpha}$ غ̇ $\pi o ́ \mu \varepsilon v \alpha$.

 $\mathrm{E}, \mathrm{Z}$, ©̀s tò A тpòs tò B , oữ $\omega$ s tò $\Gamma$ rpòs tò $\Delta$, xaì tò E
 $\mathrm{A}, \Gamma, \mathrm{E} \pi \rho o ̀ s ~ \tau \grave{\alpha} \mathrm{~B}, \Delta, \mathrm{Z}$.

 $\tau \dot{\alpha} \Lambda, \mathrm{M}, \mathrm{N}$.







## Proposition $12^{\dagger}$

If there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following.


Let there be any number of magnitudes whatsoever, $A, B, C, D, E, F$, (which are) proportional, (so that) as $A$ (is) to $B$, so $C$ (is) to $D$, and $E$ to $F$. I say that as $A$ is to $B$, so $A, C, E$ (are) to $B, D, F$.

For let the equal multiples $G, H, K$ have been taken of $A, C, E$ (respectively), and the other random equal multiples $L, M, N$ of $B, D, F$ (respectively).

And since as $A$ is to $B$, so $C$ (is) to $D$, and $E$ to $F$, and the equal multiples $G, H, K$ have been taken of $A, C, E$ (respectively), and the other random equal multiples $L$, $M, N$ of $B, D, F$ (respectively), thus if $G$ exceeds $L$ then $H$ also exceeds $M$, and $K$ (exceeds) $N$, and if ( $G$ is) equal (to $L$ then $H$ is also) equal (to $M$, and $K$ to $N$ ),




 है้ $\tau \widetilde{\omega} \nu \mu \varepsilon \gamma \varepsilon \vartheta \widetilde{\omega} \nu$ ह̀vós, $\tau о \sigma \alpha \cup \tau \alpha \pi \lambda \alpha ́ \sigma ı \alpha$ है $\sigma \tau \alpha l$ x $\alpha \grave{\iota} \tau \grave{\alpha} \pi \alpha ́ \nu \tau \alpha$

 tò $\mathrm{A} \pi \rho o ̀ s ~ \tau o ̀ ~ \mathrm{~B}$, oút $\omega \varsigma$ т̀̀ $\mathrm{A}, \Gamma, \mathrm{E} \pi \rho o ̀ s ~ \tau \grave{\alpha} \mathrm{~B}, \Delta, \mathrm{Z}$.



and if ( $G$ is) less (than $L$ then $H$ is also) less (than $M$, and $K$ than $N$ ) [Def. 5.5]. And, hence, if $G$ exceeds $L$ then $G, H, K$ also exceed $L, M, N$, and if ( $G$ is) equal (to $L$ then $G, H, K$ are also) equal (to $L, M, N$ ) and if ( $G$ is) less (than $L$ then $G, H, K$ are also) less (than $L, M, N)$. And $G$ and $G, H, K$ are equal multiples of $A$ and $A, C, E$ (respectively), inasmuch as if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second) [Prop. 5.1]. So, for the same (reasons), $L$ and $L, M, N$ are also equal multiples of $B$ and $B, D, F$ (respectively). Thus, as $A$ is to $B$, so $A, C, E$ (are) to $B, D, F$ (respectively).

Thus, if there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \alpha^{\prime}:: \beta: \beta^{\prime}:: \gamma: \gamma^{\prime}$ etc. then $\alpha: \alpha^{\prime}::(\alpha+\beta+\gamma+\cdots):\left(\alpha^{\prime}+\beta^{\prime}+\gamma^{\prime}+\cdots\right)$.

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' ' '.
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 Z.


 тò $\delta$ è тoũ $\mathrm{E} \pi \circ \lambda \lambda \alpha \pi \lambda \alpha ́ \sigma เ o \nu ~ \tau o u ̃ ~ \tau o u ̃ ~ Z ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma i ́ o u ~ o u ̉ \chi ~$







## Proposition $13^{\dagger}$

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the third (magnitude) has a greater ratio to the fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth.


For let a first (magnitude) $A$ have the same ratio to a second $B$ that a third $C$ (has) to a fourth $D$, and let the third (magnitude) $C$ have a greater ratio to the fourth $D$ than a fifth $E$ (has) to a sixth $F$. I say that the first (magnitude) $A$ will also have a greater ratio to the second $B$ than the fifth $E$ (has) to the sixth $F$.

For since there are some equal multiples of $C$ and $E$, and other random equal multiples of $D$ and $F$, (for which) the multiple of $C$ exceeds the (multiple) of $D$, and the multiple of $E$ does not exceed the multiple of $F$ [Def. 5.7], let them have been taken. And let $G$ and $H$ be equal multiples of $C$ and $E$ (respectively), and $K$ and $L$ other random equal multiples of $D$ and $F$ (respectively), such that $G$ exceeds $K$, but $H$ does not exceed $L$. And as many times as $G$ is (divisible) by $C$, so many times let $M$ be (divisible) by $A$. And as many times as $K$ (is divisible)








 tò E tpòs tò Z .




by $D$, so many times let $N$ be (divisible) by $B$.
And since as $A$ is to $B$, so $C$ (is) to $D$, and the equal multiples $M$ and $G$ have been taken of $A$ and $C$ (respectively), and the other random equal multiples $N$ and $K$ of $B$ and $D$ (respectively), thus if $M$ exceeds $N$ then $G$ exceeds $K$, and if ( $M$ is) equal (to $N$ then $G$ is also) equal (to $K$ ), and if ( $M$ is) less (than $N$ then $G$ is also) less (than $K$ ) [Def. 5.5]. And $G$ exceeds $K$. Thus, $M$ also exceeds $N$. And $H$ does not exceeds $L$. And $M$ and $H$ are equal multiples of $A$ and $E$ (respectively), and $N$ and $L$ other random equal multiples of $B$ and $F$ (respectively). Thus, $A$ has a greater ratio to $B$ than $E$ (has) to $F$ [Def. 5.7].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and a third (magnitude) has a greater ratio to a fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ and $\gamma: \delta>\epsilon: \zeta$ then $\alpha: \beta>\epsilon: \zeta$.

## ${ }^{\prime} \delta^{\prime}$.














 $\mu \varepsilon$ ц̌̆óv દ̀бтı тò B тои̃ $\Delta$.

 ěवтal หaì tò B toũ $\Delta$.





## Proposition $14^{\dagger}$

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth).


For let a first (magnitude) $A$ have the same ratio to a second $B$ that a third $C$ (has) to a fourth $D$. And let $A$ be greater than $C$. I say that $B$ is also greater than $D$.

For since $A$ is greater than $C$, and $B$ (is) another random [magnitude], $A$ thus has a greater ratio to $B$ than $C$ (has) to $B$ [Prop. 5.8]. And as $A$ (is) to $B$, so $C$ (is) to $D$. Thus, $C$ also has a greater ratio to $D$ than $C$ (has) to $B$. And that (magnitude) to which the same (magnitude) has a greater ratio is the lesser [Prop. 5.10]. Thus, $D$ (is) less than $B$. Hence, $B$ is greater than $D$.

So, similarly, we can show that even if $A$ is equal to $C$ then $B$ will also be equal to $D$, and even if $A$ is less than $C$ then $B$ will also be less than $D$.

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is)
equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth). (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ then $\alpha \gtreqless \gamma$ as $\beta \gtreqless \delta$.
$\iota \varepsilon^{\prime}$.
入óүov $\lambda \eta \varphi \vartheta \varepsilon ́ v \tau \alpha ~ \varkappa \alpha \tau \alpha \lambda \lambda \eta \lambda \alpha$.


 трòs tò $\Delta \mathrm{E}$.












 tò AB тpòs tò $\Delta \mathrm{E}$.


${ }^{\dagger}$ In modern notation, this proposition reads that $\alpha: \beta:: m \alpha: m \beta$.

$$
i q^{\prime}
$$

 ย$\sigma \tau \alpha\llcorner$.
 $\mathrm{A} \pi \rho o ̀ \varsigma ~ \tau o ̀ ~ B, ~ o u ̛ \tau \omega \varsigma ~ \tau o ̀ ~ \Gamma ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \Delta \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~ o ̛ \tau \iota ~ \varkappa \alpha l ~ \varepsilon ́ v \alpha \lambda \lambda \lambda \xi ~$
 $\Delta$.

 $\Theta$.

## Proposition $15^{\dagger}$

Parts have the same ratio as similar multiples, taken in corresponding order.


For let $A B$ and $D E$ be equal multiples of $C$ and $F$ (respectively). I say that as $C$ is to $F$, so $A B$ (is) to $D E$.

For since $A B$ and $D E$ are equal multiples of $C$ and $F$ (respectively), thus as many magnitudes as there are in $A B$ equal to $C$, so many (are there) also in $D E$ equal to $F$. Let $A B$ have been divided into (magnitudes) $A G$, $G H, H B$, equal to $C$, and $D E$ into (magnitudes) $D K$, $K L, L E$, equal to $F$. So, the number of (magnitudes) $A G, G H, H B$ will equal the number of (magnitudes) $D K, K L, L E$. And since $A G, G H, H B$ are equal to one another, and $D K, K L, L E$ are also equal to one another, thus as $A G$ is to $D K$, so $G H$ (is) to $K L$, and $H B$ to $L E$ [Prop. 5.7]. And, thus (for proportional magnitudes), as one of the leading (magnitudes) will be to one of the following, so all of the leading (magnitudes will be) to all of the following [Prop. 5.12]. Thus, as $A G$ is to $D K$, so $A B$ (is) to $D E$. And $A G$ is equal to $C$, and $D K$ to $F$. Thus, as $C$ is to $F$, so $A B$ (is) to $D E$.

Thus, parts have the same ratio as similar multiples, taken in corresponding order. (Which is) the very thing it was required to show.

## Proposition $16^{\dagger}$

If four magnitudes are proportional then they will also be proportional alternately.

Let $A, B, C$ and $D$ be four proportional magnitudes, (such that) as $A$ (is) to $B$, so $C$ (is) to $D$. I say that they will also be [proportional] alternately, (so that) as $A$ (is) to $C$, so $B$ (is) to $D$.

For let the equal multiples $E$ and $F$ have been taken of $A$ and $B$ (respectively), and the other random equal multiples $G$ and $H$ of $C$ and $D$ (respectively).















 oútcs tò B тpòs tò $\Delta$.




And since $E$ and $F$ are equal multiples of $A$ and $B$ (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as $A$ is to $B$, so $E$ (is) to $F$. But as $A$ (is) to $B$, so $C$ (is) to $D$. And, thus, as $C$ (is) to $D$, so $E$ (is) to $F$ [Prop. 5.11]. Again, since $G$ and $H$ are equal multiples of $C$ and $D$ (respectively), thus as $C$ is to $D$, so $G$ (is) to $H$ [Prop. 5.15]. But as $C$ (is) to $D$, [so] $E$ (is) to $F$. And, thus, as $E$ (is) to $F$, so $G$ (is) to $H$ [Prop. 5.11]. And if four magnitudes are proportional, and the first is greater than the third then the second will also be greater than the fourth, and if (the first is) equal (to the third then the second will also be) equal (to the fourth), and if (the first is) less (than the third then the second will also be) less (than the fourth) [Prop. 5.14]. Thus, if $E$ exceeds $G$ then $F$ also exceeds $H$, and if ( $E$ is) equal (to $G$ then $F$ is also) equal (to $H$ ), and if ( $E$ is) less (than $G$ then $F$ is also) less (than $H$ ). And $E$ and $F$ are equal multiples of $A$ and $B$ (respectively), and $G$ and $H$ other random equal multiples of $C$ and $D$ (respectively). Thus, as $A$ is to $C$, so $B$ (is) to $D$ [Def. 5.5].

Thus, if four magnitudes are proportional then they will also be proportional alternately. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ then $\alpha: \gamma:: \beta: \delta$.
ఢ'.
 àvá入oүov हैஎтal.

 $\Delta \mathrm{Z}$, ஸ́s tò AB тpòs tò BE , oưt $\omega$ ц tò $\Gamma \Delta$ tpòs tò $\Delta \mathrm{Z}$.
 tò EB , oưt $\omega$ s tò $\Gamma \mathrm{Z}$ тpòs tò $\Delta \mathrm{Z}$.






## Proposition $17^{\dagger}$

If composed magnitudes are proportional then they will also be proportional (when) separarted.


Let $A B, B E, C D$, and $D F$ be composed magnitudes (which are) proportional, (so that) as $A B$ (is) to $B E$, so $C D$ (is) to $D F$. I say that they will also be proportional (when) separated, (so that) as $A E$ (is) to $E B$, so $C F$ (is) to $D F$.

For let the equal multiples $G H, H K, L M$, and $M N$ have been taken of $A E, E B, C F$, and $F D$ (respectively), and the other random equal multiples $K O$ and $N P$ of $E B$ and $F D$ (respectively).











 $\pi \rho o ̀ s ~ \tau o ̀ ~ B E, ~ o u ́ \tau \omega \varsigma ~ t o ̀ ~ \Gamma \Delta ~ \pi \rho o ̀ s ~ t o ̀ ~ \Delta Z, ~ x \alpha i ̀ ~ \varepsilon ̋ \lambda \eta \pi \tau \tau \alpha l ~ \tau \widetilde{\omega} \nu$

















And since $G H$ and $H K$ are equal multiples of $A E$ and $E B$ (respectively), $G H$ and $G K$ are thus equal multiples of $A E$ and $A B$ (respectively) [Prop. 5.1]. But $G H$ and $L M$ are equal multiples of $A E$ and $C F$ (respectively). Thus, $G K$ and $L M$ are equal multiples of $A B$ and $C F$ (respectively). Again, since $L M$ and $M N$ are equal multiples of $C F$ and $F D$ (respectively), $L M$ and $L N$ are thus equal multiples of $C F$ and $C D$ (respectively) [Prop. 5.1]. And $L M$ and $G K$ were equal multiples of $C F$ and $A B$ (respectively). Thus, $G K$ and $L N$ are equal multiples of $A B$ and $C D$ (respectively). Thus, $G K, L N$ are equal multiples of $A B, C D$. Again, since $H K$ and $M N$ are equal multiples of $E B$ and $F D$ (respectively), and $K O$ and $N P$ are also equal multiples of $E B$ and $F D$ (respectively), then, added together, $H O$ and $M P$ are also equal multiples of $E B$ and $F D$ (respectively) [Prop. 5.2]. And since as $A B$ (is) to $B E$, so $C D$ (is) to $D F$, and the equal multiples $G K, L N$ have been taken of $A B, C D$, and the equal multiples $H O, M P$ of $E B, F D$, thus if $G K$ exceeds $H O$ then $L N$ also exceeds $M P$, and if ( $G K$ is) equal (to $H O$ then $L N$ is also) equal (to $M P$ ), and if ( $G K$ is) less (than $H O$ then $L N$ is also) less (than $M P$ ) [Def. 5.5]. So let $G K$ exceed $H O$, and thus, $H K$ being taken away from both, $G H$ exceeds $K O$. But (we saw that) if $G K$ was exceeding $H O$ then $L N$ was also exceeding $M P$. Thus, $L N$ also exceeds $M P$, and, $M N$ being taken away from both, $L M$ also exceeds $N P$. Hence, if $G H$ exceeds $K O$ then $L M$ also exceeds $N P$. So, similarly, we can show that even if $G H$ is equal to $K O$ then $L M$ will also be equal to $N P$, and even if ( $G H$ is) less (than $K O$ then $L M$ will also be) less (than $N P$ ). And $G H, L M$ are equal multiples of $A E, C F$, and $K O, N P$ other random equal multiples of $E B, F D$. Thus, as $A E$ is to $E B$, so $C F$ (is) to $F D$ [Def. 5.5].

Thus, if composed magnitudes are proportional then they will also be proportional (when) separarted. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha+\beta: \beta:: \gamma+\delta: \delta$ then $\alpha: \beta:: \gamma: \delta$.

$$
i \eta^{\prime}
$$

 $\alpha \alpha^{\alpha} \alpha^{\prime} o \gamma o v$ है $\sigma \tau \alpha$.

 ஸ́s tò $\mathrm{AE} \pi p o ̀ s ~ t o ̀ ~ E B, ~ o u ̈ \tau \omega s ~ t o ̀ ~ \Gamma Z ~ \pi p o ̀ s ~ t o ̀ ~ Z \Delta \cdot ~ \lambda e ́ \gamma \omega, ~$


## Proposition $18^{\dagger}$

If separated magnitudes are proportional then they will also be proportional (when) composed.


Let $A E, E B, C F$, and $F D$ be separated magnitudes (which are) proportional, (so that) as $A E$ (is) to $E B$, so $C F$ (is) to $F D$. I say that they will also be proportional
oütccs tò $\Gamma \Delta$ т $\quad$ òs tò $\mathrm{Z} \Delta$ ．
 тpòs tò $\Delta \mathrm{Z}$ ，है $\sigma \tau \alpha \iota$ ©ंs tò AB тpòs tò BE ，oữ $\omega$ s tò $\Gamma \Delta$

 tò AB трòs тò BE ，oút tws tò $\Gamma \Delta$ тpòs tò $\Delta \mathrm{H}$ ，$\sigma \cup \gamma \varkappa \varepsilon i ́ \mu \varepsilon v \alpha$








 वैpo．


（when）composed，（so that）as $A B$（is）to $B E$ ，so $C D$（is） to $F D$ ．

For if（it is）not（the case that）as $A B$ is to $B E$ ，so $C D$（is）to $F D$ ，then it will surely be（the case that）as $A B$（is）to $B E$ ，so $C D$ is either to some（magnitude）less than $D F$ ，or（some magnitude）greater（than $D F$ ）．${ }^{\ddagger}$

Let it，first of all，be to（some magnitude）less（than $D F$ ），（namely）$D G$ ．And since composed magnitudes are proportional，（so that）as $A B$ is to $B E$ ，so $C D$（is）to $D G$ ，they will thus also be proportional（when）separated ［Prop．5．17］．Thus，as $A E$ is to $E B$ ，so $C G$（is）to $G D$ ． But it was also assumed that as $A E$（is）to $E B$ ，so $C F$ （is）to $F D$ ．Thus，（it is）also（the case that）as $C G$（is） to $G D$ ，so $C F$（is）to $F D$［Prop．5．11］．And the first （magnitude）$C G$（is）greater than the third $C F$ ．Thus， the second（magnitude）$G D$（is）also greater than the fourth $F D$［Prop．5．14］．But（it is）also less．The very thing is impossible．Thus，（it is）not（the case that）as $A B$ is to $B E$ ，so $C D$（is）to less than $F D$ ．Similarly，we can show that neither（is it the case）to greater（than $F D$ ）． Thus，（it is the case）to the same（as $F D$ ）．

Thus，if separated magnitudes are proportional then they will also be proportional（when）composed．（Which is）the very thing it was required to show．
${ }^{\dagger}$ In modern notation，this proposition reads that if $\alpha: \beta:: \gamma: \delta$ then $\alpha+\beta: \beta:: \gamma+\delta: \delta$ ．
${ }^{\ddagger}$ Here，Euclid assumes，without proof，that a fourth magnitude proportional to three given magnitudes can always be found．

## しิ＇．


 ő入ov．



 $\pi \rho o ̀ s ~ o ̈ \lambda o v ~ t o ̀ ~ \Gamma \Delta . ~$







 ö入ov 七ò $\Gamma \Delta$ ．


## Proposition $19^{\dagger}$

If as the whole is to the whole so the（part）taken away is to the（part）taken away then the remainder to the remainder will also be as the whole（is）to the whole．


For let the whole $A B$ be to the whole $C D$ as the（part） taken away $A E$（is）to the（part）taken away $C F$ ．I say that the remainder $E B$ to the remainder $F D$ will also be as the whole $A B$（is）to the whole $C D$ ．

For since as $A B$ is to $C D$ ，so $A E$（is）to $C F$ ，（it is） also（the case），alternately，（that）as $B A$（is）to $A E$ ，so $D C$（is）to $C F$［Prop．5．16］．And since composed magni－ tudes are proportional then they will also be proportional （when）separated，（so that）as $B E$（is）to $E A$ ，so $D F$（is） to $C F$［Prop．5．17］．Also，alternately，as $B E$（is）to $D F$ ， so $E A$（is）to $F C$［Prop．5．16］．And it was assumed that as $A E$（is）to $C F$ ，so the whole $A B$（is）to the whole $C D$ ． And，thus，as the remainder $E B$（is）to the remainder $F D$ ，so the whole $A B$ will be to the whole $C D$ ．


[Kai ह̇л


 ГZ• $x \alpha i ́ l$ ह̇ $\sigma \tau \iota \nu ~ \alpha ่ \nu \alpha \sigma \tau \rho \varepsilon ́ \psi \alpha \nu \tau \iota] . ~$

## По́pıб $\alpha$.





Thus, if as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole. [(Which is) the very thing it was required to show.]
[And since it was shown (that) as $A B$ (is) to $C D$, so $E B$ (is) to $F D$, (it is) also (the case), alternately, (that) as $A B$ (is) to $B E$, so $C D$ (is) to $F D$. Thus, composed magnitudes are proportional. And it was shown (that) as $B A$ (is) to $A E$, so $D C$ (is) to $C F$. And (the latter) is converted (from the former).]

## Corollary ${ }^{\ddagger}$

So (it is) clear, from this, that if composed magnitudes are proportional then they will also be proportional (when) converted. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ then $\alpha: \beta:: \alpha-\gamma: \beta-\delta$.
${ }^{\ddagger}$ In modern notation, this corollary reads that if $\alpha: \beta:: \gamma: \delta$ then $\alpha: \alpha-\beta:: \gamma: \gamma-\delta$.

$$
x^{\prime} .
$$















 ò̀ tò $\Gamma$ tpòs tò B , ảvár $\alpha \lambda \lambda \iota$ oütcos tò Z tpòs tò E • xaì tò $\Delta$ äp $\alpha$ трòs tò $\mathrm{E} \mu \varepsilon$ íhova $\lambda o ́ \gamma o v$ है $\chi \varepsilon \iota$ ท̈ $\pi \varepsilon \rho$ tò Z трòs tò E .




## Proposition $20^{\dagger}$

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).


Let $A, B$, and $C$ be three magnitudes, and $D, E, F$ other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, (so that) as $A$ (is) to $B$, so $D$ (is) to $E$, and as $B$ (is) to $C$, so $E$ (is) to $F$. And let $A$ be greater than $C$, via equality. I say that $D$ will also be greater than $F$. And if ( $A$ is) equal (to $C$ then $D$ will also be) equal (to $F$ ). And if ( $A$ is) less (than $C$ then $D$ will also be) less (than $F$ ).

For since $A$ is greater than $C$, and $B$ some other (magnitude), and the greater (magnitude) has a greater ratio than the lesser to the same (magnitude) [Prop. 5.8], $A$ thus has a greater ratio to $B$ than $C$ (has) to $B$. But as $A$ (is) to $B$, [so] $D$ (is) to $E$. And, inversely, as $C$ (is) to $B$, so $F$ (is) to $E$ [Prop. 5.7 corr.]. Thus, $D$ also has a greater ratio to $E$ than $F$ (has) to $E$ [Prop. 5.13]. And for (mag-

है $\lambda \alpha \tau \tau \circ \nu$, है $\lambda \alpha \tau \tau \circ \nu$.





nitudes) having a ratio to the same (magnitude), that having the greater ratio is greater [Prop. 5.10]. Thus, $D$ (is) greater than $F$. Similarly, we can show that even if $A$ is equal to $C$ then $D$ will also be equal to $F$, and even if ( $A$ is) less (than $C$ then $D$ will also be) less (than $F$ ).

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third, then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And (if the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \delta: \epsilon$ and $\beta: \gamma:: \epsilon: \zeta$ then $\alpha \gtreqless \gamma$ as $\delta \gtreqless \zeta$.

## $\chi \alpha^{\prime}$.












 हैخ $\lambda \tau \tau \circ \nu$, है $\lambda \alpha \tau \tau \circ \nu$.


 ठè tò $\Gamma$ тpòs tò $\mathrm{B}, \alpha \dot{\alpha} \nu \alpha ́ \pi \alpha \lambda \iota \nu ~ o u ̈ \tau \omega \varsigma ~ \tau o ̀ ~ E ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \Delta . ~ x \alpha \grave{~}$










## Proposition $21^{\dagger}$

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).


Let $A, B$, and $C$ be three magnitudes, and $D, E, F$ other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as $A$ (is) to $B$, so $E$ (is) to $F$, and as $B$ (is) to $C$, so $D$ (is) to $E$. And let $A$ be greater than $C$, via equality. I say that $D$ will also be greater than $F$. And if ( $A$ is) equal (to $C$ then $D$ will also be) equal (to $F$ ). And if ( $A$ is) less (than $C$ then $D$ will also be) less (than $F$ ).

For since $A$ is greater than $C$, and $B$ some other (magnitude), $A$ thus has a greater ratio to $B$ than $C$ (has) to $B$ [Prop. 5.8]. But as $A$ (is) to $B$, so $E$ (is) to $F$. And, inversely, as $C$ (is) to $B$, so $E$ (is) to $D$ [Prop. 5.7 corr.]. Thus, $E$ also has a greater ratio to $F$ than $E$ (has) to $D$ [Prop. 5.13]. And that (magnitude) to which the same (magnitude) has a greater ratio is (the) lesser (magnitude) [Prop. 5.10]. Thus, $F$ is less than $D$. Thus, $D$ is greater than $F$. Similarly, we can show that even if $A$ is equal to $C$ then $D$ will also be equal to $F$, and even if ( $A$ is) less (than $C$ then $D$ will also be) less (than $F$ ).


Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \epsilon: \zeta$ and $\beta: \gamma:: \delta: \epsilon$ then $\alpha \gtreqless \gamma$ as $\delta \gtreqless \zeta$.

$$
x \beta^{\prime} .
$$







 ס乇̀ tò $\mathrm{B} \pi \rho o ̀ s ~ \tau o ̀ ~ \Gamma, ~ o u ̛ \tau \omega s ~ \tau o ̀ ~ E ~ \pi p o ̀ s ~ \tau o ̀ ~ Z \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~ o ̛ \tau ı ~ \chi \alpha \grave{~}$



 M, N.
















## Proposition $22^{\dagger}$

If there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.


Let there be any number of magnitudes whatsoever, $A, B, C$, and (some) other (magnitudes), $D, E, F$, of equal number to them, (which are) in the same ratio taken two by two, (so that) as $A$ (is) to $B$, so $D$ (is) to $E$, and as $B$ (is) to $C$, so $E$ (is) to $F$. I say that they will also be in the same ratio via equality. (That is, as $A$ is to $C$, so $D$ is to $F$.)

For let the equal multiples $G$ and $H$ have been taken of $A$ and $D$ (respectively), and the other random equal multiples $K$ and $L$ of $B$ and $E$ (respectively), and the yet other random equal multiples $M$ and $N$ of $C$ and $F$ (respectively).

And since as $A$ is to $B$, so $D$ (is) to $E$, and the equal multiples $G$ and $H$ have been taken of $A$ and $D$ (respectively), and the other random equal multiples $K$ and $L$ of $B$ and $E$ (respectively), thus as $G$ is to $K$, so $H$ (is) to $L$ [Prop. 5.4]. And, so, for the same (reasons), as $K$ (is) to $M$, so $L$ (is) to $N$. Therefore, since $G, K$, and $M$ are three magnitudes, and $H, L$, and $N$ other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, thus, via equality, if $G$ exceeds $M$ then $H$ also exceeds $N$, and if ( $G$ is) equal (to $M$ then $H$ is also) equal (to $N$ ), and if ( $G$ is) less (than $M$ then $H$ is also) less (than $N$ ) [Prop. 5.20]. And $G$ and $H$ are equal multiples of $A$ and $D$ (respectively), and $M$ and $N$ other random equal multiples of $C$ and $F$ (respectively). Thus, as $A$ is to $C$, so $D$ (is) to $F$ [Def. 5.5].

Thus, if there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by
two, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \epsilon: \zeta$ and $\beta: \gamma:: \zeta: \eta$ and $\gamma: \delta:: \eta: \theta$ then $\alpha: \delta:: \epsilon: \theta$.

$$
x \gamma^{\prime} .
$$







 tò B , oüt tos tò E tpòs tò Z , ís ôè tò B tpòs tò $\Gamma$, oưtcus
 tò $\Delta$ tpòs tò Z .

 $\Lambda, \mathrm{M}, \mathrm{N}$.









 tò B tpòs tò $\Delta$, oüt tos tò $\Theta$ tpòs tò K . $\dot{\alpha} \lambda \lambda^{\prime}$ 'ús tò B тpòs


 $\Lambda$ тpòs tò M . $\dot{\alpha} \lambda \lambda^{\prime}$ ' $\omega$ s tò $\Gamma$ tpòs tò E , oüt tus tò $\Theta$ тpòs








 tò A tpòs tò $\Gamma$, oưt t s tò $\Delta$ tpòs tò Z .



## Proposition $23^{\dagger}$

If there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality.


Let $A, B$, and $C$ be three magnitudes, and $D, E$ and $F$ other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as $A$ (is) to $B$, so $E$ (is) to $F$, and as $B$ (is) to $C$, so $D$ (is) to $E$. I say that as $A$ is to $C$, so $D$ (is) to $F$.

Let the equal multiples $G, H$, and $K$ have been taken of $A, B$, and $D$ (respectively), and the other random equal multiples $L, M$, and $N$ of $C, E$, and $F$ (respectively).

And since $G$ and $H$ are equal multiples of $A$ and $B$ (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as $A$ (is) to $B$, so $G$ (is) to $H$. And, so, for the same (reasons), as $E$ (is) to $F$, so $M$ (is) to $N$. And as $A$ is to $B$, so $E$ (is) to $F$. And, thus, as $G$ (is) to $H$, so $M$ (is) to $N$ [Prop. 5.11]. And since as $B$ is to $C$, so $D$ (is) to $E$, also, alternately, as $B$ (is) to $D$, so $C$ (is) to $E$ [Prop. 5.16]. And since $H$ and $K$ are equal multiples of $B$ and $D$ (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as $B$ is to $D$, so $H$ (is) to $K$. But, as $B$ (is) to $D$, so $C$ (is) to $E$. And, thus, as $H$ (is) to $K$, so $C$ (is) to $E$ [Prop. 5.11]. Again, since $L$ and $M$ are equal multiples of $C$ and $E$ (respectively), thus as $C$ is to $E$, so $L$ (is) to $M$ [Prop. 5.15]. But, as $C$ (is) to $E$, so $H$ (is) to $K$. And, thus, as $H$ (is) to $K$, so $L$ (is) to $M$ [Prop. 5.11]. Also, alternately, as $H$ (is) to $L$, so $K$ (is) to $M$ [Prop. 5.16]. And it was also shown (that) as $G$ (is) to $H$, so $M$ (is) to $N$. Therefore, since $G, H$, and $L$ are three magnitudes, and $K, M$, and $N$ other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, and their proportion is perturbed, thus, via equality, if $G$ exceeds $L$ then $K$ also exceeds $N$, and if ( $G$ is) equal (to $L$ then $K$ is also) equal (to $N$ ), and if ( $G$ is) less (than $L$ then $K$ is also) less (than $N$ ) [Prop. 5.21]. And $G$ and $K$ are equal multiples of $A$ and $D$ (respectively), and $L$ and $N$ of $C$ and


${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \epsilon: \zeta$ and $\beta: \gamma:: \delta: \epsilon$ then $\alpha: \gamma:: \delta: \zeta$.
$F$ (respectively). Thus, as $A$ (is) to $C$, so $D$ (is) to $F$ [Def. 5.5].

Thus, if there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show.

$$
x \delta^{\prime}
$$








 $\pi \varepsilon ́ \mu \pi \tau o v ~ \tau o ̀ ~ B H ~ \pi p o ̀ \varsigma ~ \delta \varepsilon u ́ \tau \varepsilon p o v ~ \tau o ̀ ~ \Gamma ~ \tau o ̀ v ~ \alpha u ́ t o ̀ v ~ \lambda o ́ \gamma o v ~ \chi \alpha \grave{~}$


 Z.








 $\pi \rho o ̀ s ~ \tau o ̀ ~ \Gamma, ~ o u ́ \tau \omega \varsigma ~ t o ̀ ~ \Delta \Theta ~ \pi \rho o ̀ s ~ t o ̀ ~ Z . ~ . ~$







F
For let a first (magnitude) $A B$ have the same ratio to a second $C$ that a third $D E$ (has) to a fourth $F$. And let a fifth (magnitude) $B G$ also have the same ratio to the second $C$ that a sixth $E H$ (has) to the fourth $F$. I say that the first (magnitude) and the fifth, added together, $A G$, will also have the same ratio to the second $C$ that the third (magnitude) and the sixth, (added together), $D H$, (has) to the fourth $F$.

For since as $B G$ is to $C$, so $E H$ (is) to $F$, thus, inversely, as $C$ (is) to $B G$, so $F$ (is) to $E H$ [Prop. 5.7 corr.]. Therefore, since as $A B$ is to $C$, so $D E$ (is) to $F$, and as $C$ (is) to $B G$, so $F$ (is) to $E H$, thus, via equality, as $A B$ is to $B G$, so $D E$ (is) to $E H$ [Prop. 5.22]. And since separated magnitudes are proportional then they will also be proportional (when) composed [Prop. 5.18]. Thus, as $A G$ is to $G B$, so $D H$ (is) to $H E$. And, also, as $B G$ is to $C$, so $E H$ (is) to $F$. Thus, via equality, as $A G$ is to $C$, so $D H$ (is) to $F$ [Prop. 5.22].

Thus, if a first (magnitude) has to a second the same ratio that a third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and the sixth (added
together, have) to the fourth. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ and $\epsilon: \beta:: \zeta: \delta$ then $\alpha+\epsilon: \beta:: \gamma+\zeta: \delta$.
$\chi \varepsilon^{\prime}$.



















 $\mathrm{AB}, \mathrm{Z} \mu \varepsilon i \zeta \rho \nu \alpha \tau \widetilde{\omega} \Gamma \Delta$, E.




Proposition $25^{\dagger}$
If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes).


Let $A B, C D, E$, and $F$ be four proportional magnitudes, (such that) as $A B$ (is) to $C D$, so $E$ (is) to $F$. And let $A B$ be the greatest of them, and $F$ the least. I say that $A B$ and $F$ is greater than $C D$ and $E$.

For let $A G$ be made equal to $E$, and $C H$ equal to $F$.
[In fact,] since as $A B$ is to $C D$, so $E$ (is) to $F$, and $E$ (is) equal to $A G$, and $F$ to $C H$, thus as $A B$ is to $C D$, so $A G$ (is) to $C H$. And since the whole $A B$ is to the whole $C D$ as the (part) taken away $A G$ (is) to the (part) taken away $C H$, thus the remainder $G B$ will also be to the remainder $H D$ as the whole $A B$ (is) to the whole $C D$ [Prop. 5.19]. And $A B$ (is) greater than $C D$. Thus, $G B$ (is) also greater than $H D$. And since $A G$ is equal to $E$, and $C H$ to $F$, thus $A G$ and $F$ is equal to $C H$ and $E$. And [since] if [equal (magnitudes) are added to unequal (magnitudes) then the wholes are unequal, thus if] $A G$ and $F$ are added to $G B$, and $C H$ and $E$ to $H D-G B$ and $H D$ being unequal, and $G B$ greater-it is inferred that $A B$ and $F$ (is) greater than $C D$ and $E$.

Thus, if four magnitudes are proportional then the (sum of the) largest and the smallest of them is greater than the (sum of the) remaining two (magnitudes). (Which is) the very thing it was required to show.

[^10]
## ELEMENTS BOOK 6

## Similar Figures

${ }^{\circ}$ Opor.

 $\pi \lambda \varepsilon u p \alpha{ }_{c} \alpha \dot{\alpha} v \alpha ́ \lambda o \gamma o v$.

 прòs tò ề $\lambda \alpha \tau$ òv.



$$
\alpha^{\prime} .
$$







 тò ГZ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о v$.



















## Definitions

1. Similar rectilinear figures are those (which) have (their) angles separately equal and the (corresponding) sides about the equal angles proportional.
2. A straight-line is said to have been cut in extreme and mean ratio when as the whole is to the greater segment so the greater (segment is) to the lesser.
3. The height of any figure is the (straight-line) drawn from the vertex perpendicular to the base.

## Proposition $1^{\dagger}$

Triangles and parallelograms which are of the same height are to one another as their bases.


Let $A B C$ and $A C D$ be triangles, and $E C$ and $C F$ parallelograms, of the same height $A C$. I say that as base $B C$ is to base $C D$, so triangle $A B C$ (is) to triangle $A C D$, and parallelogram $E C$ to parallelogram $C F$.

For let the (straight-line) $B D$ have been produced in each direction to points $H$ and $L$, and let [any number] (of straight-lines) $B G$ and $G H$ be made equal to base $B C$, and any number (of straight-lines) $D K$ and $K L$ equal to base $C D$. And let $A G, A H, A K$, and $A L$ have been joined.

And since $C B, B G$, and $G H$ are equal to one another, triangles $A H G, A G B$, and $A B C$ are also equal to one another [Prop. 1.38]. Thus, as many times as base $H C$ is (divisible by) base $B C$, so many times is triangle $A H C$ also (divisible) by triangle $A B C$. So, for the same (reasons), as many times as base $L C$ is (divisible) by base $C D$, so many times is triangle $A L C$ also (divisible) by triangle $A C D$. And if base $H C$ is equal to base $C L$ then triangle $A H C$ is also equal to triangle $A C L$ [Prop. 1.38]. And if base $H C$ exceeds base $C L$ then triangle $A H C$ also exceeds triangle $A C L . \ddagger$ And if ( $H C$ is) less (than $C L$ then $A H C$ is also) less (than $A C L$ ). So, their being four magnitudes, two bases, $B C$ and $C D$, and two trian-




 tò $А Г \Delta$ тpírcuvov.



 трír $\omega v$ ov $\pi \rho o ̀ s ~ \tau o ̀ ~ А Г \Delta ~ \tau \rho i ́ \gamma \omega \nu o v, ~ o u ̛ \tau \omega \varsigma ~ \tau o ̀ ~ E \Gamma ~ \pi \alpha \rho \alpha \lambda-~$





 $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о ́ \gamma \rho \alpha \mu \mu о \nu$.



gles, $A B C$ and $A C D$, equal multiples have been taken of base $B C$ and triangle $A B C$-(namely), base $H C$ and triangle $A H C$-and other random equal multiples of base $C D$ and triangle $A D C$-(namely), base $L C$ and triangle $A L C$. And it has been shown that if base $H C$ exceeds base $C L$ then triangle $A H C$ also exceeds triangle $A L C$, and if ( $H C$ is) equal (to $C L$ then $A H C$ is also) equal (to $A L C$ ), and if ( $H C$ is) less (than $C L$ then $A H C$ is also) less (than $A L C$ ). Thus, as base $B C$ is to base $C D$, so triangle $A B C$ (is) to triangle $A C D$ [Def. 5.5]. And since parallelogram $E C$ is double triangle $A B C$, and parallelogram $F C$ is double triangle $A C D$ [Prop. 1.34], and parts have the same ratio as similar multiples [Prop. 5.15], thus as triangle $A B C$ is to triangle $A C D$, so parallelogram $E C$ (is) to parallelogram $F C$. In fact, since it was shown that as base $B C$ (is) to $C D$, so triangle $A B C$ (is) to triangle $A C D$, and as triangle $A B C$ (is) to triangle $A C D$, so parallelogram $E C$ (is) to parallelogram $C F$, thus, also, as base $B C$ (is) to base $C D$, so parallelogram $E C$ (is) to parallelogram $F C$ [Prop. 5.11].

Thus, triangles and parallelograms which are of the same height are to one another as their bases. (Which is) the very thing it was required to show.
${ }^{\dagger}$ As is easily demonstrated, this proposition holds even when the triangles, or parallelograms, do not share a common side, and/or are not right-angled.
$\ddagger$ This is a straight-forward generalization of Prop. 1.38.

$$
\beta^{\prime} .
$$




 $\pi \lambda \varepsilon \cup \rho \alpha ́ v$.





## Proposition 2

If some straight-line is drawn parallel to one of the sides of a triangle then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle.


For let $D E$ have been drawn parallel to one of the sides $B C$ of triangle $A B C$. I say that as $B D$ is to $D A$, so $C E$ (is) to $E A$.
’Елє弓६ú $\vartheta \vartheta \omega \sigma \alpha \nu \gamma \dot{\alpha} \rho \alpha i \operatorname{BE}, \Gamma \Delta$.









 EA• xaì $\dot{\omega} \varsigma \ddot{\alpha} \rho \alpha \dot{\eta} \mathrm{B} \Delta \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \Delta \mathrm{~A}$, oữ $\tau \omega \varsigma \dot{\eta}$ ГЕ $\pi \rho o ̀ s ~ \tau \dot{\eta} \nu$ EA.


 $\pi \alpha p \alpha ́ \lambda \lambda \eta \lambda o ́ s ~ \varepsilon ̇ \sigma \tau \iota \nu \dot{\eta} \Delta \mathrm{E} \tau \tilde{n} \mathrm{~B} \mathrm{\Gamma}$.










 $\pi \alpha p \alpha \lambda \lambda \eta \lambda$ оs $\alpha \rho \alpha$ ह̇のтì $\dot{\eta} \Delta \mathrm{E} \tau \tilde{n} \mathrm{~B}$.






## $\gamma^{\prime}$.






 $\gamma \omega v i \alpha \alpha$.

 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \Delta$, oút $\omega$ s $\dot{\eta}$ ВА $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu ~ А Г . ~$
 $\delta \iota \alpha \chi \vartheta \varepsilon \tau ̃ \sigma \alpha \dot{\eta} \mathrm{BA} \sigma \cup \mu \pi \iota \pi \tau \varepsilon \tau \omega \alpha \cup ๋ \tau \tilde{n} \chi \alpha \tau \dot{\alpha}$ tò E .

For let $B E$ and $C D$ have been joined.
Thus, triangle $B D E$ is equal to triangle $C D E$. For they are on the same base $D E$ and between the same parallels $D E$ and $B C$ [Prop. 1.38]. And $A D E$ is some other triangle. And equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7]. Thus, as triangle $B D E$ is to [triangle] $A D E$, so triangle $C D E$ (is) to triangle $A D E$. But, as triangle $B D E$ (is) to triangle $A D E$, so (is) $B D$ to $D A$. For, having the same height-(namely), the (straight-line) drawn from $E$ perpendicular to $A B-$ they are to one another as their bases [Prop. 6.1]. So, for the same (reasons), as triangle $C D E$ (is) to $A D E$, so $C E$ (is) to $E A$. And, thus, as $B D$ (is) to $D A$, so $C E$ (is) to $E A$ [Prop. 5.11].

And so, let the sides $A B$ and $A C$ of triangle $A B C$ have been cut proportionally (such that) as $B D$ (is) to $D A$, so $C E$ (is) to $E A$. And let $D E$ have been joined. I say that $D E$ is parallel to $B C$.

For, by the same construction, since as $B D$ is to $D A$, so $C E$ (is) to $E A$, but as $B D$ (is) to $D A$, so triangle $B D E$ (is) to triangle $A D E$, and as $C E$ (is) to $E A$, so triangle $C D E$ (is) to triangle $A D E$ [Prop. 6.1], thus, also, as triangle $B D E$ (is) to triangle $A D E$, so triangle $C D E$ (is) to triangle $A D E$ [Prop. 5.11]. Thus, triangles $B D E$ and $C D E$ each have the same ratio to $A D E$. Thus, triangle $B D E$ is equal to triangle $C D E$ [Prop. 5.9]. And they are on the same base $D E$. And equal triangles, which are also on the same base, are also between the same parallels [Prop. 1.39]. Thus, $D E$ is parallel to $B C$.

Thus, if some straight-line is drawn parallel to one of the sides of a triangle, then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally, then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle. (Which is) the very thing it was required to show.

## Proposition 3

If an angle of a triangle is cut in half, and the straightline cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half.

Let $A B C$ be a triangle. And let the angle $B A C$ have been cut in half by the straight-line $A D$. I say that as $B D$ is to $C D$, so $B A$ (is) to $A C$.

For let $C E$ have been drawn through (point) $C$ parallel to $D A$. And, $B A$ being drawn through, let it meet











 $\dot{\eta} \mathrm{B} \Delta$ трòs $\tau \dot{\eta} \nu \Delta \Gamma$, oútcos $\dot{\eta} \mathrm{BA}$ тpòs tr̀v $\mathrm{A} \Gamma$.
'A $\lambda \lambda \alpha \dot{\alpha} \delta \dot{\eta}$ है $\sigma \tau \omega \dot{\omega} \varsigma \dot{\eta} \mathrm{B} \Delta$ тpòs trìv $\Delta \Gamma$, oú $\tau \omega \varsigma \dot{\eta} \mathrm{BA} \pi \rho o ̀ s$


 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Delta \Gamma$, oút $\omega \varsigma \dot{\eta} \mathrm{BA} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{~A} \Gamma, \alpha \dot{\alpha} \lambda \lambda \alpha$ x $\alpha i$ ìs $\dot{\eta} \mathrm{B} \Delta$ тpòs tì̀ $\Delta \Gamma$, oưt $\omega \varsigma$ ह̇ $\sigma \tau i v \dot{\eta} \mathrm{BA}$ тpòs tr̀v AE tpırćvou














$(C E)$ at (point) $E .{ }^{\dagger}$


And since the straight-line $A C$ falls across the parallel (straight-lines) $A D$ and $E C$, angle $A C E$ is thus equal to $C A D$ [Prop. 1.29]. But, (angle) $C A D$ is assumed (to be) equal to $B A D$. Thus, (angle) $B A D$ is also equal to $A C E$. Again, since the straight-line $B A E$ falls across the parallel (straight-lines) $A D$ and $E C$, the external angle $B A D$ is equal to the internal (angle) $A E C$ [Prop. 1.29]. And (angle) $A C E$ was also shown (to be) equal to $B A D$. Thus, angle $A C E$ is also equal to $A E C$. And, hence, side $A E$ is equal to side $A C$ [Prop. 1.6]. And since $A D$ has been drawn parallel to one of the sides $E C$ of triangle $B C E$, thus, proportionally, as $B D$ is to $D C$, so $B A$ (is) to $A E$ [Prop. 6.2]. And $A E$ (is) equal to $A C$. Thus, as $B D$ (is) to $D C$, so $B A$ (is) to $A C$.

And so, let $B D$ be to $D C$, as $B A$ (is) to $A C$. And let $A D$ have been joined. I say that angle $B A C$ has been cut in half by the straight-line $A D$.

For, by the same construction, since as $B D$ is to $D C$, so $B A$ (is) to $A C$, then also as $B D$ (is) to $D C$, so $B A$ is to $A E$. For $A D$ has been drawn parallel to one (of the sides) $E C$ of triangle $B C E$ [Prop. 6.2]. Thus, also, as $B A$ (is) to $A C$, so $B A$ (is) to $A E$ [Prop. 5.11]. Thus, $A C$ (is) equal to $A E$ [Prop. 5.9]. And, hence, angle $A E C$ is equal to $A C E$ [Prop. 1.5]. But, $A E C$ [is] equal to the external (angle) $B A D$, and $A C E$ is equal to the alternate (angle) $C A D$ [Prop. 1.29]. Thus, (angle) $B A D$ is also equal to $C A D$. Thus, angle $B A C$ has been cut in half by the straight-line $A D$.

Thus, if an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half. (Which is) the very thing it was required to show.

[^11]See Post. 5.

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 $\mu \varepsilon ̀ \nu ~ U ́ \pi o ̀ ~ A B \Gamma ~ \gamma \omega v i ́ \alpha \nu ~ \tau n ̃ ~ U ́ \pi o ̀ ~ \Delta \Gamma E, ~ \tau \grave{\eta} \nu ~ \delta e ̀ ~ U ́ \pi o ̀ ~ B A \Gamma ~ \tau n ̃ ~ U ́ \pi o ̀ ~$




















 $\dot{\omega} \varsigma \mu \varepsilon ̀ \nu \dot{\eta} \mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{~B} \mathrm{\Gamma}$, oưt $\omega \varsigma \dot{\eta} \Delta \Gamma \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \mathrm{E}$, $\dot{\omega} \varsigma$






## Proposition 4

In equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.


Let $A B C$ and $D C E$ be equiangular triangles, having angle $A B C$ equal to $D C E$, and (angle) $B A C$ to $C D E$, and, further, (angle) $A C B$ to $C E D$. I say that in triangles $A B C$ and $D C E$ the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.

Let $B C$ be placed straight-on to $C E$. And since angles $A B C$ and $A C B$ are less than two right-angles [Prop 1.17], and $A C B$ (is) equal to $D E C$, thus $A B C$ and $D E C$ are less than two right-angles. Thus, $B A$ and $E D$, being produced, will meet [C.N. 5]. Let them have been produced, and let them meet at (point) $F$.

And since angle $D C E$ is equal to $A B C, B F$ is parallel to $C D$ [Prop. 1.28]. Again, since (angle) $A C B$ is equal to $D E C, A C$ is parallel to $F E$ [Prop. 1.28]. Thus, $F A C D$ is a parallelogram. Thus, $F A$ is equal to $D C$, and $A C$ to $F D$ [Prop. 1.34]. And since $A C$ has been drawn parallel to one (of the sides) $F E$ of triangle $F B E$, thus as $B A$ is to $A F$, so $B C$ (is) to $C E$ [Prop. 6.2]. And $A F$ (is) equal to $C D$. Thus, as $B A$ (is) to $C D$, so $B C$ (is) to $C E$, and, alternately, as $A B$ (is) to $B C$, so $D C$ (is) to $C E$ [Prop. 5.16]. Again, since $C D$ is parallel to $B F$, thus as $B C$ (is) to $C E$, so $F D$ (is) to $D E$ [Prop. 6.2]. And $F D$ (is) equal to $A C$. Thus, as $B C$ is to $C E$, so $A C$ (is) to $D E$, and, alternately, as $B C$ (is) to $C A$, so $C E$ (is) to $E D$ [Prop. 6.2]. Therefore, since it was shown that as $A B$ (is) to $B C$, so $D C$ (is) to $C E$, and as $B C$ (is) to $C A$, so $C E$ (is) to $E D$, thus, via equality, as $B A$ (is) to $A C$, so $C D$ (is) to $D E$ [Prop. 5.22].

Thus, in equiangular triangles the sides about the equal angles are proportional, and those (sides) subtend-

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ing equal angles correspond．（Which is）the very thing it was required to show．

## Proposition 5

If two triangles have proportional sides then the trian－ gles will be equiangular，and will have the angles which corresponding sides subtend equal．


Let $A B C$ and $D E F$ be two triangles having propor－ tional sides，（so that）as $A B$（is）to $B C$ ，so $D E$（is）to $E F$ ，and as $B C$（is）to $C A$ ，so $E F$（is）to $F D$ ，and，fur－ ther，as $B A$（is）to $A C$ ，so $E D$（is）to $D F$ ．I say that triangle $A B C$ is equiangular to triangle $D E F$ ，and（that the triangles）will have the angles which corresponding sides subtend equal．（That is），（angle）$A B C$（equal）to $D E F, B C A$ to $E F D$ ，and，further，$B A C$ to $E D F$ ．

For let（angle）$F E G$ ，equal to angle $A B C$ ，and（an－ gle）$E F G$ ，equal to $A C B$ ，have been constructed on the straight－line $E F$ at the points $E$ and $F$ on it（respectively） ［Prop．1．23］．Thus，the remaining（angle）at $A$ is equal to the remaining（angle）at $G$（Prop．1．32］．

Thus，triangle $A B C$ is equiangular to［triangle］$E G F$ ． Thus，for triangles $A B C$ and $E G F$ ，the sides about the equal angles are proportional，and（those）sides subtend－ ing equal angles correspond［Prop．6．4］．Thus，as $A B$ is to $B C$ ，［so］$G E$（is）to $E F$ ．But，as $A B$（is）to $B C$ ， so，it was assumed，（is）$D E$ to $E F$ ．Thus，as $D E$（is）to $E F$ ，so $G E$（is）to $E F$［Prop．5．11］．Thus，$D E$ and $G E$ each have the same ratio to $E F$ ．Thus，$D E$ is equal to $G E$［Prop．5．9］．So，for the same（reasons），$D F$ is also equal to $G F$ ．Therefore，since $D E$ is equal to $E G$ ，and $E F$（is）common，the two（sides）$D E, E F$ are equal to the two（sides）$G E, E F$（respectively）．And base $D F$ ［is］equal to base $F G$ ．Thus，angle $D E F$ is equal to angle $G E F$［Prop．1．8］，and triangle $D E F$（is）equal to triangle $G E F$ ，and the remaining angles（are）equal to the remaining angles which the equal sides subtend ［Prop．1．4］．Thus，angle $D F E$ is also equal to $G F E$ ，and
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 $\pi \lambda \varepsilon \cup p \alpha i ̀$ ن́ $\pi о \tau \varepsilon$ ívouøレ.


 $\gamma \omega v i \alpha s$ тàs $\pi \lambda \varepsilon u p \alpha ̀ s ~ \dot{\alpha} v \alpha ́ \lambda o \gamma o v, ~ \dot{\omega} \varsigma ~ \tau \eta ̀ \nu ~ B A ~ \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ A \Gamma, ~$


















(angle) $E D F$ to $E G F$. And since (angle) $F E D$ is equal to $G E F$, and (angle) $G E F$ to $A B C$, angle $A B C$ is thus also equal to $D E F$. So, for the same (reasons), (angle) $A C B$ is also equal to $D F E$, and, further, the (angle) at $A$ to the (angle) at $D$. Thus, triangle $A B C$ is equiangular to triangle $D E F$.

Thus, if two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

## Proposition 6

If two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.


Let $A B C$ and $D E F$ be two triangles having one angle, $B A C$, equal to one angle, $E D F$ (respectively), and the sides about the equal angles proportional, (so that) as $B A$ (is) to $A C$, so $E D$ (is) to $D F$. I say that triangle $A B C$ is equiangular to triangle $D E F$, and will have angle $A B C$ equal to $D E F$, and (angle) $A C B$ to $D F E$.

For let (angle) $F D G$, equal to each of $B A C$ and $E D F$, and (angle) $D F G$, equal to $A C B$, have been constructed on the straight-line $A F$ at the points $D$ and $F$ on it (respectively) [Prop. 1.23]. Thus, the remaining angle at $B$ is equal to the remaining angle at $G$ [Prop. 1.32].

Thus, triangle $A B C$ is equiangular to triangle $D G F$. Thus, proportionally, as $B A$ (is) to $A C$, so $G D$ (is) to $D F$ [Prop. 6.4]. And it was also assumed that as $B A$ is) to $A C$, so $E D$ (is) to $D F$. And, thus, as $E D$ (is) to $D F$, so $G D$ (is) to $D F$ [Prop. 5.11]. Thus, $E D$ (is) equal to $D G$ [Prop. 5.9]. And $D F$ (is) common. So, the two (sides) $E D, D F$ are equal to the two (sides) $G D$, $D F$ (respectively). And angle $E D F$ [is] equal to angle $G D F$. Thus, base $E F$ is equal to base $G F$, and triangle $D E F$ is equal to triangle $G D F$, and the remaining angles










## $\zeta^{\prime}$.







"E $\sigma \tau \omega$ ठúo $\tau \rho i ́ \gamma \omega \nu \alpha$ 就 $\mathrm{AB} \mathrm{\Gamma}, \Delta \mathrm{EZ} \mu i \alpha \nu \gamma \omega v i ́ \alpha \nu \mu \iota \tilde{\alpha}$















will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, (angle) $D F G$ is equal to $D F E$, and (angle) $D G F$ to $D E F$. But, (angle) $D F G$ is equal to $A C B$. Thus, (angle) $A C B$ is also equal to $D F E$. And (angle) $B A C$ was also assumed (to be) equal to $E D F$. Thus, the remaining (angle) at $B$ is equal to the remaining (angle) at $E$ [Prop. 1.32]. Thus, triangle $A B C$ is equiangular to triangle $D E F$.

Thus, if two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

## Proposition 7

If two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides are proportional equal.


Let $A B C$ and $D E F$ be two triangles having one angle, $B A C$, equal to one angle, $E D F$ (respectively), and the sides about (some) other angles, $A B C$ and $D E F$ (respectively), proportional, (so that) as $A B$ (is) to $B C$, so $D E$ (is) to $E F$, and the remaining (angles) at $C$ and $F$, first of all, both less than right-angles. I say that triangle $A B C$ is equiangular to triangle $D E F$, and (that) angle $A B C$ will be equal to $D E F$, and (that) the remaining (angle) at $C$ (will be) manifestly equal to the remaining (angle) at $F$.

For if angle $A B C$ is not equal to (angle) $D E F$ then one of them is greater. Let $A B C$ be greater. And let (angle) $A B G$, equal to (angle) $D E F$, have been constructed on the straight-line $A B$ at the point $B$ on it [Prop. 1.23].

And since angle $A$ is equal to (angle) $D$, and (angle) $A B G$ to $D E F$, the remaining (angle) $A G B$ is thus equal
























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to the remaining (angle) $D F E$ [Prop. 1.32]. Thus, triangle $A B G$ is equiangular to triangle $D E F$. Thus, as $A B$ is to $B G$, so $D E$ (is) to $E F$ [Prop. 6.4]. And as $D E$ (is) to $E F$, [so] it was assumed (is) $A B$ to $B C$. Thus, $A B$ has the same ratio to each of $B C$ and $B G$ [Prop. 5.11]. Thus, $B C$ (is) equal to $B G$ [Prop. 5.9]. And, hence, the angle at $C$ is equal to angle $B G C$ [Prop. 1.5]. And the angle at $C$ was assumed (to be) less than a right-angle. Thus, (angle) $B G C$ is also less than a right-angle. Hence, the adjacent angle to it, $A G B$, is greater than a right-angle [Prop. 1.13]. And $(A G B)$ was shown to be equal to the (angle) at $F$. Thus, the (angle) at $F$ is also greater than a right-angle. But it was assumed (to be) less than a rightangle. The very thing is absurd. Thus, angle $A B C$ is not unequal to (angle) $D E F$. Thus, (it is) equal. And the (angle) at $A$ is also equal to the (angle) at $D$. And thus the remaining (angle) at $C$ is equal to the remaining (angle) at $F$ [Prop. 1.32]. Thus, triangle $A B C$ is equiangular to triangle $D E F$.

But, again, let each of the (angles) at $C$ and $F$ be assumed (to be) not less than a right-angle. I say, again, that triangle $A B C$ is equiangular to triangle $D E F$ in this case also.

For, with the same construction, we can similarly show that $B C$ is equal to $B G$. Hence, also, the angle at $C$ is equal to (angle) $B G C$. And the (angle) at $C$ (is) not less than a right-angle. Thus, $B G C$ (is) not less than a right-angle either. So, in triangle $B G C$ the (sum of) two angles is not less than two right-angles. The very thing is impossible [Prop. 1.17]. Thus, again, angle $A B C$ is not unequal to $D E F$. Thus, (it is) equal. And the (angle) at $A$ is also equal to the (angle) at $D$. Thus, the remaining (angle) at $C$ is equal to the remaining (angle) at $F$ [Prop. 1.32]. Thus, triangle $A B C$ is equiangular to triangle $D E F$.

Thus, if two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides (are) proportional equal. (Which is) the very thing it was required to show.

## Proposition 8

If, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the triangles around the perpendicular are similar to the whole (triangle), and to one another.

Let $A B C$ be a right-angled triangle having the angle $B A C$ a right-angle, and let $A D$ have been drawn from


















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$A$, perpendicular to $B C$ [Prop. 1.12]. I say that triangles $A B D$ and $A D C$ are each similar to the whole (triangle) $A B C$ and, further, to one another.


For since (angle) $B A C$ is equal to $A D B$-for each (are) right-angles-and the (angle) at $B$ (is) common to the two triangles $A B C$ and $A B D$, the remaining (angle) $A C B$ is thus equal to the remaining (angle) $B A D$ [Prop. 1.32]. Thus, triangle $A B C$ is equiangular to triangle $A B D$. Thus, as $B C$, subtending the right-angle in triangle $A B C$, is to $B A$, subtending the right-angle in triangle $A B D$, so the same $A B$, subtending the angle at $C$ in triangle $A B C$, (is) to $B D$, subtending the equal (angle) $B A D$ in triangle $A B D$, and, further, (so is) $A C$ to $A D$, (both) subtending the angle at $B$ common to the two triangles [Prop. 6.4]. Thus, triangle $A B C$ is equiangular to triangle $A B D$, and has the sides about the equal angles proportional. Thus, triangle $A B C$ [is] similar to triangle $A B D$ [Def. 6.1]. So, similarly, we can show that triangle $A B C$ is also similar to triangle $A D C$. Thus, [triangles] $A B D$ and $A D C$ are each similar to the whole (triangle) $A B C$.

So I say that triangles $A B D$ and $A D C$ are also similar to one another.

For since the right-angle $B D A$ is equal to the rightangle $A D C$, and, indeed, (angle) $B A D$ was also shown (to be) equal to the (angle) at $C$, thus the remaining (angle) at $B$ is also equal to the remaining (angle) $D A C$ [Prop. 1.32]. Thus, triangle $A B D$ is equiangular to triangle $A D C$. Thus, as $B D$, subtending (angle) $B A D$ in triangle $A B D$, is to $D A$, subtending the (angle) at $C$ in triangle $A D C$, (which is) equal to (angle) $B A D$, so (is) the same $A D$, subtending the angle at $B$ in triangle $A B D$, to $D C$, subtending (angle) $D A C$ in triangle $A D C$, (which is) equal to the (angle) at $B$, and, further, (so is) $B A$ to $A C$, (each) subtending right-angles [Prop. 6.4]. Thus, triangle $A B D$ is similar to triangle $A D C$ [Def. 6.1].

Thus, if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base

## По́рıбиа.





${ }^{\dagger}$ In other words, the perpendicular is the geometric mean of the pieces.

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 $\varepsilon \cup ̉ \vartheta \varepsilon i ̃ \alpha \dot{\eta} A \Gamma \gamma \omega v i \alpha \alpha \nu \pi \varepsilon \rho เ \varepsilon ́ \chi O \cup \sigma \alpha \mu \varepsilon \tau \grave{\alpha} \tau \tilde{\eta} \varsigma \mathrm{AB} \tau \cup \chi O \cup ̃ \sigma \alpha \nu \cdot x \alpha \grave{\iota}$

 $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ оs $\alpha \cup \tilde{n} \tilde{\eta} \eta \not \chi \vartheta \omega \dot{\eta} \Delta \mathrm{Z}$.







## $i^{\prime}$.



then the triangles around the perpendicular are similar to the whole (triangle), and to one another. [(Which is) the very thing it was required to show.]

## Corollary

So (it is) clear, from this, that if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the (straight-line so) drawn is in mean proportion to the pieces of the base. ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 9

To cut off a prescribed part from a given straight-line.


Let $A B$ be the given straight-line. So it is required to cut off a prescribed part from $A B$.

So let a third (part) have been prescribed. [And] let some straight-line $A C$ have been drawn from (point) $A$, encompassing a random angle with $A B$. And let a random point $D$ have been taken on $A C$. And let $D E$ and $E C$ be made equal to $A D$ [Prop. 1.3]. And let $B C$ have been joined. And let $D F$ have been drawn through $D$ parallel to it [Prop. 1.31].

Therefore, since $F D$ has been drawn parallel to one of the sides, $B C$, of triangle $A B C$, then, proportionally, as $C D$ is to $D A$, so $B F$ (is) to $F A$ [Prop. 6.2]. And $C D$ (is) double $D A$. Thus, $B F$ (is) also double $F A$. Thus, $B A$ (is) triple $A F$.

Thus, the prescribed third part, $A F$, has been cut off from the given straight-line, $A B$. (Which is) the very thing it was required to do.

## Proposition 10

To cut a given uncut straight-line similarly to a given cut (straight-line).



 $\tau \widetilde{\omega} \nu \Delta$, $\mathrm{E} \tau \tilde{\eta} \mathrm{B} \Gamma \pi \alpha \rho \alpha \dot{\lambda} \lambda \eta \lambda$ ol $\eta \chi \chi \vartheta \omega \sigma \alpha \nu \alpha i \quad \Delta \mathrm{Z}, \mathrm{EH}, \delta \iota \alpha ̀ ~ \delta \check{~}$ то兀̃ $\Delta \tau \tilde{\eta} \mathrm{AB} \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda о \varsigma \tilde{\eta} \chi \vartheta \omega \dot{\eta} \Delta \Theta K$.










 $\pi \rho o ̀ s ~ \tau \eta ̀ \nu \Delta \mathrm{~A}$, oüt $\omega \varsigma \dot{\eta} \mathrm{HZ} \pi \rho o ̀ s ~ \tau \eta ̀ \nu \mathrm{ZA}$.



## $\alpha^{\prime}$.


"Eбт $\omega \sigma \alpha \nu$ גi $\delta 0 \vartheta \varepsilon 亢 ̃ \sigma \alpha l ~[\delta u ́ o ~ \varepsilon u ́ v \varepsilon i ̃ \alpha l] ~ \alpha i ~ B A, ~ А Г ~ \chi \alpha i ~$










Let $A B$ be the given uncut straight-line, and $A C$ a (straight-line) cut at points $D$ and $E$, and let $(A C)$ be laid down so as to encompass a random angle (with $A B$ ). And let $C B$ have been joined. And let $D F$ and $E G$ have been drawn through (points) $D$ and $E$ (respectively), parallel to $B C$, and let $D H K$ have been drawn through (point) $D$, parallel to $A B$ [Prop. 1.31].

Thus, $F H$ and $H B$ are each parallelograms. Thus, $D H$ (is) equal to $F G$, and $H K$ to $G B$ [Prop. 1.34]. And since the straight-line $H E$ has been drawn parallel to one of the sides, $K C$, of triangle $D K C$, thus, proportionally, as $C E$ is to $E D$, so $K H$ (is) to $H D$ [Prop. 6.2]. And $K H$ (is) equal to $B G$, and $H D$ to $G F$. Thus, as $C E$ is to $E D$, so $B G$ (is) to $G F$. Again, since $F D$ has been drawn parallel to one of the sides, $G E$, of triangle $A G E$, thus, proportionally, as $E D$ is to $D A$, so $G F$ (is) to $F A$ [Prop. 6.2]. And it was also shown that as $C E$ (is) to $E D$, so $B G$ (is) to $G F$. Thus, as $C E$ is to $E D$, so $B G$ (is) to $G F$, and as $E D$ (is) to $D A$, so $G F$ (is) to $F A$.

Thus, the given uncut straight-line, $A B$, has been cut similarly to the given cut straight-line, $A C$. (Which is) the very thing it was required to do.

## Proposition 11

To find a third (straight-line) proportional to two given straight-lines.

Let $B A$ and $A C$ be the [two] given [straight-lines], and let them be laid down encompassing a random angle. So it is required to find a third (straight-line) proportional to $B A$ and $A C$. For let ( $B A$ and $A C$ ) have been produced to points $D$ and $E$ (respectively), and let $B D$ be made equal to $A C$ [Prop. 1.3]. And let $B C$ have been joined. And let $D E$ have been drawn through (point) $D$ parallel to it [Prop. 1.31].

Therefore, since $B C$ has been drawn parallel to one of the sides $D E$ of triangle $A D E$, proportionally, as $A B$ is to $B D$, so $A C$ (is) to $C E$ [Prop. 6.2]. And $B D$ (is) equal




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 $\tau \tilde{\eta} \varsigma \mathrm{H} \Theta \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o s \alpha u ̋ \tau \tilde{\eta} \eta \eta \chi \vartheta \omega$ ठı̀ тоũ E $\dot{\eta} \mathrm{EZ}$.






to $A C$. Thus, as $A B$ is to $A C$, so $A C$ (is) to $C E$.


Thus, a third (straight-line), $C E$, has been found (which is) proportional to the two given straight-lines, $A B$ and $A C$. (Which is) the very thing it was required to do.

## Proposition 12

To find a fourth (straight-line) proportional to three given straight-lines.


Let $A, B$, and $C$ be the three given straight-lines. So it is required to find a fourth (straight-line) proportional to $A, B$, and $C$.

Let the two straight-lines $D E$ and $D F$ be set out encompassing the [random] angle $E D F$. And let $D G$ be made equal to $A$, and $G E$ to $B$, and, further, $D H$ to $C$ [Prop. 1.3]. And $G H$ being joined, let $E F$ have been drawn through (point) $E$ parallel to it [Prop. 1.31].

Therefore, since $G H$ has been drawn parallel to one of the sides $E F$ of triangle $D E F$, thus as $D G$ is to $G E$, so $D H$ (is) to $H F$ [Prop. 6.2]. And $D G$ (is) equal to $A$, and $G E$ to $B$, and $D H$ to $C$. Thus, as $A$ is to $B$, so $C$ (is)

$$
\begin{aligned}
& \text { ' } \gamma \text { '. }
\end{aligned}
$$


 $\mathrm{AB}, \mathrm{B} \mathrm{\Gamma} \mu \varepsilon ́ \sigma \eta \nu \dot{\alpha} \nu \alpha ́ \lambda o \gamma o \nu \pi \rho o \sigma \varepsilon \cup \rho \varepsilon i ̃ \nu$.









to $H F$.
Thus, a fourth (straight-line), $H F$, has been found (which is) proportional to the three given straight-lines, $A, B$, and $C$. (Which is) the very thing it was required to do.

## Proposition 13

To find the (straight-line) in mean proportion to two given straight-lines. ${ }^{\dagger}$


Let $A B$ and $B C$ be the two given straight-lines. So it is required to find the (straight-line) in mean proportion to $A B$ and $B C$.

Let ( $A B$ and $B C$ ) be laid down straight-on (with respect to one another), and let the semi-circle $A D C$ have been drawn on $A C$ [Prop. 1.10]. And let $B D$ have been drawn from (point) $B$, at right-angles to $A C$ [Prop. 1.11]. And let $A D$ and $D C$ have been joined.

And since $A D C$ is an angle in a semi-circle, it is a right-angle [Prop. 3.31]. And since, in the right-angled triangle $A D C$, the (straight-line) $D B$ has been drawn from the right-angle perpendicular to the base, $D B$ is thus the mean proportional to the pieces of the base, $A B$ and $B C$ [Prop. 6.8 corr.].

Thus, $D B$ has been found (which is) in mean proportion to the two given straight-lines, $A B$ and $B C$. (Which is) the very thing it was required to do.
${ }^{\dagger}$ In other words, to find the geometric mean of two given straight-lines.

$$
\delta^{\prime}
$$


 $\gamma \omega v i ́ \omega \nu \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o \gamma \rho \alpha ́ \mu \mu \omega \nu \alpha \nu \tau \iota \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma \iota \nu$ ai $\pi \lambda \varepsilon \cup p \alpha i ~ \alpha i$



 $\lambda \varepsilon ́ \gamma \omega$, ő $\tau \tau \tau \widetilde{\omega} \nu \mathrm{AB}, B \Gamma$ àv $\tau \iota \tau \varepsilon \pi o ́ v \vartheta \alpha \sigma \iota \nu$ 人i $\pi \lambda \varepsilon \cup p \alpha i \quad \alpha i \quad \pi \varepsilon p i ̀$
 BE , oüt $\tau \omega \stackrel{\grave{\eta}}{\mathrm{HB}} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{BZ}$.

## Proposition 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let $A B$ and $B C$ be equal and equiangular parallelograms having the angles at $B$ equal. And let $D B$ and $B E$ be laid down straight-on (with respect to one another). Thus, $F B$ and $B G$ are also straight-on (with respect to one another) [Prop. 1.14]. I say that the sides of $A B$ and





 ZE , oüt $\omega \varsigma \dot{\eta} \mathrm{HB} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{BZ} \cdot x \alpha i$ © $\omega$ s $\alpha \rho \alpha \dot{\eta} \Delta \mathrm{B} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu$
 $\lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega \nu \dot{\alpha} \nu \tau \iota \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma เ \nu$ גi $\pi \lambda \varepsilon \cup p \alpha i ̀ \alpha i ~ \pi \varepsilon p i ̀ ~ \tau \grave{\alpha} \varsigma$ hै $\sigma \alpha \varsigma$ ү $\omega$ vías.
'A $\lambda \lambda \dot{\alpha} \delta \dot{\eta}$ है $\sigma \tau \omega \dot{\omega} \varsigma \dot{\eta} \Delta \mathrm{B}$ трòs t $\grave{\eta} \nu \mathrm{BE}$, oư $\tau \omega \varsigma \dot{\eta} \mathrm{HB} \pi \rho o ̀ s$
 ВГ тар $\alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$.

 AB тар $\alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu \pi \rho o ̀ s ~ t o ̀ ~ Z E ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda \lambda o ́ \gamma \rho \alpha \mu \mu o \nu, ~ i ́ s ~$


 $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о ́ \gamma \rho \alpha \mu \mu о \nu \tau \widetilde{\varphi}$ ВГ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$.

 iбoү $\omega v i ́ \omega \nu ~ \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega \nu \dot{\alpha} \nu \tau เ \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma เ \nu ~ \alpha i ~ \pi \lambda \varepsilon \cup \rho \alpha \grave{~}$


## $\iota \varepsilon^{\prime}$.









$B C$ about the equal angles are reciprocally proportional, that is to say, that as $D B$ is to $B E$, so $G B$ (is) to $B F$.


For let the parallelogram $F E$ have been completed. Therefore, since parallelogram $A B$ is equal to parallelogram $B C$, and $F E$ (is) some other (parallelogram), thus as (parallelogram) $A B$ is to $F E$, so (parallelogram) $B C$ (is) to $F E$ [Prop. 5.7]. But, as (parallelogram) $A B$ (is) to $F E$, so $D B$ (is) to $B E$, and as (parallelogram) $B C$ (is) to $F E$, so $G B$ (is) to $B F$ [Prop. 6.1]. Thus, also, as $D B$ (is) to $B E$, so $G B$ (is) to $B F$. Thus, in parallelograms $A B$ and $B C$ the sides about the equal angles are reciprocally proportional.

And so, let $D B$ be to $B E$, as $G B$ (is) to $B F$. I say that parallelogram $A B$ is equal to parallelogram $B C$.

For since as $D B$ is to $B E$, so $G B$ (is) to $B F$, but as $D B$ (is) to $B E$, so parallelogram $A B$ (is) to parallelogram $F E$, and as $G B$ (is) to $B F$, so parallelogram $B C$ (is) to parallelogram $F E$ [Prop. 6.1], thus, also, as (parallelogram) $A B$ (is) to $F E$, so (parallelogram) $B C$ (is) to $F E$ [Prop. 5.11]. Thus, parallelogram $A B$ is equal to parallelogram $B C$ [Prop. 5.9].

Thus, in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal. (Which is) the very thing it was required to show.

## Proposition 15

In equal triangles also having one angle equal to one (angle) the sides about the equal angles are reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal.

Let $A B C$ and $A D E$ be equal triangles having one angle equal to one (angle), (namely) $B A C$ (equal) to $D A E$. I say that, in triangles $A B C$ and $A D E$, the sides about the
$\dot{\eta} \mathrm{EA} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{AB}$.













 $\mathrm{A} \Delta \mathrm{E}$ трıүஸ́vఱ.
 $\pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ A \Delta, ~ o u ̈ \tau \omega s ~ \grave{\eta} \mathrm{EA} \pi \rho o ̀ s ~ \tau \eta ̀ \nu \mathrm{AB}$, $\dot{\alpha} \lambda \lambda^{\circ} \dot{\omega} \varsigma \mu \varepsilon ̀ \nu$













## $\iota^{\prime}$.




equal angles are reciprocally proportional, that is to say, that as $C A$ is to $A D$, so $E A$ (is) to $A B$.


For let $C A$ be laid down so as to be straight-on (with respect) to $A D$. Thus, $E A$ is also straight-on (with respect) to $A B$ [Prop. 1.14]. And let $B D$ have been joined.

Therefore, since triangle $A B C$ is equal to triangle $A D E$, and $B A D$ (is) some other (triangle), thus as triangle $C A B$ is to triangle $B A D$, so triangle $E A D$ (is) to triangle $B A D$ [Prop. 5.7]. But, as (triangle) $C A B$ (is) to $B A D$, so $C A$ (is) to $A D$, and as (triangle) $E A D$ (is) to $B A D$, so $E A$ (is) to $A B$ [Prop. 6.1]. And thus, as $C A$ (is) to $A D$, so $E A$ (is) to $A B$. Thus, in triangles $A B C$ and $A D E$ the sides about the equal angles (are) reciprocally proportional.

And so, let the sides of triangles $A B C$ and $A D E$ be reciprocally proportional, and (thus) let $C A$ be to $A D$, as $E A$ (is) to $A B$. I say that triangle $A B C$ is equal to triangle $A D E$.

For, $B D$ again being joined, since as $C A$ is to $A D$, so $E A$ (is) to $A B$, but as $C A$ (is) to $A D$, so triangle $A B C$ (is) to triangle $B A D$, and as $E A$ (is) to $A B$, so triangle $E A D$ (is) to triangle $B A D$ [Prop. 6.1], thus as triangle $A B C$ (is) to triangle $B A D$, so triangle $E A D$ (is) to triangle $B A D$. Thus, (triangles) $A B C$ and $E A D$ each have the same ratio to $B A D$. Thus, [triangle] $A B C$ is equal to triangle $E A D$ [Prop. 5.9].

Thus, in equal triangles also having one angle equal to one (angle) the sides about the equal angles (are) reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal. (Which is) the very thing it was required to show.

## Proposition 16

If four straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rect-




E ط







 $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu \alpha$ ．

 $\dot{\eta} \mathrm{AB}$ трòs $\tau \grave{\eta} \nu \Gamma \Delta$ ，oü $\tau \omega \varsigma \dot{\eta} \Gamma \Theta \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{AH}$ ．七テั้ BH ，










 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \Delta$ ，oüt $\omega \varsigma \dot{\eta} \mathrm{E} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{Z}$ ．




 $\pi \alpha p \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \alpha \mu \omega \nu \dot{\alpha} \nu \tau \iota \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma \omega \nu$ 人i $\pi \lambda \varepsilon \cup p \alpha i ̀ \alpha i \quad \pi \varepsilon p i ̀ \tau \alpha \varsigma$









angle contained by the（two）outermost is equal to the rectangle contained by the middle（two）then the four straight－lines will be proportional．

$\mathrm{E} \longmapsto$
F
Let $A B, C D, E$ ，and $F$ be four proportional straight－ lines，（such that）as $A B$（is）to $C D$ ，so $E$（is）to $F$ ．I say that the rectangle contained by $A B$ and $F$ is equal to the rectangle contained by $C D$ and $E$ ．
［For］let $A G$ and $C H$ have been drawn from points $A$ and $C$ at right－angles to the straight－lines $A B$ and $C D$ （respectively）［Prop．1．11］．And let $A G$ be made equal to $F$ ，and $C H$ to $E$［Prop．1．3］．And let the parallelograms $B G$ and $D H$ have been completed．

And since as $A B$ is to $C D$ ，so $E$（is）to $F$ ，and $E$ （is）equal $C H$ ，and $F$ to $A G$ ，thus as $A B$ is to $C D$ ，so $C H$（is）to $A G$ ．Thus，in the parallelograms $B G$ and $D H$ the sides about the equal angles are reciprocally propor－ tional．And those equiangular parallelograms in which the sides about the equal angles are reciprocally propor－ tional are equal［Prop．6．14］．Thus，parallelogram $B G$ is equal to parallelogram $D H$ ．And $B G$ is the（rectangle contained）by $A B$ and $F$ ．For $A G$（is）equal to $F$ ．And $D H$（is）the（rectangle contained）by $C D$ and $E$ ．For $E$ （is）equal to $C H$ ．Thus，the rectangle contained by $A B$ and $F$ is equal to the rectangle contained by $C D$ and $E$ ．

And so，let the rectangle contained by $A B$ and $F$ be equal to the rectangle contained by $C D$ and $E$ ．I say that the four straight－lines will be proportional，（so that）as $A B$（is）to $C D$ ，so $E$（is）to $F$ ．

For，with the same construction，since the（rectangle contained）by $A B$ and $F$ is equal to the（rectangle con－ tained）by $C D$ and $E$ ．And $B G$ is the（rectangle con－ tained）by $A B$ and $F$ ．For $A G$ is equal to $F$ ．And $D H$ （is）the（rectangle contained）by $C D$ and $E$ ．For $C H$ （is）equal to $E . B G$ is thus equal to $D H$ ．And they are equiangular．And in equal and equiangular parallel－ ograms the sides about the equal angles are reciprocally proportional［Prop．6．14］．Thus，as $A B$ is to $C D$ ，so $C H$ （is）to $A G$ ．And $C H$（is）equal to $E$ ，and $A G$ to $F$ ．Thus， as $A B$ is to $C D$ ，so $E$（is）to $F$ ．

Thus，if four straight－lines are proportional then the rectangle contained by the（two）outermost is equal to the rectangle contained by the middle（two）．And if the rectangle contained by the（two）outermost is equal to

## $\zeta^{\prime}$







$\Gamma$



























the rectangle contained by the middle (two) then the four straight-lines will be proportional. (Which is) the very thing it was required to show.

## Proposition 17

If three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional.


Let $A, B$ and $C$ be three proportional straight-lines, (such that) as $A$ (is) to $B$, so $B$ (is) to $C$. I say that the rectangle contained by $A$ and $C$ is equal to the square on $B$.

Let $D$ be made equal to $B$ [Prop. 1.3].
And since as $A$ is to $B$, so $B$ (is) to $C$, and $B$ (is) equal to $D$, thus as $A$ is to $B$, (so) $D$ (is) to $C$. And if four straight-lines are proportional then the [rectangle] contained by the (two) outermost is equal to the rectangle contained by the middle (two) [Prop. 6.16]. Thus, the (rectangle contained) by $A$ and $C$ is equal to the (rectangle contained) by $B$ and $D$. But, the (rectangle contained) by $B$ and $D$ is the (square) on $B$. For $B$ (is) equal to $D$. Thus, the rectangle contained by $A$ and $C$ is equal to the square on $B$.

And so, let the (rectangle contained) by $A$ and $C$ be equal to the (square) on $B$. I say that as $A$ is to $B$, so $B$ (is) to $C$.

For, with the same construction, since the (rectangle contained) by $A$ and $C$ is equal to the (square) on $B$. But, the (square) on $B$ is the (rectangle contained) by $B$ and $D$. For $B$ (is) equal to $D$. The (rectangle contained) by $A$ and $C$ is thus equal to the (rectangle contained) by $B$ and $D$. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four straight-lines are proportional [Prop. 6.16]. Thus, as $A$ is to $B$, so $D$ (is) to $C$. And $B$ (is) equal to $D$. Thus, as $A$ (is) to $B$, so $B$ (is) to $C$.

Thus, if three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional. (Which is) the very thing it was required to
show.

## Proposition 18

To describe a rectilinear figure similar, and similarly laid down, to a given rectilinear figure on a given straight-line.


Let $A B$ be the given straight-line, and $C E$ the given rectilinear figure. So it is required to describe a rectilinear figure similar, and similarly laid down, to the rectilinear figure $C E$ on the straight-line $A B$.

Let $D F$ have been joined, and let $G A B$, equal to the angle at $C$, and $A B G$, equal to (angle) $C D F$, have been constructed on the straight-line $A B$ at the points $A$ and $B$ on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) $C F D$ is equal to $A G B$ [Prop. 1.32]. Thus, triangle $F C D$ is equiangular to triangle $G A B$. Thus, proportionally, as $F D$ is to $G B$, so $F C$ (is) to $G A$, and $C D$ to $A B$ [Prop. 6.4]. Again, let $B G H$, equal to angle $D F E$, and $G B H$ equal to (angle) $F D E$, have been constructed on the straight-line $B G$ at the points $G$ and $B$ on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at $E$ is equal to the remaining (angle) at $H$ [Prop. 1.32]. Thus, triangle $F D E$ is equiangular to triangle $G H B$. Thus, proportionally, as $F D$ is to $G B$, so $F E$ (is) to $G H$, and $E D$ to $H B$ [Prop. 6.4]. And it was also shown (that) as $F D$ (is) to $G B$, so $F C$ (is) to $G A$, and $C D$ to $A B$. Thus, also, as $F C$ (is) to $A G$, so $C D$ (is) to $A B$, and $F E$ to $G H$, and, further, $E D$ to $H B$. And since angle $C F D$ is equal to $A G B$, and $D F E$ to $B G H$, thus the whole (angle) $C F E$ is equal to the whole (angle) $A G H$. So, for the same (reasons), (angle) $C D E$ is also equal to $A B H$. And the (angle) at $C$ is also equal to the (angle) at $A$, and the (angle) at $E$ to the (angle) at $H$. Thus, (figure) $A H$ is equiangular to $C E$. And (the two figures) have the sides about their equal angles proportional. Thus, the rectilinear figure $A H$ is similar to the rectilinear figure $C E$ [Def. 6.1].

Thus, the rectilinear figure $A H$, similar, and similarly laid down, to the given rectilinear figure $C E$ has been constructed on the given straight-line $A B$. (Which is) the
$1 \vartheta^{\prime}$.
 $\tau \widetilde{\omega} \nu \dot{\delta} \mu о \lambda o ́ \gamma \omega \nu \pi \lambda \varepsilon \cup \rho \tilde{\omega} \nu$.



 тñ $\mathrm{EZ} \cdot \lambda \varepsilon ́ \gamma \omega$, öть тò AB трír $\omega$ vov $\pi \rho o ̀ \varsigma ~ \tau o ̀ ~ ~ \triangle E Z ~ \tau p i ́ \gamma \omega v o v ~$













 $\pi \rho \omega ́ \tau \eta ~ \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ \tau р i ́ t \eta \nu ~ \delta \iota \pi \lambda \alpha \sigma i ́ o v \alpha ~ \lambda o ́ \gamma o v ~ ह ै \chi \varepsilon ા ~ \eta ̆ \pi \varepsilon \rho ~ \pi \rho o ̀ s ~$

 oútcus tò $\mathrm{AB} \mathrm{\Gamma}$ тpírcuvov $\pi \rho o ̀ s ~ \tau o ̀ ~ A B H ~ \tau p i ́ r \omega v o v . ~ \chi \alpha \grave{~ \tau o ̀ ~}$







## Пópıб $\mu$.



very thing it was required to do.

## Proposition 19

Similar triangles are to one another in the squared ${ }^{\dagger}$ ratio of (their) corresponding sides.


Let $A B C$ and $D E F$ be similar triangles having the angle at $B$ equal to the (angle) at $E$, and $A B$ to $B C$, as $D E$ (is) to $E F$, such that $B C$ corresponds to $E F$. I say that triangle $A B C$ has a squared ratio to triangle $D E F$ with respect to (that side) $B C$ (has) to $E F$.

For let a third (straight-line), $B G$, have been taken (which is) proportional to $B C$ and $E F$, so that as $B C$ (is) to $E F$, so $E F$ (is) to $B G$ [Prop. 6.11]. And let $A G$ have been joined.

Therefore, since as $A B$ is to $B C$, so $D E$ (is) to $E F$, thus, alternately, as $A B$ is to $D E$, so $B C$ (is) to $E F$ [Prop. 5.16]. But, as $B C$ (is) to $E F$, so $E F$ is to $B G$. And, thus, as $A B$ (is) to $D E$, so $E F$ (is) to $B G$. Thus, for triangles $A B G$ and $D E F$, the sides about the equal angles are reciprocally proportional. And those triangles having one (angle) equal to one (angle) for which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.15]. Thus, triangle $A B G$ is equal to triangle $D E F$. And since as $B C$ (is) to $E F$, so $E F$ (is) to $B G$, and if three straight-lines are proportional then the first has a squared ratio to the third with respect to the second [Def. 5.9], $B C$ thus has a squared ratio to $B G$ with respect to (that) $C B$ (has) to $E F$. And as $C B$ (is) to $B G$, so triangle $A B C$ (is) to triangle $A B G$ [Prop. 6.1]. Thus, triangle $A B C$ also has a squared ratio to (triangle) $A B G$ with respect to (that side) $B C$ (has) to $E F$. And triangle $A B G$ (is) equal to triangle $D E F$. Thus, triangle $A B C$ also has a squared ratio to triangle $D E F$ with respect to (that side) $B C$ (has) to $E F$.

Thus, similar triangles are to one another in the squared ratio of (their) corresponding sides. [(Which is) the very thing it was required to show].

## Corollary

So it is clear, from this, that if three straight-lines are proportional, then as the first is to the third, so the figure


† Literally, "double".

$$
x^{\prime} .
$$



 $\pi \lambda \varepsilon u p \dot{\alpha} \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu$ ó $\mu o ́ \lambda o \gamma o v \pi \lambda \varepsilon u p \alpha ́ \nu$.

 $\delta \varepsilon ̀ ~ \varepsilon ้ \sigma \tau \omega ~ \dot{\eta} \mathrm{AB} \tau \tilde{n} \mathrm{ZH} \cdot \lambda \varepsilon ́ \gamma \omega$, ǒ $\tau \iota \tau \dot{\alpha} \mathrm{AB} \mathrm{\Gamma} \Delta \mathrm{E}, \mathrm{ZH} \mathrm{ZK}$

 $\pi \rho o ̀ s ~ \tau o ̀ ~ Z H \Theta K \Lambda ~ \pi o \lambda u ́ \gamma \omega \nu o v ~ \delta ı \pi \lambda \alpha \sigma i ́ o v \alpha ~ \lambda o ́ \gamma o v ~ ह ै \chi ~ \chi \varepsilon ا ~ \eta ̈ \pi \varepsilon p ~ \dot{\eta}$ $\mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\nu} \nu \mathrm{ZH}$.
'Е $\pi \varepsilon \zeta \varepsilon u ́ \chi \vartheta \omega \sigma \alpha \nu \alpha i \mathrm{BE}, \mathrm{E}, \mathrm{H} \Lambda, \Lambda \Theta$.











 $\tau \widetilde{\omega} \nu \pi o \lambda u \gamma \omega \dot{\omega} \omega \nu \nu$ ह̇бтi้ $\dot{\omega} \varsigma \dot{\eta} \mathrm{AB}$ трòs $\mathrm{B} \mathrm{\Gamma}$, oút $\tau \varsigma \dot{\eta} \mathrm{ZH} \pi \rho o ̀ s$







(described) on the first (is) to the similar, and similarly described, (figure) on the second. (Which is) the very thing it was required to show.

## Proposition 20

Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side.


Let $A B C D E$ and $F G H K L$ be similar polygons, and let $A B$ correspond to $F G$. I say that polygons $A B C D E$ and $F G H K L$ can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and (that) polygon $A B C D E$ has a squared ratio to polygon $F G H K L$ with respect to that $A B$ (has) to $F G$.

Let $B E, E C, G L$, and $L H$ have been joined.
And since polygon $A B C D E$ is similar to polygon $F G H K L$, angle $B A E$ is equal to angle $G F L$, and as $B A$ is to $A E$, so $G F$ (is) to $F L$ [Def. 6.1]. Therefore, since $A B E$ and $F G L$ are two triangles having one angle equal to one angle and the sides about the equal angles proportional, triangle $A B E$ is thus equiangular to triangle $F G L$ [Prop. 6.6]. Hence, (they are) also similar [Prop. 6.4, Def. 6.1]. Thus, angle $A B E$ is equal to (angle) $F G L$. And the whole (angle) $A B C$ is equal to the whole (angle) $F G H$, on account of the similarity of the polygons. Thus, the remaining angle $E B C$ is equal to $L G H$. And since, on account of the similarity of triangles $A B E$ and $F G L$, as $E B$ is to $B A$, so $L G$ (is) to $G F$, but also, on account of the similarity of the polygons, as $A B$ is to $B C$, so $F G$ (is) to $G H$, thus, via equality, as $E B$ is to $B C$, so $L G$ (is) to $G H$ [Prop. 5.22], and the sides about the equal angles, $E B C$ and $L G H$, are proportional. Thus, triangle $E B C$ is equiangular to triangle $L G H$ [Prop. 6.6]. Hence, triangle $E B C$ is also similar to triangle $L G H$ [Prop. 6.4, Def. 6.1]. So, for the same (reasons), triangle $E C D$ is also similar
tò $\pi \lambda \tilde{\eta} \vartheta \circ \varsigma$.


 ơtı tò $\mathrm{AB} \mathrm{\Gamma} \Delta \mathrm{E} \pi 0 \lambda u ́ \gamma \omega v o v ~ \pi \rho o ̀ s ~ t o ̀ ~ Z H \Theta K \Lambda ~ \pi o \lambda u ́ \gamma \omega v o v ~$












 $\pi \rho o ̀ \varsigma ~ M B, ~ o u ̈ \tau \omega \varsigma ~ \dot{\eta} \mathrm{ZN} \pi \rho o ̀ s \mathrm{NH}, \dot{\omega} \varsigma \delta \dot{\varepsilon} \dot{\eta} \mathrm{BM} \pi \rho o ̀ s ~ M \Gamma$,

 tò ABM [ $\tau \rho i \not \gamma \omega \nu o v]$ rpòs tò $\mathrm{MB} \mathrm{\Gamma}$, xaì tò AME rpòs tò


 тpír $\omega$ vov $\pi \rho o ̀ s ~ t o ̀ ~ B M \Gamma, ~ o u ̛ \tau \omega s ~ t o ̀ ~ A B E ~ \pi p o ̀ s ~ t o ̀ ~ Г В Е . ~ \alpha \lambda \lambda ' ~$






 трí $\omega$ vov $\pi \rho o ̀ s ~ \tau o ̀ ~ Z H \Lambda ~ \tau p i ́ \gamma \omega v o v, ~ o u ́ \tau \omega s ~ \tau o ̀ ~ B E \Gamma ~ \tau p i ́ \gamma \omega v o v ~$
 $\tau \widetilde{\omega} \nu \mathrm{B} \Delta$, HK , ötı xaì $\omega$ s tò $\mathrm{BE} \Gamma$ трí $\gamma \omega \nu$ ov $\pi \rho o ̀ s ~ t o ̀ ~ ~ \Lambda H \Theta ~$

 oútcus tò $\mathrm{EB} \mathrm{\Gamma}$ трòs tò $\Lambda \mathrm{H} \Theta$, xai हैtı тò $\mathrm{E} \Gamma \Delta$ тpòs tò $\Lambda \Theta \mathrm{K}$,



 ABE трí $\omega$ vov $\pi \rho o ̀ s ~ t o ̀ ~ Z H \Lambda ~ \tau p i ́ \gamma \omega \nu o v ~ \delta ı \pi \lambda \alpha \sigma i o v \alpha ~ \lambda o ́ \gamma o v ~$ है $\chi \varepsilon \iota ~ \grave{\eta} \pi \varepsilon \rho \dot{\eta} \mathrm{AB}$ ó $\mu o ́ \lambda o \gamma o s ~ \pi \lambda \varepsilon u p \alpha ̀ ~ \pi \rho o ̀ s ~ t \grave{\eta} \nu ~ Z H ~ o ́ \mu o ́ \lambda o \gamma o v ~$

 $\pi \rho o ̀ s ~ \tau o ̀ ~ Z H \Theta K \Lambda ~ \pi o \lambda u ́ \gamma \omega \nu o \nu ~ \delta ı \pi \lambda \alpha \sigma i o v \alpha ~ \lambda o ́ \gamma o v ~ ह ै \chi \varepsilon ı ~ \eta ̈ \pi \varepsilon \rho ~ \dot{~}$ AB ó $\mu o ́ \lambda o \gamma o s ~ \pi \lambda \varepsilon u p \alpha ̀ ~ \pi \rho o ̀ s ~ \tau \grave{\nu ~ Z H ~ o ́ \mu o ́ \lambda o \gamma o v ~ \pi \lambda \varepsilon u p \alpha ́ \nu . ~}$


to triangle $L H K$. Thus, the similar polygons $A B C D E$ and $F G H K L$ have been divided into equal numbers of similar triangles.

I also say that (the triangles) correspond (in proportion) to the wholes. That is to say, the triangles are proportional: $A B E, E B C$, and $E C D$ are the leading (magnitudes), and their (associated) following (magnitudes are) $F G L, L G H$, and $L H K$ (respectively). (I) also (say) that polygon $A B C D E$ has a squared ratio to polygon $F G H K L$ with respect to (that) a corresponding side (has) to a corresponding side-that is to say, (side) $A B$ to $F G$.

For let $A C$ and $F H$ have been joined. And since angle $A B C$ is equal to $F G H$, and as $A B$ is to $B C$, so $F G$ (is) to $G H$, on account of the similarity of the polygons, triangle $A B C$ is equiangular to triangle $F G H$ [Prop. 6.6]. Thus, angle $B A C$ is equal to $G F H$, and (angle) $B C A$ to $G H F$. And since angle $B A M$ is equal to $G F N$, and (angle) $A B M$ is also equal to $F G N$ (see earlier), the remaining (angle) $A M B$ is thus also equal to the remaining (angle) $F N G$ [Prop. 1.32]. Thus, triangle $A B M$ is equiangular to triangle $F G N$. So, similarly, we can show that triangle $B M C$ is also equiangular to triangle $G N H$. Thus, proportionally, as $A M$ is to $M B$, so $F N$ (is) to $N G$, and as $B M$ (is) to $M C$, so $G N$ (is) to $N H$ [Prop. 6.4]. Hence, also, via equality, as $A M$ (is) to $M C$, so $F N$ (is) to $N H$ [Prop. 5.22]. But, as $A M$ (is) to $M C$, so [triangle] $A B M$ is to $M B C$, and $A M E$ to $E M C$. For they are to one another as their bases [Prop. 6.1]. And as one of the leading (magnitudes) is to one of the following (magnitudes), so (the sum of) all the leading (magnitudes) is to (the sum of) all the following (magnitudes) [Prop. 5.12]. Thus, as triangle $A M B$ (is) to $B M C$, so (triangle) $A B E$ (is) to $C B E$. But, as (triangle) $A M B$ (is) to $B M C$, so $A M$ (is) to $M C$. Thus, also, as $A M$ (is) to $M C$, so triangle $A B E$ (is) to triangle $E B C$. And so, for the same (reasons), as $F N$ (is) to $N H$, so triangle $F G L$ (is) to triangle $G L H$. And as $A M$ is to $M C$, so $F N$ (is) to $N H$. Thus, also, as triangle $A B E$ (is) to triangle $B E C$, so triangle $F G L$ (is) to triangle $G L H$, and, alternately, as triangle $A B E$ (is) to triangle $F G L$, so triangle $B E C$ (is) to triangle $G L H$ [Prop. 5.16]. So, similarly, we can also show, by joining $B D$ and $G K$, that as triangle $B E C$ (is) to triangle $L G H$, so triangle $E C D$ (is) to triangle $L H K$. And since as triangle $A B E$ is to triangle $F G L$, so (triangle) $E B C$ (is) to $L G H$, and, further, (triangle) $E C D$ to $L H K$, and also as one of the leading (magnitudes is) to one of the following, so (the sum of) all the leading (magnitudes is) to (the sum of) all the following [Prop. 5.12], thus as triangle $A B E$ is to triangle $F G L$, so polygon $A B C D E$ (is) to polygon $F G H K L$. But, triangle $A B E$ has a squared ratio

 бєїそ $\alpha l]$.

## По́рьб $\alpha$.






$\chi \alpha^{\prime}$.
 ő $\mu$ оь .










to triangle $F G L$ with respect to (that) the corresponding side $A B$ (has) to the corresponding side $F G$. For, similar triangles are in the squared ratio of corresponding sides [Prop. 6.14]. Thus, polygon $A B C D E$ also has a squared ratio to polygon $F G H K L$ with respect to (that) the corresponding side $A B$ (has) to the corresponding side $F G$.

Thus, similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side. [(Which is) the very thing it was required to show].

## Corollary

And, in the same manner, it can also be shown for [similar] quadrilaterals that they are in the squared ratio of (their) corresponding sides. And it was also shown for triangles. Hence, in general, similar rectilinear figures are also to one another in the squared ratio of (their) corresponding sides. (Which is) the very thing it was required to show.

## Proposition 21

(Rectilinear figures) similar to the same rectilinear figure are also similar to one another.


Let each of the rectilinear figures $A$ and $B$ be similar to (the rectilinear figure) $C$. I say that $A$ is also similar to $B$.

For since $A$ is similar to $C,(A)$ is equiangular to $(C)$, and has the sides about the equal angles proportional [Def. 6.1]. Again, since $B$ is similar to $C,(B)$ is equiangular to $(C)$, and has the sides about the equal angles proportional [Def. 6.1]. Thus, $A$ and $B$ are each equiangular to $C$, and have the sides about the equal angles



$$
x \beta^{\prime} .
$$




 हैбovtal.

 $\mathrm{H} \Theta, \dot{\omega} \varsigma \dot{\eta} \mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \Delta$, oüt $\omega \varsigma \dot{\eta} \mathrm{EZ} \pi \rho o ̀ s ~ \tau \dot{\eta} \nu \mathrm{H} \Theta$, x $\alpha \grave{l}$



 $\mathrm{N} \Theta$.






 tò KAB тpòs tò $\Lambda \Gamma \Delta$, oút $\omega$ s tò MZ тpòs tò $\mathrm{N} \Theta$.





proportional [hence, $A$ is also equiangular to $B$, and has the sides about the equal angles proportional]. Thus, $A$ is similar to $B$ [Def. 6.1]. (Which is) the very thing it was required to show.

## Proposition 22

If four straight-lines are proportional then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional.


Let $A B, C D, E F$, and $G H$ be four proportional straight-lines, (such that) as $A B$ (is) to $C D$, so $E F$ (is) to $G H$. And let the similar, and similarly laid out, rectilinear figures $K A B$ and $L C D$ have been described on $A B$ and $C D$ (respectively), and the similar, and similarly laid out, rectilinear figures $M F$ and $N H$ on $E F$ and $G H$ (respectively). I say that as $K A B$ is to $L C D$, so $M F$ (is) to $N H$.

For let a third (straight-line) $O$ have been taken (which is) proportional to $A B$ and $C D$, and a third (straight-line) $P$ proportional to $E F$ and $G H$ [Prop. 6.11]. And since as $A B$ is to $C D$, so $E F$ (is) to $G H$, and as $C D$ (is) to $O$, so $G H$ (is) to $P$, thus, via equality, as $A B$ is to $O$, so $E F$ (is) to $P$ [Prop. 5.22]. But, as $A B$ (is) to $O$, so [also] $K A B$ (is) to $L C D$, and as $E F$ (is) to $P$, so $M F$ (is) to $N H$ [Prop. 5.19 corr.]. And, thus, as $K A B$ (is) to $L C D$, so $M F$ (is) to $N H$.

And so let $K A B$ be to $L C D$, as $M F$ (is) to $N H$. I say also that as $A B$ is to $C D$, so $E F$ (is) to $G H$. For if as $A B$ is to $C D$, so $E F$ (is) not to $G H$, let $A B$ be to $C D$, as $E F$
 عưษú रрациоь tò $\Sigma \mathrm{P}$ ．











 EZ $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{H} \Theta$ ．





（is）to $Q R$［Prop．6．12］．And let the rectilinear figure $S R$ ， similar，and similarly laid down，to either of $M F$ or $N H$ ， have been described on $Q R$［Props．6．18，6．21］．

Therefore，since as $A B$ is to $C D$ ，so $E F$（is）to $Q R$ ， and the similar，and similarly laid out，（rectilinear fig－ ures）$K A B$ and $L C D$ have been described on $A B$ and $C D$（respectively），and the similar，and similarly laid out， （rectilinear figures）$M F$ and $S R$ on $E F$ and $Q R$（re－ sespectively），thus as $K A B$ is to $L C D$ ，so $M F$（is）to $S R$（see above）．And it was also assumed that as $K A B$ （is）to $L C D$ ，so $M F$（is）to $N H$ ．Thus，also，as $M F$（is） to $S R$ ，so $M F$（is）to $N H$［Prop．5．11］．Thus，$M F$ has the same ratio to each of $N H$ and $S R$ ．Thus，$N H$ is equal to $S R$［Prop．5．9］．And it is also similar，and similarly laid out，to it．Thus，$G H$（is）equal to $Q R .^{\dagger}$ And since $A B$ is to $C D$ ，as $E F$（is）to $Q R$ ，and $Q R$（is）equal to $G H$ ，thus as $A B$ is to $C D$ ，so $E F$（is）to $G H$ ．

Thus，if four straight－lines are proportional，then sim－ ilar，and similarly described，rectilinear figures（drawn） on them will also be proportional．And if similar，and similarly described，rectilinear figures（drawn）on them are proportional then the straight－lines themselves will also be proportional．（Which is）the very thing it was required to show．
${ }^{\dagger}$ Here，Euclid assumes，without proof，that if two similar figures are equal then any pair of corresponding sides is also equal．

$$
x \gamma^{\prime}
$$

 тòv $\sigma \cup \gamma \varkappa \varepsilon i ́ \mu \varepsilon \nu o \nu$ ย̇x $\tau \widetilde{\omega} \nu \pi \lambda \varepsilon \cup p \widetilde{\omega} \nu$ ．










Oi äpa $\lambda o ́ \gamma o l ~ \tau n ̃ s ~ \tau \varepsilon ~ K ~ \pi p o ̀ s ~ \tau \eta ̀ \nu ~ \Lambda ~ x \alpha i ̀ ~ \tau n ̃ s ~ \Lambda ~ \pi p o ̀ s ~$

 M 入óүos $\sigma u ́ \gamma \varkappa \varepsilon เ \tau \alpha l ~ ह ै \chi ~ \tau \varepsilon ~ \tau о u ̃ ~ \tau ท ̃ \varsigma ~ K ~ \pi p o ̀ s ~ \Lambda ~ \lambda o ́ \gamma o u ~ \chi \alpha \grave{~}$


 $\Gamma \Theta, \alpha \dot{\alpha} \lambda \lambda^{\prime} \dot{\omega} \varsigma \dot{\eta} \mathrm{B} \Gamma$ трòs $\tau \dot{\eta} \nu \Gamma \mathrm{H}$ ，oüt $\omega \varsigma \dot{\eta} \mathrm{K}$ тpòs $\tau \grave{\eta} \nu \Lambda$ ，

 pa入入ク入óүpau

## Proposition 23

Equiangular parallelograms have to one another the ratio compounded ${ }^{\dagger}$ out of（the ratios of）their sides．

Let $A C$ and $C F$ be equiangular parallelograms having angle $B C D$ equal to $E C G$ ．I say that parallelogram $A C$ has to parallelogram $C F$ the ratio compounded out of （the ratios of）their sides．

For let $B C$ be laid down so as to be straight－on to $C G$ ．Thus，$D C$ is also straight－on to $C E$［Prop．1．14］． And let the parallelogram $D G$ have been completed．And let some straight－line $K$ have been laid down．And let it be contrived that as $B C$（is）to $C G$ ，so $K$（is）to $L$ ，and as $D C$（is）to $C E$ ，so $L$（is）to $M$［Prop．6．12］．

Thus，the ratios of $K$ to $L$ and of $L$ to $M$ are the same as the ratios of the sides，（namely），$B C$ to $C G$ and $D C$ to $C E$（respectively）．But，the ratio of $K$ to $M$ is com－ pounded out of the ratio of $K$ to $L$ and（the ratio）of $L$ to $M$ ．Hence，$K$ also has to $M$ the ratio compounded out of（the ratios of）the sides（of the parallelograms）． And since as $B C$ is to $C G$ ，so parallelogram $A C$（is）to $C H$［Prop．6．1］，but as $B C$（is）to $C G$ ，so $K$（is）to $L$ ， thus，also，as $K$（is）to $L$ ，so（parallelogram）$A C$（is）to $C H$ ．Again，since as $D C$（is）to $C E$ ，so parallelogram
 tò $\Gamma \Theta \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu$ тpòs tò $\Gamma Z \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o v$. غ̇л
 $\dot{\eta} \Lambda \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ M, ~ o u ̛ \tau \omega s ~ \tau o ̀ ~ \Gamma \Theta ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu ~ \pi p o ̀ s ~ \tau o ̀ ~$



 $\tau \widetilde{\omega} \nu \pi \lambda \varepsilon \cup \rho \widetilde{\omega} \nu$.



$C H$ (is) to $C F$ [Prop. 6.1], but as $D C$ (is) to $C E$, so $L$ (is) to $M$, thus, also, as $L$ (is) to $M$, so parallelogram $C H$ (is) to parallelogram $C F$. Therefore, since it was shown that as $K$ (is) to $L$, so parallelogram $A C$ (is) to parallelogram $C H$, and as $L$ (is) to $M$, so parallelogram $C H$ (is) to parallelogram $C F$, thus, via equality, as $K$ is to $M$, so (parallelogram) $A C$ (is) to parallelogram $C F$ [Prop. 5.22]. And $K$ has to $M$ the ratio compounded out of (the ratios of) the sides (of the parallelograms). Thus, (parallelogram) $A C$ also has to (parallelogram) $C F$ the ratio compounded out of (the ratio of) their sides.


Thus, equiangular parallelograms have to one another the ratio compounded out of (the ratio of) their sides. (Which is) the very thing it was required to show.
† In modern terminology, if two ratios are "compounded" then they are multiplied together.

## $x \delta^{\prime}$.

Паvтòs $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ оүра́ $\mu \mu о \cup \tau \grave{\alpha} \pi \varepsilon p i ̀ \tau \grave{\eta} \nu \delta \iota \alpha ́ \mu \varepsilon \tau \rho о \nu \pi \alpha-$


 ӨK• $\lambda \varepsilon ́ \gamma \omega$, őтı غ́x $\alpha \tau \varepsilon \rho о \nu \tau \widetilde{\omega} \nu \mathrm{EH}, \Theta \mathrm{K} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega \nu$

'Eлєi $\gamma \alpha ̀ \rho ~ \tau \rho ı \gamma \omega ́ v o u ~ \tau o u ̃ ~ A B \Gamma ~ \pi \alpha p \grave{\alpha} \mu i \alpha \alpha \nu \tau \widetilde{\omega} \nu \pi \lambda \varepsilon \cup \rho \widetilde{\omega} \nu$







 EA $\pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ А Н . ~ \tau \widetilde{\omega} \nu \alpha \not \rho \alpha ~ А В Г \Delta, ~ Е Н ~ \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega \nu$




## Proposition 24

In any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another.

Let $A B C D$ be a parallelogram, and $A C$ its diagonal. And let $E G$ and $H K$ be parallelograms about $A C$. I say that the parallelograms $E G$ and $H K$ are each similar to the whole (parallelogram) $A B C D$, and to one another.

For since $E F$ has been drawn parallel to one of the sides $B C$ of triangle $A B C$, proportionally, as $B E$ is to $E A$, so $C F$ (is) to $F A$ [Prop. 6.2]. Again, since $F G$ has been drawn parallel to one (of the sides) $C D$ of triangle $A C D$, proportionally, as $C F$ is to $F A$, so $D G$ (is) to $G A$ [Prop. 6.2]. But, as $C F$ (is) to $F A$, so it was also shown (is) $B E$ to $E A$. And thus as $B E$ (is) to $E A$, so $D G$ (is) to $G A$. And, thus, compounding, as $B A$ (is) to $A E$, so $D A$ (is) to $A G$ [Prop. 5.18]. And, alternately, as $B A$ (is) to $A D$, so $E A$ (is) to $A G$ [Prop. 5.16]. Thus, in parallelograms $A B C D$ and $E G$ the sides about the common angle $B A D$ are proportional. And since $G F$ is parallel to $D C$, angle $A F G$ is equal to $D C A$ [Prop. 1.29].


 каі ö入ov tò АВГవ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu \tau \widetilde{\varphi} \mathrm{EH} \pi \alpha \rho \alpha \lambda \lambda \eta$ -


 $\Gamma \mathrm{B}$, oüt





 АВГ $\Delta \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma р \alpha ́ \mu \mu о \nu \tau \widetilde{\varphi}$ ЕН т $\pi \alpha \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$. $\delta \dot{\alpha}$

 $\pi \alpha p \alpha \lambda \lambda \eta \lambda о \gamma р \alpha ́ \mu \mu \omega \nu \tau \widetilde{\mu}$ АВГ $\Delta$ [ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega]$ ö $\mu о$ เóv
 ö $\mu o \alpha \alpha \cdot x \alpha i$ тò EH $\alpha p \alpha \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu \tau \widetilde{\varphi} \Theta K \pi \alpha p \alpha \lambda \lambda \eta$ -






## $\chi \varepsilon^{\prime}$.

 そ้ซov tò $\alpha$ đitò $\sigma \cup \sigma \tau ท ́ \sigma \alpha \sigma \vartheta \alpha$.

And angle $D A C$ (is) common to the two triangles $A D C$ and $A G F$. Thus, triangle $A D C$ is equiangular to triangle $A G F$ [Prop. 1.32]. So, for the same (reasons), triangle $A C B$ is equiangular to triangle $A F E$, and the whole parallelogram $A B C D$ is equiangular to parallelogram $E G$. Thus, proportionally, as $A D$ (is) to $D C$, so $A G$ (is) to $G F$, and as $D C$ (is) to $C A$, so $G F$ (is) to $F A$, and as $A C$ (is) to $C B$, so $A F$ (is) to $F E$, and, further, as $C B$ (is) to $B A$, so $F E$ (is) to $E A$ [Prop. 6.4]. And since it was shown that as $D C$ is to $C A$, so $G F$ (is) to $F A$, and as $A C$ (is) to $C B$, so $A F$ (is) to $F E$, thus, via equality, as $D C$ is to $C B$, so $G F$ (is) to $F E$ [Prop. 5.22]. Thus, in parallelograms $A B C D$ and $E G$ the sides about the equal angles are proportional. Thus, parallelogram $A B C D$ is similar to parallelogram $E G$ [Def. 6.1]. So, for the same (reasons), parallelogram $A B C D$ is also similar to parallelogram $K H$. Thus, parallelograms $E G$ and $H K$ are each similar to [parallelogram] $A B C D$. And (rectilinear figures) similar to the same rectilinear figure are also similar to one another [Prop. 6.21]. Thus, parallelogram $E G$ is also similar to parallelogram $H K$.


Thus, in any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another. (Which is) the very thing it was required to show.

## Proposition 25

To construct a single (rectilinear figure) similar to a given rectilinear figure, and equal to a different given rectilinear figure.





 $\pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu ~ \tau o ̀ ~ Г М ~ ह ُ ้ ~ \gamma \omega v i ́ \alpha ~ \tau \tilde{n} ~ u ́ \pi o ̀ ~ Z Г E, ~ \eta ̈ ~ \varepsilon ̇ \sigma \tau \iota ~$








 АВГ трír $\omega$ vov $\pi \rho o ̀ s ~ t o ̀ ~ K H \Theta ~ \tau р i ́ \gamma \omega v o v . ~ \alpha ̀ \lambda \lambda \alpha ̀ ~ x \alpha i ̀ ~ \omega ̀ s ~ \dot{\eta}$ В $\pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ Г Z, ~ o u ́ \tau \omega s ~ \tau o ̀ ~ B E ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p a \mu \mu o v ~ \pi p o ̀ s ~ \tau o ̀ ~ E Z ~$





 $\tau \widetilde{\varphi} \mathrm{EZ} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \alpha \mu \varphi$, . $\alpha \lambda \lambda \dot{\alpha}$ тò $\mathrm{EZ} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu$



 हैठ์เ $\pi \circ เ ท ั \sigma \alpha l$.

## $\chi G^{\prime}$.

${ }^{\circ} \mathrm{E} \alpha \nu \dot{\alpha} \pi o ̀ ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu о \cup \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu о \nu \dot{\alpha} \varphi \alpha l-$







Let $A B C$ be the given rectilinear figure to which it is required to construct a similar (rectilinear figure), and $D$ the (rectilinear figure) to which (the constructed figure) is required (to be) equal. So it is required to construct a single (rectilinear figure) similar to $A B C$, and equal to D.

For let the parallelogram $B E$, equal to triangle $A B C$, have been applied to (the straight-line) $B C$ [Prop. 1.44], and the parallelogram $C M$, equal to $D$, (have been applied) to (the straight-line) $C E$, in the angle $F C E$, which is equal to $C B L$ [Prop. 1.45]. Thus, $B C$ is straight-on to $C F$, and $L E$ to $E M$ [Prop. 1.14]. And let the mean proportion $G H$ have been taken of $B C$ and $C F$ [Prop. 6.13]. And let $K G H$, similar, and similarly laid out, to $A B C$ have been described on $G H$ [Prop. 6.18].

And since as $B C$ is to $G H$, so $G H$ (is) to $C F$, and if three straight-lines are proportional then as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.], thus as $B C$ is to $C F$, so triangle $A B C$ (is) to triangle $K G H$. But, also, as $B C$ (is) to $C F$, so parallelogram $B E$ (is) to parallelogram $E F$ [Prop. 6.1]. And, thus, as triangle $A B C$ (is) to triangle $K G H$, so parallelogram $B E$ (is) to parallelogram $E F$. Thus, alternately, as triangle $A B C$ (is) to parallelogram $B E$, so triangle $K G H$ (is) to parallelogram $E F$ [Prop. 5.16]. And triangle $A B C$ (is) equal to parallelogram $B E$. Thus, triangle $K G H$ (is) also equal to parallelogram $E F$. But, parallelogram $E F$ is equal to $D$. Thus, $K G H$ is also equal to $D$. And $K G H$ is also similar to $A B C$.

Thus, a single (rectilinear figure) $K G H$ has been constructed (which is) similar to the given rectilinear figure $A B C$, and equal to a different given (rectilinear figure) $D$. (Which is) the very thing it was required to do.

## Proposition 26

If from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole.

For, from parallelogram $A B C D$, let (parallelogram)








 «$p \alpha \dot{\eta}$ HA $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu$ AK, oưt $\omega \varsigma \dot{\eta}$ HA $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu$ AE. $\dot{\eta}$ HA



 $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о ́ \gamma \rho \alpha \mu \mu о \nu \tau \widetilde{\varphi} \mathrm{AZ} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega$.





$$
x \zeta^{\prime} .
$$




 $\beta \alpha \lambda \lambda o ́ \mu \varepsilon v o v[\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o \nu]$ ö $\mu o เ o \nu$ oैv $\tau \widetilde{̣}$ ह̀ $\lambda \lambda \varepsilon \varepsilon^{\prime} \mu \mu \alpha \nu \tau \iota$.



 o้тו $\pi \alpha \dot{\nu} \tau \omega \nu \tau \widetilde{\omega} \nu \tau \alpha \rho \alpha ̀ ~ \tau \grave{\nu} \nu \mathrm{AB} \pi \alpha \rho \alpha \beta \alpha \lambda \lambda о \mu \varepsilon ́ v \omega \nu \pi \alpha \rho \alpha \lambda \lambda \eta$ -


$A F$ have been subtracted (which is) similar, and similarly laid out, to $A B C D$, having the common angle $D A B$ with it. I say that $A B C D$ is about the same diagonal as $A F$.


For (if) not, then, if possible, let $A H C$ be [ $A B C D$ 's] diagonal. And producing $G F$, let it have been drawn through to (point) $H$. And let $H K$ have been drawn through (point) $H$, parallel to either of $A D$ or $B C$ [Prop. 1.31].

Therefore, since $A B C D$ is about the same diagonal as $K G$, thus as $D A$ is to $A B$, so $G A$ (is) to $A K$ [Prop. 6.24]. And, on account of the similarity of $A B C D$ and $E G$, also, as $D A$ (is) to $A B$, so $G A$ (is) to $A E$. Thus, also, as $G A$ (is) to $A K$, so $G A$ (is) to $A E$. Thus, $G A$ has the same ratio to each of $A K$ and $A E$. Thus, $A E$ is equal to $A K$ [Prop. 5.9], the lesser to the greater. The very thing is impossible. Thus, $A B C D$ is not not about the same diagonal as $A F$. Thus, parallelogram $A B C D$ is about the same diagonal as parallelogram $A F$.

Thus, if from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole. (Which is) the very thing it was required to show.

## Proposition 27

Of all the parallelograms applied to the same straightline, and falling short by parallelogrammic figures similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line) which (is) similar to (that parallelogram) by which it falls short.

Let $A B$ be a straight-line, and let it have been cut in half at (point) $C$ [Prop. 1.10]. And let the parallelogram $A D$ have been applied to the straight-line $A B$, falling short by the parallelogrammic figure $D B$ (which is) applied to half of $A B$-that is to say, $C B$. I say that of all the parallelograms applied to $A B$, and falling short by


 тò $\mathrm{A} \Delta$ тoṽ AZ .
















$$
x n^{\prime} .
$$











[parallelogrammic] figures similar, and similarly laid out, to $D B$, the greatest is $A D$. For let the parallelogram $A F$ have been applied to the straight-line $A B$, falling short by the parallelogrammic figure $F B$ (which is) similar, and similarly laid out, to $D B$. I say that $A D$ is greater than $A F$.


For since parallelogram $D B$ is similar to parallelogram $F B$, they are about the same diagonal [Prop. 6.26]. Let their (common) diagonal $D B$ have been drawn, and let the (rest of the) figure have been described.

Therefore, since (complement) $C F$ is equal to (complement) $F E$ [Prop. 1.43], and (parallelogram) $F B$ is common, the whole (parallelogram) CH is thus equal to the whole (parallelogram) $K E$. But, (parallelogram) $C H$ is equal to $C G$, since $A C$ (is) also (equal) to $C B$ [Prop. 6.1]. Thus, (parallelogram) $G C$ is also equal to $E K$. Let (parallelogram) $C F$ have been added to both. Thus, the whole (parallelogram) $A F$ is equal to the gnomon $L M N$. Hence, parallelogram $D B$-that is to say, $A D$-is greater than parallelogram $A F$.

Thus, for all parallelograms applied to the same straight-line, and falling short by a parallelogrammic figure similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line). (Which is) the very thing it was required to show.

## Proposition $28^{\dagger}$

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) falling short by a parallelogrammic figure similar to a given (parallelogram). It is necessary for the given rectilinear figure [to which it is required to apply an equal (parallelogram)] not to be greater than the (parallelogram) described on half (of the straight-line) and similar to the deficit.

Let $A B$ be the given straight-line, and $C$ the given rectilinear figure to which the (parallelogram) applied to





 тò EBZH, каі $\sigma \cup \mu \pi \varepsilon \pi \lambda \eta \rho \omega \dot{\omega} \vartheta \omega$ тò $\mathrm{AH} \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu$.












 $\dot{\eta} \mathrm{HO}, x \alpha i$ би $\pi \tau \varepsilon \pi \lambda \eta \rho \omega ́ \sigma \vartheta \omega$ тò $\Xi Н О П ~ \pi \alpha р \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о v . ~$
















$A B$ is required (to be) equal, [being] not greater than the (parallelogram) described on half of $A B$ and similar to the deficit, and $D$ the (parallelogram) to which the deficit is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure $C$, to the straight-line $A B$, falling short by a parallelogrammic figure which is similar to $D$.


Let $A B$ have been cut in half at point $E$ [Prop. 1.10], and let (parallelogram) $E B F G$, (which is) similar, and similarly laid out, to (parallelogram) $D$, have been described on $E B$ [Prop. 6.18]. And let parallelogram $A G$ have been completed.

Therefore, if $A G$ is equal to $C$ then the thing prescribed has happened. For a parallelogram $A G$, equal to the given rectilinear figure $C$, has been applied to the given straight-line $A B$, falling short by a parallelogrammic figure $G B$ which is similar to $D$. And if not, let $H E$ be greater than $C$. And $H E$ (is) equal to $G B$ [Prop. 6.1]. Thus, $G B$ (is) also greater than $C$. So, let (parallelogram) $K L M N$ have been constructed (so as to be) both similar, and similarly laid out, to $D$, and equal to the excess by which $G B$ is greater than $C$ [Prop. 6.25]. But, $G B$ [is] similar to $D$. Thus, $K M$ is also similar to $G B$ [Prop. 6.21]. Therefore, let $K L$ correspond to $G E$, and $L M$ to $G F$. And since (parallelogram) $G B$ is equal to (figure) $C$ and (parallelogram) $K M, G B$ is thus greater than $K M$. Thus, $G E$ is also greater than $K L$, and $G F$ than $L M$. Let $G O$ be made equal to $K L$, and $G P$ to $L M$ [Prop. 1.3]. And let the parallelogram $O G P Q$ have been completed. Thus, [GQ] is equal and similar to $K M$ [but, $K M$ is similar to $G B]$. Thus, $G Q$ is also similar to $G B$ [Prop. 6.21]. Thus, $G Q$ and $G B$ are about the same diagonal [Prop. 6.26]. Let $G Q B$ be their (common) diagonal, and let the (remainder of the) figure have been described.

Therefore, since $B G$ is equal to $C$ and $K M$, of which $G Q$ is equal to $K M$, the remaining gnomon $U W V$ is thus equal to the remainder $C$. And since (the complement) $P R$ is equal to (the complement) $O S$ [Prop. 1.43], let (parallelogram) $Q B$ have been added to both. Thus, the whole (parallelogram) $P B$ is equal to the whole (par-
 $\pi o เ \eta ̃ \sigma \alpha$.
allelogram) $O B$. But, $O B$ is equal to $T E$, since side $A E$ is equal to side $E B$ [Prop. 6.1]. Thus, $T E$ is also equal to $P B$. Let (parallelogram) $O S$ have been added to both. Thus, the whole (parallelogram) $T S$ is equal to the gnomon $V W U$. But, gnomon $V W U$ was shown (to be) equal to $C$. Therefore, (parallelogram) $T S$ is also equal to (figure) $C$.

Thus, the parallelogram $S T$, equal to the given rectilinear figure $C$, has been applied to the given straightline $A B$, falling short by the parallelogrammic figure $Q B$, which is similar to $D$ [inasmuch as $Q B$ is similar to $G Q$ [Prop. 6.24]]. (Which is) the very thing it was required to do.


#### Abstract

${ }^{\dagger}$ This proposition is a geometric solution of the quadratic equation $x^{2}-\alpha x+\beta=0$. Here, $x$ is the ratio of a side of the deficit to the corresponding side of figure $D, \alpha$ is the ratio of the length of $A B$ to the length of that side of figure $D$ which corresponds to the side of the deficit running along $A B$, and $\beta$ is the ratio of the areas of figures $C$ and $D$. The constraint corresponds to the condition $\beta<\alpha^{2} / 4$ for the equation to have real roots. Only the smaller root of the equation is found. The larger root can be found by a similar method.


## $\chi \vartheta^{\prime}$.


 $\rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$ о́ $\mu$ oí $\varphi$ т $\widetilde{\varphi}$ бо才є́vть.













 $\dot{\eta}$ ZEN, $\chi \alpha \grave{~} \sigma \cup \mu \pi \varepsilon \pi \lambda \eta \rho \omega ́ \sigma \vartheta \omega$ tò MN . тò MN 关 $\alpha \alpha \tau \widetilde{\varphi} \mathrm{H}$


## Proposition $29^{\dagger}$

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) overshooting by a parallelogrammic figure similar to a given (parallelogram).


Let $A B$ be the given straight-line, and $C$ the given rectilinear figure to which the (parallelogram) applied to $A B$ is required (to be) equal, and $D$ the (parallelogram) to which the excess is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure $C$, to the given straight-line $A B$, overshooting by a parallelogrammic figure similar to $D$.

Let $A B$ have been cut in half at (point) $E$ [Prop. 1.10], and let the parallelogram $B F$, (which is) similar, and similarly laid out, to $D$, have been described on $E B$ [Prop. 6.18]. And let (parallelogram) $G H$ have been constructed (so as to be) both similar, and similarly laid out, to $D$, and equal to the sum of $B F$ and $C$ [Prop. 6.25]. And let $K H$ correspond to $F L$, and $K G$ to $F E$. And since (parallelogram) $G H$ is greater than (parallelogram) $F B$,

 Z $\Xi$, xai $\chi \alpha \tau \alpha \gamma \varepsilon \gamma \rho \alpha ́ \varphi \vartheta \vartheta \omega$ tò $\sigma \chi \tilde{\eta} \mu \alpha$.


 そ̈бos. xal غ̇л






 $\pi o เ \eta ̃ \sigma \alpha l$.
$K H$ is thus also greater than $F L$, and $K G$ than $F E$. Let $F L$ and $F E$ have been produced, and let $F L M$ be (made) equal to $K H$, and $F E N$ to $K G$ [Prop. 1.3]. And let (parallelogram) $M N$ have been completed. Thus, $M N$ is equal and similar to $G H$. But, $G H$ is similar to $E L$. Thus, $M N$ is also similar to $E L$ [Prop. 6.21]. $E L$ is thus about the same diagonal as $M N$ [Prop. 6.26]. Let their (common) diagonal $F O$ have been drawn, and let the (remainder of the) figure have been described.

And since (parallelogram) $G H$ is equal to (parallelogram) $E L$ and (figure) $C$, but $G H$ is equal to (parallelogram) $M N, M N$ is thus also equal to $E L$ and $C$. Let $E L$ have been subtracted from both. Thus, the remaining gnomon $X W V$ is equal to (figure) $C$. And since $A E$ is equal to $E B$, (parallelogram) $A N$ is also equal to (parallelogram) $N B$ [Prop. 6.1], that is to say, (parallelogram) $L P$ [Prop. 1.43]. Let (parallelogram) $E O$ have been added to both. Thus, the whole (parallelogram) $A O$ is equal to the gnomon $V W X$. But, the gnomon $V W X$ is equal to (figure) $C$. Thus, (parallelogram) $A O$ is also equal to (figure) $C$.

Thus, the parallelogram $A O$, equal to the given rectilinear figure $C$, has been applied to the given straightline $A B$, overshooting by the parallelogrammic figure $Q P$ which is similar to $D$, since $P Q$ is also similar to $E L$ [Prop. 6.24]. (Which is) the very thing it was required to do.

[^12]
## $\lambda^{\prime}$.

入óүоข тєцعіั.




## Proposition $30^{\dagger}$

To cut a given finite straight-line in extreme and mean ratio.


Let $A B$ be the given finite straight-line. So it is required to cut the straight-line $A B$ in extreme and mean




Tعтрáү(c)






 $\dot{\eta} \mathrm{AE} \tau \tilde{n} \mathrm{E} \mathrm{EB}$.

 тоıท̃баи.
${ }^{\dagger}$ This method of cutting a straight-line is sometimes called the "Golden Section"-see Prop. 2.11.
$\lambda \alpha^{\prime}$.






 а́vaүpapouévols.









ratio.
Let the square $B C$ have been described on $A B$ [Prop. 1.46], and let the parallelogram $C D$, equal to $B C$, have been applied to $A C$, overshooting by the figure $A D$ (which is) similar to $B C$ [Prop. 6.29].

And $B C$ is a square. Thus, $A D$ is also a square. And since $B C$ is equal to $C D$, let (rectangle) $C E$ have been subtracted from both. Thus, the remaining (rectangle) $B F$ is equal to the remaining (square) $A D$. And it is also equiangular to it. Thus, the sides of $B F$ and $A D$ about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as $F E$ is to $E D$, so $A E$ (is) to $E B$. And $F E$ (is) equal to $A B$, and $E D$ to $A E$. Thus, as $B A$ is to $A E$, so $A E$ (is) to $E B$. And $A B$ (is) greater than $A E$. Thus, $A E$ (is) also greater than $E B$ [Prop. 5.14].

Thus, the straight-line $A B$ has been cut in extreme and mean ratio at $E$, and $A E$ is its greater piece. (Which is) the very thing it was required to do.

## Proposition 31

In right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle.


Let $A B C$ be a right-angled triangle having the angle $B A C$ a right-angle. I say that the figure (drawn) on $B C$ is equal to the (sum of the) similar, and similarly described, figures on $B A$ and $A C$.

Let the perpendicular $A D$ have been drawn [Prop. 1.12].

Therefore, since, in the right-angled triangle $A B C$, the (straight-line) $A D$ has been drawn from the rightangle at $A$ perpendicular to the base $B C$, the triangles $A B D$ and $A D C$ about the perpendicular are similar to the whole (triangle) $A B C$, and to one another [Prop. 6.8]. And since $A B C$ is similar to $A B D$, thus














$$
\lambda \beta^{\prime}
$$






"E $\sigma \tau \omega$ ठúo $\tau p i ́ \gamma \omega \nu \alpha$ $\tau \alpha ̀ ~ А В \Gamma, ~, ~ Г Г Е ~ \tau \grave{\alpha} \varsigma ~ \delta v ́ o ~ \pi \lambda \varepsilon u p \alpha ̀ \varsigma ~ \tau \grave{\alpha} \varsigma$










as $C B$ is to $B A$, so $A B$ (is) to $B D$ [Def. 6.1]. And since three straight-lines are proportional, as the first is to the third, so the figure (drawn) on the first is to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. Thus, as $C B$ (is) to $B D$, so the figure (drawn) on $C B$ (is) to the similar, and similarly described, (figure) on $B A$. And so, for the same (reasons), as $B C$ (is) to $C D$, so the figure (drawn) on $B C$ (is) to the (figure) on $C A$. Hence, also, as $B C$ (is) to $B D$ and $D C$, so the figure (drawn) on $B C$ (is) to the (sum of the) similar, and similarly described, (figures) on $B A$ and $A C$ [Prop. 5.24]. And $B C$ is equal to $B D$ and $D C$. Thus, the figure (drawn) on $B C$ (is) also equal to the (sum of the) similar, and similarly described, figures on $B A$ and $A C$ [Prop. 5.9].

Thus, in right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle. (Which is) the very thing it was required to show.

## Proposition 32

If two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another).


Let $A B C$ and $D C E$ be two triangles having the two sides $B A$ and $A C$ proportional to the two sides $D C$ and $D E$-so that as $A B$ (is) to $A C$, so $D C$ (is) to $D E$-and (having side) $A B$ parallel to $D C$, and $A C$ to $D E$. I say that (side) $B C$ is straight-on to $C E$.

For since $A B$ is parallel to $D C$, and the straight-line $A C$ has fallen across them, the alternate angles $B A C$ and $A C D$ are equal to one another [Prop. 1.29]. So, for the same (reasons), $C D E$ is also equal to $A C D$. And, hence, $B A C$ is equal to $C D E$. And since $A B C$ and $D C E$ are two triangles having the one angle at $A$ equal to the one





 Ú $\pi o ̀ ~ В А Г, ~ А Г В, ~ Г В А ~ 彳 ̛ \sigma \alpha ı ~ \varepsilon i \sigma i ́ v . ~ \alpha ̀ \lambda \lambda ’ ~ \alpha i ~ \cup ́ \pi o ̀ ~ В А Г, ~ А В Г, ~$




 $\Gamma$ Е.





$$
\lambda \gamma^{\prime}
$$






 $\mathrm{E} \Theta \mathrm{Z}, \pi \rho o ̀ s ~ \delta e ̀ ~ \tau \alpha i ̃ ̆ ~ \pi \varepsilon p l \varphi \varepsilon p \varepsilon i \alpha l s ~ \alpha i ~ u ́ \pi o ̀ ~ B A Г, ~ E \Delta Z \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~$

 $\mathrm{BA} \Gamma$ тpòs tì̀ Ú $\pi$ ò $\mathrm{E} \Delta \mathrm{Z}$.

 ঠ$\eta \pi о \tau o u ̃ \nu ~ \alpha i ~ Z M, ~ M N, ~ \varkappa \alpha i ~ \varepsilon ่ \pi \varepsilon \zeta \varepsilon u ́ \chi \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ H K, ~ H \Lambda, ~ \Theta M, ~$ $\Theta \mathrm{N}$.




angle at $D$, and the sides about the equal angles proportional, (so that) as $B A$ (is) to $A C$, so $C D$ (is) to $D E$, triangle $A B C$ is thus equiangular to triangle $D C E$ [Prop. 6.6]. Thus, angle $A B C$ is equal to $D C E$. And (angle) $A C D$ was also shown (to be) equal to $B A C$. Thus, the whole (angle) $A C E$ is equal to the two (angles) $A B C$ and $B A C$. Let $A C B$ have been added to both. Thus, $A C E$ and $A C B$ are equal to $B A C, A C B$, and $C B A$. But, $B A C, A B C$, and $A C B$ are equal to two right-angles [Prop. 1.32]. Thus, $A C E$ and $A C B$ are also equal to two right-angles. Thus, the two straight-lines $B C$ and $C E$, not lying on the same side, make adjacent angles $A C E$ and $A C B$ (whose sum is) equal to two right-angles with some straight-line $A C$, at the point $C$ on it. Thus, $B C$ is straight-on to $C E$ [Prop. 1.14].

Thus, if two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another). (Which is) the very thing it was required to show.

## Proposition 33

In equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences.


Let $A B C$ and $D E F$ be equal circles, and let $B G C$ and $E H F$ be angles at their centers, $G$ and $H$ (respectively), and $B A C$ and $E D F$ (angles) at their circumferences. I say that as circumference $B C$ is to circumference $E F$, so angle $B G C$ (is) to $E H F$, and (angle) $B A C$ to $E D F$.

For let any number whatsoever of consecutive (circumferences), $C K$ and $K L$, be made equal to circumference $B C$, and any number whatsoever, $F M$ and $M N$, to circumference $E F$. And let $G K, G L, H M$, and $H N$ have been joined.

Therefore, since circumferences $B C, C K$, and $K L$ are equal to one another, angles $B G C, C G K$, and $K G L$ are also equal to one another [Prop. 3.27]. Thus, as many times as circumference $B L$ is (divisible) by $B C$, so many














 ВНГ $\gamma \omega v i ́ \alpha ~ \pi \rho o ̀ s ~ \tau \grave{\eta} v ~ u ́ \pi o ̀ ~ E \Theta Z . ~ \alpha ̀ \lambda \lambda ’ ~ \dot{\omega} \varsigma ~ \dot{\eta}$ ú $\pi o ̀ ~ B H \Gamma ~ \gamma \omega v i ́ \alpha ~$

 $\pi \rho o ̀ \varsigma ~ \tau \eta ̀ \nu ~ E Z ~ \pi \varepsilon \rho เ \varphi \varepsilon ́ p \varepsilon ı \alpha \nu, ~ o u ́ \tau \omega \varsigma ~ \eta ั ~ \tau \varepsilon ~ u ́ \pi o ̀ ~ В Н Г ~ \gamma \omega v i ́ \alpha ~ \pi \rho o ̀ \varsigma ~$





times is angle $B G L$ also (divisible) by $B G C$. And so, for the same (reasons), as many times as circumference $N E$ is (divisible) by $E F$, so many times is angle $N H E$ also (divisible) by $E H F$. Thus, if circumference $B L$ is equal to circumference $E N$ then angle $B G L$ is also equal to $E H N$ [Prop. 3.27], and if circumference $B L$ is greater than circumference $E N$ then angle $B G L$ is also greater than $E H N,^{\dagger}$ and if ( $B L$ is) less (than $E N$ then $B G L$ is also) less (than $E H N$ ). So there are four magnitudes, two circumferences $B C$ and $E F$, and two angles $B G C$ and $E H F$. And equal multiples have been taken of circumference $B C$ and angle $B G C$, (namely) circumference $B L$ and angle $B G L$, and of circumference $E F$ and angle $E H F$, (namely) circumference $E N$ and angle $E H N$. And it has been shown that if circumference $B L$ exceeds circumference $E N$ then angle $B G L$ also exceeds angle $E H N$, and if ( $B L$ is) equal (to $E N$ then $B G L$ is also) equal (to $E H N$ ), and if ( $B L$ is) less (than $E N$ then $B G L$ is also) less (than $E H N$ ). Thus, as circumference $B C$ (is) to $E F$, so angle $B G C$ (is) to $E H F$ [Def. 5.5]. But as angle $B G C$ (is) to $E H F$, so (angle) $B A C$ (is) to $E D F$ [Prop. 5.15]. For the former (are) double the latter (respectively) [Prop. 3.20]. Thus, also, as circumference $B C$ (is) to circumference $E F$, so angle $B G C$ (is) to $E H F$, and $B A C$ to $E D F$.

Thus, in equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences. (Which is) the very thing it was required to show.

[^13]
## ELEMENTS BOOK 7

## Elementary Number Theory

[^14]
## "Opor.





$\delta^{\prime}$. Мép $\delta$ 它, öт $\alpha \nu \mu \grave{\eta} \chi \alpha \tau \alpha \mu \varepsilon \tau \rho \tilde{\eta}$.







 $\mu \varepsilon \tau \rho о$ и́ $\varepsilon$ vos $\chi \alpha \tau \grave{\alpha} \pi \varepsilon \rho เ \sigma \sigma o ̀ v ~ \grave{\alpha} \rho เ \vartheta \mu o ́ v . ~$









 $\lambda \alpha \pi \lambda \alpha \sigma \iota \zeta$ ó $\mu \varepsilon \nu о \varsigma$, , кגi $\gamma \varepsilon ́ v \eta \tau \alpha i ́ ~ \tau ı \varsigma . ~$















 $\stackrel{\omega}{\omega}$

## Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.
2. And a number (is) a multitude composed of units. ${ }^{\dagger}$
3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater. ${ }^{\ddagger}$
4. But (the lesser is) parts (of the greater) when it does not measure it. ${ }^{\S}$
5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.
6. An even number is one (which can be) divided in half.
7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.
8. An even-times-even number is one (which is) measured by an even number according to an even number. ${ }^{\top}$
9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.*
10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number. ${ }^{\$}$
11. A prime ${ }^{\|}$number is one (which is) measured by a unit alone.
12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.
13. A composite number is one (which is) measured by some number.
14. And numbers composite to one another are those (which are) measured by some number as a common measure.
15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.
16. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.
17. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.
18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.
19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.
20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.
21. Similar plane and solid numbers are those having proportional sides.
22. A perfect number is that which is equal to its own parts. ${ }^{\dagger \dagger}$
$\dagger$ In other words, a "number" is a positive integer greater than unity.
$\ddagger$ In other words, a number $a$ is part of another number $b$ if there exists some number $n$ such that $n a=b$.
§ In other words, a number $a$ is parts of another number $b$ (where $a<b$ ) if there exist distinct numbers, $m$ and $n$, such that $n a=m b$.
${ }^{\text {4 }}$ In other words. an even-times-even number is the product of two even numbers.

* In other words, an even-times-odd number is the product of an even and an odd number.
\$ In other words, an odd-times-odd number is the product of two odd numbers.
|| Literally, "first".
${ }^{\dagger \dagger}$ In other words, a perfect number is equal to the sum of its own factors.

$$
\alpha^{\prime}
$$






$\Delta u ́ o ~ \gamma \dot{\alpha} p[\alpha \dot{\nu i \sigma \omega \nu] ~ \alpha ́ p ı \vartheta \mu \widetilde{\omega} \nu \tau \widetilde{\omega} \nu \mathrm{AB}, \Gamma \Delta \dot{\alpha} \nu \vartheta \cup \varphi \alpha l-~}$







 ó ठè $\mathrm{H} \Gamma$ тòv $\mathrm{Z} \Theta \mu \varepsilon \tau \rho \widetilde{\omega} \nu \lambda \varepsilon เ \tau \varepsilon ́ \tau \omega ~ \mu о \nu \alpha ́ \delta \alpha ~ \tau \eta ̀ \nu ~ \Theta A . ~$






## Proposition 1

Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.


For two [unequal] numbers, $A B$ and $C D$, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains. I say that $A B$ and $C D$ are prime to one another-that is to say, that a unit alone measures (both) $A B$ and $C D$.

For if $A B$ and $C D$ are not prime to one another then some number will measure them. Let (some number) measure them, and let it be $E$. And let $C D$ measuring $B F$ leave $F A$ less than itself, and let $A F$ measuring $D G$ leave $G C$ less than itself, and let $G C$ measuring $F H$ leave a unit, $H A$.

In fact, since $E$ measures $C D$, and $C D$ measures $B F$, $E$ thus also measures $B F .^{\dagger}$ And ( $E$ ) also measures the whole of $B A$. Thus, $(E)$ will also measure the remainder





$A F{ }^{\ddagger}$ And $A F$ measures $D G$. Thus, $E$ also measures $D G$. And $(E)$ also measures the whole of $D C$. Thus, $(E)$ will also measure the remainder $C G$. And $C G$ measures $F H$. Thus, $E$ also measures $F H$. And ( $E$ ) also measures the whole of $F A$. Thus, $(E)$ will also measure the remaining unit $A H$, (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers $A B$ and $C D$. Thus, $A B$ and $C D$ are prime to one another. (Which is) the very thing it was required to show.
${ }^{\dagger}$ Here, use is made of the unstated common notion that if $a$ measures $b$, and $b$ measures $c$, then $a$ also measures $c$, where all symbols denote numbers.
${ }^{\ddagger}$ Here, use is made of the unstated common notion that if $a$ measures $b$, and $a$ measures part of $b$, then $a$ also measures the remainder of $b$, where all symbols denote numbers.
$\beta^{\prime}$.







Eỉ $\mu \varepsilon ̀ v ~ o u ̛ ้ \nu ~ o ́ ~ \Gamma \Delta ~ \tau o ̀ v ~ A B ~ \mu \varepsilon \tau р \varepsilon и ̃, ~ \mu \varepsilon \tau р \varepsilon і ̃ ~ ס e ̀ ~ \chi \alpha i ̀ ~ \varepsilon ́ \alpha u \tau o ́ v, ~ o ́ ~$







 BE $\mu \varepsilon \tau \rho \widetilde{\omega} \nu \lambda \varepsilon เ \tau \varepsilon ́ \tau \omega ~ \varepsilon ̇ \alpha \cup \tau o u ̃ ~ \varepsilon ̇ \lambda \alpha ́ \sigma \sigma o v \alpha ~ \tau o ̀ v ~ E A, ~ o ́ ~ \delta e ̀ ~ E A ~ \tau o ̀ v ~$








## Proposition 2

To find the greatest common measure of two given numbers (which are) not prime to one another.


Let $A B$ and $C D$ be the two given numbers (which are) not prime to one another. So it is required to find the greatest common measure of $A B$ and $C D$.

In fact, if $C D$ measures $A B, C D$ is thus a common measure of $C D$ and $A B$, (since $C D$ ) also measures itself. And (it is) manifest that (it is) also the greatest (common measure). For nothing greater than $C D$ can measure $C D$.

But if $C D$ does not measure $A B$ then some number will remain from $A B$ and $C D$, the lesser being continually subtracted, in turn, from the greater, which will measure the (number) preceding it. For a unit will not be left. But if not, $A B$ and $C D$ will be prime to one another [Prop. 7.1]. The very opposite thing was assumed. Thus, some number will remain which will measure the (number) preceding it. And let $C D$ measuring $B E$ leave $E A$ less than itself, and let $E A$ measuring $D F$ leave $F C$ less than itself, and let $C F$ measure $A E$. Therefore, since $C F$ measures $A E$, and $A E$ measures $D F, C F$ will thus also measure $D F$. And it also measures itself. Thus, it will












## По́pıб $\alpha$.





## $\gamma^{\prime}$.

Tрı $\widetilde{\omega} \nu \dot{\alpha} \rho เ \vartheta \mu \widetilde{\omega} \nu ~ \delta o \vartheta \varepsilon ́ v \tau \omega \nu ~ \mu \grave{\eta} \pi \rho \dot{\omega} \tau \omega \nu \pi \rho o ̀ s ~ \alpha \dot{\alpha} \lambda \lambda \dot{\eta} \lambda o u s ~ \tau o ̀ ~$



 $\mu$ е́тpov عن́pعĩ.




also measure the whole of $C D$. And $C D$ measures $B E$. Thus, $C F$ also measures $B E$. And it also measures $E A$. Thus, it will also measure the whole of $B A$. And it also measures $C D$. Thus, $C F$ measures (both) $A B$ and $C D$. Thus, $C F$ is a common measure of $A B$ and $C D$. So I say that (it is) also the greatest (common measure). For if $C F$ is not the greatest common measure of $A B$ and $C D$ then some number which is greater than $C F$ will measure the numbers $A B$ and $C D$. Let it (so) measure ( $A B$ and $C D$ ), and let it be $G$. And since $G$ measures $C D$, and $C D$ measures $B E, G$ thus also measures $B E$. And it also measures the whole of $B A$. Thus, it will also measure the remainder $A E$. And $A E$ measures $D F$. Thus, $G$ will also measure $D F$. And it also measures the whole of $D C$. Thus, it will also measure the remainder $C F$, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than $C F$ cannot measure the numbers $A B$ and $C D$. Thus, $C F$ is the greatest common measure of $A B$ and $C D$. [(Which is) the very thing it was required to show].

## Corollary

So it is manifest, from this, that if a number measures two numbers then it will also measure their greatest common measure. (Which is) the very thing it was required to show.

## Proposition 3

To find the greatest common measure of three given numbers (which are) not prime to one another.


Let $A, B$, and $C$ be the three given numbers (which are) not prime to one another. So it is required to find the greatest common measure of $A, B$, and $C$.

For let the greatest common measure, $D$, of the two (numbers) $A$ and $B$ have been taken [Prop. 7.2]. So $D$ either measures, or does not measure, $C$. First of all, let it measure ( $C$ ). And it also measures $A$ and $B$. Thus, $D$



 $\mu \varepsilon ́ \gamma เ \sigma \tau o \nu ~ \chi o เ v o ̀ \nu ~ \mu \varepsilon ́ \tau p o \nu ~ \mu \varepsilon \tau p \eta ́ \sigma \varepsilon ı . ~ \tau o ̀ ~ \delta غ ̀ ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B} \mu \varepsilon ́ \gamma เ \sigma \tau o \nu$犭oเvòv $\mu \varepsilon ́ \tau \rho o \nu ~ \varepsilon ̇ \sigma \tau i \nu ~ o ́ ~ \Delta \cdot ~ o ́ ~ E ~ a ̆ p \alpha ~ \tau o ̀ \nu ~ \Delta \mu \varepsilon \tau р \varepsilon 亢 ̃ ~ o ́ ~ \mu \varepsilon i ́ \zeta \omega \nu ~$
 ג́pıখ $\tau \widetilde{\omega} \nu \mathrm{A}, \mathrm{B}, \Gamma \mu \varepsilon ́ \gamma เ \sigma \tau o ́ v$ ह̀ $\sigma \tau \iota$ xoเvòv $\mu \varepsilon ́ \tau \rho o \nu$.

Mウ̀ $\mu \varepsilon \tau \rho \varepsilon i ́ \tau \omega$ ò̀̀ ó $\Delta$ тòv $\Gamma \cdot \lambda \varepsilon ́ \gamma \omega \pi \rho \widetilde{\omega} \tau o v$ ，őтı oi $\Gamma, \Delta$

 ס̀̀ toùs $\mathrm{A}, \mathrm{B}, \Gamma \mu \varepsilon \tau \rho \tilde{\omega} \nu$ xaì toùs $\mathrm{A}, \mathrm{B} \mu \varepsilon \tau \rho \eta \dot{\sigma} \varepsilon \iota$ ，x $\alpha i$ tò










 $\mu \varepsilon ́ \gamma เ \sigma \tau o v ~ \chi o l v o ̀ v ~ \mu \varepsilon ́ \tau p o v ~ \mu \varepsilon \tau p \eta ́ \sigma \varepsilon ı . ~ \tau o ̀ ~ \delta e ̀ ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B} \mu \varepsilon ́ \gamma เ \sigma \tau o v$




 ג́pıখ

$\delta^{\prime}$ ．
 グтоا $\mu \varepsilon ́ \rho o s ~ \varepsilon ̇ \sigma \tau i \nu ~ \eta ̀ ~ \mu \varepsilon ́ p \eta . ~$


measures $A, B$ ，and $C$ ．Thus，$D$ is a common measure of $A, B$ ，and $C$ ．So I say that（it is）also the greatest （common measure）．For if $D$ is not the greatest common measure of $A, B$ ，and $C$ then some number greater than $D$ will measure the numbers $A, B$ ，and $C$ ．Let it（so） measure（ $A, B$ ，and $C$ ），and let it be $E$ ．Therefore，since $E$ measures $A, B$ ，and $C$ ，it will thus also measure $A$ and $B$ ．Thus，it will also measure the greatest common mea－ sure of $A$ and $B$［Prop． 7.2 corr．］．And $D$ is the greatest common measure of $A$ and $B$ ．Thus，$E$ measures $D$ ，the greater（measuring）the lesser．The very thing is impossi－ ble．Thus，some number which is greater than $D$ cannot measure the numbers $A, B$ ，and $C$ ．Thus，$D$ is the great－ est common measure of $A, B$ ，and $C$ ．

So let $D$ not measure $C$ ．I say，first of all，that $C$ and $D$ are not prime to one another．For since $A, B, C$ are not prime to one another，some number will measure them．So the（number）measuring $A, B$ ，and $C$ will also measure $A$ and $B$ ，and it will also measure the greatest common measure，$D$ ，of $A$ and $B$［Prop． 7.2 corr．］．And it also measures $C$ ．Thus，some number will measure the numbers $D$ and $C$ ．Thus，$D$ and $C$ are not prime to one another．Therefore，let their greatest common measure， $E$ ，have been taken［Prop．7．2］．And since $E$ measures $D$ ，and $D$ measures $A$ and $B, E$ thus also measures $A$ and $B$ ．And it also measures $C$ ．Thus，$E$ measures $A, B$ ， and $C$ ．Thus，$E$ is a common measure of $A, B$ ，and $C$ ．So I say that（it is）also the greatest（common measure）．For if $E$ is not the greatest common measure of $A, B$ ，and $C$ then some number greater than $E$ will measure the num－ bers $A, B$ ，and $C$ ．Let it（so）measure（ $A, B$ ，and $C$ ），and let it be $F$ ．And since $F$ measures $A, B$ ，and $C$ ，it also measures $A$ and $B$ ．Thus，it will also measure the great－ est common measure of $A$ and $B$［Prop． 7.2 corr．］．And $D$ is the greatest common measure of $A$ and $B$ ．Thus，$F$ measures $D$ ．And it also measures $C$ ．Thus，$F$ measures $D$ and $C$ ．Thus，it will also measure the greatest com－ mon measure of $D$ and $C$［Prop． 7.2 corr．］．And $E$ is the greatest common measure of $D$ and $C$ ．Thus，$F$ measures $E$ ，the greater（measuring）the lesser．The very thing is impossible．Thus，some number which is greater than $E$ does not measure the numbers $A, B$ ，and $C$ ．Thus，$E$ is the greatest common measure of $A, B$ ，and $C$ ．（Which is）the very thing it was required to show．

## Proposition 4

Any number is either part or parts of any（other）num－ ber，the lesser of the greater．

Let $A$ and $B C$ be two numbers，and let $B C$ be the lesser．I say that $B C$ is either part or parts of $A$ ．



 тoũ A .


 A $\mu \varepsilon \tau \rho \varepsilon \tilde{\imath}, \mu \varepsilon ́ \rho o \varsigma ~ \varepsilon ̇ \sigma \tau i \nu ~ o ́ ~ B \Gamma ~ \tau о u ̃ ~ A . ~ \varepsilon i ̉ ~ \delta e ̀ ~ o u ̛, ~ \varepsilon i ̉ \lambda n ́ \varphi \vartheta \omega ~ \tau \widetilde{\omega} \nu$







$\varepsilon^{\prime}$.
 גủtò $\mu \varepsilon ́ p o s ~ n ̃, ~ x \alpha i ~ \sigma u v \alpha \mu \varphi o ́ t \varepsilon p o s ~ \sigma u v a \mu \varphi o \tau \varepsilon ́ p o u ~ t o ̀ ~ \alpha u ̉ t o ̀ ~$ $\mu \varepsilon ́ p o \varsigma$ है $\sigma \tau \alpha l$, ö $\pi \varepsilon \rho$ ó हǐऽ тоũ ह́vós.



For $A$ and $B C$ are either prime to one another, or not. Let $A$ and $B C$, first of all, be prime to one another. So separating $B C$ into its constituent units, each of the units in $B C$ will be some part of $A$. Hence, $B C$ is parts of $A$.


So let $A$ and $B C$ be not prime to one another. So $B C$ either measures, or does not measure, $A$. Therefore, if $B C$ measures $A$ then $B C$ is part of $A$. And if not, let the greatest common measure, $D$, of $A$ and $B C$ have been taken [Prop. 7.2], and let $B C$ have been divided into $B E$, $E F$, and $F C$, equal to $D$. And since $D$ measures $A, D$ is a part of $A$. And $D$ is equal to each of $B E, E F$, and $F C$. Thus, $B E, E F$, and $F C$ are also each part of $A$. Hence, $B C$ is parts of $A$.

Thus, any number is either part or parts of any (other) number, the lesser of the greater. (Which is) the very thing it was required to show.

## Proposition $5^{\dagger}$

If a number is part of a number, and another (number) is the same part of another, then the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.


For let a number $A$ be part of a [number] $B C$, and

 $\mathrm{B} \mathrm{\Gamma}, \mathrm{EZ}$ тò aủtò $\mu \varepsilon ́ \rho o s ~ \varepsilon ̇ \sigma \tau i ́ v, ~ o ̈ \pi \varepsilon \rho ~ o ̀ ~ A ~ \tau o u ̃ ~ В Г . ~$
'Eлモì үáp, ô $\mu \varepsilon ́ p o s ~ \varepsilon ̇ \sigma \tau i ̀ ~ o ́ ~ A ~ \tau o u ̃ ~ B \Gamma, ~ \tau o ̀ ~ \alpha u ̉ \tau o ̀ ~ \mu \varepsilon ́ p o s ~ \varepsilon ̉ \sigma \tau i ̀ ~$











 тои̃ $\mathrm{B} \mathrm{\Gamma}, \mathrm{EZ}$ •ٌँ
another (number) $D$ (be) the same part of another (number) $E F$ that $A$ (is) of $B C$. I say that the sum $A, D$ is also the same part of the sum $B C, E F$ that $A$ (is) of $B C$.

For since which(ever) part $A$ is of $B C, D$ is the same part of $E F$, thus as many numbers as are in $B C$ equal to $A$, so many numbers are also in $E F$ equal to $D$. Let $B C$ have been divided into $B G$ and $G C$, equal to $A$, and $E F$ into $E H$ and $H F$, equal to $D$. So the multitude of (divisions) $B G, G C$ will be equal to the multitude of (divisions) $E H, H F$. And since $B G$ is equal to $A$, and $E H$ to $D$, thus $B G, E H$ (is) also equal to $A, D$. So, for the same (reasons), $G C, H F$ (is) also (equal) to $A, D$. Thus, as many numbers as [are] in $B C$ equal to $A$, so many are also in $B C, E F$ equal to $A, D$. Thus, as many times as $B C$ is (divisible) by $A$, so many times is the sum $B C, E F$ also (divisible) by the sum $A, D$. Thus, which(ever) part $A$ is of $B C$, the sum $A, D$ is also the same part of the sum $B C, E F$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a=(1 / n) b$ and $c=(1 / n) d$ then $(a+c)=(1 / n)(b+d)$, where all symbols denote numbers.

$$
\tau^{\prime}
$$

 $\alpha u ̉ \tau \dot{\alpha} \mu \varepsilon ́ \rho \eta ~ \eta ̆, ~ x \alpha i ̀ ~ \sigma u v \alpha \mu \varphi o ́ \tau \varepsilon \rho о s ~ \sigma u v \alpha \mu \varphi о \tau \varepsilon ́ \rho о и ~ \tau \alpha ̀ ~ \alpha u ̉ \tau \alpha ̀ ~$










 $\tau \widetilde{\omega} \nu \Delta \Theta, ~ \Theta E . ~ x \alpha i ̀ ~ \varepsilon ่ ~ \pi \varepsilon i ́, ~ o ̋ ~ \mu \varepsilon ́ p o s ~ \varepsilon ̇ \sigma \tau i \nu ~ o ́ ~ A H ~ \tau о u ̃ ~ \Gamma, ~ \tau o ̀ ~$

## Proposition $6^{\dagger}$

If a number is parts of a number, and another (number) is the same parts of another, then the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.


For let a number $A B$ be parts of a number $C$, and another (number) $D E$ (be) the same parts of another (number) $F$ that $A B$ (is) of $C$. I say that the sum $A B, D E$ is also the same parts of the sum $C, F$ that $A B$ (is) of $C$.

For since which(ever) parts $A B$ is of $C, D E$ (is) also the same parts of $F$, thus as many parts of $C$ as are in $A B$, so many parts of $F$ are also in $D E$. Let $A B$ have been divided into the parts of $C, A G$ and $G B$, and $D E$ into the parts of $F, D H$ and $H E$. So the multitude of (divisions) $A G, G B$ will be equal to the multitude of (divisions) $D H$,







$H E$ ．And since which（ever）part $A G$ is of $C, D H$ is also the same part of $F$ ，thus which（ever）part $A G$ is of $C$ ， the sum $A G, D H$ is also the same part of the sum $C, F$ ［Prop．7．5］．And so，for the same（reasons），which（ever） part $G B$ is of $C$ ，the sum $G B, H E$ is also the same part of the sum $C, F$ ．Thus，which（ever）parts $A B$ is of $C$ ， the sum $A B, D E$ is also the same parts of the sum $C, F$ ． （Which is）the very thing it was required to show．
${ }^{\dagger}$ In modern notation，this proposition states that if $a=(m / n) b$ and $c=(m / n) d$ then $(a+c)=(m / n)(b+d)$ ，where all symbols denote numbers．

## $\zeta^{\prime}$.


 ӧтєр ó ö入oऽ тоธ̃ ö入ou．

## A E B


＇Apıখ

 ő入ou тои̃ $\Gamma \Delta$ ．















## Proposition $7^{\dagger}$

If a number is that part of a number that a（part） taken away（is）of a（part）taken away then the remain－ der will also be the same part of the remainder that the whole（is）of the whole．


For let a number $A B$ be that part of a number $C D$ that a（part）taken away $A E$（is）of a part taken away $C F$ ．I say that the remainder $E B$ is also the same part of the remainder $F D$ that the whole $A B$（is）of the whole $C D$ ．

For which（ever）part $A E$ is of $C F$ ，let $E B$ also be the same part of $C G$ ．And since which（ever）part $A E$ is of $C F, E B$ is also the same part of $C G$ ，thus which（ever） part $A E$ is of $C F, A B$ is also the same part of $G F$ ［Prop．7．5］．And which（ever）part $A E$ is of $C F, A B$ is also assumed（to be）the same part of $C D$ ．Thus，also， which（ever）part $A B$ is of $G F,(A B)$ is also the same part of $C D$ ．Thus，$G F$ is equal to $C D$ ．Let $C F$ have been subtracted from both．Thus，the remainder $G C$ is equal to the remainder $F D$ ．And since which（ever）part $A E$ is of $C F, E B$［is］also the same part of $G C$ ，and $G C$（is） equal to $F D$ ，thus which（ever）part $A E$ is of $C F, E B$ is also the same part of $F D$ ．But，which（ever）part $A E$ is of $C F, A B$ is also the same part of $C D$ ．Thus，the remain－ der $E B$ is also the same part of the remainder $F D$ that the whole $A B$（is）of the whole $C D$ ．（Which is）the very thing it was required to show．
${ }^{\dagger}$ In modern notation，this proposition states that if $a=(1 / n) b$ and $c=(1 / n) d$ then $(a-c)=(1 / n)(b-d)$ ，where all symbols denote numbers．

$$
\eta^{\prime}
$$


 $\alpha \not \pi \varepsilon \rho ~ o ́ ~ o ̈ \lambda о \varsigma ~ \tau о \tilde{~} \partial \circ \lambda o u$ ．

## Proposition $8^{\dagger}$

If a number is those parts of a number that a（part） taken away（is）of a（part）taken away then the remain－ der will also be the same parts of the remainder that the



 ő $\lambda$ ou тoũ $\Gamma \Delta$.
 тoũ $\Gamma \Delta$, 七̀̀ $\alpha \cup ̛ \tau \grave{\alpha} \mu \varepsilon ́ p \eta ~ \varepsilon ̇ \sigma \tau i ~ x \alpha i ̀ ~ o ̀ ~ A E ~ \tau o u ̃ ~ \Gamma Z . ~ \delta ı n \rho \eta ́ \sigma \vartheta \omega ~ o ́ ~$




 ó HM. ô a̋pa $\mu$ ह́pos ह̇ $\sigma \tau i v ~ o ́ ~ H K ~ \tau o u ̃ ~ \Gamma \Delta, ~ \tau o ̀ ~ \alpha u ̉ \tau o ̀ ~ \mu \varepsilon ́ p o s ~ \varepsilon ̇ \sigma \tau i ~$














whole (is) of the whole.


For let a number $A B$ be those parts of a number $C D$ that a (part) taken away $A E$ (is) of a (part) taken away $C F$. I say that the remainder $E B$ is also the same parts of the remainder $F D$ that the whole $A B$ (is) of the whole $C D$.

For let $G H$ be laid down equal to $A B$. Thus, which(ever) parts $G H$ is of $C D, A E$ is also the same parts of $C F$. Let $G H$ have been divided into the parts of $C D, G K$ and $K H$, and $A E$ into the part of $C F, A L$ and $L E$. So the multitude of (divisions) $G K, K H$ will be equal to the multitude of (divisions) $A L, L E$. And since which(ever) part $G K$ is of $C D, A L$ is also the same part of $C F$, and $C D$ (is) greater than $C F, G K$ (is) thus also greater than $A L$. Let $G M$ be made equal to $A L$. Thus, which (ever) part $G K$ is of $C D, G M$ is also the same part of $C F$. Thus, the remainder $M K$ is also the same part of the remainder $F D$ that the whole $G K$ (is) of the whole $C D$ [Prop. 7.5]. Again, since which(ever) part $K H$ is of $C D, E L$ is also the same part of $C F$, and $C D$ (is) greater than $C F, H K$ (is) thus also greater than $E L$. Let $K N$ be made equal to $E L$. Thus, which(ever) part $K H$ (is) of $C D, K N$ is also the same part of $C F$. Thus, the remainder $N H$ is also the same part of the remainder $F D$ that the whole $K H$ (is) of the whole $C D$ [Prop. 7.5]. And the remainder $M K$ was also shown to be the same part of the remainder $F D$ that the whole $G K$ (is) of the whole $C D$. Thus, the sum $M K, N H$ is the same parts of $D F$ that the whole $H G$ (is) of the whole $C D$. And the sum $M K, N H$ (is) equal to $E B$, and $H G$ to $B A$. Thus, the remainder $E B$ is also the same parts of the remainder $F D$ that the whole $A B$ (is) of the whole $C D$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a=(m / n) b$ and $c=(m / n) d$ then $(a-c)=(m / n)(b-d)$, where all symbols denote numbers.

$$
\vartheta^{\prime}
$$



 ठєúтєроऽ тои̃ тєта́คтоu.

## Proposition $9^{\dagger}$

If a number is part of a number, and another (number) is the same part of another, also, alternately, which(ever) part, or parts, the first (number) is of the third, the second (number) will also be the same part, or









 $\mathrm{H} \Gamma \tau \widetilde{\varphi} \pi \lambda \eta \vartheta \vartheta \varepsilon \iota \tau \widetilde{\omega} \mathrm{E} \Theta, \Theta \mathrm{Z}$.




 $\mathrm{E} \Theta$ ท̀ $\mu \varepsilon ́ \rho \eta$, tò $\alpha u ̛ \tau o ̀ ~ \mu \varepsilon ́ p o s ~ \varepsilon ̇ \sigma \tau i ~ x \alpha \grave{~} \sigma u v \alpha \mu \varphi o ́ \tau \varepsilon \rho о \varsigma ~ o ́ ~ B \Gamma ~$




the same parts, of the fourth.


For let a number $A$ be part of a number $B C$, and another (number) $D$ (be) the same part of another $E F$ that $A$ (is) of $B C$. I say that, also, alternately, which(ever) part, or parts, $A$ is of $D, B C$ is also the same part, or parts, of $E F$.

For since which(ever) part $A$ is of $B C, D$ is also the same part of $E F$, thus as many numbers as are in $B C$ equal to $A$, so many are also in $E F$ equal to $D$. Let $B C$ have been divided into $B G$ and $G C$, equal to $A$, and $E F$ into $E H$ and $H F$, equal to $D$. So the multitude of (divisions) $B G, G C$ will be equal to the multitude of (divisions) $E H, H F$.

And since the numbers $B G$ and $G C$ are equal to one another, and the numbers $E H$ and $H F$ are also equal to one another, and the multitude of (divisions) $B G, G C$ is equal to the multitude of (divisions) $E H, H C$, thus which(ever) part, or parts, $B G$ is of $E H, G C$ is also the same part, or the same parts, of $H F$. And hence, which(ever) part, or parts, $B G$ is of $E H$, the sum $B C$ is also the same part, or the same parts, of the sum $E F$ [Props. 7.5, 7.6]. And $B G$ (is) equal to $A$, and $E H$ to $D$. Thus, which(ever) part, or parts, $A$ is of $D, B C$ is also the same part, or the same parts, of $E F$. (Which is) the very thing it was required to show.

[^15]$$
\therefore^{\prime} .
$$


 tò aủtò $\mu$ и́pos.


 ó $\Gamma$ тои̃ Z ท̆ тò aủtò $\mu$ épos.

## Proposition $10^{\dagger}$

If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

For let a number $A B$ be parts of a number $C$, and another (number) $D E$ (be) the same parts of another $F$. I say that, also, alternately, which(ever) parts, or part,











 $\alpha u ̋ \tau o ̀ ~ \mu \varepsilon ́ p o s ~ \varepsilon ́ \sigma \tau i ~ x \alpha i ̀ ~ o ̀ ~ H B ~ \tau o u ̃ ~ \Theta E ~ \ddot{\eta} \tau \alpha ̀ ~ \alpha u ̉ \tau \alpha ̀ ~ \mu \varepsilon ́ p \eta ~ x \alpha i ̀ ~ o ̈ ~$






$A B$ is of $D E, C$ is also the same parts, or the same part, of $F$.


For since which(ever) parts $A B$ is of $C, D E$ is also the same parts of $F$, thus as many parts of $C$ as are in $A B$, so many parts of $F$ (are) also in $D E$. Let $A B$ have been divided into the parts of $C, A G$ and $G B$, and $D E$ into the parts of $F, D H$ and $H E$. So the multitude of (divisions) $A G, G B$ will be equal to the multitude of (divisions) $D H, H E$. And since which(ever) part $A G$ is of $C, D H$ is also the same part of $F$, also, alternately, which(ever) part, or parts, $A G$ is of $D H, C$ is also the same part, or the same parts, of $F$ [Prop. 7.9]. And so, for the same (reasons), which(ever) part, or parts, $G B$ is of $H E, C$ is also the same part, or the same parts, of $F$ [Prop. 7.9]. And so [which(ever) part, or parts, $A G$ is of $D H, G B$ is also the same part, or the same parts, of $H E$. And thus, which(ever) part, or parts, $A G$ is of $D H, A B$ is also the same part, or the same parts, of $D E$ [Props. 7.5, 7.6]. But, which(ever) part, or parts, $A G$ is of $D H, C$ was also shown (to be) the same part, or the same parts, of $F$. And, thus] which(ever) parts, or part, $A B$ is of $D E$, $C$ is also the same parts, or the same part, of $F$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a=(m / n) b$ and $c=(m / n) d$ then if $a=(k / l) c$ then $b=(k / l) d$, where all symbols denote numbers.
$\alpha^{\prime}$.

 ő入оv.


 tòv $\Gamma \Delta$.

## Proposition 11

If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, then the remainder will also be to the remainder as the whole (is) to the whole.

Let the whole $A B$ be to the whole $C D$ as the (part) taken away $A E$ (is) to the (part) taken away $C F$. I say that the remainder $E B$ is to the remainder $F D$ as the whole $A B$ (is) to the whole $C D$.








(For) since as $A B$ is to $C D$, so $A E$ (is) to $C F$, thus which(ever) part, or parts, $A B$ is of $C D, A E$ is also the same part, or the same parts, of $C F$ [Def. 7.20]. Thus, the remainder $E B$ is also the same part, or parts, of the remainder $F D$ that $A B$ (is) of $C D$ [Props. 7.7, 7.8]. Thus, as $E B$ is to $F D$, so $A B$ (is) to $C D$ [Def. 7.20]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a: b:: c: d$ then $a: b:: a-c: b-d$, where all symbols denote numbers.

$$
\beta^{\prime}
$$















## Proposition $12^{\dagger}$

If any multitude whatsoever of numbers are proportional then as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.


Let any multitude whatsoever of numbers, $A, B, C$, $D$, be proportional, (such that) as $A$ (is) to $B$, so $C$ (is) to $D$. I say that as $A$ is to $B$, so $A, C$ (is) to $B, D$.

For since as $A$ is to $B$, so $C$ (is) to $D$, thus which(ever) part, or parts, $A$ is of $B, C$ is also the same part, or parts, of $D$ [Def. 7.20]. Thus, the sum $A, C$ is also the same part, or the same parts, of the sum $B, D$ that $A$ (is) of $B$ [Props. 7.5, 7.6]. Thus, as $A$ is to $B$, so $A, C$ (is) to $B, D$ [Def. 7.20]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a: b:: c: d$ then $a: b:: a+c: b+d$, where all symbols denote numbers.
$i \gamma^{\prime}$.
 $\alpha \alpha^{\prime} \lambda$ oүov है́oovtal.



 трòs tòv $\Delta$.







Proposition $13^{\dagger}$
If four numbers are proportional then they will also be proportional alternately.


Let the four numbers $A, B, C$, and $D$ be proportional, (such that) as $A$ (is) to $B$, so $C$ (is) to $D$. I say that they will also be proportional alternately, (such that) as $A$ (is) to $C$, so $B$ (is) to $D$.

For since as $A$ is to $B$, so $C$ (is) to $D$, thus which(ever) part, or parts, $A$ is of $B, C$ is also the same part, or the same parts, of $D$ [Def. 7.20]. Thus, alterately, which(ever) part, or parts, $A$ is of $C, B$ is also the same part, or the same parts, of $D$ [Props. 7.9, 7.10]. Thus, as $A$ is to $C$, so $B$ (is) to $D$ [Def. 7.20]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a: b:: c: d$ then $a: c:: b: d$, where all symbols denote numbers.

$$
\delta^{\prime}
$$






 $\Delta, \mathrm{E}, \mathrm{Z}$, ©̊s $\mu \mathrm{e} v$ ó A tpòs tòv B , oữ $\omega$ s ò $\Delta$ tpòs tòv E ,

 Z.




## Proposition $14^{\dagger}$

If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.


Let there be any multitude of numbers whatsoever, $A$, $B, C$, and (some) other (numbers), $D, E, F$, of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as $A$ (is) to $B$, so $D$ (is) to $E$, and as $B$ (is) to $C$, so $E$ (is) to $F$. I say that also, via equality, as $A$ is to $C$, so $D$ (is) to $F$.

For since as $A$ is to $B$, so $D$ (is) to $E$, thus, alternately, as $A$ is to $D$, so $B$ (is) to $E$ [Prop. 7.13]. Again, since as $B$ is to $C$, so $E$ (is) to $F$, thus, alternately, as $B$ is





to $E$, so $C$ (is) to $F$ [Prop. 7.13]. And as $B$ (is) to $E$, so $A$ (is) to $D$. Thus, also, as $A$ (is) to $D$, so $C$ (is) to $F$. Thus, alternately, as $A$ is to $C$, so $D$ (is) to $F$ [Prop. 7.13]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a: b:: d: e$ and $b: c:: e: f$ then $a: c:: d: f$, where all symbols denote numbers.

$$
\iota \varepsilon^{\prime} \text {. }
$$







 $\mu \varepsilon \tau \rho \varepsilon i ̃ ~ \varkappa \alpha \grave{~ o ́ ~ B \Gamma ~ \tau o ̀ v ~} \mathrm{EZ}$.







 $\mathrm{H} \Theta, ~ \Theta \Gamma \mu о \nu \alpha ́ \delta \omega \nu \tau \widetilde{\varphi} \pi \lambda \dot{\eta} \vartheta \varepsilon \iota \tau \widetilde{\omega} \nu \mathrm{EK}, \mathrm{K} \Lambda, \Lambda \mathrm{Z} \dot{\alpha} \rho \imath \vartheta \mu \widetilde{\omega} \nu$,











## Proposition 15

If a unit measures some number, and another number measures some other number as many times, then, also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.


For let a unit $A$ measure some number $B C$, and let another number $D$ measure some other number $E F$ as many times. I say that, also, alternately, the unit $A$ also measures the number $D$ as many times as $B C$ (measures) $E F$.

For since the unit $A$ measures the number $B C$ as many times as $D$ (measures) $E F$, thus as many units as are in $B C$, so many numbers are also in $E F$ equal to $D$. Let $B C$ have been divided into its constituent units, $B G, G H$, and $H C$, and $E F$ into the (divisions) $E K, K L$, and $L F$, equal to $D$. So the multitude of (units) $B G$, $G H, H C$ will be equal to the multitude of (divisions) $E K, K L, L F$. And since the units $B G, G H$, and $H C$ are equal to one another, and the numbers $E K, K L$, and $L F$ are also equal to one another, and the multitude of the (units) $B G, G H, H C$ is equal to the multitude of the numbers $E K, K L, L F$, thus as the unit $B G$ (is) to the number $E K$, so the unit $G H$ will be to the number $K L$, and the unit $H C$ to the number $L F$. And thus, as one of the leading (numbers is) to one of the following, so (the sum of) all of the leading will be to (the sum of) all of the following [Prop. 7.12]. Thus, as the unit $B G$ (is) to the number $E K$, so $B C$ (is) to $E F$. And the unit $B G$ (is) equal to the unit $A$, and the number $E K$ to the number $D$. Thus, as the unit $A$ is to the number $D$, so $B C$ (is) to $E F$. Thus, the unit $A$ measures the number $D$ as many times as $B C$ (measures) $E F$ [Def. 7.20]. (Which is) the very thing it was required to show.

[^16]$1 f^{\prime}$.



$\Gamma$








 ó A тòv $\Gamma$. $\pi \alpha ́ \lambda \iota \iota$, غ̇ $\pi \varepsilon \grave{\jmath}$ ó B тòv $\mathrm{A} \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Delta ~$







## Proposition $16^{\dagger}$

If two numbers multiplying one another make some (numbers) then the (numbers) generated from them will be equal to one another.


Let $A$ and $B$ be two numbers. And let $A$ make $C$ (by) multiplying $B$, and let $B$ make $D$ (by) multiplying $A$. I say that $C$ is equal to $D$.

For since $A$ has made $C$ (by) multiplying $B, B$ thus measures $C$ according to the units in $A$ [Def. 7.15]. And the unit $E$ also measures the number $A$ according to the units in it. Thus, the unit $E$ measures the number $A$ as many times as $B$ (measures) $C$. Thus, alternately, the unit $E$ measures the number $B$ as many times as $A$ (measures) $C$ [Prop. 7.15]. Again, since $B$ has made $D$ (by) multiplying $A, A$ thus measures $D$ according to the units in $B$ [Def. 7.15]. And the unit $E$ also measures $B$ according to the units in it. Thus, the unit $E$ measures the number $B$ as many times as $A$ (measures) $D$. And the unit $E$ was measuring the number $B$ as many times as $A$ (measures) $C$. Thus, $A$ measures each of $C$ and $D$ an equal number of times. Thus, $C$ is equal to $D$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that $a b=b a$, where all symbols denote numbers.

$$
\zeta^{\prime}
$$


 $\pi \lambda \alpha \sigma \iota \alpha \sigma \vartheta \varepsilon$ โัбเข.
$\mathrm{A} \longmapsto$


 тòv $\Gamma$, oütcos ò $\Delta$ тpòs tòv E .
'Eлєi ү̀̀p ò A тòv $\mathrm{B} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Delta \pi \varepsilon \pi о i ́ \eta \chi \varepsilon \nu, ~$


## Proposition $17{ }^{\dagger}$

If a number multiplying two numbers makes some (numbers) then the (numbers) generated from them will have the same ratio as the multiplied (numbers).

$F \longmapsto$
For let the number $A$ make (the numbers) $D$ and $E$ (by) multiplying the two numbers $B$ and $C$ (respectively). I say that as $B$ is to $C$, so $D$ (is) to $E$.

For since $A$ has made $D$ (by) multiplying $B, B$ thus measures $D$ according to the units in $A$ [Def. 7.15]. And







the unit $F$ also measures the number $A$ according to the units in it. Thus, the unit $F$ measures the number $A$ as many times as $B$ (measures) $D$. Thus, as the unit $F$ is to the number $A$, so $B$ (is) to $D$ [Def. 7.20]. And so, for the same (reasons), as the unit $F$ (is) to the number $A$, so $C$ (is) to $E$. And thus, as $B$ (is) to $D$, so $C$ (is) to $E$. Thus, alternately, as $B$ is to $C$, so $D$ (is) to $E$ [Prop. 7.13]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $d=a b$ and $e=a c$ then $d: e:: b: c$, where all symbols denote numbers.

$$
i \eta^{\prime} .
$$


 тоїऽ $\pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \sigma \iota \nu$.


B


 $\mathrm{A} \pi \rho o ̀ s ~ \tau o ̀ v \mathrm{~B}$, oút $\omega \varsigma$ ò $\Delta$ tpòs tòv E .







## Proposition $18^{\dagger}$

If two numbers multiplying some number make some (other numbers) then the (numbers) generated from them will have the same ratio as the multiplying (numbers).


For let the two numbers $A$ and $B$ make (the numbers) $D$ and $E$ (respectively, by) multiplying some number $C$. I say that as $A$ is to $B$, so $D$ (is) to $E$.

For since $A$ has made $D$ (by) multiplying $C, C$ has thus also made $D$ (by) multiplying $A$ [Prop. 7.16]. So, for the same (reasons), $C$ has also made $E$ (by) multiplying $B$. So the number $C$ has made $D$ and $E$ (by) multiplying the two numbers $A$ and $B$ (respectively). Thus, as $A$ is to $B$, so $D$ (is) to $E$ [Prop. 7.17]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this propositions states that if $a c=d$ and $b c=e$ then $a: b:: d: e$, where all symbols denote numbers.

## ${ }^{19}$.




 $\tau \varepsilon ́ \sigma \sigma \alpha \sigma \rho \varepsilon \varsigma ~ \alpha ̉ p ı \vartheta \mu o i ̀ ~ \alpha ̀ \alpha \alpha \lambda o \gamma o v ~ ह ै \sigma o v \tau \alpha l . ~$





## Proposition $19^{\dagger}$

If four number are proportional then the number created from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third then the four numbers will be proportional.

Let $A, B, C$, and $D$ be four proportional numbers, (such that) as $A$ (is) to $B$, so $C$ (is) to $D$. And let $A$ make $E$ (by) multiplying $D$, and let $B$ make $F$ (by) multiplying $C$. I say that $E$ is equal to $F$.





 ©́s ó $\Gamma$ tpòs tòv $\Delta$, oütcus ò A tpòs tòv B. xaì ís äp $\alpha$
 тòv $\Gamma \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma ~ t o ̀ v ~ H ~ \pi \varepsilon \pi o i ́ \eta \ell \varepsilon v, ~ \alpha ̀ \lambda \lambda \alpha ̀ ~ \mu \grave{\nu} \nu ~ \chi \alpha i ̀ ~ o ́ ~$



 tòv E . xaì cos äp $\alpha$ ó H tpòs tòv E , oútcus ó H tpòs tòv











For let $A$ make $G$ (by) multiplying $C$. Therefore, since $A$ has made $G$ (by) multiplying $C$, and has made $E$ (by) multiplying $D$, the number $A$ has made $G$ and $E$ by multiplying the two numbers $C$ and $D$ (respectively). Thus, as $C$ is to $D$, so $G$ (is) to $E$ [Prop. 7.17]. But, as $C$ (is) to $D$, so $A$ (is) to $B$. Thus, also, as $A$ (is) to $B$, so $G$ (is) to $E$. Again, since $A$ has made $G$ (by) multiplying $C$, but, in fact, $B$ has also made $F$ (by) multiplying $C$, the two numbers $A$ and $B$ have made $G$ and $F$ (respectively, by) multiplying some number $C$. Thus, as $A$ is to $B$, so $G$ (is) to $F$ [Prop. 7.18]. But, also, as $A$ (is) to $B$, so $G$ (is) to $E$. And thus, as $G$ (is) to $E$, so $G$ (is) to $F$. Thus, $G$ has the same ratio to each of $E$ and $F$. Thus, $E$ is equal to $F$ [Prop. 5.9].

So, again, let $E$ be equal to $F$. I say that as $A$ is to $B$, so $C$ (is) to $D$.

For, with the same construction, since $E$ is equal to $F$, thus as $G$ is to $E$, so $G$ (is) to $F$ [Prop. 5.7]. But, as $G$ (is) to $E$, so $C$ (is) to $D$ [Prop. 7.17]. And as $G$ (is) to $F$, so $A$ (is) to $B$ [Prop. 7.18]. And, thus, as $A$ (is) to $B$, so $C$ (is) to $D$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition reads that if $a: b:: c: d$ then $a d=b c$, and vice versa, where all symbols denote numbers.

## $x^{\prime}$.





 A $\mu \varepsilon \tau р \varepsilon і ̃ ~ \nsim \alpha i ̀ ~ o ́ ~ E Z ~ т o ̀ v ~ B . ~ . ~$

## Proposition 20

The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

For let $C D$ and $E F$ be the least numbers having the same ratio as $A$ and $B$ (respectively). I say that $C D$ measures $A$ the same number of times as $E F$ (measures) $B$.








 $\pi \lambda \tilde{\eta} \vartheta \circ \varsigma \tau \widetilde{\omega} \nu \Gamma H, H \Delta \tau \widetilde{\varphi} \pi \lambda \eta \dot{\eta} \vartheta \varepsilon \tau \widetilde{\omega} \nu \mathrm{E} \Theta, \Theta Z$, है $\sigma \tau \iota \nu \alpha \sim \alpha \dot{\omega}$










$x \alpha^{\prime}$.



 aủtoĭs.

Ei $\gamma \dot{\alpha} \rho \mu \dot{\eta}$, है $\sigma o v \tau \alpha i ́ ~ \tau เ \nu \varepsilon \varsigma ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}$ ह̀入 $\alpha \sigma \sigma o v \varepsilon \varsigma ~ \dot{\alpha} \rho เ \vartheta \mu o i$



For $C D$ is not parts of $A$. For, if possible, let it be (parts of $A$ ). Thus, $E F$ is also the same parts of $B$ that $C D$ (is) of $A$ [Def. 7.20, Prop. 7.13]. Thus, as many parts of $A$ as are in $C D$, so many parts of $B$ are also in $E F$. Let $C D$ have been divided into the parts of $A, C G$ and $G D$, and $E F$ into the parts of $B, E H$ and $H F$. So the multitude of (divisions) $C G, G D$ will be equal to the multitude of (divisions) $E H, H F$. And since the numbers $C G$ and $G D$ are equal to one another, and the numbers $E H$ and $H F$ are also equal to one another, and the multitude of (divisions) $C G, G D$ is equal to the multitude of (divisions) $E H, H F$, thus as $C G$ is to $E H$, so $G D$ (is) to $H F$. Thus, as one of the leading (numbers is) to one of the following, so will (the sum of) all of the leading (numbers) be to (the sum of) all of the following [Prop. 7.12]. Thus, as $C G$ is to $E H$, so $C D$ (is) to $E F$. Thus, $C G$ and $E H$ are in the same ratio as $C D$ and $E F$, being less than them. The very thing is impossible. For $C D$ and $E F$ were assumed (to be) the least of those (numbers) having the same ratio as them. Thus, $C D$ is not parts of $A$. Thus, (it is) a part (of $A$ ) [Prop. 7.4]. And $E F$ is the same part of $B$ that $C D$ (is) of $A$ [Def. 7.20, Prop 7.13]. Thus, $C D$ measures $A$ the same number of times that $E F$ (measures) $B$. (Which is) the very thing it was required to show.

## Proposition 21

Numbers prime to one another are the least of those (numbers) having the same ratio as them.

Let $A$ and $B$ be numbers prime to one another. I say that $A$ and $B$ are the least of those (numbers) having the same ratio as them.

For if not then there will be some numbers less than $A$ and $B$ which are in the same ratio as $A$ and $B$. Let them be $C$ and $D$.

















$$
x \beta^{\prime} .
$$




"E $\quad \tau \omega \sigma \alpha \nu$ ह̀入 $\alpha ́ \chi เ \sigma \tau o l ~ \alpha ́ p ı \vartheta \mu o i ̀ ~ \tau \widetilde{\omega} \nu ~ \tau o ̀ v ~ \alpha u ̉ \tau o ̀ v ~ \lambda o ́ \gamma o v ~$
 $\dot{\alpha} \lambda \lambda \eta_{n} \lambda$ ous عíбí.





Therefore, since the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following- $C$ thus measures $A$ the same number of times that $D$ (measures) $B$ [Prop. 7.20]. So as many times as $C$ measures $A$, so many units let there be in $E$. Thus, $D$ also measures $B$ according to the units in $E$. And since $C$ measures $A$ according to the units in $E, E$ thus also measures $A$ according to the units in $C$ [Prop. 7.16]. So, for the same (reasons), $E$ also measures $B$ according to the units in $D$ [Prop. 7.16]. Thus, $E$ measures $A$ and $B$, which are prime to one another. The very thing is impossible. Thus, there cannot be any numbers less than $A$ and $B$ which are in the same ratio as $A$ and $B$. Thus, $A$ and $B$ are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

## Proposition 22

The least numbers of those (numbers) having the same ratio as them are prime to one another.


Let $A$ and $B$ be the least numbers of those (numbers) having the same ratio as them. I say that $A$ and $B$ are prime to one another.

For if they are not prime to one another then some number will measure them. Let it (so measure them), and let it be $C$. And as many times as $C$ measures $A$, so
 $\tau \widetilde{\varphi}$ E.
 $\Gamma$ ้้p $\alpha$ тòv $\Delta \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma เ \alpha ́ \sigma \alpha s ~ \tau o ̀ v ~ A ~ \pi \varepsilon \pi o i ́ \eta \chi \varepsilon v . ~ \delta ı \alpha ̀ ~ \tau \grave{\alpha}$
 ápıখ






$$
x \gamma^{\prime}
$$















## $\chi \delta^{\prime}$.



many units let there be in $D$. And as many times as $C$ measures $B$, so many units let there be in $E$.

Since $C$ measures $A$ according to the units in $D, C$ has thus made $A$ (by) multiplying $D$ [Def. 7.15]. So, for the same (reasons), $C$ has also made $B$ (by) multiplying $E$. So the number $C$ has made $A$ and $B$ (by) multiplying the two numbers $D$ and $E$ (respectively). Thus, as $D$ is to $E$, so $A$ (is) to $B$ [Prop. 7.17]. Thus, $D$ and $E$ are in the same ratio as $A$ and $B$, being less than them. The very thing is impossible. Thus, some number does not measure the numbers $A$ and $B$. Thus, $A$ and $B$ are prime to one another. (Which is) the very thing it was required to show.

## Proposition 23

If two numbers are prime to one another then a number measuring one of them will be prime to the remaining (one).


Let $A$ and $B$ be two numbers (which are) prime to one another, and let some number $C$ measure $A$. I say that $C$ and $B$ are also prime to one another.

For if $C$ and $B$ are not prime to one another then [some] number will measure $C$ and $B$. Let it (so) measure (them), and let it be $D$. Since $D$ measures $C$, and $C$ measures $A, D$ thus also measures $A$. And ( $D$ ) also measures $B$. Thus, $D$ measures $A$ and $B$, which are prime to one another. The very thing is impossible. Thus, some number does not measure the numbers $C$ and $B$. Thus, $C$ and $B$ are prime to one another. (Which is) the very thing it was required to show.

## Proposition 24

If two numbers are prime to some number then the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).

$\Delta u ́ o ~ \gamma a ̀ p ~ \alpha ́ p ı \vartheta \mu o i ̀ ~ o i ~ A, ~ B ~ \pi p o ́ s ~ \tau ı v \alpha ~ \alpha ́ p ı \vartheta \mu o ̀ v ~ \tau o ̀ v ~ \Gamma ~ \pi \rho \widetilde{̃ \tau т ь ~}$










 $\tau \widetilde{\omega} \nu$ 和











## $\chi \varepsilon^{\prime}$.







For let $A$ and $B$ be two numbers (which are both) prime to some number $C$. And let $A$ make $D$ (by) multiplying $B$. I say that $C$ and $D$ are prime to one another.

For if $C$ and $D$ are not prime to one another then [some] number will measure $C$ and $D$. Let it (so) measure them, and let it be $E$. And since $C$ and $A$ are prime to one another, and some number $E$ measures $C, A$ and $E$ are thus prime to one another [Prop. 7.23]. So as many times as $E$ measures $D$, so many units let there be in $F$. Thus, $F$ also measures $D$ according to the units in $E$ [Prop. 7.16]. Thus, $E$ has made $D$ (by) multiplying $F$ [Def. 7.15]. But, in fact, $A$ has also made $D$ (by) multiplying $B$. Thus, the (number created) from (multiplying) $E$ and $F$ is equal to the (number created) from (multiplying) $A$ and $B$. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four numbers are proportional [Prop. 6.15]. Thus, as $E$ is to $A$, so $B$ (is) to $F$. And $A$ and $E$ (are) prime (to one another). And (numbers) prime (to one another) are also the least (of those numbers having the same ratio) [Prop. 7.21]. And the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $E$ measures $B$. And it also measures $C$. Thus, $E$ measures $B$ and $C$, which are prime to one another. The very thing is impossible. Thus, some number cannot measure the numbers $C$ and $D$. Thus, $C$ and $D$ are prime to one another. (Which is) the very thing it was required to show.

## Proposition 25

If two numbers are prime to one another then the number created from (squaring) one of them will be prime to the remaining (number).

Let $A$ and $B$ be two numbers (which are) prime to









$$
x G^{\prime}
$$


 $\pi \rho o ̀ s ~ \alpha \dot{\alpha} \lambda \eta$ ท́入ous है $\sigma o v \tau \alpha l$.


Zط



 трòs $\alpha \lambda \lambda$ ń $\lambda$ ous عíбív.








one another. And let $A$ make $C$ (by) multiplying itself. I say that $B$ and $C$ are prime to one another.


For let $D$ be made equal to $A$. Since $A$ and $B$ are prime to one another, and $A$ (is) equal to $D, D$ and $B$ are thus also prime to one another. Thus, $D$ and $A$ are each prime to $B$. Thus, the (number) created from (multilying) $D$ and $A$ will also be prime to $B$ [Prop. 7.24]. And $C$ is the number created from (multiplying) $D$ and $A$. Thus, $C$ and $B$ are prime to one another. (Which is) the very thing it was required to show.

## Proposition 26

If two numbers are both prime to each of two numbers then the (numbers) created from (multiplying) them will also be prime to one another.


For let two numbers, $A$ and $B$, both be prime to each of two numbers, $C$ and $D$. And let $A$ make $E$ (by) multiplying $B$, and let $C$ make $F$ (by) multiplying $D$. I say that $E$ and $F$ are prime to one another.

For since $A$ and $B$ are each prime to $C$, the (number) created from (multiplying) $A$ and $B$ will thus also be prime to $C$ [Prop. 7.24]. And $E$ is the (number) created from (multiplying) $A$ and $B$. Thus, $E$ and $C$ are prime to one another. So, for the same (reasons), $E$ and $D$ are also prime to one another. Thus, $C$ and $D$ are each prime to $E$. Thus, the (number) created from (multiplying) $C$ and $D$ will also be prime to $E$ [Prop. 7.24]. And $F$ is the (number) created from (multiplying) $C$ and $D$. Thus, $E$ and $F$ are prime to one another. (Which is) the very thing it was required to show.

$$
\chi \zeta^{\prime} .
$$





 тои̃то бицß $\alpha$ íveı].


 ठદ̀ $\Gamma \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Delta \pi o เ \varepsilon i ́ \tau \omega, ~ o ́ ~ \delta દ ̀ ~ B ~ \varepsilon ́ \alpha u \tau o ̀ \nu ~ \mu \varepsilon ̀ v ~$ $\pi \circ \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ v ~ E ~ \pi o เ \varepsilon i ́ \tau \omega, ~ \tau o ̀ v ~ \delta e ̀ ~ E ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma เ \alpha ́ \sigma \alpha \varsigma ~$
















## Proposition $27^{\dagger}$

If two numbers are prime to one another and each makes some (number by) multiplying itself then the numbers created from them will be prime to one another, and if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be prime to one another [and this always happens with the extremes].


Let $A$ and $B$ be two numbers prime to one another, and let $A$ make $C$ (by) multiplying itself, and let it make $D$ (by) multiplying $C$. And let $B$ make $E$ (by) multiplying itself, and let it make $F$ by multiplying $E$. I say that $C$ and $E$, and $D$ and $F$, are prime to one another.

For since $A$ and $B$ are prime to one another, and $A$ has made $C$ (by) multiplying itself, $C$ and $B$ are thus prime to one another [Prop. 7.25]. Therefore, since $C$ and $B$ are prime to one another, and $B$ has made $E$ (by) multiplying itself, $C$ and $E$ are thus prime to one another [Prop. 7.25]. Again, since $A$ and $B$ are prime to one another, and $B$ has made $E$ (by) multiplying itself, $A$ and $E$ are thus prime to one another [Prop. 7.25]. Therefore, since the two numbers $A$ and $C$ are both prime to each of the two numbers $B$ and $E$, the (number) created from (multiplying) $A$ and $C$ is thus prime to the (number created) from (multiplying) $B$ and $E$ [Prop. 7.26]. And $D$ is the (number created) from (multiplying) $A$ and $C$, and $F$ the (number created) from (multiplying) $B$ and $E$. Thus, $D$ and $F$ are prime to one another. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In modern notation, this proposition states that if $a$ is prime to $b$, then $a^{2}$ is also prime to $b^{2}$, as well as $a^{3}$ to $b^{3}$, etc., where all symbols denote numbers.

$$
x r^{\prime}
$$






## Proposition 28

If two numbers are prime to one another then their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them then the original numbers will also be prime to one another.

 ous oi $\mathrm{AB}, \mathrm{B} \mathrm{\Gamma} \cdot \lambda \varepsilon ́ \gamma \omega$, ơtı xall $\sigma u v \alpha \mu \varphi o ́ \tau \varepsilon \rho о \varsigma ~ o ́ ~ A \Gamma ~ \pi \rho o ̀ s ~$



 $\mu \varepsilon \tau \rho \eta ́ \sigma \varepsilon \iota$. $\mu \varepsilon \tau \rho \varepsilon \imath ̃ ~ \delta e ̀ ~ x \alpha i ̀ ~ \tau o ̀ \nu ~ B A \cdot ~ o ́ ~ \Delta ~ \alpha ̈ p \alpha ~ \tau o u ̀ s ~ A B, ~ В Г ~ \mu \varepsilon-~$



 غ́x $\alpha \tau \varepsilon \rho о \nu \tau \widetilde{\omega} \nu \mathrm{AB}, ~ В Г ~ т р \tilde{\omega} \tau o ́ s ~ \varepsilon ̇ \sigma \tau \iota \nu . ~$










$$
\chi v^{\prime}
$$




 ǒtı oi $\mathrm{B}, \mathrm{A} \pi \rho \widetilde{\omega} \tau o l ~ \pi \rho o ̀ s ~ \alpha ~ \alpha \lambda \lambda \eta ́ \lambda o u s ~ \varepsilon i \sigma i ́ v . ~$









D
For let the two numbers, $A B$ and $B C$, (which are) prime to one another, be laid down together. I say that their sum $A C$ is also prime to each of $A B$ and $B C$.

For if $C A$ and $A B$ are not prime to one another then some number will measure $C A$ and $A B$. Let it (so) measure (them), and let it be $D$. Therefore, since $D$ measures $C A$ and $A B$, it will thus also measure the remainder $B C$. And it also measures $B A$. Thus, $D$ measures $A B$ and $B C$, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers $C A$ and $A B$. Thus, $C A$ and $A B$ are prime to one another. So, for the same (reasons), $A C$ and $C B$ are also prime to one another. Thus, $C A$ is prime to each of $A B$ and $B C$.

So, again, let $C A$ and $A B$ be prime to one another. I say that $A B$ and $B C$ are also prime to one another.

For if $A B$ and $B C$ are not prime to one another then some number will measure $A B$ and $B C$. Let it (so) measure (them), and let it be $D$. And since $D$ measures each of $A B$ and $B C$, it will thus also measure the whole of $C A$. And it also measures $A B$. Thus, $D$ measures $C A$ and $A B$, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers $A B$ and $B C$. Thus, $A B$ and $B C$ are prime to one another. (Which is) the very thing it was required to show.

## Proposition 29

Every prime number is prime to every number which it does not measure.


Let $A$ be a prime number, and let it not measure $B$. I say that $B$ and $A$ are prime to one another. For if $B$ and $A$ are not prime to one another then some number will measure them. Let $C$ measure (them). Since $C$ measures $B$, and $A$ does not measure $B, C$ is thus not the same as $A$. And since $C$ measures $B$ and $A$, it thus also measures $A$, which is prime, (despite) not being the same as it. The very thing is impossible. Thus, some number cannot measure (both) $B$ and $A$. Thus, $A$ and $B$ are prime to one another. (Which is) the very thing it was required to

## $\lambda^{\prime}$.






 $\Delta \cdot \lambda \varepsilon ́ \gamma \omega$, ơтı ó $\Delta$ ह́v $\nu \tau \widetilde{\omega} \nu \mathrm{A}, \mathrm{B} \mu \varepsilon \tau \rho \varepsilon \tau ̃$.














$\lambda \alpha^{\prime}$.
 трعїтац.



show.

## Proposition 30

If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).


For let two numbers $A$ and $B$ make $C$ (by) multiplying one another, and let some prime number $D$ measure $C$. I say that $D$ measures one of $A$ and $B$.

For let it not measure $A$. And since $D$ is prime, $A$ and $D$ are thus prime to one another [Prop. 7.29]. And as many times as $D$ measures $C$, so many units let there be in $E$. Therefore, since $D$ measures $C$ according to the units $E, D$ has thus made $C$ (by) multiplying $E$ [Def. 7.15]. But, in fact, $A$ has also made $C$ (by) multiplying $B$. Thus, the (number created) from (multiplying) $D$ and $E$ is equal to the (number created) from (multiplying) $A$ and $B$. Thus, as $D$ is to $A$, so $B$ (is) to $E$ [Prop. 7.19]. And $D$ and $A$ (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $D$ measures $B$. So, similarly, we can also show that if $(D)$ does not measure $B$ then it will measure $A$. Thus, $D$ measures one of $A$ and $B$. (Which is) the very thing it was required to show.

## Proposition 31

Every composite number is measured by some prime number.

Let $A$ be a composite number. I say that $A$ is measured by some prime number.

For since $A$ is composite, some number will measure it. Let it (so) measure ( $A$ ), and let it be $B$. And if $B$










 $\mu \varepsilon \tau р \grave{\sigma} \sigma \varepsilon$.




$$
\lambda \beta^{\prime} .
$$







Ei $\mu$ èv oưv $\pi \rho \tilde{\tau} \tau o ́ s ~ \varepsilon ̇ \sigma \tau \omega ~ o ́ ~ A, ~ \gamma \varepsilon \gamma o v o ̀ s ~ \alpha ̀ \nu ~ \varepsilon i n ~ \tau o ́ ~$





$$
\lambda \gamma^{\prime}
$$



"Eбт ${ }^{\text {T }}$
 toĩs $\mathrm{A}, \mathrm{B}, \Gamma$.



is prime then that which was prescribed has happened. And if ( $B$ is) composite then some number will measure it. Let it (so) measure ( $B$ ), and let it be $C$. And since $C$ measures $B$, and $B$ measures $A, C$ thus also measures $A$. And if $C$ is prime then that which was prescribed has happened. And if ( $C$ is) composite then some number will measure it. So, in this manner of continued investigation, some prime number will be found which will measure (the number preceding it, which will also measure $A$ ). And if (such a number) cannot be found then an infinite (series of) numbers, each of which is less than the preceding, will measure the number $A$. The very thing is impossible for numbers. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure $A$.


Thus, every composite number is measured by some prime number. (Which is) the very thing it was required to show.

## Proposition 32

Every number is either prime or is measured by some prime number.

## A

Let $A$ be a number. I say that $A$ is either prime or is measured by some prime number.

In fact, if $A$ is prime then that which was prescribed has happened. And if (it is) composite then some prime number will measure it [Prop. 7.31].

Thus, every number is either prime or is measured by some prime number. (Which is) the very thing it was required to show.

## Proposition 33

To find the least of those (numbers) having the same ratio as any given multitude of numbers.

Let $A, B$, and $C$ be any given multitude of numbers. So it is required to find the least of those (numbers) having the same ratio as $A, B$, and $C$.

For $A, B$, and $C$ are either prime to one another, or not. In fact, if $A, B$, and $C$ are prime to one another then they are the least of those (numbers) having the same ratio as them [Prop. 7.22].

 $\mu \varepsilon ́ t \rho o \nu ~ o ́ ~ \Delta, \chi \alpha i ̀ ~ o ́ \sigma \alpha ́ x ı s ~ o ̀ ~ \Delta ~ \varepsilon ̌ x \alpha \sigma \tau o \nu ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma \mu \varepsilon \tau \rho \varepsilon \tilde{\imath}$,



























## $\lambda \delta^{\prime}$.

 ápıひuóv.
"E $\sigma \tau \omega \sigma \alpha \nu$ oí $\delta$ o७


And if not, let the greatest common measure, $D$, of $A, B$, and $C$ have be taken [Prop. 7.3]. And as many times as $D$ measures $A, B, C$, so many units let there be in $E, F, G$, respectively. And thus $E, F, G$ measure $A, B, C$, respectively, according to the units in $D$ [Prop. 7.15]. Thus, $E, F, G$ measure $A, B, C$ (respectively) an equal number of times. Thus, $E, F, G$ are in the same ratio as $A, B, C$ (respectively) [Def. 7.20]. So I say that (they are) also the least (of those numbers having the same ratio as $A, B, C$ ). For if $E, F, G$ are not the least of those (numbers) having the same ratio as $A$, $B, C$ (respectively), then there will be [some] numbers less than $E, F, G$ which are in the same ratio as $A, B, C$ (respectively). Let them be $H, K, L$. Thus, $H$ measures $A$ the same number of times that $K, L$ also measure $B$, $C$, respectively. And as many times as $H$ measures $A$, so many units let there be in $M$. Thus, $K, L$ measure $B$, $C$, respectively, according to the units in $M$. And since $H$ measures $A$ according to the units in $M, M$ thus also measures $A$ according to the units in $H$ [Prop. 7.15]. So, for the same (reasons), $M$ also measures $B, C$ according to the units in $K$, $L$, respectively. Thus, $M$ measures $A, B$, and $C$. And since $H$ measures $A$ according to the units in $M, H$ has thus made $A$ (by) multiplying $M$. So, for the same (reasons), $E$ has also made $A$ (by) multiplying $D$. Thus, the (number created) from (multiplying) $E$ and $D$ is equal to the (number created) from (multiplying) $H$ and $M$. Thus, as $E$ (is) to $H$, so $M$ (is) to $D$ [Prop. 7.19]. And $E$ (is) greater than $H$. Thus, $M$ (is) also greater than $D$ [Prop. 5.13]. And ( $M$ ) measures $A, B$, and $C$. The very thing is impossible. For $D$ was assumed (to be) the greatest common measure of $A, B$, and $C$. Thus, there cannot be any numbers less than $E$, $F, G$ which are in the same ratio as $A, B, C$ (respectively). Thus, $E, F, G$ are the least of (those numbers) having the same ratio as $A, B, C$ (respectively). (Which is) the very thing it was required to show.

## Proposition 34

To find the least number which two given numbers (both) measure.

Let $A$ and $B$ be the two given numbers. So it is re-










 $\pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ t o ̀ v ~ \Delta ~ \pi \varepsilon \pi o i ́ \eta \ell \varepsilon v, ~ o ́ ~ o ̀ ̀ ~ B ~ t o ̀ v ~ Z ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma ~$










 $\mu \varepsilon \tau р \varepsilon і ั \tau \alpha$.




quired to find the least number which they (both) measure.


For $A$ and $B$ are either prime to one another, or not. Let them, first of all, be prime to one another. And let $A$ make $C$ (by) multiplying $B$. Thus, $B$ has also made $C$ (by) multiplying $A$ [Prop. 7.16]. Thus, $A$ and $B$ (both) measure $C$. So I say that ( $C$ ) is also the least (number which they both measure). For if not, $A$ and $B$ will (both) measure some (other) number which is less than $C$. Let them (both) measure $D$ (which is less than $C$ ). And as many times as $A$ measures $D$, so many units let there be in $E$. And as many times as $B$ measures $D$, so many units let there be in $F$. Thus, $A$ has made $D$ (by) multiplying $E$, and $B$ has made $D$ (by) multiplying $F$. Thus, the (number created) from (multiplying) $A$ and $E$ is equal to the (number created) from (multiplying) $B$ and $F$. Thus, as $A$ (is) to $B$, so $F$ (is) to $E$ [Prop. 7.19]. And $A$ and $B$ are prime (to one another), and prime (numbers) are the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, $B$ measures $E$, as the following (number measuring) the following. And since $A$ has made $C$ and $D$ (by) multiplying $B$ and $E$ (respectively), thus as $B$ is to $E$, so $C$ (is) to $D$ [Prop. 7.17]. And $B$ measures $E$. Thus, $C$ also measures $D$, the greater (measuring) the lesser. The very thing is impossible. Thus, $A$ and $B$ do not (both) measure some number which is less than $C$. Thus, $C$ is the least (number) which is measured by (both) $A$ and $B$.


So let $A$ and $B$ be not prime to one another. And let the least numbers, $F$ and $E$, have been taken having the same ratio as $A$ and $B$ (respectively) [Prop. 7.33].



 Г. $\mu \varepsilon \tau р \varepsilon i ́ \tau \omega \sigma \alpha \nu$ тòv $\Delta$. xג̀ ó $\sigma \alpha ́ x ı \varsigma ~ \mu \varepsilon ̀ \nu ~ o ̀ ~ A ~ \tau o ̀ \nu ~ \Delta ~ \mu \varepsilon \tau р \varepsilon і ̃, ~$









 A тoùs $\mathrm{E}, \mathrm{H} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma$ тoùs $\Gamma, \Delta \pi \varepsilon \pi o i ́ \eta \chi \varepsilon \nu$, है $\sigma \tau \iota$






## $\lambda \varepsilon^{\prime}$.

 ن́ $\pi^{\prime} \alpha \cup ๋ \tau \widetilde{\omega} \nu \mu \varepsilon \tau \rho о \cup ́ \mu \varepsilon \nu о \varsigma ~ \tau o ̀ \nu ~ \alpha u ̉ \tau o ̀ v ~ \mu \varepsilon \tau р ท ́ \sigma \varepsilon เ . ~$


 $\mu \varepsilon \tau р \varepsilon і ̃$.








Thus, the (number created) from (multiplying) $A$ and $E$ is equal to the (number created) from (multiplying) $B$ and $F$ [Prop. 7.19]. And let $A$ make $C$ (by) multiplying $E$. Thus, $B$ has also made $C$ (by) multiplying $F$. Thus, $A$ and $B$ (both) measure $C$. So I say that ( $C$ ) is also the least (number which they both measure). For if not, $A$ and $B$ will (both) measure some number which is less than $C$. Let them (both) measure $D$ (which is less than $C$ ). And as many times as $A$ measures $D$, so many units let there be in $G$. And as many times as $B$ measures $D$, so many units let there be in $H$. Thus, $A$ has made $D$ (by) multiplying $G$, and $B$ has made $D$ (by) multiplying $H$. Thus, the (number created) from (multiplying) $A$ and $G$ is equal to the (number created) from (multiplying) $B$ and $H$. Thus, as $A$ is to $B$, so $H$ (is) to $G$ [Prop. 7.19]. And as $A$ (is) to $B$, so $F$ (is) to $E$. Thus, also, as $F$ (is) to $E$, so $H$ (is) to $G$. And $F$ and $E$ are the least (numbers having the same ratio as $A$ and $B$ ), and the least (numbers) measure those (numbers) having the same ratio an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, $E$ measures $G$. And since $A$ has made $C$ and $D$ (by) multiplying $E$ and $G$ (respectively), thus as $E$ is to $G$, so $C$ (is) to $D$ [Prop. 7.17]. And $E$ measures $G$. Thus, $C$ also measures $D$, the greater (measuring) the lesser. The very thing is impossible. Thus, $A$ and $B$ do not (both) measure some (number) which is less than $C$. Thus, $C$ (is) the least (number) which is measured by (both) $A$ and $B$. (Which is) the very thing it was required to show.

## Proposition 35

If two numbers (both) measure some number then the least (number) measured by them will also measure the same (number).


For let two numbers, $A$ and $B$, (both) measure some number $C D$, and (let) $E$ (be the) least (number measured by both $A$ and $B$ ). I say that $E$ also measures $C D$.

For if $E$ does not measure $C D$ then let $E$ leave $C F$ less than itself (in) measuring $D F$. And since $A$ and $B$ (both) measure $E$, and $E$ measures $D F, A$ and $B$ will thus also measure $D F$. And ( $A$ and $B$ ) also measure the whole of $C D$. Thus, they will also measure the remainder $C F$, which is less than $E$. The very thing is impossible. Thus, $E$ cannot not measure $C D$. Thus, ( $E$ ) measures

$$
\lambda \varepsilon^{\prime} .
$$







 прótepov. $\mu \varepsilon \tau \rho o и ̃ \sigma 兀 ~$ dè xaì oî $\mathrm{A}, \mathrm{B}$ tòv $\Delta$ oi $\mathrm{A}, \mathrm{B}, \Gamma$

 той $\Delta$. $\mu \varepsilon \tau \rho \varepsilon i ́ \tau \omega \sigma \alpha \nu ~ \tau o ̀ v ~ E . ~ غ ̀ л \varepsilon i ̀ ~ o i ~ A, ~ B, ~ Г ~ \tau o ̀ v ~ E ~ \mu \varepsilon \tau р о и ̆ \sigma \tau, ~$























$(C D)$. (Which is) the very thing it was required to show.

## Proposition 36

To find the least number which three given numbers (all) measure.

Let $A, B$, and $C$ be the three given numbers. So it is required to find the least number which they (all) measure.


For let the least (number), $D$, measured by the two (numbers) $A$ and $B$ have been taken [Prop. 7.34]. So $C$ either measures, or does not measure, $D$. Let it, first of all, measure ( $D$ ). And $A$ and $B$ also measure $D$. Thus, $A, B$, and $C$ (all) measure $D$. So I say that ( $D$ is) also the least (number measured by $A, B$, and $C$ ). For if not, $A, B$, and $C$ will (all) measure [some] number which is less than $D$. Let them measure $E$ (which is less than $D$ ). Since $A, B$, and $C$ (all) measure $E$ then $A$ and $B$ thus also measure $E$. Thus, the least (number) measured by $A$ and $B$ will also measure [ $E$ ] [Prop. 7.35]. And $D$ is the least (number) measured by $A$ and $B$. Thus, $D$ will measure $E$, the greater (measuring) the lesser. The very thing is impossible. Thus, $A, B$, and $C$ cannot (all) measure some number which is less than $D$. Thus, $A, B$, and $C$ (all) measure the least (number) $D$.

So, again, let $C$ not measure $D$. And let the least number, $E$, measured by $C$ and $D$ have been taken [Prop. 7.34]. Since $A$ and $B$ measure $D$, and $D$ measures $E, A$ and $B$ thus also measure $E$. And $C$ also measures [ $E$ ]. Thus, $A, B$, and $C$ [also] measure $E$. So I say that ( $E$ is) also the least (number measured by $A, B$, and $C$ ). For if not, $A, B$, and $C$ will (all) measure some (number) which is less than $E$. Let them measure $F$ (which is less than $E$ ). Since $A, B$, and $C$ (all) measure $F, A$ and $B$ thus also measure $F$. Thus, the least (number) measured by $A$ and $B$ will also measure $F$ [Prop. 7.35]. And $D$ is the least (number) measured by $A$ and $B$. Thus, $D$ measures $F$. And $C$ also measures $F$. Thus, $D$ and $C$ (both) measure $F$. Hence, the least (number) measured by $D$ and $C$ will also measure $F$ [Prop. 7.35]. And $E$

## $\lambda \zeta^{\prime}$.





## $\Delta \longmapsto$














$$
\lambda \eta^{\prime} .
$$

 $\mu \varepsilon \tau \rho \eta \vartheta \dot{\eta} \sigma \varepsilon \tau \alpha \iota \tau \widetilde{\varphi} \mu \dot{\varepsilon} \rho \varepsilon \iota$.


 $\mu \varepsilon \tau р \varepsilon$ นั.


is the least (number) measured by $C$ and $D$. Thus, $E$ measures $F$, the greater (measuring) the lesser. The very thing is impossible. Thus, $A, B$, and $C$ cannot measure some number which is less than $E$. Thus, $E$ (is) the least (number) which is measured by $A, B$, and $C$. (Which is) the very thing it was required to show.

## Proposition 37

If a number is measured by some number then the (number) measured will have a part called the same as the measuring (number).


For let the number $A$ be measured by some number $B$. I say that $A$ has a part called the same as $B$.

For as many times as $B$ measures $A$, so many units let there be in $C$. Since $B$ measures $A$ according to the units in $C$, and the unit $D$ also measures $C$ according to the units in it, the unit $D$ thus measures the number $C$ as many times as $B$ (measures) $A$. Thus, alternately, the unit $D$ measures the number $B$ as many times as $C$ (measures) $A$ [Prop. 7.15]. Thus, which(ever) part the unit $D$ is of the number $B, C$ is also the same part of $A$. And the unit $D$ is a part of the number $B$ called the same as it (i.e., a $B$ th part). Thus, $C$ is also a part of $A$ called the same as $B$ (i.e., $C$ is the $B$ th part of $A$ ). Hence, $A$ has a part $C$ which is called the same as $B$ (i.e., $A$ has a $B$ th part). (Which is) the very thing it was required to show.

## Proposition 38

If a number has any part whatever then it will be measured by a number called the same as the part.


For let the number $A$ have any part whatever, $B$. And let the [number] $C$ be called the same as the part $B$ (i.e., $B$ is the $C$ th part of $A$ ). I say that $C$ measures $A$.

For since $B$ is a part of $A$ called the same as $C$, and the unit $D$ is also a part of $C$ called the same as it (i.e.,




$\lambda \vartheta^{\prime}$.





 трои́




 $\alpha \dot{\alpha} เ \vartheta \mu \widetilde{\omega} \nu \mu \varepsilon \tau \rho \eta \vartheta \dot{\eta} \sigma \varepsilon \tau \alpha \iota ~ \tau о і ̃ \varsigma ~ A, ~ B, ~ Г ~ \mu \varepsilon ́ p \varepsilon \sigma เ \nu . ~ \tau о і ̃ \varsigma ~ \delta غ ̀ ~ A, ~ B, ~$




$D$ is the $C$ th part of $C$ ), thus which(ever) part the unit $D$ is of the number $C, B$ is also the same part of $A$. Thus, the unit $D$ measures the number $C$ as many times as $B$ (measures) $A$. Thus, alternately, the unit $D$ measures the number $B$ as many times as $C$ (measures) $A$ [Prop. 7.15]. Thus, $C$ measures $A$. (Which is) the very thing it was required to show.

## Proposition 39

To find the least number that will have given parts.


Let $A, B$, and $C$ be the given parts. So it is required to find the least number which will have the parts $A, B$, and $C$ (i.e., an $A$ th part, a $B$ th part, and a $C$ th part).

For let $D, E$, and $F$ be numbers having the same names as the parts $A, B$, and $C$ (respectively). And let the least number, $G$, measured by $D, E$, and $F$, have been taken [Prop. 7.36].

Thus, $G$ has parts called the same as $D, E$, and $F$ [Prop. 7.37]. And $A, B$, and $C$ are parts called the same as $D, E$, and $F$ (respectively). Thus, $G$ has the parts $A$, $B$, and $C$. So I say that $(G)$ is also the least (number having the parts $A, B$, and $C$ ). For if not, there will be some number less than $G$ which will have the parts $A$, $B$, and $C$. Let it be $H$. Since $H$ has the parts $A, B$, and $C, H$ will thus be measured by numbers called the same as the parts $A, B$, and $C$ [Prop. 7.38]. And $D, E$, and $F$ are numbers called the same as the parts $A, B$, and $C$ (respectively). Thus, $H$ is measured by $D, E$, and $F$. And $(H)$ is less than $G$. The very thing is impossible. Thus, there cannot be some number less than $G$ which will have the parts $A, B$, and $C$. (Which is) the very thing it was required to show.

## ELEMENTS BOOK 8

## Continued Proportion

[^17]
## $\alpha^{\prime}$.










Ei $\gamma \dot{\alpha} \rho \mu \dot{\eta}$ ，हैठ $\tau \omega \sigma \alpha \nu$ ह̇入 $\alpha \tau \tau o v \varepsilon \varsigma ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma, \Delta$ oi E ，














## $\beta^{\prime}$.





 $\lambda о ́ \gamma \varphi$.
 $\pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Gamma ~ \pi o เ \varepsilon i ́ \tau \omega, ~ \tau o ̀ \nu ~ \delta e ̀ ~ B ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Delta ~$
 каі है兀ぃ ó A тoùs $\Gamma, \Delta$ ， $\mathrm{E} \pi$ т $\lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma ~ \tau o u ̀ s ~ Z, ~ H, ~ \Theta ~$


## Proposition 1

If there are any multitude whatsoever of continuously proportional numbers，and the outermost of them are prime to one another，then the（numbers）are the least of those（numbers）having the same ratio as them．


Let $A, B, C, D$ be any multitude whatsoever of con－ tinuously proportional numbers．And let the outermost of them，$A$ and $D$ ，be prime to one another．I say that $A, B, C, D$ are the least of those（numbers）having the same ratio as them．

For if not，let $E, F, G, H$ be less than $A, B, C, D$ （respectively），being in the same ratio as them．And since $A, B, C, D$ are in the same ratio as $E, F, G, H$ ，and the multitude［of $A, B, C, D$ ］is equal to the multitude［of $E$ ， $F, G, H]$ ，thus，via equality，as $A$ is to $D$ ，（so）$E$（is）to $H$ ［Prop．7．14］．And $A$ and $D$（are）prime（to one another）． And prime（numbers are）also the least of those（numbers having the same ratio as them）［Prop．7．21］．And the least numbers measure those（numbers）having the same ratio（as them）an equal number of times，the greater （measuring）the greater，and the lesser the lesser－that is to say，the leading（measuring）the leading，and the following the following［Prop．7．20］．Thus，$A$ measures $E$ ，the greater（measuring）the lesser．The very thing is impossible．Thus，$E, F, G, H$ ，being less than $A, B, C$ ， $D$ ，are not in the same ratio as them．Thus，$A, B, C, D$ are the least of those（numbers）having the same ratio as them．（Which is）the very thing it was required to show．

## Proposition 2

To find the least numbers，as many as may be pre－ scribed，（which are）continuously proportional in a given ratio．

Let the given ratio，（expressed）in the least numbers， be that of $A$ to $B$ ．So it is required to find the least num－ bers，as many as may be prescribed，（which are）in the ratio of $A$ to $B$ ．

Let four（numbers）have been prescribed．And let $A$ make $C$（by）multiplying itself，and let it make $D$（by） multiplying $B$ ．And，further，let $B$ make $E$（by）multiply－ ing itself．And，further，let $A$ make $F, G, H$（by）mul－ tiplying $C, D, E$ ．And let $B$ make $K$（by）multiplying E．








 äpa ó $\Gamma$ т $\quad$ òs tòv $\Delta$, ó $\Delta$ tpòs tòv E . xai èreì ó A toùs $\Gamma$,




 трòs tòv $\Theta$. $\dot{\alpha} \lambda \lambda^{\lambda} \dot{\omega} \varsigma$ s ó $\Delta$ tpòs tòv E , ó A tpòs tòv B . xaì




 tòv $\Theta$ xaì ó $\Theta$ tpòs tòv K oi $\Gamma, \Delta$, E äp $\alpha$ xaì oi Z, H ,















And since $A$ has made $C$ (by) multiplying itself, and has made $D$ (by) multiplying $B$, thus as $A$ is to $B$, [so] $C$ (is) to $D$ [Prop. 7.17]. Again, since $A$ has made $D$ (by) multiplying $B$, and $B$ has made $E$ (by) multiplying itself, $A, B$ have thus made $D, E$, respectively, (by) multiplying $B$. Thus, as $A$ is to $B$, so $D$ (is) to $E$ [Prop. 7.18]. But, as $A$ (is) to $B$, (so) $C$ (is) to $D$. And thus as $C$ (is) to $D$, (so) $D$ (is) to $E$. And since $A$ has made $F, G$ (by) multiplying $C, D$, thus as $C$ is to $D$, [so] $F$ (is) to $G$ [Prop. 7.17]. And as $C$ (is) to $D$, so $A$ was to $B$. And thus as $A$ (is) to $B$, (so) $F$ (is) to $G$. Again, since $A$ has made $G, H$ (by) multiplying $D, E$, thus as $D$ is to $E$, (so) $G$ (is) to $H$ [Prop. 7.17]. But, as $D$ (is) to $E$, (so) $A$ (is) to $B$. And thus as $A$ (is) to $B$, so $G$ (is) to $H$. And since $A, B$ have made $H, K$ (by) multiplying $E$, thus as $A$ is to $B$, so $H$ (is) to $K$. But, as $A$ (is) to $B$, so $F$ (is) to $G$, and $G$ to $H$. And thus as $F$ (is) to $G$, so $G$ (is) to $H$, and $H$ to $K$. Thus, $C, D, E$ and $F, G, H, K$ are (both continuously) proportional in the ratio of $A$ to $B$. So I say that (they are) also the least (sets of numbers continuously proportional in that ratio). For since $A$ and $B$ are the least of those (numbers) having the same ratio as them, and the least of those (numbers) having the same ratio are prime to one another [Prop. 7.22], $A$ and $B$ are thus prime to one another. And $A, B$ have made $C, E$, respectively, (by) multiplying themselves, and have made $F, K$ by multiplying $C, E$, respectively. Thus, $C, E$ and $F, K$ are prime to one another [Prop. 7.27]. And if there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them [Prop. 8.1]. Thus, $C$, $D, E$ and $F, G, H, K$ are the least of those (continuously proportional sets of numbers) having the same ratio as $A$ and $B$. (Which is) the very thing it was required to show.

## 



 xúßot.

$$
\gamma^{\prime}
$$


 $\pi \rho \widetilde{\omega} \tau o l ~ \pi \rho o ̀ s ~ \alpha ̀ \lambda \lambda \eta ́ \lambda o u s ~ \varepsilon i \sigma i ́ v . ~$







 N, $\Xi$.








 $\Delta \tau \widetilde{\mu} \pi \lambda \hat{\eta} \vartheta \varepsilon \iota \tau \tau \widetilde{\omega} \Lambda, \mathrm{M}, \mathrm{N}, \Xi, \varepsilon \approx \chi \alpha \sigma \tau o s \ddot{\alpha} \rho \alpha \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma$,





## Corollary

So it is clear, from this, that if three continuously proportional numbers are the least of those (numbers) having the same ratio as them then the outermost of them are square, and, if four (numbers), cube.

## Proposition 3

If there are any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them then the outermost of them are prime to one another.


Let $A, B, C, D$ be any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that the outermost of them, $A$ and $D$, are prime to one another.

For let the two least (numbers) $E, F$ (which are) in the same ratio as $A, B, C, D$ have been taken [Prop. 7.33]. And the three (least numbers) $G, H, K$ [Prop. 8.2]. And (so on), successively increasing by one, until the multitude of (numbers) taken is made equal to the multitude of $A, B, C, D$. Let them have been taken, and let them be $L, M, N, O$.

And since $E$ and $F$ are the least of those (numbers) having the same ratio as them they are prime to one another [Prop. 7.22]. And since $E, F$ have made $G, K$, respectively, (by) multiplying themselves [Prop. 8.2 corr.], and have made $L, O$ (by) multiplying $G, K$, respectively, $G, K$ and $L, O$ are thus also prime to one another [Prop. 7.27]. And since $A, B, C, D$ are the least of those (numbers) having the same ratio as them, and $L, M, N$, $O$ are also the least (of those numbers having the same ratio as them), being in the same ratio as $A, B, C, D$, and the multitude of $A, B, C, D$ is equal to the multitude of $L, M, N, O$, thus $A, B, C, D$ are equal to $L, M, N, O$, respectively. Thus, $A$ is equal to $L$, and $D$ to $O$. And $L$ and $O$ are prime to one another. Thus, $A$ and $D$ are also prime to one another. (Which is) the very thing it was

## $\delta^{\prime}$.

 ג́pıখ入óүols.




 $\Gamma \pi \rho o ̀ s ~ t o ̀ v \Delta x \alpha i l ~ ह ै \tau \iota ~ \tau \widetilde{̣}$ toũ E трòs tòv Z .













 $\Delta$ xaì ह̀v т巛̛ тoũ E тpòs tòv Z خóүols, है $\sigma \tau \omega \sigma \alpha \nu$ oí $\mathrm{N}, \Xi$,












required to show.

## Proposition 4

For any multitude whatsoever of given ratios, (expressed) in the least numbers, to find the least numbers continuously proportional in these given ratios.


Let the given ratios, (expressed) in the least numbers, be the (ratios) of $A$ to $B$, and of $C$ to $D$, and, further, of $E$ to $F$. So it is required to find the least numbers continuously proportional in the ratio of $A$ to $B$, and of $C$ to $B$, and, further, of $E$ to $F$.

For let the least number, $G$, measured by (both) $B$ and $C$ have be taken [Prop. 7.34]. And as many times as $B$ measures $G$, so many times let $A$ also measure $H$. And as many times as $C$ measures $G$, so many times let $D$ also measure $K$. And $E$ either measures, or does not measure, $K$. Let it, first of all, measure ( $K$ ). And as many times as $E$ measures $K$, so many times let $F$ also measure $L$. And since $A$ measures $H$ the same number of times that $B$ also (measures) $G$, thus as $A$ is to $B$, so $H$ (is) to $G$ [Def. 7.20, Prop. 7.13]. And so, for the same (reasons), as $C$ (is) to $D$, so $G$ (is) to $K$, and, further, as $E$ (is) to $F$, so $K$ (is) to $L$. Thus, $H, G, K, L$ are continuously proportional in the ratio of $A$ to $B$, and of $C$ to $D$, and, further, of $E$ to $F$. So I say that (they are) also the least (numbers continuously proportional in these ratios). For if $H, G, K$, $L$ are not the least numbers continuously proportional in the ratios of $A$ to $B$, and of $C$ to $D$, and of $E$ to $F$, let $N$, $O, M, P$ be (the least such numbers). And since as $A$ is to $B$, so $N$ (is) to $O$, and $A$ and $B$ are the least (numbers which have the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20], $B$ thus measures $O$. So, for the same (reasons), $C$ also measures $O$. Thus, $B$ and $C$ (both) measure $O$. Thus, the least number measured by (both) $B$ and $C$ will also measure $O$ [Prop. 7.35]. And $G$ (is) the least number measured by (both) $B$ and $C$.
































Thus, $G$ measures $O$, the greater (measuring) the lesser. The very thing is impossible. Thus, there cannot be any numbers less than $H, G, K, L$ (which are) continuously (proportional) in the ratio of $A$ to $B$, and of $C$ to $D$, and, further, of $E$ to $F$.


So let $E$ not measure $K$. And let the least number, $M$, measured by (both) $E$ and $K$ have been taken [Prop. 7.34]. And as many times as $K$ measures $M$, so many times let $H, G$ also measure $N, O$, respectively. And as many times as $E$ measures $M$, so many times let $F$ also measure $P$. Since $H$ measures $N$ the same number of times as $G$ (measures) $O$, thus as $H$ is to $G$, so $N$ (is) to $O$ [Def. 7.20, Prop. 7.13]. And as $H$ (is) to $G$, so $A$ (is) to $B$. And thus as $A$ (is) to $B$, so $N$ (is) to $O$. And so, for the same (reasons), as $C$ (is) to $D$, so $O$ (is) to $M$. Again, since $E$ measures $M$ the same number of times as $F$ (measures) $P$, thus as $E$ is to $F$, so $M$ (is) to $P$ [Def. 7.20, Prop. 7.13]. Thus, $N, O, M, P$ are continuously proportional in the ratios of $A$ to $B$, and of $C$ to $D$, and, further, of $E$ to $F$. So I say that (they are) also the least (numbers) in the ratios of $A B, C D$, $E F$. For if not, then there will be some numbers less than $N, O, M, P$ (which are) continuously proportional in the ratios of $A B, C D, E F$. Let them be $Q, R, S$, $T$. And since as $Q$ is to $R$, so $A$ (is) to $B$, and $A$ and $B$ (are) the least (numbers having the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20], $B$ thus measures $R$. So, for the same (reasons), $C$ also measures $R$. Thus, $B$ and $C$ (both) measure $R$. Thus, the least (number) measured by (both) $B$ and $C$ will also measure $R$ [Prop. 7.35]. And $G$ is the least number measured by (both) $B$ and $C$. Thus, $G$ measures $R$. And as $G$ is to $R$, so $K$ (is) to $S$. Thus,





## $\varepsilon^{\prime}$.

 $\sigma \cup \gamma \varkappa \varepsilon \dot{\prime} \mu \varepsilon \nu \circ \nu$ غ่ $\chi \tau \widetilde{\omega} \nu \pi \lambda \varepsilon \cup \rho \widetilde{\omega} \nu$.



 $\pi \lambda \varepsilon \cup \rho \widetilde{\sim} \nu$.


 tòv E , oứtus tòv H tpòs tòv $\Theta$, ìs ठè tòv $\Delta$ tpòs tòv Z ,
 тòv $\Lambda \pi$ тьєít $\omega$.



 $\mathrm{H} \pi \rho o ̀ \varsigma ~ \tau o ̀ v ~ \Theta, ~ o u ́ \tau \omega \varsigma ~ o ́ ~ A ~ \pi \rho o ̀ s ~ \tau o ̀ v ~ \Lambda . ~ \pi \alpha ́ \lambda ı \imath, ~ \varepsilon ̇ \pi \varepsilon i ̀ ~ o ́ ~ E ~ \tau o ̀ v ~$





 [oưt $\omega \varsigma$ ] ó A rpòs tòv B. ó סè H tpòs tòv K 入óүov है $\chi \varepsilon \iota$
$K$ also measures $S$ [Def. 7.20]. And $E$ also measures $S$ [Prop. 7.20]. Thus, $E$ and $K$ (both) measure $S$. Thus, the least (number) measured by (both) $E$ and $K$ will also measure $S$ [Prop. 7.35]. And $M$ is the least (number) measured by (both) $E$ and $K$. Thus, $M$ measures $S$, the greater (measuring) the lesser. The very thing is impossible. Thus there cannot be any numbers less than $N, O$, $M, P$ (which are) continuously proportional in the ratios of $A$ to $B$, and of $C$ to $D$, and, further, of $E$ to $F$. Thus, $N, O, M, P$ are the least (numbers) continuously proportional in the ratios of $A B, C D, E F$. (Which is) the very thing it was required to show.

## Proposition 5

Plane numbers have to one another the ratio compounded ${ }^{\dagger}$ out of (the ratios of) their sides.


Let $A$ and $B$ be plane numbers, and let the numbers $C, D$ be the sides of $A$, and (the numbers) $E, F$ (the sides) of $B$. I say that $A$ has to $B$ the ratio compounded out of (the ratios of) their sides.

For given the ratios which $C$ has to $E$, and $D$ (has) to $F$, let the least numbers, $G, H, K$, continuously proportional in the ratios $C E, D F$ have been taken [Prop. 8.4], so that as $C$ is to $E$, so $G$ (is) to $H$, and as $D$ (is) to $F$, so $H$ (is) to $K$. And let $D$ make $L$ (by) multiplying $E$.

And since $D$ has made $A$ (by) multiplying $C$, and has made $L$ (by) multiplying $E$, thus as $C$ is to $E$, so $A$ (is) to $L$ [Prop. 7.17]. And as $C$ (is) to $E$, so $G$ (is) to $H$. And thus as $G$ (is) to $H$, so $A$ (is) to $L$. Again, since $E$ has made $L$ (by) multiplying $D$ [Prop. 7.16], but, in fact, has also made $B$ (by) multiplying $F$, thus as $D$ is to $F$, so $L$ (is) to $B$ [Prop. 7.17]. But, as $D$ (is) to $F$, so $H$ (is) to $K$. And thus as $H$ (is) to $K$, so $L$ (is) to $B$. And it was also shown that as $G$ (is) to $H$, so $A$ (is) to $L$. Thus, via equality, as $G$ is to $K$, [so] $A$ (is) to $B$ [Prop. 7.14]. And $G$ has to $K$ the ratio compounded out of (the ratios of) the sides (of $A$ and $B$ ). Thus, $A$ also has to $B$ the ratio compounded out of (the ratios of) the sides (of $A$ and $B$ ).

 ठєїそац.
$\dagger$ i.e., multiplied.

$$
\varsigma^{\prime} .
$$


 $\mu \varepsilon \tau \rho \grave{\eta} \sigma \varepsilon$.


H
$\Theta$
 $\Gamma, \Delta, \mathrm{E}$, ó ठ̀̀ A tòv $\mathrm{B} \mu \grave{\eta} \mu \varepsilon \tau \rho \varepsilon i \tau \omega \cdot \lambda \varepsilon ́ \gamma \omega$, ơtı oủס̀̀ $\alpha \lambda \lambda o \varsigma$





 $\lambda o ́ \gamma o v ~ \varepsilon ̇ \chi o ́ v \tau \omega \nu ~ \tau o i ̃ s ~ A, ~ B, ~ \Gamma ~ o i ~ Z, ~ H, ~ \Theta . ~ x \alpha i ̀ ~ \varepsilon ̇ \pi \varepsilon i ̀ ~ o i ~ Z, ~$











## $\zeta^{\prime}$.

${ }^{\circ} \mathrm{E} \dot{\alpha} \nu$ ஸ̃бเ

(Which is) the very thing it was required to show.

## Proposition 6

If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other (number) will measure any other (number) either.


Let $A, B, C, D, E$ be any multitude whatsoever of continuously proportional numbers, and let $A$ not measure $B$. I say that no other (number) will measure any other (number) either.

Now, (it is) clear that $A, B, C, D, E$ do not successively measure one another. For $A$ does not even measure $B$. So I say that no other (number) will measure any other (number) either. For, if possible, let $A$ measure $C$. And as many (numbers) as are $A, B, C$, let so many of the least numbers, $F, G, H$, have been taken of those (numbers) having the same ratio as $A, B, C$ [Prop. 7.33]. And since $F, G, H$ are in the same ratio as $A, B, C$, and the multitude of $A, B, C$ is equal to the multitude of $F$, $G, H$, thus, via equality, as $A$ is to $C$, so $F$ (is) to $H$ [Prop. 7.14]. And since as $A$ is to $B$, so $F$ (is) to $G$, and $A$ does not measure $B, F$ does not measure $G$ either [Def. 7.20]. Thus, $F$ is not a unit. For a unit measures all numbers. And $F$ and $H$ are prime to one another [Prop. 8.3] [and thus $F$ does not measure $H$ ]. And as $F$ is to $H$, so $A$ (is) to $C$. And thus $A$ does not measure $C$ either [Def. 7.20]. So, similarly, we can show that no other (number) can measure any other (number) either. (Which is) the very thing it was required to show.

## Proposition 7

If there are any multitude whatsoever of [continuously] proportional numbers, and the first measures the

 $\Delta$, ó ò̀ A tòv $\Delta \mu \varepsilon \tau \rho \varepsilon i ́ \tau \omega \cdot \lambda \varepsilon ́ \gamma \omega$, ơtı xà ò A tòv $\mathrm{B} \mu \varepsilon \tau \rho \varepsilon і ̃ . ~$




$$
\eta^{\prime}
$$




 $\alpha \nu \alpha ́ \lambda o \gamma o v ~ \varepsilon ̇ \mu \pi \varepsilon \sigma о и ̃ \nu \tau \alpha \iota$

$\mathrm{K} \longmapsto$
$\Lambda \longmapsto$
$\Delta$ úo $\gamma \grave{\alpha} \rho \dot{\alpha} p ı \vartheta \mu \widetilde{\omega} \nu \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B} \mu \varepsilon \tau \alpha \xi \dot{u} \chi \alpha \tau \dot{\alpha}$ tò $\sigma \cup v \varepsilon \chi$ ह̀ऽ













last, then (the first) will also measure the second.

$\mathrm{B} \longmapsto$


Let $A, B, C, D$ be any number whatsoever of continuously proportional numbers. And let $A$ measure $D$. I say that $A$ also measures $B$.

For if $A$ does not measure $B$ then no other (number) will measure any other (number) either [Prop. 8.6]. But $A$ measures $D$. Thus, $A$ also measures $B$. (Which is) the very thing it was required to show.

## Proposition 8

If between two numbers there fall (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall in between (any two numbers) having the same ratio [as them] in continued proportion.
$L \longmapsto$

For let the numbers, $C$ and $D$, fall between two numbers, $A$ and $B$, in continued proportion, and let it have been contrived (that) as $A$ (is) to $B$, so $E$ (is) to $F$. I say that, as many numbers as have fallen in between $A$ and $B$ in continued proportion, so many (numbers) will also fall in between $E$ and $F$ in continued proportion.

For as many as $A, B, C, D$ are in multitude, let so many of the least numbers, $G, H, K, L$, having the same ratio as $A, B, C, D$, have been taken [Prop. 7.33]. Thus, the outermost of them, $G$ and $L$, are prime to one another [Prop. 8.3]. And since $A, B, C, D$ are in the same ratio as $G, H, K, L$, and the multitude of $A, B, C, D$ is equal to the multitude of $G, H, K, L$, thus, via equality, as $A$ is to $B$, so $G$ (is) to $L$ [Prop. 7.14]. And as $A$ (is) to $B$, so






 $\tau \widetilde{\omega} \mathrm{M}, \mathrm{N} \mu \varepsilon \tau \rho \varepsilon i ́ \tau \omega$ oi $\mathrm{H}, \Theta, \mathrm{K}, \Lambda$ äp $\alpha$ тoùs $\mathrm{E}, \mathrm{M}, \mathrm{N}, \mathrm{Z}$





 ג́pıখ


## $\vartheta^{\prime}$.







 B, xaì عis aủtoùs $\mu \varepsilon \tau \alpha \xi i ̀ ~ x \alpha \tau \alpha ̀ ~ \tau o ̀ ~ \sigma u v \varepsilon \chi e ̀ s ~ \alpha ̀ v a ́ \lambda o \gamma o v ~$


 B x $\alpha \grave{\imath} \tau \tilde{\eta} \varsigma ~ \mu o v \alpha ́ \delta o s ~ \mu \varepsilon \tau \alpha \xi ் ~ x \alpha \tau \alpha ̀ ~ \tau o ̀ ~ \sigma u v e \chi દ ̀ \varsigma ~ \alpha ~ \alpha \nu \alpha ́ \lambda o \gamma o v ~$ غ่ $\mu \pi \varepsilon \sigma о$ ũvтаl.


$E$ (is) to $F$. And thus as $G$ (is) to $L$, so $E$ (is) to $F$. And $G$ and $L$ (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser-that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $G$ measures $E$ the same number of times as $L$ (measures) $F$. So as many times as $G$ measures $E$, so many times let $H, K$ also measure $M$, $N$, respectively. Thus, $G, H, K, L$ measure $E, M, N$, $F$ (respectively) an equal number of times. Thus, $G, H$, $K, L$ are in the same ratio as $E, M, N, F$ [Def. 7.20]. But, $G, H, K, L$ are in the same ratio as $A, C, D, B$. Thus, $A, C, D, B$ are also in the same ratio as $E, M, N$, $F$. And $A, C, D, B$ are continuously proportional. Thus, $E, M, N, F$ are also continuously proportional. Thus, as many numbers as have fallen in between $A$ and $B$ in continued proportion, so many numbers have also fallen in between $E$ and $F$ in continued proportion. (Which is) the very thing it was required to show.

## Proposition 9

If two numbers are prime to one another and there fall in between them (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall between each of them and a unit in continued proportion.


Let $A$ and $B$ be two numbers (which are) prime to one another, and let the (numbers) $C$ and $D$ fall in between them in continued proportion. And let the unit $E$ be set out. I say that, as many numbers as have fallen in between $A$ and $B$ in continued proportion, so many (numbers) will also fall between each of $A$ and $B$ and the unit in continued proportion.

For let the least two numbers, $F$ and $G$, which are in the ratio of $A, C, D, B$, have been taken [Prop. 7.33].
 $\pi \lambda \dot{\eta} \vartheta \varepsilon \iota \tau \widetilde{\omega} \nu \mathrm{A}, \Gamma, \Delta$, B. عỉn' $\varphi \vartheta \omega \sigma \alpha \nu, \chi \alpha i$ है $\sigma \tau \omega \sigma \alpha \nu$ oi $\mathrm{M}, \mathrm{N}$,
























 H tpòs tòv $\Lambda$ xaì ó $\Lambda$ tpòs tòv B . ơool äp $\alpha$ हís toùs $\mathrm{A}, \mathrm{B}$





## $i^{\prime}$.




 tò $\sigma u v \varepsilon \chi$ èऽ $\dot{\alpha} v \alpha ́ \lambda o \gamma o v ~ \varepsilon ̇ ่ \mu \pi \varepsilon \sigma o u ̃ \nu \tau \alpha l . ~$







And the (least) three (numbers), $H, K, L$. And so on, successively increasing by one, until the multitude of the (least numbers taken) is made equal to the multitude of $A, C, D, B$ [Prop. 8.2]. Let them have been taken, and let them be $M, N, O, P$. So (it is) clear that $F$ has made $H$ (by) multiplying itself, and has made $M$ (by) multiplying $H$. And $G$ has made $L$ (by) multiplying itself, and has made $P$ (by) multiplying $L$ [Prop. 8.2 corr.]. And since $M, N, O, P$ are the least of those (numbers) having the same ratio as $F, G$, and $A, C, D, B$ are also the least of those (numbers) having the same ratio as $F, G$ [Prop. 8.2], and the multitude of $M, N, O, P$ is equal to the multitude of $A, C, D, B$, thus $M, N, O, P$ are equal to $A, C, D, B$, respectively. Thus, $M$ is equal to $A$, and $P$ to $B$. And since $F$ has made $H$ (by) multiplying itself, $F$ thus measures $H$ according to the units in $F$ [Def. 7.15]. And the unit $E$ also measures $F$ according to the units in it. Thus, the unit $E$ measures the number $F$ as many times as $F$ (measures) $H$. Thus, as the unit $E$ is to the number $F$, so $F$ (is) to $H$ [Def. 7.20]. Again, since $F$ has made $M$ (by) multiplying $H, H$ thus measures $M$ according to the units in $F$ [Def. 7.15]. And the unit $E$ also measures the number $F$ according to the units in it. Thus, the unit $E$ measures the number $F$ as many times as $H$ (measures) $M$. Thus, as the unit $E$ is to the number $F$, so $H$ (is) to $M$ [Prop. 7.20]. And it was shown that as the unit $E$ (is) to the number $F$, so $F$ (is) to $H$. And thus as the unit $E$ (is) to the number $F$, so $F$ (is) to $H$, and $H$ (is) to $M$. And $M$ (is) equal to $A$. Thus, as the unit $E$ is to the number $F$, so $F$ (is) to $H$, and $H$ to $A$. And so, for the same (reasons), as the unit $E$ (is) to the number $G$, so $G$ (is) to $L$, and $L$ to $B$. Thus, as many (numbers) as have fallen in between $A$ and $B$ in continued proportion, so many numbers have also fallen between each of $A$ and $B$ and the unit $E$ in continued proportion. (Which is) the very thing it was required to show.

## Proposition 10

If (some) numbers fall between each of two numbers and a unit in continued proportion then, as many (numbers) as fall between each of the (two numbers) and the unit in continued proportion, so many (numbers) will also fall in between the (two numbers) themselves in continued proportion.

For let the numbers $D, E$ and $F, G$ fall between the numbers $A$ and $B$ (respectively) and the unit $C$ in continued proportion. I say that, as many numbers as have fallen between each of $A$ and $B$ and the unit $C$ in continued proportion, so many will also fall in between $A$ and $B$ in continued proportion.


















 $\Theta$ tpòs tòv $H$. xaì cos äp $\alpha$ ó E tpòs tòv $\Theta$, oütcus ó $\Theta$ tpòs






 A tpòs tòv $\mathrm{K} \cdot$ xaì cos äp $\alpha$ ó A tpòs tòv K , oüt tos ó K tpòs


 ó $\Delta$ tpòs tòv Z xaì ís äpa ó $\Delta$ tpòs tòv Z , oütcus ó $\Lambda$

 tòv K , oüt $\omega$ ऽ ó K tpòs tòv $\Lambda$ xaì ò $\Lambda$ tpòs tòv B . oi A ,






For let $D$ make $H$ (by) multiplying $F$. And let $D, F$ make $K, L$, respectively, by multiplying $H$.

As since as the unit $C$ is to the number $D$, so $D$ (is) to $E$, the unit $C$ thus measures the number $D$ as many times as $D$ (measures) $E$ [Def. 7.20]. And the unit $C$ measures the number $D$ according to the units in $D$. Thus, the number $D$ also measures $E$ according to the units in $D$. Thus, $D$ has made $E$ (by) multiplying itself. Again, since as the [unit] $C$ is to the number $D$, so $E$ (is) to $A$, the unit $C$ thus measures the number $D$ as many times as $E$ (measures) $A$ [Def. 7.20]. And the unit $C$ measures the number $D$ according to the units in $D$. Thus, $E$ also measures $A$ according to the units in $D$. Thus, $D$ has made $A$ (by) multiplying $E$. And so, for the same (reasons), $F$ has made $G$ (by) multiplying itself, and has made $B$ (by) multiplying $G$. And since $D$ has made $E$ (by) multiplying itself, and has made $H$ (by) multiplying $F$, thus as $D$ is to $F$, so $E$ (is) to $H$ [Prop 7.17]. And so, for the same reasons, as $D$ (is) to $F$, so $H$ (is) to $G$ [Prop. 7.18]. And thus as $E$ (is) to $H$, so $H$ (is) to $G$. Again, since $D$ has made $A, K$ (by) multiplying $E, H$, respectively, thus as $E$ is to $H$, so $A$ (is) to $K$ [Prop 7.17]. But, as $E$ (is) to $H$, so $D$ (is) to $F$. And thus as $D$ (is) to $F$, so $A$ (is) to $K$. Again, since $D, F$ have made $K, L$, respectively, (by) multiplying $H$, thus as $D$ is to $F$, so $K$ (is) to $L$ [Prop. 7.18]. But, as $D$ (is) to $F$, so $A$ (is) to $K$. And thus as $A$ (is) to $K$, so $K$ (is) to $L$. Further, since $F$ has made $L, B$ (by) multiplying $H, G$, respectively, thus as $H$ is to $G$, so $L$ (is) to $B$ [Prop 7.17]. And as $H$ (is) to $G$, so $D$ (is) to $F$. And thus as $D$ (is) to $F$, so $L$ (is) to $B$. And it was also shown that as $D$ (is) to $F$, so $A$ (is) to $K$, and $K$ to $L$. And thus as $A$ (is) to $K$, so $K$ (is) to $L$, and $L$ to $B$. Thus, $A, K$, $L, B$ are successively in continued proportion. Thus, as many numbers as fall between each of $A$ and $B$ and the unit $C$ in continued proportion, so many will also fall in between $A$ and $B$ in continued proportion. (Which is) the very thing it was required to show.


## $\alpha^{\prime}$.


 $\pi \lambda \alpha \sigma$ iov $\alpha$ 入óүov عै $\chi \varepsilon l ~ ク ̆ \pi \varepsilon \rho ~ \dot{\eta} \pi \lambda \varepsilon u p \alpha ̀ ~ \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \pi \lambda \varepsilon u \rho \alpha ́ \nu$.


E
 $\pi \lambda \varepsilon \cup p \alpha ̀ ~ \varepsilon ै \sigma \tau \omega ~ o ́ ~ \Gamma, ~ \tau о \tilde{~} \delta \varepsilon ̀ ~ B ~ o ́ \Delta \cdot \lambda \varepsilon ́ \gamma \omega$ ，ơ $\tau \tau \tau \widetilde{\omega} \mathrm{A}, \mathrm{B}$ हiॅร $\mu \varepsilon ́ \sigma o s ~ \alpha ̀ v \alpha ́ \lambda o \gamma o ́ v ~ \varepsilon ̇ \sigma \tau ו \nu ~ \alpha ̉ p ı \vartheta \mu o ́ s, ~ x \alpha i ~ o ̀ ~ A ~ \pi p o ̀ s ~ t o ̀ v ~ B ~ \delta t-~$ $\pi \lambda \alpha \sigma$ iova $\lambda$ óүov है $\chi \varepsilon ا$ ク̆ $\pi \varepsilon \rho$ ó $\Gamma$ трòs tòv $\Delta$ ．












 тòv $\Delta$ ．ó A «̈p $\pi \rho o ̀ s ~ \tau o ̀ v ~ B ~ \delta \iota \pi \lambda \alpha \sigma i o v \alpha ~ \lambda o ́ \gamma o v ~ ह ै \chi \varepsilon ા ~ ク ゙ \pi \varepsilon \rho ~ \dot{~}$


## Proposition 11

There exists one number in mean proportion to two （given）square numbers．${ }^{\dagger}$ And（one）square（number） has to the（other）square（number）a squared ${ }^{\ddagger}$ ratio with respect to（that）the side（of the former has）to the side （of the latter）．


Let $A$ and $B$ be square numbers，and let $C$ be the side of $A$ ，and $D$（the side）of $B$ ．I say that there exists one number in mean proportion to $A$ and $B$ ，and that $A$ has to $B$ a squared ratio with respect to（that）$C$（has）to $D$ ．

For let $C$ make $E$（by）multiplying $D$ ．And since $A$ is square，and $C$ is its side，$C$ has thus made $A$（by）multi－ plying itself．And so，for the same（reasons），$D$ has made $B$（by）multiplying itself．Therefore，since $C$ has made $A$ ， $E$（by）multiplying $C, D$ ，respectively，thus as $C$ is to $D$ ， so $A$（is）to $E$［Prop．7．17］．And so，for the same（rea－ sons），as $C$（is）to $D$ ，so $E$（is）to $B$［Prop．7．18］．And thus as $A$（is）to $E$ ，so $E$（is）to $B$ ．Thus，one number （namely，$E$ ）is in mean proportion to $A$ and $B$ ．

So I say that $A$ also has to $B$ a squared ratio with respect to（that）$C$（has）to $D$ ．For since $A, E, B$ are three（continuously）proportional numbers，$A$ thus has to $B$ a squared ratio with respect to（that）$A$（has）to $E$ ［Def．5．9］．And as $A$（is）to $E$ ，so $C$（is）to $D$ ．Thus，$A$ has to $B$ a squared ratio with respect to（that）side $C$（has） to（side）$D$ ．（Which is）the very thing it was required to show．

[^18]$$
\beta^{\prime}
$$
 x $\alpha$ 人̀ ó $x u ́ \beta o s ~ \pi \rho o ̀ s ~ \tau o ̀ v ~ x u ́ \beta o v ~ \tau p ı \pi \lambda \alpha \sigma i o v \alpha ~ \lambda o ́ \gamma o v ~ ह ै \chi \varepsilon ા ~ ク ゙ \pi \varepsilon \rho ~ \dot{\eta}$ $\pi \lambda \varepsilon u p \alpha ̀ \pi \rho o ̀ s ~ \tau \grave{\nu} \nu \pi \lambda \varepsilon u p \alpha ́ \nu$.
＂E $\sigma \tau \omega \sigma \alpha \nu$ xúßol $\alpha$ рı७นoì oi A，B xaì тoũ $\mu \varepsilon ̀ v ~ A ~ \pi \lambda \varepsilon u p \alpha ̀ ~$

 $\lambda o ́ \gamma o v$ है $\chi \varepsilon \iota$ ク̆ $\pi \varepsilon \rho$ ó $\Gamma$ тро̀s tòv $\Delta$ ．

## Proposition 12

There exist two numbers in mean proportion to two （given）cube numbers．${ }^{\dagger}$ And（one）cube（number）has to the（other）cube（number）a cubed ${ }^{\ddagger}$ ratio with respect to（that）the side（of the former has）to the side（of the latter）．

Let $A$ and $B$ be cube numbers，and let $C$ be the side of $A$ ，and $D$（the side）of $B$ ．I say that there exist two numbers in mean proportion to $A$ and $B$ ，and that $A$ has

$\mathrm{K} \longmapsto$


 Z $\pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma$ غ́x $\alpha \tau \varepsilon \rho o \nu \tau \widetilde{\omega} \nu \Theta, \mathrm{~K} \pi o เ \varepsilon i ́ \tau \omega$.



 $\Delta$ غ́autòv $\mu \varepsilon ̀ v ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ v ~ H ~ \pi \varepsilon \pi o i ́ \eta \chi \varepsilon \nu, ~ \tau o ̀ \nu ~ \delta \grave{~} \mathrm{H}$










 $\mathrm{Z}, \mathrm{H} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma$ غ́x $\alpha \tau \varepsilon \rho \circ \nu \tau \widetilde{\omega} \mathrm{K}, \mathrm{B} \pi \varepsilon \pi o \neq \eta \chi \varepsilon \nu$, है $\sigma \tau \iota$

 tòv $\Delta$, oưt $\omega$ s ơ тع A тpòs tòv $\Theta$ xai ò $\Theta$ tpòs tòv K xaì
 oi $\Theta, \mathrm{K}$.



 $\Gamma \pi \rho o ̀ s ~ \tau o ̀ v ~ \Delta \cdot ~ x \alpha i ̀ ~ o ́ ~ A ~[a ̈ p \alpha] ~ \pi p o ̀ s ~ t o ̀ v ~ B ~ \tau p ı \pi \lambda \alpha \sigma i ́ o v \alpha ~ \lambda o ́ \gamma o v ~$

to $B$ a cubed ratio with respect to (that) $C$ (has) to $D$.


For let $C$ make $E$ (by) multiplying itself, and let it make $F$ (by) multiplying $D$. And let $D$ make $G$ (by) multiplying itself, and let $C, D$ make $H, K$, respectively, (by) multiplying $F$.

And since $A$ is cube, and $C$ (is) its side, and $C$ has made $E$ (by) multiplying itself, $C$ has thus made $E$ (by) multiplying itself, and has made $A$ (by) multiplying $E$. And so, for the same (reasons), $D$ has made $G$ (by) multiplying itself, and has made $B$ (by) multiplying $G$. And since $C$ has made $E, F$ (by) multiplying $C, D$, respectively, thus as $C$ is to $D$, so $E$ (is) to $F$ [Prop. 7.17]. And so, for the same (reasons), as $C$ (is) to $D$, so $F$ (is) to $G$ [Prop. 7.18]. Again, since $C$ has made $A, H$ (by) multiplying $E, F$, respectively, thus as $E$ is to $F$, so $A$ (is) to $H$ [Prop. 7.17]. And as $E$ (is) to $F$, so $C$ (is) to $D$. And thus as $C$ (is) to $D$, so $A$ (is) to $H$. Again, since $C, D$ have made $H, K$, respectively, (by) multiplying $F$, thus as $C$ is to $D$, so $H$ (is) to $K$ [Prop. 7.18]. Again, since $D$ has made $K, B$ (by) multiplying $F, G$, respectively, thus as $F$ is to $G$, so $K$ (is) to $B$ [Prop. 7.17]. And as $F$ (is) to $G$, so $C$ (is) to $D$. And thus as $C$ (is) to $D$, so $A$ (is) to $H$, and $H$ to $K$, and $K$ to $B$. Thus, $H$ and $K$ are two (numbers) in mean proportion to $A$ and $B$.

So I say that $A$ also has to $B$ a cubed ratio with respect to (that) $C$ (has) to $D$. For since $A, H, K, B$ are four (continuously) proportional numbers, $A$ thus has to $B$ a cubed ratio with respect to (that) $A$ (has) to $H$ [Def. 5.10]. And as $A$ (is) to $H$, so $C$ (is) to $D$. And [thus] $A$ has to $B$ a cubed ratio with respect to (that) $C$ (has) to $D$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, between two given cube numbers there exist two numbers in continued proportion.
\# Literally, "triple".

$$
\mathfrak{\prime} \gamma^{\prime} \text {. }
$$








## Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some (number by) multiplying itself, then the (numbers) created from them will (also) be (continuously) proportional. And if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also
$\Gamma$, ©́s ó A тpòs tòv B , oưt $\mathrm{A}, \mathrm{B}, \Gamma$ غ́autoùs $\mu \varepsilon ̀ \nu ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \nu \tau \varepsilon \varsigma ~ \tau o u ̀ s ~, ~, ~ E, ~ Z ~$




'O $\mu$ è̀ $\gamma \dot{\alpha} \rho \mathrm{A}$ tòv $\mathrm{B} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma เ \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Lambda ~ \pi о เ \varepsilon i ́ \tau \omega, ~$










 $\tau \widetilde{\omega} \nu \mathrm{E}, \Xi, \mathrm{Z} \pi \lambda \eta \dot{\eta} \vartheta \varepsilon$, $\tau o ̀ ~ \delta \grave{\varepsilon} \tau \widetilde{\omega} \nu \mathrm{H}, \mathrm{M}, \mathrm{N}, \Theta \tau \widetilde{\varphi} \tau \widetilde{\omega} \nu \Theta, \mathrm{O}$,
 E tpòs tòv Z , ís $\delta$ è ó H tpòs tòv $\Theta$, oữ $\omega$ s ó $\Theta$ tpòs tòv


$$
\delta^{\prime}
$$

'Е $\alpha \nu \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \circ \varsigma ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu о \nu \mu \varepsilon \tau \rho \tilde{n}$, , каi $\dot{\eta} \pi \lambda \varepsilon \cup \rho \alpha \dot{\alpha} \tau \dot{\eta} \nu$


"E $\sigma \tau \omega \sigma \alpha \nu \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o l \alpha \dot{\alpha} \downarrow \imath \mu o i ̀ ~ o i ~ A, ~ B, ~ \pi \lambda \varepsilon \cup p \alpha i ̀ ~ \delta e ̀ ~ \alpha u ̉ \tau \widetilde{\omega} \nu$
 $\Gamma$ тòv $\Delta \mu \varepsilon \tau р \varepsilon$ โ̃.
be (continuously) proportional [and this always happens with the extremes].

Let $A, B, C$ be any multitude whatsoever of continuously proportional numbers, (such that) as $A$ (is) to $B$, so $B$ (is) to $C$. And let $A, B, C$ make $D, E, F$ (by) multiplying themselves, and let them make $G, H, K$ (by) multiplying $D, E, F$. I say that $D, E, F$ and $G, H, K$ are continuously proportional.


For let $A$ make $L$ (by) multiplying $B$. And let $A, B$ make $M, N$, respectively, (by) multiplying $L$. And, again, let $B$ make $O$ (by) multiplying $C$. And let $B, C$ make $P$, $Q$, respectively, (by) multplying $O$.

So, similarly to the above, we can show that $D, L$, $E$ and $G, M, N, H$ are continuously proportional in the ratio of $A$ to $B$, and, further, (that) $E, O, F$ and $H, P, Q$, $K$ are continuously proportional in the ratio of $B$ to $C$. And as $A$ is to $B$, so $B$ (is) to $C$. And thus $D, L, E$ are in the same ratio as $E, O, F$, and, further, $G, M, N, H$ (are in the same ratio) as $H, P, Q, K$. And the multitude of $D, L, E$ is equal to the multitude of $E, O, F$, and that of $G, M, N, H$ to that of $H, P, Q, K$. Thus, via equality, as $D$ is to $E$, so $E$ (is) to $F$, and as $G$ (is) to $H$, so $H$ (is) to $K$ [Prop. 7.14]. (Which is) the very thing it was required to show.

## Proposition 14

If a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number).

Let $A$ and $B$ be square numbers, and let $C$ and $D$ be their sides (respectively). And let $A$ measure $B$. I say that $C$ also measures $D$.






 $\mu \varepsilon \tau р \varepsilon$ І̃.







 ঠєĭ̧ $\alpha$.

$$
\iota \varepsilon^{\prime} .
$$





 $\Delta \mu \varepsilon \tau \rho \varepsilon \tilde{\text { in. }}$



For let $C$ make $E$ (by) multiplying $D$. Thus, $A, E$, $B$ are continuously proportional in the ratio of $C$ to $D$ [Prop. 8.11]. And since $A, E, B$ are continuously proportional, and $A$ measures $B, A$ thus also measures $E$ [Prop. 8.7]. And as $A$ is to $E$, so $C$ (is) to $D$. Thus, $C$ also measures $D$ [Def. 7.20].

So, again, let $C$ measure $D$. I say that $A$ also measures $B$.

For similarly, with the same construction, we can show that $A, E, B$ are continuously proportional in the ratio of $C$ to $D$. And since as $C$ is to $D$, so $A$ (is) to $E$, and $C$ measures $D, A$ thus also measures $E$ [Def. 7.20]. And $A, E, B$ are continuously proportional. Thus, $A$ also measures $B$.

Thus, if a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number). (Which is) the very thing it was required to show.

## Proposition 15

If a cube number measures a(nother) cube number then the side (of the former) will also measure the side (of the latter). And if the side (of a cube number) measures the side (of another cube number) then the (former) cube (number) will also measure the (latter) cube (number).

For let the cube number $A$ measure the cube (number) $B$, and let $C$ be the side of $A$, and $D$ (the side) of $B$. I say that $C$ measures $D$.

$\mathrm{F} \longmapsto$
For let $C$ make $E$ (by) multiplying itself. And let

 Z $\pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma เ \alpha ́ \sigma \alpha \varsigma ~ \varepsilon ́ x \alpha ́ \tau \varepsilon р о \nu ~ \tau \widetilde{\omega} \nu ~ \Theta, ~ K ~ \pi о เ \varepsilon i ́ \tau \omega . ~ \varphi \alpha \nu \varepsilon р o ̀ v ~$





 $\mu \varepsilon \tau \rho \dot{\gamma} \sigma \varepsilon$.


 трòs tòv $\Delta$, oưtws ó A тpòs tòv $\Theta$, xaì ó A äp $\alpha$ tòv $\Theta$


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|'
```



 $\mu \varepsilon \tau \rho \grave{\eta} \sigma \varepsilon$.

${ }^{3}$ E $\sigma \tau \omega \sigma \alpha \nu \tau \varepsilon \tau \rho \alpha ̀ \gamma \omega \nu o l ~ \dot{\alpha} \rho \imath \vartheta \mu o i ̀ ~ o i ~ A, ~ B, ~ \pi \lambda \varepsilon u p \alpha i ̀ ~ \delta e ̀ ~ \alpha u ̉ \tau \widetilde{\omega} \nu$
 ó $\Gamma$ tòv $\Delta \mu \varepsilon \tau \rho \varepsilon і ̃$.


 тòv $\mathrm{B} \mu \varepsilon \tau р$ ท́бєı.

Eî үàp $\mu \varepsilon \tau \rho \varepsilon 亢 ̃ ~ o ́ ~ A ~ \tau o ̀ v ~ B, ~ \mu \varepsilon \tau p \eta ́ \sigma \varepsilon ı ~ \chi \alpha \grave{~ o ́ ~} \Gamma$ tòv $\Delta$. oủ



$$
\zeta^{\prime} .
$$




$D$ make $G$ (by) multiplying itself. And, further, [let] $C$ [make] $F$ (by) multiplying $D$, and let $C, D$ make $H, K$, respectively, (by) multiplying $F$. So it is clear that $E, F$, $G$ and $A, H, K, B$ are continuously proportional in the ratio of $C$ to $D$ [Prop. 8.12]. And since $A, H, K, B$ are continuously proportional, and $A$ measures $B,(A)$ thus also measures $H$ [Prop. 8.7]. And as $A$ is to $H$, so $C$ (is) to $D$. Thus, $C$ also measures $D$ [Def. 7.20].

And so let $C$ measure $D$. I say that $A$ will also measure $B$.

For similarly, with the same construction, we can show that $A, H, K, B$ are continuously proportional in the ratio of $C$ to $D$. And since $C$ measures $D$, and as $C$ is to $D$, so $A$ (is) to $H, A$ thus also measures $H$ [Def. 7.20]. Hence, $A$ also measures $B$. (Which is) the very thing it was required to show.

## Proposition 16

If a square number does not measure a(nother) square number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a square number) does not measure the side (of another square number) then the (former) square (number) will not measure the (latter) square (number) either.


Let $A$ and $B$ be square numbers, and let $C$ and $D$ be their sides (respectively). And let $A$ not measure $B$. I say that $C$ does not measure $D$ either.

For if $C$ measures $D$ then $A$ will also measure $B$ [Prop. 8.14]. And $A$ does not measure $B$. Thus, $C$ will not measure $D$ either.
[So], again, let $C$ not measure $D$. I say that $A$ will not measure $B$ either.

For if $A$ measures $B$ then $C$ will also measure $D$ [Prop. 8.14]. And $C$ does not measure $D$. Thus, $A$ will not measure $B$ either. (Which is) the very thing it was required to show.

## Proposition 17

If a cube number does not measure a(nother) cube number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a cube number) does not measure the side (of another cube number) then the (former) cube (number) will not measure the (latter) cube (number) either.


 $\lambda \varepsilon ́ \gamma \omega$, ơтı ó $\Gamma$ tòv $\Delta$ oủ $\mu \varepsilon \tau р$ ท́бєı.


 тòv $\mathrm{B} \mu \varepsilon \tau \rho \dot{\text { ńбє. }}$




$$
i \eta^{\prime}
$$





 $\mu \varepsilon ̀ v ~ A ~ \pi \lambda \varepsilon u p \alpha i ̀ ~ ह ै \sigma \tau \omega \sigma \alpha \nu ~ o i ~ \Gamma, \Delta ~ \alpha ́ p ı \vartheta \mu o i ́, ~ \tau o u ̃ ~ \delta \varepsilon ̀ ~ B ~ o i ~ E, ~$




 $\pi \lambda \varepsilon u p \alpha ̀ ~ \pi \rho o ̀ s ~ \tau i ̀ ̀ ~ o ́ \mu o ́ \lambda o \gamma o v ~[\pi \lambda \varepsilon u p \alpha ́ v] . ~$





 $\mu \varepsilon ̀ \nu \Gamma \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu \mathrm{~A} \pi \varepsilon \pi o i ́ \eta \chi \varepsilon \nu$, тòv $\delta$ ह̀ $\mathrm{E} \pi o \lambda \lambda \alpha-$
 oút $\omega \varsigma$ ó A $\pi \rho o ̀ s ~ \tau o ̀ v ~ H . ~ \alpha ̀ \lambda \lambda ’ ~ i ̀ s ~ o ́ ~ \Gamma ~ \pi p o ̀ s ~ \tau o ̀ v ~ E, ~[o u ́ \tau \omega \varsigma] ~$







For let the cube number $A$ not measure the cube number $B$. And let $C$ be the side of $A$, and $D$ (the side) of $B$. I say that $C$ will not measure $D$.

For if $C$ measures $D$ then $A$ will also measure $B$ [Prop. 8.15]. And $A$ does not measure $B$. Thus, $C$ does not measure $D$ either.

And so let $C$ not measure $D$. I say that $A$ will not measure $B$ either.

For if $A$ measures $B$ then $C$ will also measure $D$ [Prop. 8.15]. And $C$ does not measure $D$. Thus, $A$ will not measure $B$ either. (Which is) the very thing it was required to show.

## Proposition 18

There exists one number in mean proportion to two similar plane numbers. And (one) plane (number) has to the (other) plane (number) a squared ${ }^{\dagger}$ ratio with respect to (that) a corresponding side (of the former has) to a corresponding side (of the latter).


Let $A$ and $B$ be two similar plane numbers. And let the numbers $C, D$ be the sides of $A$, and $E, F$ (the sides) of $B$. And since similar numbers are those having proportional sides [Def. 7.21], thus as $C$ is to $D$, so $E$ (is) to $F$. Therefore, I say that there exists one number in mean proportion to $A$ and $B$, and that $A$ has to $B$ a squared ratio with respect to that $C$ (has) to $E$, or $D$ to $F$-that is to say, with respect to (that) a corresponding side (has) to a corresponding [side].

For since as $C$ is to $D$, so $E$ (is) to $F$, thus, alternately, as $C$ is to $E$, so $D$ (is) to $F$ [Prop. 7.13]. And since $A$ is plane, and $C, D$ its sides, $D$ has thus made $A$ (by) multiplying $C$. And so, for the same (reasons), $E$ has made $B$ (by) multiplying $F$. So let $D$ make $G$ (by) multiplying $E$. And since $D$ has made $A$ (by) multiplying $C$, and has made $G$ (by) multiplying $E$, thus as $C$ is to $E$, so $A$ (is) to $G$ [Prop. 7.17]. But as $C$ (is) to $E$, [so] $D$ (is) to $F$. And thus as $D$ (is) to $F$, so $A$ (is) to $G$. Again, since $E$ has made $G$ (by) multiplying $D$, and has made $B$ (by) multiplying $F$, thus as $D$ is to $F$, so $G$ (is) to $B$ [Prop. 7.17]. And it was also shown that as $D$ (is) to $F$, so $A$ (is) to $G$. And thus as $A$ (is) to $G$, so $G$ (is) to $B$. Thus, $A, G, B$ are



 है $\chi \varepsilon เ ~ \eta ̈ \pi \varepsilon \rho ~ \dot{\eta}$ ó $\mu o ́ \lambda о \gamma о \varsigma ~ \pi \lambda \varepsilon u p \alpha ̀ ~ \pi \rho o ̀ s ~ t \grave{\eta} \nu ~ o ́ \mu o ́ \lambda o \gamma o v ~ \pi \lambda \varepsilon u p \alpha ́ \nu, ~$






$\dagger$ Literally, "double".

## เง'.

$\Delta u ́ o ~ o ́ \mu o i ́ \omega \nu ~ \sigma \tau \varepsilon р \varepsilon \widetilde{\omega} \nu ~ \alpha ̇ \rho ı \vartheta \mu \widetilde{\omega} \nu ~ \delta u ́ o ~ \mu \varepsilon ́ \sigma o l ~ \alpha ̀ v \alpha ́ \lambda o \gamma o v ~$

 óцó入oүov $\pi \lambda \varepsilon u p \alpha ́ v$.

"Eбт $\omega \sigma \alpha \nu$ ठúo ơ $\mu$ oıol $\sigma \tau \varepsilon p \varepsilon o i ̀ ~ o i ̂ ~ A, ~ B, ~ x \alpha i ̀ ~ \tau o u ̃ ~ \mu \varepsilon ̀ \nu ~ A ~$ $\pi \lambda \varepsilon \cup p \alpha i$ है $\sigma \tau \omega \sigma \alpha \nu$ oi $\Gamma, \Delta, \mathrm{E}$, тoũ סè B oi $\mathrm{Z}, \mathrm{H}, \Theta$. xaì દ̇ $\pi \varepsilon \grave{\imath}$

 ó $\Delta$ tpòs đòv E , oú $\tau \omega \varsigma$ ò H тpòs $\tau o ̀ v ~ \Theta . ~ \lambda \varepsilon ́ \gamma \omega, ~ o ้ \tau \iota ~ \tau \widetilde{\omega} \nu \mathrm{~A}$,

 $\Delta$ tpòs tòv H xail हैtı ó E tpòs tòv $\Theta$.
'O $\Gamma$ ү $\alpha$ р тòv $\Delta \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ t o ̀ v ~ K ~ \pi o เ \varepsilon i ́ \tau \omega, ~ o ́ ~ \delta e ̀ ~$










continously proportional. Thus, there exists one number (namely, $G$ ) in mean proportion to $A$ and $B$.

So I say that $A$ also has to $B$ a squared ratio with respect to (that) a corresponding side (has) to a corresponding side-that is to say, with respect to (that) $C$ (has) to $E$, or $D$ to $F$. For since $A, G, B$ are continuously proportional, $A$ has to $B$ a squared ratio with respect to (that $A$ has) to $G$ [Prop. 5.9]. And as $A$ is to $G$, so $C$ (is) to $E$, and $D$ to $F$. And thus $A$ has to $B$ a squared ratio with respect to (that) $C$ (has) to $E$, or $D$ to $F$. (Which is) the very thing it was required to show.

## Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed ${ }^{\dagger}$ ratio with respect to (that) a corresponding side (has) to a corresponding side.


Let $A$ and $B$ be two similar solid numbers, and let $C, D, E$ be the sides of $A$, and $F, G, H$ (the sides) of $B$. And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as $C$ is to $D$, so $F$ (is) to $G$, and as $D$ (is) to $E$, so $G$ (is) to $H$. I say that two numbers fall (between) $A$ and $B$ in mean proportion, and (that) $A$ has to $B$ a cubed ratio with respect to (that) $C$ (has) to $F$, and $D$ to $G$, and, further, $E$ to $H$.

For let $C$ make $K$ (by) multiplying $D$, and let $F$ make $L$ (by) multiplying $G$. And since $C, D$ are in the same ratio as $F, G$, and $K$ is the (number created) from (multiplying) $C, D$, and $L$ the (number created) from (multiplying) $F, G$, [thus] $K$ and $L$ are similar plane numbers [Def. 7.21]. Thus, there exits one number in mean proportion to $K$ and $L$ [Prop. 8.18]. Let it be $M$. Thus, $M$ is the (number created) from (multiplying) $D, F$, as shown in the theorem before this (one). And since $D$ has made $K$ (by) multiplying $C$, and has made $M$ (by) multiplying $F$, thus as $C$ is to $F$, so $K$ (is) to $M$ [Prop. 7.17]. But, as

















 غ่x $\alpha \tau \varepsilon \rho \circ \varsigma \tau \widetilde{\omega} \nu \mathrm{E}, \Theta$ тòv $\mathrm{M} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma 1 \alpha ́ \sigma \alpha \varsigma ~ \dot{\varepsilon} x \alpha ́ \tau \varepsilon \rho o \nu \tau \widetilde{\omega} \nu$







 xaì ó $\Delta$ тpòs tòv H xai ó E трòs tòv $\Theta$. xai $\dot{\omega} s$ äp $\alpha$ ó $\Gamma$ тpòs tòv Z xaì ò $\Delta$ тpòs tòv H каì ó E тpòs tòv $\Theta$, oưt $\tau \omega$









 tòv H xai है $\tau \iota$ ó E трòs tòv $\Theta$. xail ó A «̈p тpòs tòv B




$\dagger$ Literally, "triple".

$$
x^{\prime} .
$$



$K$ (is) to $M$, (so) $M$ (is) to $L$. Thus, $K, M, L$ are continuously proportional in the ratio of $C$ to $F$. And since as $C$ is to $D$, so $F$ (is) to $G$, thus, alternately, as $C$ is to $F$, so $D$ (is) to $G$ [Prop. 7.13]. And so, for the same (reasons), as $D$ (is) to $G$, so $E$ (is) to $H$. Thus, $K, M, L$ are continuously proportional in the ratio of $C$ to $F$, and of $D$ to $G$, and, further, of $E$ to $H$. So let $E, H$ make $N, O$, respectively, (by) multiplying $M$. And since $A$ is solid, and $C$, $D, E$ are its sides, $E$ has thus made $A$ (by) multiplying the (number created) from (multiplying) $C, D$. And $K$ is the (number created) from (multiplying) $C, D$. Thus, $E$ has made $A$ (by) multiplying $K$. And so, for the same (reasons), $H$ has made $B$ (by) multiplying $L$. And since $E$ has made $A$ (by) multiplying $K$, but has, in fact, also made $N$ (by) multiplying $M$, thus as $K$ is to $M$, so $A$ (is) to $N$ [Prop. 7.17]. And as $K$ (is) to $M$, so $C$ (is) to $F$, and $D$ to $G$, and, further, $E$ to $H$. And thus as $C$ (is) to $F$, and $D$ to $G$, and $E$ to $H$, so $A$ (is) to $N$. Again, since $E, H$ have made $N, O$, respectively, (by) multiplying $M$, thus as $E$ is to $H$, so $N$ (is) to $O$ [Prop. 7.18]. But, as $E$ (is) to $H$, so $C$ (is) to $F$, and $D$ to $G$. And thus as $C$ (is) to $F$, and $D$ to $G$, and $E$ to $H$, so (is) $A$ to $N$, and $N$ to $O$. Again, since $H$ has made $O$ (by) multiplying $M$, but has, in fact, also made $B$ (by) multiplying $L$, thus as $M$ (is) to $L$, so $O$ (is) to $B$ [Prop. 7.17]. But, as $M$ (is) to $L$, so $C$ (is) to $F$, and $D$ to $G$, and $E$ to $H$. And thus as $C$ (is) to $F$, and $D$ to $G$, and $E$ to $H$, so not only (is) $O$ to $B$, but also $A$ to $N$, and $N$ to $O$. Thus, $A, N, O$, $B$ are continuously proportional in the aforementioned ratios of the sides.

So I say that $A$ also has to $B$ a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side-that is to say, with respect to (that) the number $C$ (has) to $F$, or $D$ to $G$, and, further, $E$ to $H$. For since $A$, $N, O, B$ are four continuously proportional numbers, $A$ thus has to $B$ a cubed ratio with respect to (that) $A$ (has) to $N$ [Def. 5.10]. But, as $A$ (is) to $N$, so it was shown (is) $C$ to $F$, and $D$ to $G$, and, further, $E$ to $H$. And thus $A$ has to $B$ a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side-that is to say, with respect to (that) the number $C$ (has) to $F$, and $D$ to $G$, and, further, $E$ to $H$. (Which is) the very thing it was required to show.

## Proposition 20

If one number falls between two numbers in mean proportion then the numbers will be similar plane (num-


























bers).
For let one number $C$ fall between the two numbers $A$ and $B$ in mean proportion. I say that $A$ and $B$ are similar plane numbers.

[For] let the least numbers, $D$ and $E$, having the same ratio as $A$ and $C$ have been taken [Prop. 7.33]. Thus, $D$ measures $A$ as many times as $E$ (measures) $C$ [Prop. 7.20]. So as many times as $D$ measures $A$, so many units let there be in $F$. Thus, $F$ has made $A$ (by) multiplying $D$ [Def. 7.15]. Hence, $A$ is plane, and $D$, $F$ (are) its sides. Again, since $D$ and $E$ are the least of those (numbers) having the same ratio as $C$ and $B$, $D$ thus measures $C$ as many times as $E$ (measures) $B$ [Prop. 7.20]. So as many times as $E$ measures $B$, so many units let there be in $G$. Thus, $E$ measures $B$ according to the units in $G$. Thus, $G$ has made $B$ (by) multiplying $E$ [Def. 7.15]. Thus, $B$ is plane, and $E, G$ are its sides. Thus, $A$ and $B$ are (both) plane numbers. So I say that (they are) also similar. For since $F$ has made $A$ (by) multiplying $D$, and has made $C$ (by) multiplying $E$, thus as $D$ is to $E$, so $A$ (is) to $C$-that is to say, $C$ to $B$ [Prop. 7.17]. ${ }^{\dagger}$ Again, since $E$ has made $C, B$ (by) multiplying $F, G$, respectively, thus as $F$ is to $G$, so $C$ (is) to $B$ [Prop. 7.17]. And as $C$ (is) to $B$, so $D$ (is) to $E$. And thus as $D$ (is) to $E$, so $F$ (is) to $G$. And, alternately, as $D$ (is) to $F$, so $E$ (is) to $G$ [Prop. 7.13]. Thus, $A$ and $B$ are similar plane numbers. For their sides are proportional [Def. 7.21]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This part of the proof is defective, since it is not demonstrated that $F \times E=C$. Furthermore, it is not necessary to show that $D: E:: A: C$, because this is true by hypothesis.

$$
x \alpha^{\prime} .
$$






 $\lambda o ́ \gamma o v$ è $\chi$ óvtcov toĭs $\mathrm{A}, \Gamma, \Delta$ tpeĩs oi $\mathrm{E}, \mathrm{Z}$, H. oi äpa äxpor




## Proposition 21

If two numbers fall between two numbers in mean proportion then the (latter) are similar solid (numbers).

For let the two numbers $C$ and $D$ fall between the two numbers $A$ and $B$ in mean proportion. I say that $A$ and $B$ are similar solid (numbers).

For let the three least numbers $E, F, G$ having the same ratio as $A, C, D$ have been taken [Prop. 8.2]. Thus, the outermost of them, $E$ and $G$, are prime to one another [Prop. 8.3]. And since one number, $F$, has fallen (between) $E$ and $G$ in mean proportion, $E$ and $G$ are










 ó E tòv $\mathrm{A} \mu \varepsilon \tau \rho \varepsilon \imath \tau ~ x \alpha i ̀ ~ o ́ ~ H ~ \tau o ̀ v ~ \Delta . ~ o ́ \sigma \alpha ́ x ı s ~ o \grave{\eta}$ ó E tòv A









 $\tau \widetilde{\nu} \Lambda, \mathrm{M} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma ~ \tau o ̀ \nu \mathrm{~B} \pi \varepsilon \pi о$ í $\chi \varepsilon \varepsilon$ ．$\sigma \tau \varepsilon \rho \varepsilon o ̀ \varsigma ~ « ้ \rho \alpha$




 трòs tòv $\Xi$ ，ó A трòs tòv $\Gamma$ ，тoutéのтเレ ó E тpòs tòv Z ．

 ó N тpòs tòv $\Xi$ ．xaí $\varepsilon i \sigma ル ~ o i ~ \mu e ̀ v ~ \Theta, ~ K, ~ N ~ \pi \lambda \varepsilon u p \alpha i ~ \tau o u ̃ ~ A, ~$
thus similar plane numbers［Prop．8．20］．Therefore，let $H, K$ be the sides of $E$ ，and $L, M$（the sides）of $G$ ．Thus， it is clear from the（proposition）before this（one）that $E$ ， $F, G$ are continuously proportional in the ratio of $H$ to $L$ ，and of $K$ to $M$ ．And since $E, F, G$ are the least（num－ bers）having the same ratio as $A, C, D$ ，and the multitude of $E, F, G$ is equal to the multitude of $A, C, D$ ，thus， via equality，as $E$ is to $G$ ，so $A$（is）to $D$［Prop．7．14］． And $E$ and $G$（are）prime（to one another），and prime （numbers）are also the least（of those numbers having the same ratio as them）［Prop．7．21］，and the least（num－ bers）measure those（numbers）having the same ratio as them an equal number of times，the greater（measuring） the greater，and the lesser the lesser－that is to say，the leading（measuring）the leading，and the following the following［Prop．7．20］．Thus，$E$ measures $A$ the same number of times as $G$（measures）$D$ ．So as many times as $E$ measures $A$ ，so many units let there be in $N$ ．Thus，$N$ has made $A$（by）multiplying $E$［Def．7．15］．And $E$ is the （number created）from（multiplying）$H$ and $K$ ．Thus，$N$ has made $A$（by）multiplying the（number created）from （multiplying）$H$ and $K$ ．Thus，$A$ is solid，and its sides are $H, K, N$ ．Again，since $E, F, G$ are the least（numbers） having the same ratio as $C, D, B$ ，thus $E$ measures $C$ the same number of times as $G$（measures）$B$［Prop．7．20］． So as many times as $E$ measures $C$ ，so many units let there be in $O$ ．Thus，$G$ measures $B$ according to the units in $O$ ．Thus，$O$ has made $B$（by）multiplying $G$ ．And $G$ is the（number created）from（multiplying）$L$ and $M$ ． Thus，$O$ has made $B$（by）multiplying the（number cre－ ated）from（multiplying）$L$ and $M$ ．Thus，$B$ is solid，and its sides are $L, M, O$ ．Thus，$A$ and $B$ are（both）solid．

［So］I say that（they are）also similar．For since $N, O$ have made $A, C$（by）multiplying $E$ ，thus as $N$ is to $O$ ，so $A$（is）to $C$－that is to say，$E$ to $F$［Prop．7．18］．But，as $E$（is）to $F$ ，so $H$（is）to $L$ ，and $K$ to $M$ ．And thus as $H$ （is）to $L$ ，so $K$（is）to $M$ ，and $N$ to $O$ ．And $H, K, N$ are the sides of $A$ ，and $L, M, O$ the sides of $B$ ．Thus，$A$ and


${ }^{\dagger}$ The Greek text has " $O, L, M$ ", which is obviously a mistake.

$$
x \beta^{\prime}
$$





 тєтрáү $\omega$ vós દ̇எтıレ.



$x \gamma^{\prime}$.








$\chi \delta^{\prime}$.




$B$ are similar solid numbers [Def. 7.21]. (Which is) the very thing it was required to show.

## Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.


Let $A, B, C$ be three continuously proportional numbers, and let the first $A$ be square. I say that the third $C$ is also square.

For since one number, $B$, is in mean proportion to $A$ and $C, A$ and $C$ are thus similar plane (numbers) [Prop. 8.20]. And $A$ is square. Thus, $C$ is also square [Def. 7.21]. (Which is) the very thing it was required to show.

## Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.


Let $A, B, C, D$ be four continuously proportional numbers, and let $A$ be cube. I say that $D$ is also cube.

For since two numbers, $B$ and $C$, are in mean proportion to $A$ and $D, A$ and $D$ are thus similar solid numbers [Prop. 8.21]. And $A$ (is) cube. Thus, $D$ (is) also cube [Def. 7.21]. (Which is) the very thing it was required to show.

## Proposition 24

If two numbers have to one another the ratio which a square number (has) to a(nother) square number, and the first is square, then the second will also be square.


For let two numbers, $A$ and $B$, have to one another










## $\chi \varepsilon^{\prime}$.





$\Delta u ́ o ~ \gamma a ̀ p ~ \alpha ́ p ı \vartheta \mu o i ̀ ~ o i ~ A, ~ B ~ \pi p o ̀ s ~ \alpha ́ \alpha \lambda \eta ́ \lambda o u s ~ \lambda o ́ \gamma o v ~$











## $\chi \bar{q}^{\prime}$.






the ratio which the square number $C$ (has) to the square number $D$. And let $A$ be square. I say that $B$ is also square.

For since $C$ and $D$ are square, $C$ and $D$ are thus similar plane (numbers). Thus, one number falls (between) $C$ and $D$ in mean proportion [Prop. 8.18]. And as $C$ is to $D$, (so) $A$ (is) to $B$. Thus, one number also falls (between) $A$ and $B$ in mean proportion [Prop. 8.8]. And $A$ is square. Thus, $B$ is also square [Prop. 8.22]. (Which is) the very thing it was required to show.

## Proposition 25

If two numbers have to one another the ratio which a cube number (has) to a(nother) cube number, and the first is cube, then the second will also be cube.


For let two numbers, $A$ and $B$, have to one another the ratio which the cube number $C$ (has) to the cube number $D$. And let $A$ be cube. [So] I say that $B$ is also cube.

For since $C$ and $D$ are cube (numbers), $C$ and $D$ are (thus) similar solid (numbers). Thus, two numbers fall (between) $C$ and $D$ in mean proportion [Prop. 8.19]. And as many (numbers) as fall in between $C$ and $D$ in continued proportion, so many also (fall) in (between) those (numbers) having the same ratio as them (in continued proportion) [Prop. 8.8]. And hence two numbers fall (between) $A$ and $B$ in mean proportion. Let $E$ and $F$ (so) fall. Therefore, since the four numbers $A, E, F, B$ are continuously proportional, and $A$ is cube, $B$ (is) thus also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

## Proposition 26

Similar plane numbers have to one another the ratio which (some) square number (has) to a(nother) square number.


Let $A$ and $B$ be similar plane numbers. I say that $A$ has to $B$ the ratio which (some) square number (has) to










$$
x \zeta^{\prime} .
$$





 àpıখuóv.




 E tpòs tòv $\Theta$, oütus ó A tpòs tòv B • xai ò A äp $\alpha$ tpòs


a(nother) square number.
For since $A$ and $B$ are similar plane numbers, one number thus falls (between) $A$ and $B$ in mean proportion [Prop. 8.18]. Let it (so) fall, and let it be $C$. And let the least numbers, $D, E, F$, having the same ratio as $A, C, B$ have been taken [Prop. 8.2]. The outermost of them, $D$ and $F$, are thus square [Prop. 8.2 corr.]. And since as $D$ is to $F$, so $A$ (is) to $B$, and $D$ and $F$ are square, $A$ thus has to $B$ the ratio which (some) square number (has) to a(nother) square number. (Which is) the very thing it was required to show.

## Proposition 27

Similar solid numbers have to one another the ratio which (some) cube number (has) to a(nother) cube number.


Let $A$ and $B$ be similar solid numbers. I say that $A$ has to $B$ the ratio which (some) cube number (has) to a(nother) cube number.

For since $A$ and $B$ are similar solid (numbers), two numbers thus fall (between) $A$ and $B$ in mean proportion [Prop. 8.19]. Let $C$ and $D$ have (so) fallen. And let the least numbers, $E, F, G, H$, having the same ratio as $A$, $C, D, B$, (and) equal in multitude to them, have been taken [Prop. 8.2]. Thus, the outermost of them, $E$ and $H$, are cube [Prop. 8.2 corr.]. And as $E$ is to $H$, so $A$ (is) to $B$. And thus $A$ has to $B$ the ratio which (some) cube number (has) to a(nother) cube number. (Which is) the very thing it was required to show.

## ELEMENTS BOOK 9

Applications of Number Theory $\ddagger$

[^19]
## $\alpha^{\prime}$.




$\Gamma$

 A tòv $\mathrm{B} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha s$ tòv $\Gamma \pi o \iota \varepsilon i \tau \omega \cdot \lambda \varepsilon ́ \gamma \omega$, ơ $\tau \iota$ ó $\Gamma$ тeтра́ү $\omega \nu$ 人́s ह̇ $\tau \tau \iota$.












## $\beta^{\prime}$.

'Е $\mathrm{\alpha} \nu$ ठ тєтра́ $\gamma \omega \nu$,











## Proposition 1

If two similar plane numbers make some (number by) multiplying one another then the created (number) will be square.


D
Let $A$ and $B$ be two similar plane numbers, and let $A$ make $C$ (by) multiplying $B$. I say that $C$ is square.

For let $A$ make $D$ (by) multiplying itself. $D$ is thus square. Therefore, since $A$ has made $D$ (by) multiplying itself, and has made $C$ (by) multiplying $B$, thus as $A$ is to $B$, so $D$ (is) to $C$ [Prop. 7.17]. And since $A$ and $B$ are similar plane numbers, one number thus falls (between) $A$ and $B$ in mean proportion [Prop. 8.18]. And if (some) numbers fall between two numbers in continued proportion then, as many (numbers) as fall in (between) them (in continued proportion), so many also (fall) in (between numbers) having the same ratio (as them in continued proportion) [Prop. 8.8]. And hence one number falls (between) $D$ and $C$ in mean proportion. And $D$ is square. Thus, $C$ (is) also square [Prop. 8.22]. (Which is) the very thing it was required to show.

## Proposition 2

If two numbers make a square (number by) multiplying one another then they are similar plane numbers.


Let $A$ and $B$ be two numbers, and let $A$ make the square (number) $C$ (by) multiplying $B$. I say that $A$ and $B$ are similar plane numbers.

For let $A$ make $D$ (by) multiplying itself. Thus, $D$ is square. And since $A$ has made $D$ (by) multiplying itself, and has made $C$ (by) multiplying $B$, thus as $A$ is to $B$, so $D$ (is) to $C$ [Prop. 7.17]. And since $D$ is square, and $C$ (is) also, $D$ and $C$ are thus similar plane numbers. Thus, one (number) falls (between) $D$ and $C$ in mean propor-






## $\gamma^{\prime}$.




 $\pi о เ \varepsilon i \tau \omega \cdot \lambda \varepsilon ́ \gamma \omega$, ơт兀 ó B xúßos દ̇бтiv.
 $\pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma$ тòv $\Delta$ тоเฉít $\omega$. ழ








 äpa $\dot{\eta}$ uovàs $\pi p o ̀ s ~ \tau o ̀ v ~ \Gamma, ~ o u ̈ t \omega s ~ o ́ ~ \Gamma ~ \pi p o ̀ s ~ t o ̀ v ~ \Delta ~ x \alpha i ̀ ~ o ́ ~ \Delta ~$











$\delta^{\prime}$.


tion [Prop. 8.18]. And as $D$ is to $C$, so $A$ (is) to $B$. Thus, one (number) also falls (between) $A$ and $B$ in mean proportion [Prop. 8.8]. And if one (number) falls (between) two numbers in mean proportion then [the] numbers are similar plane (numbers) [Prop. 8.20]. Thus, $A$ and $B$ are similar plane (numbers). (Which is) the very thing it was required to show.

## Proposition 3

If a cube number makes some (number by) multiplying itself then the created (number) will be cube.

A


For let the cube number $A$ make $B$ (by) multiplying itself. I say that $B$ is cube.

For let the side $C$ of $A$ have been taken. And let $C$ make $D$ by multiplying itself. So it is clear that $C$ has made $A$ (by) multiplying $D$. And since $C$ has made $D$ (by) multiplying itself, $C$ thus measures $D$ according to the units in it [Def. 7.15]. But, in fact, a unit also measures $C$ according to the units in it [Def. 7.20]. Thus, as a unit is to $C$, so $C$ (is) to $D$. Again, since $C$ has made $A$ (by) multiplying $D, D$ thus measures $A$ according to the units in $C$. And a unit also measures $C$ according to the units in it. Thus, as a unit is to $C$, so $D$ (is) to $A$. But, as a unit (is) to $C$, so $C$ (is) to $D$. And thus as a unit (is) to $C$, so $C$ (is) to $D$, and $D$ to $A$. Thus, two numbers, $C$ and $D$, have fallen (between) a unit and the number $A$ in continued mean proportion. Again, since $A$ has made $B$ (by) multiplying itself, $A$ thus measures $B$ according to the units in it. And a unit also measures $A$ according to the units in it. Thus, as a unit is to $A$, so $A$ (is) to $B$. And two numbers have fallen (between) a unit and $A$ in mean proportion. Thus two numbers will also fall (between) $A$ and $B$ in mean proportion [Prop. 8.8]. And if two (numbers) fall (between) two numbers in mean proportion, and the first (number) is cube, then the second will also be cube [Prop. 8.23]. And $A$ is cube. Thus, $B$ is also cube. (Which is) the very thing it was required to show.

## Proposition 4

If a cube number makes some (number by) multiplying a(nother) cube number then the created (number)


Kúßos $\gamma \dot{\alpha} \rho \dot{\alpha} \rho ı \vartheta \mu o ̀ s ~ o ́ ~ A ~ x u ́ \beta o v ~ \alpha ́ p ı \vartheta \mu o ̀ v ~ t o ̀ v ~ B ~ \pi o \lambda \lambda \alpha-~$











$$
\varepsilon^{\prime} .
$$

'Eג̀ $x \cup ́ \beta o s ~ \alpha ̉ p ı \vartheta \mu o ̀ s ~ \alpha ̉ p ı \vartheta \mu o ́ v ~ \tau ı \nu \alpha ~ \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma เ \alpha ́ \sigma \alpha s ~ x u ́ \beta o v ~$ $\pi o เ \tilde{n}, x \alpha i$ ó $\pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \vartheta \varepsilon i \varsigma ~ \varkappa u ́ \beta o \varsigma ~ ह ै \sigma \tau \alpha l . ~$













$$
\tau^{\prime}
$$


will be cube.


For let the cube number $A$ make $C$ (by) multiplying the cube number $B$. I say that $C$ is cube.

For let $A$ make $D$ (by) multiplying itself. Thus, $D$ is cube [Prop. 9.3]. And since $A$ has made $D$ (by) multiplying itself, and has made $C$ (by) multiplying $B$, thus as $A$ is to $B$, so $D$ (is) to $C$ [Prop. 7.17]. And since $A$ and $B$ are cube, $A$ and $B$ are similar solid (numbers). Thus, two numbers fall (between) $A$ and $B$ in mean proportion [Prop. 8.19]. Hence, two numbers will also fall (between) $D$ and $C$ in mean proportion [Prop. 8.8]. And $D$ is cube. Thus, $C$ (is) also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

## Proposition 5

If a cube number makes a(nother) cube number (by) multiplying some (number) then the (number) multiplied will also be cube.


For let the cube number $A$ make the cube (number) $C$ (by) multiplying some number $B$. I say that $B$ is cube.

For let $A$ make $D$ (by) multiplying itself. $D$ is thus cube [Prop. 9.3]. And since $A$ has made $D$ (by) multiplying itself, and has made $C$ (by) multiplying $B$, thus as $A$ is to $B$, so $D$ (is) to $C$ [Prop. 7.17]. And since $D$ and $C$ are (both) cube, they are similar solid (numbers). Thus, two numbers fall (between) $D$ and $C$ in mean proportion [Prop. 8.19]. And as $D$ is to $C$, so $A$ (is) to $B$. Thus, two numbers also fall (between) $A$ and $B$ in mean proportion [Prop. 8.8]. And $A$ is cube. Thus, $B$ is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

## Proposition 6

If a number makes a cube (number by) multiplying

аủròs xúßos है $\sigma \tau \alpha$.

$\Gamma$












 tòv A , oüt $\omega \varsigma$ ó A tpòs tòv B raì cis àp $\alpha$ ó A tpòs tòv B ,






## $\zeta^{\prime}$.














itself then it itself will also be cube.


For let the number $A$ make the cube (number) $B$ (by) multiplying itself. I say that $A$ is also cube.

For let $A$ make $C$ (by) multiplying $B$. Therefore, since $A$ has made $B$ (by) multiplying itself, and has made $C$ (by) multiplying $B, C$ is thus cube. And since $A$ has made $B$ (by) multiplying itself, $A$ thus measures $B$ according to the units in $(A)$. And a unit also measures $A$ according to the units in it. Thus, as a unit is to $A$, so $A$ (is) to $B$. And since $A$ has made $C$ (by) multiplying $B, B$ thus measures $C$ according to the units in $A$. And a unit also measures $A$ according to the units in it. Thus, as a unit is to $A$, so $B$ (is) to $C$. But, as a unit (is) to $A$, so $A$ (is) to $B$. And thus as $A$ (is) to $B$, (so) $B$ (is) to $C$. And since $B$ and $C$ are cube, they are similar solid (numbers). Thus, there exist two numbers in mean proportion (between) $B$ and $C$ [Prop. 8.19]. And as $B$ is to $C$, (so) $A$ (is) to $B$. Thus, there also exist two numbers in mean proportion (between) $A$ and $B$ [Prop. 8.8]. And $B$ is cube. Thus, $A$ is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

## Proposition 7

If a composite number makes some (number by) multiplying some (other) number then the created (number) will be solid.


E
For let the composite number $A$ make $C$ (by) multiplying some number $B$. I say that $C$ is solid.

For since $A$ is a composite (number), it will be measured by some number. Let it be measured by $D$. And, as many times as $D$ measures $A$, so many units let there be in $E$. Therefore, since $D$ measures $A$ according to the units in $E, E$ has thus made $A$ (by) multiplying $D$ [Def. 7.15]. And since $A$ has made $C$ (by) multiplying $B$, and $A$ is the (number created) from (multiplying) $D, E$, the (number created) from (multiplying) $D, E$ has thus


## $\eta^{\prime}$.







 ov oi $\mathrm{A}, \mathrm{B}, \Gamma, \Delta, \mathrm{E}, \mathrm{Z} \cdot \lambda \varepsilon ́ \gamma \omega$, ơ兀ı ó $\mu$ èv 七pítos $\dot{\alpha} \pi o ̀$


 $\pi \varepsilon ́ \nu \tau \varepsilon \delta \downarrow \alpha \lambda \varepsilon i ́ \pi o \nu \tau \varepsilon \varsigma \pi \alpha ́ \nu \tau \varepsilon \varsigma$.










 غ̇л


 äp $\alpha$ тòv $\mathrm{B} \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ v ~ \Gamma ~ \pi \varepsilon \pi o i ́ \eta \varkappa \varepsilon \nu . ~ ह ̀ \pi \varepsilon i ̀ ~ o u ̛ ้ ~ o ́ ~$
 $\pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \dot{\alpha} \sigma \alpha \varsigma$ тòv $\Gamma \pi \varepsilon \pi о$ ín $\varkappa \varepsilon \nu, \chi u ́ \beta o \varsigma ~ \alpha ̈ p \alpha ~ \varepsilon ̇ \sigma \tau i v ~ o ́ ~ \Gamma . ~ \chi \alpha \grave{~}$

made $C$ (by) multiplying $B$. Thus, $C$ is solid, and its sides are $D, E, B$. (Which is) the very thing it was required to show.

## Proposition 8

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one (number), and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two (numbers), and the seventh (will be) both cube and square, and (all) those (numbers after that) which leave an interval of five (numbers).


Let any multitude whatsoever of numbers, $A, B, C$, $D, E, F$, be continuously proportional, (starting) from a unit. I say that the third from the unit, $B$, is square, and all those (numbers after that) which leave an interval of one (number). And the fourth (from the unit), $C$, (is) cube, and all those (numbers after that) which leave an interval of two (numbers). And the seventh (from the unit), $F$, (is) both cube and square, and all those (numbers after that) which leave an interval of five (numbers).

For since as the unit is to $A$, so $A$ (is) to $B$, the unit thus measures the number $A$ the same number of times as $A$ (measures) $B$ [Def. 7.20]. And the unit measures the number $A$ according to the units in it. Thus, $A$ also measures $B$ according to the units in $A$. $A$ has thus made $B$ (by) multiplying itself [Def. 7.15]. Thus, $B$ is square. And since $B, C, D$ are continuously proportional, and $B$ is square, $D$ is thus also square [Prop. 8.22]. So, for the same (reasons), $F$ is also square. So, similarly, we can also show that all those (numbers after that) which leave an interval of one (number) are square. So I also say that the fourth (number) from the unit, $C$, is cube, and all those (numbers after that) which leave an interval of two (numbers). For since as the unit is to $A$, so $B$ (is) to $C$, the unit thus measures the number $A$ the same number of times that $B$ (measures) $C$. And the unit measures the





## $\vartheta^{\prime}$ ．
















 $\pi \alpha ́ \nu \tau \varepsilon \varsigma ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \circ$ ยi $i \sigma เ \nu$.
 xúßol عiб⿱宀⿱一兀口儿．





number $A$ according to the units in $A$ ．And thus $B$ mea－ sures $C$ according to the units in $A . A$ has thus made $C$ （by）multiplying $B$ ．Therefore，since $A$ has made $B$（by） multiplying itself，and has made $C$（by）multiplying $B, C$ is thus cube．And since $C, D, E, F$ are continuously pro－ portional，and $C$ is cube，$F$ is thus also cube［Prop．8．23］． And it was also shown（to be）square．Thus，the seventh （number）from the unit is（both）cube and square．So， similarly，we can show that all those（numbers after that） which leave an interval of five（numbers）are（both）cube and square．（Which is）the very thing it was required to show．

## Proposition 9

If any multitude whatsoever of numbers is continu－ ously proportional，（starting）from a unit，and the（num－ ber）after the unit is square，then all the remaining（num－ bers）will also be square．And if the（number）after the unit is cube，then all the remaining（numbers）will also be cube．


Let any multitude whatsoever of numbers，$A, B, C$ ， $D, E, F$ ，be continuously proportional，（starting）from a unit．And let the（number）after the unit，$A$ ，be square．I say that all the remaining（numbers）will also be square．

In fact，it has（already）been shown that the third （number）from the unit，$B$ ，is square，and all those（num－ bers after that）which leave an interval of one（number） ［Prop．9．8］．［So］I say that all the remaining（num－ bers）are also square．For since $A, B, C$ are continu－ ously proportional，and $A$（is）square，$C$ is［thus］also square［Prop．8．22］．Again，since $B, C, D$ are［also］con－ tinuously proportional，and $B$ is square，$D$ is［thus］also square［Prop．8．22］．So，similarly，we can show that all the remaining（numbers）are also square．

And so let $A$ be cube．I say that all the remaining （numbers）are also cube．

In fact，it has（already）been shown that the fourth （number）from the unit，$C$ ，is cube，and all those（num－ bers after that）which leave an interval of two（numbers）









## $i^{\prime}$.






 $\pi \alpha ́ \nu \tau \omega \nu$.




 $\lambda \varepsilon \iota \pi o ́ v \tau \omega \nu]$.









[Prop. 9.8]. [So] I say that all the remaining (numbers) are also cube. For since as the unit is to $A$, so $A$ (is) to $B$, the unit thus measures $A$ the same number of times as $A$ (measures) $B$. And the unit measures $A$ according to the units in it. Thus, $A$ also measures $B$ according to the units in ( $A$ ). $A$ has thus made $B$ (by) multiplying itself. And $A$ is cube. And if a cube number makes some (number by) multiplying itself then the created (number) is cube [Prop. 9.3]. Thus, $B$ is also cube. And since the four numbers $A, B, C, D$ are continuously proportional, and $A$ is cube, $D$ is thus also cube [Prop. 8.23]. So, for the same (reasons), $E$ is also cube, and, similarly, all the remaining (numbers) are cube. (Which is) the very thing it was required to show.

## Proposition 10

If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the (number) after the unit is not square, then no other (number) will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one (number). And if the (number) after the unit is not cube, then no other (number) will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two (numbers).


F
Let any multitude whatsoever of numbers, $A, B, C$, $D, E, F$, be continuously proportional, (starting) from a unit. And let the (number) after the unit, $A$, not be square. I say that no other (number) will be square either, apart from the third from the unit [and (all) those (numbers after that) which leave an interval of one (number)].

For, if possible, let $C$ be square. And $B$ is also square [Prop. 9.8]. Thus, $B$ and $C$ have to one another (the) ratio which (some) square number (has) to (some other) square number. And as $B$ is to $C$, (so) $A$ (is) to $B$. Thus, $A$ and $B$ have to one another (the) ratio which (some) square number has to (some other) square number. Hence, $A$ and $B$ are similar plane (numbers)

тои̃ трítou $\alpha \pi o ̀ ~ \tau \tilde{\eta} \varsigma \mu o \nu \alpha ́ \delta o \varsigma ~ x \alpha i ̀ \tau \widetilde{\omega} \nu$ हैv $\alpha \delta \iota \alpha \lambda \varepsilon เ \pi o ́ v \tau \omega \nu$.

 x $\alpha i$ т $\widetilde{\nu} \nu$ ठ́́o $\delta \iota \alpha \lambda \varepsilon เ \pi o ́ v \tau \omega \nu$.

 тòv $\Delta$, ó B тpòs tòv $\Gamma$ • xal ó B äp тpòs tòv $\Gamma$ 入óүov है $\chi \varepsilon ا$,











## $\alpha^{\prime}$.





 $\alpha{ }^{\alpha} \alpha^{\lambda}$ orov oi $\mathrm{B}, \Gamma, \Delta, \mathrm{E} \cdot \lambda \varepsilon ́ \gamma \omega$, ơтı $\tau \widetilde{\omega} \nu \mathrm{B}, \Gamma, \Delta, \mathrm{E}$ ó





[Prop. 8.26]. And $B$ is square. Thus, $A$ is also square. The very opposite thing was assumed. $C$ is thus not square. So, similarly, we can show that no other (number is) square either, apart from the third from the unit, and (all) those (numbers after that) which leave an interval of one (number).

And so let $A$ not be cube. I say that no other (number) will be cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers).

For, if possible, let $D$ be cube. And $C$ is also cube [Prop. 9.8]. For it is the fourth (number) from the unit. And as $C$ is to $D$, (so) $B$ (is) to $C$. And $B$ thus has to $C$ the ratio which (some) cube (number has) to (some other) cube (number). And $C$ is cube. Thus, $B$ is also cube [Props. 7.13, 8.25]. And since as the unit is to $A$, (so) $A$ (is) to $B$, and the unit measures $A$ according to the units in it, $A$ thus also measures $B$ according to the units in ( $A$ ). Thus, $A$ has made the cube (number) $B$ (by) multiplying itself. And if a number makes a cube (number by) multiplying itself then it itself will be cube [Prop. 9.6]. Thus, $A$ (is) also cube. The very opposite thing was assumed. Thus, $D$ is not cube. So, similarly, we can show that no other (number) is cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers). (Which is) the very thing it was required to show.

## Proposition 11

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser (number) measures a greater according to some existing (number) among the proportional numbers.


Let any multitude whatsoever of numbers, $B, C, D$, $E$, be continuously proportional, (starting) from the unit $A$. I say that, for $B, C, D, E$, the least (number), $B$, measures $E$ according to some (one) of $C, D$.

For since as the unit $A$ is to $B$, so $D$ (is) to $E$, the unit $A$ thus measures the number $B$ the same number of times as $D$ (measures) $E$. Thus, alternately, the unit $A$


人⿱㇒㠯刂七иоїц．

## Пópıбиа．





$$
\beta^{\prime}
$$











 ó E тòv $\Delta \mu \varepsilon \tau \rho \varepsilon і ̃, ~ \mu \varepsilon \tau \rho \varepsilon i ́ \tau \omega ~ \alpha u ̉ \tau o ̀ v ~ \chi \alpha \tau \grave{\alpha} ~ \tau o ̀ \nu ~ Z . ~ o ́ ~ E ~ « ̈ p \alpha ~$








 $\mathrm{H} \cdot \dot{\delta} \mathrm{E} \not \ddot{\alpha}^{\rho} \alpha$ тòv $\mathrm{H} \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Gamma \pi \varepsilon \pi o i ́ \eta \chi \varepsilon \nu . ~ \dot{\alpha} \lambda \lambda \grave{\alpha}$ $\mu \grave{\imath} \nu$ ठı̀̀ 七ò $\pi \rho o ̀ ~ \tau o u ́ \tau o u ~ x \alpha i ̀ ~ o ́ ~ A ~ \tau o ̀ v ~ B ~ \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ v ~$



measures $D$ the same number of times as $B$（measures） $E$［Prop．7．15］．And the unit $A$ measures $D$ according to the units in it．Thus，$B$ also measures $E$ according to the units in $D$ ．Hence，the lesser（number）$B$ measures the greater $E$ according to some existing number among the proportional numbers（namely，$D$ ）．

## Corollary

And（it is）clear that what（ever relative）place the measuring（number）has from the unit，the（number） according to which it measures has the same（relative） place from the measured（number），in（the direction of the number）before it．（Which is）the very thing it was required to show．

## Proposition 12

If any multitude whatsoever of numbers is continu－ ously proportional，（starting）from a unit，then however many prime numbers the last（number）is measured by， the（number）next to the unit will also be measured by the same（prime numbers）．


Let any multitude whatsoever of numbers，$A, B, C$ ， $D$ ，be（continuously）proportional，（starting）from a unit． I say that however many prime numbers $D$ is measured by，$A$ will also be measured by the same（prime numbers）．

For let $D$ be measured by some prime number $E$ ．I say that $E$ measures $A$ ．For（suppose it does）not．$E$ is prime，and every prime number is prime to every num－ ber which it does not measure［Prop．7．29］．Thus，$E$ and $A$ are prime to one another．And since $E$ measures $D$ ，let it measure it according to $F$ ．Thus，$E$ has made $D$（by）multiplying $F$ ．Again，since $A$ measures $D$ ac－ cording to the units in $C$［Prop． 9.11 corr．］，$A$ has thus made $D$（by）multiplying $C$ ．But，in fact，$E$ has also made $D$（by）multiplying $F$ ．Thus，the（number cre－ ated）from（multiplying）$A, C$ is equal to the（number created）from（multiplying）$E, F$ ．Thus，as $A$ is to $E$ ， （so）$F$（is）to $C$［Prop．7．19］．And $A$ and $E$（are）prime （to one another），and（numbers）prime（to one another are）also the least（of those numbers having the same ra－ tio as them）［Prop．7．21］，and the least（numbers）mea－ sure those（numbers）having the same ratio as them an equal number of times，the leading（measuring）the lead－


 тòv $\Theta \pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma ı \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ B ~ \pi \varepsilon \pi o i ́ \eta \chi \varepsilon \nu . ~ \alpha ̀ \lambda \lambda \grave{\alpha} \mu \grave{\eta} \nu x \alpha i$ ó A










 xaì тòv $\Delta \cdot$ ó E äp $\alpha$ тoùs $\mathrm{A}, \Delta \mu \varepsilon \tau \rho \varepsilon і ̃ . ~ o ́ \mu о i ́ \omega \varsigma ~ \delta \grave{\eta} \delta \varepsilon i ́ \xi о \mu \varepsilon \nu$,



## ${ }^{\prime} \gamma^{\prime}$.







 $\tau р \eta \vartheta \dot{\eta} \sigma \varepsilon \tau \alpha \iota \pi \alpha \rho غ े \xi \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, Г$.
ing, and the following the following [Prop. 7.20]. Thus, $E$ measures $C$. Let it measure it according to $G$. Thus, $E$ has made $C$ (by) multiplying $G$. But, in fact, via the (proposition) before this, $A$ has also made $C$ (by) multiplying $B$ [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) $A, B$ is equal to the (number created) from (multiplying) $E, G$. Thus, as $A$ is to $E$, (so) $G$ (is) to $B$ [Prop. 7.19]. And $A$ and $E$ (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $E$ measures $B$. Let it measure it according to $H$. Thus, $E$ has made $B$ (by) multiplying $H$. But, in fact, $A$ has also made $B$ (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) $E, H$ is equal to the (square) on $A$. Thus, as $E$ is to $A$, (so) $A$ (is) to $H$ [Prop. 7.19]. And $A$ and $E$ are prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $E$ measures $A$, as the leading (measuring the) leading. But, in fact, $(E)$ also does not measure $(A)$. The very thing (is) impossible. Thus, $E$ and $A$ are not prime to one another. Thus, (they are) composite (to one another). And (numbers) composite (to one another) are (both) measured by some [prime] number [Def. 7.14]. And since $E$ is assumed (to be) prime, and a prime (number) is not measured by another number (other) than itself [Def. 7.11], $E$ thus measures (both) $A$ and $E$. Hence, $E$ measures $A$. And it also measures $D$. Thus, $E$ measures (both) $A$ and $D$. So, similarly, we can show that however many prime numbers $D$ is measured by, $A$ will also be measured by the same (prime numbers). (Which is) the very thing it was required to show.

## Proposition 13

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is prime, then the greatest (number) will be measured by no [other] (numbers) except (numbers) existing among the proportional numbers.

Let any multitude whatsoever of numbers, $A, B, C$, $D$, be continuously proportional, (starting) from a unit. And let the (number) after the unit, $A$, be prime. I say








 ن́ $\varphi^{\prime}$ ย̇т





 عiॅऽ $\tau \widetilde{\omega} \nu \mathrm{A}, \mathrm{B}, \Gamma$ тòv $\Delta \mu \varepsilon \tau \rho \varepsilon \imath ̃ \varkappa \alpha \tau \alpha ́ ~ \tau \imath \nu \alpha ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma \cdot \chi \alpha \grave{~} \dot{o}$












 $\mathrm{Z} \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma$ тòv $\Delta \pi \varepsilon \pi \sigma$ ín $\chi \varepsilon \nu$. $\dot{\alpha} \lambda \lambda \grave{\alpha} \mu \eta ̀ \nu$ к $\alpha$ ì ó A тòv












that the greatest of them, $D$, will be measured by no other (numbers) except $A, B, C$.


For, if possible, let it be measured by $E$, and let $E$ not be the same as one of $A, B, C$. So it is clear that $E$ is not prime. For if $E$ is prime, and measures $D$, then it will also measure $A$, (despite $A$ ) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, $E$ is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, $E$ is measured by some prime number. So I say that it will be measured by no other prime number than $A$. For if $E$ is measured by another (prime number), and $E$ measures $D$, then this (prime number) will thus also measure $D$. Hence, it will also measure $A$, (despite $A$ ) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, $A$ measures $E$. And since $E$ measures $D$, let it measure it according to $F$. I say that $F$ is not the same as one of $A, B$, $C$. For if $F$ is the same as one of $A, B, C$, and measures $D$ according to $E$, then one of $A, B, C$ thus also measures $D$ according to $E$. But one of $A, B, C$ (only) measures $D$ according to some (one) of $A, B, C$ [Prop. 9.11]. And thus $E$ is the same as one of $A, B, C$. The very opposite thing was assumed. Thus, $F$ is not the same as one of $A, B, C$. Similarly, we can show that $F$ is measured by $A$, (by) again showing that $F$ is not prime. For if ( $F$ is prime), and measures $D$, then it will also measure $A$, (despite $A$ ) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, $F$ is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, $F$ is measured by some prime number. So I say that it will be measured by no other prime number than $A$. For if some other prime (number) measures $F$, and $F$ measures $D$, then this (prime number) will thus also measure $D$. Hence, it will also measure $A$, (despite $A$ ) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, $A$ measures $F$. And since $E$ measures $D$ according to $F, E$ has thus made $D$ (by) multiplying $F$. But, in fact, $A$ has also made $D$ (by) multiplying $C$ [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) $A, C$ is equal to the (number created) from (multiplying) $E, F$. Thus, proportionally, as $A$ is to $E$, so $F$ (is) to $C$ [Prop. 7.19]. And $A$ measures









 $\tau \widetilde{\omega} \nu \varepsilon ่ \xi \dot{\alpha} \rho \chi \tilde{\eta} \varsigma \mu \varepsilon \tau \rho o u ́ v \tau \omega \nu$.

















$E$. Thus, $F$ also measures $C$. Let it measure it according to $G$. So, similarly, we can show that $G$ is not the same as one of $A, B$, and that it is measured by $A$. And since $F$ measures $C$ according to $G, F$ has thus made $C$ (by) multiplying $G$. But, in fact, $A$ has also made $C$ (by) multiplying $B$ [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) $A, B$ is equal to the (number created) from (multiplying) $F, G$. Thus, proportionally, as $A$ (is) to $F$, so $G$ (is) to $B$ [Prop. 7.19]. And $A$ measures $F$. Thus, $G$ also measures $B$. Let it measure it according to $H$. So, similarly, we can show that $H$ is not the same as $A$. And since $G$ measures $B$ according to $H, G$ has thus made $B$ (by) multiplying $H$. But, in fact, $A$ has also made $B$ (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) $H, G$ is equal to the square on $A$. Thus, as $H$ is to $A$, (so) $A$ (is) to $G$ [Prop. 7.19]. And $A$ measures $G$. Thus, $H$ also measures $A$, (despite $A$ ) being prime (and) not being the same as it. The very thing (is) absurd. Thus, the greatest (number) $D$ cannot be measured by another (number) except (one of) $A, B$, $C$. (Which is) the very thing it was required to show.

## Proposition 14

If a least number is measured by (some) prime numbers then it will not be measured by any other prime number except (one of) the original measuring (numbers).


For let $A$ be the least number measured by the prime numbers $B, C, D$. I say that $A$ will not be measured by any other prime number except (one of) $B, C, D$.

For, if possible, let it be measured by the prime (number) $E$. And let $E$ not be the same as one of $B, C, D$. And since $E$ measures $A$, let it measure it according to $F$. Thus, $E$ has made $A$ (by) multiplying $F$. And $A$ is measured by the prime numbers $B, C, D$. And if two numbers make some (number by) multiplying one another, and some prime number measures the number created from them, then (the prime number) will also measure one of the original (numbers) [Prop. 7.30]. Thus, $B, C$, $D$ will measure one of $E, F$. In fact, they do not measure $E$. For $E$ is prime, and not the same as one of $B, C, D$. Thus, they (all) measure $F$, which is less than $A$. The very thing (is) impossible. For $A$ was assumed (to be) the least (number) measured by $B, C, D$. Thus, no prime

## เ $\varepsilon^{\prime}$

 tòv aủtòv $\lambda o ́ \gamma o v ~ \varepsilon ̇ \chi o ́ v t \omega v ~ \alpha u ̉ t o u ̃ s, ~ \delta u ́ o ~ o ́ t o l o l o u ̃ v ~ o u v-~$





 हैtı oi $\mathrm{A}, \Gamma$ тpòs tòv B .
入órov ह̇犭óvtav toĩs $\mathrm{A}, \mathrm{B}, \Gamma$ ठúo oi $\Delta \mathrm{E}, \mathrm{EZ}$. yavepòv
 тòv ס̀̀ $\mathrm{EZ} \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma ~ \tau o ̀ v ~ B ~ \pi \varepsilon \pi o i ́ n \chi \varepsilon \nu, ~ x \alpha \grave{l}$ है $\tau \iota$ ó EZ



 $\tau \widetilde{\omega} \nu \Delta \mathrm{E}, \mathrm{EZ} \pi \rho \widetilde{\omega} \tau o ́ \varsigma ~ \varepsilon ُ \sigma \tau เ \nu . ~ \alpha ̀ \lambda \lambda \grave{\alpha} \mu \grave{\nu} \nu x \alpha i ̊ ~ o ́ ~ \Delta \mathrm{E} \pi \rho o ̀ s ~ \tau o ̀ v$






















number can measure $A$ except (one of) $B, C, D$. (Which is) the very thing it was required to show.

## Proposition 15

If three continuously proportional numbers are the least of those (numbers) having the same ratio as them then two (of them) added together in any way are prime to the remaining (one).


Let $A, B, C$ be three continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that two of $A, B, C$ added together in any way are prime to the remaining (one), (that is) $A$ and $B$ (prime) to $C, B$ and $C$ to $A$, and, further, $A$ and $C$ to $B$.

Let the two least numbers, $D E$ and $E F$, having the same ratio as $A, B, C$, have been taken [Prop. 8.2]. So it is clear that $D E$ has made $A$ (by) multiplying itself, and has made $B$ (by) multiplying $E F$, and, further, $E F$ has made $C$ (by) multiplying itself [Prop. 8.2]. And since $D E, E F$ are the least (of those numbers having the same ratio as them), they are prime to one another [Prop. 7.22]. And if two numbers are prime to one another then the sum (of them) is also prime to each [Prop. 7.28]. Thus, $D F$ is also prime to each of $D E, E F$. But, in fact, $D E$ is also prime to $E F$. Thus, $D F, D E$ are (both) prime to $E F$. And if two numbers are (both) prime to some number then the (number) created from (multiplying) them is also prime to the remaining (number) [Prop. 7.24]. Hence, the (number created) from (multiplying) $F D, D E$ is prime to $E F$. Hence, the (number created) from (multiplying) $F D, D E$ is also prime to the (square) on $E F$ [Prop. 7.25]. [For if two numbers are prime to one another then the (number) created from (squaring) one of them is prime to the remaining (number).] But the (number created) from (multiplying) $F D, D E$ is the (square) on $D E$ plus the (number created) from (multiplying) $D E, E F$ [Prop. 2.3]. Thus, the (square) on $D E$ plus the (number created) from (multiplying) $D E, E F$ is prime to the (square) on $E F$. And the (square) on $D E$ is $A$, and the (number created) from (multiplying) $D E, E F$ (is) $B$, and the (square) on $E F$ (is) $C$. Thus, $A, B$ summed is prime to $C$. So, similarly, we can show that $B, C$ (summed) is also prime to $A$. So I say that $A, C$ (summed) is also prime to $B$. For since



$D F$ is prime to each of $D E, E F$ then the (square) on $D F$ is also prime to the (number created) from (multiplying) $D E, E F$ [Prop. 7.25]. But, the (sum of the squares) on $D E, E F$ plus twice the (number created) from (multiplying) $D E, E F$ is equal to the (square) on $D F$ [Prop. 2.4]. And thus the (sum of the squares) on $D E, E F$ plus twice the (rectangle contained) by $D E, E F$ [is] prime to the (rectangle contained) by $D E, E F$. By separation, the (sum of the squares) on $D E, E F$ plus once the (rectangle contained) by $D E, E F$ is prime to the (rectangle contained) by $D E, E F .^{\dagger}$ Again, by separation, the (sum of the squares) on $D E, E F$ is prime to the (rectangle contained) by $D E, E F$. And the (square) on $D E$ is $A$, and the (rectangle contained) by $D E, E F$ (is) $B$, and the (square) on $E F$ (is) $C$. Thus, $A, C$ summed is prime to $B$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ Since if $\alpha \beta$ measures $\alpha^{2}+\beta^{2}+2 \alpha \beta$ then it also measures $\alpha^{2}+\beta^{2}+\alpha \beta$, and vice versa.

$$
i q^{\prime}
$$


 $\alpha \lambda \lambda о \nu \tau เ \nu \alpha ́$.


 $\pi \rho o ̀ s ~ « ̈ \lambda \lambda o \nu \tau \tau \alpha \alpha$.









$$
\zeta^{\prime}
$$





## Proposition 16

If two numbers are prime to one another then as the first is to the second, so the second (will) not (be) to some other (number).


For let the two numbers $A$ and $B$ be prime to one another. I say that as $A$ is to $B$, so $B$ is not to some other (number).

For, if possible, let it be that as $A$ (is) to $B$, (so) $B$ (is) to $C$. And $A$ and $B$ (are) prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $A$ measures $B$, as the leading (measuring) the leading. And ( $A$ ) also measures itself. Thus, $A$ measures $A$ and $B$, which are prime to one another. The very thing (is) absurd. Thus, as $A$ (is) to $B$, so $B$ cannot be to $C$. (Which is) the very thing it was required to show.

## Proposition 17

If any multitude whatsoever of numbers is continuously proportional, and the outermost of them are prime to one another, then as the first (is) to the second, so the
tivá．






## $\mathrm{E} \longmapsto$













 бєїֹ人u．

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i n^{\prime} .
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 aủtoĩs tpítov àvá入oүov тробєupeĩ．


 д̀vव́入orov пробєиреі̃v．








last will not be to some other（number）．
Let $A, B, C, D$ be any multitude whatsoever of con－ tinuously proportional numbers．And let the outermost of them，$A$ and $D$ ，be prime to one another．I say that as $A$ is to $B$ ，so $D$（is）not to some other（number）．


For，if possible，let it be that as $A$（is）to $B$ ，so $D$ （is）to $E$ ．Thus，alternately，as $A$ is to $D$ ，（so）$B$（is） to $E$［Prop．7．13］．And $A$ and $D$ are prime（to one another）．And（numbers）prime（to one another are） also the least（of those numbers having the same ra－ tio as them）［Prop．7．21］．And the least numbers mea－ sure those（numbers）having the same ratio（as them）an equal number of times，the leading（measuring）the lead－ ing，and the following the following［Prop．7．20］．Thus， $A$ measures $B$ ．And as $A$ is to $B$ ，（so）$B$（is）to $C$ ．Thus，$B$ also measures $C$ ．And hence $A$ measures $C$［Def．7．20］． And since as $B$ is to $C$ ，（so）$C$（is）to $D$ ，and $B$ mea－ sures $C, C$ thus also measures $D$［Def．7．20］．But，$A$ was （found to be）measuring $C$ ．And hence $A$ also measures $D$ ．And（ $A$ ）also measures itself．Thus，$A$ measures $A$ and $D$ ，which are prime to one another．The very thing is impossible．Thus，as $A$（is）to $B$ ，so $D$ cannot be to some other（number）．（Which is）the very thing it was required to show．

## Proposition 18

For two given numbers，to investigate whether it is possible to find a third（number）proportional to them．

$$
\mathrm{D} \longmapsto
$$

Let $A$ and $B$ be the two given numbers．And let it be required to investigate whether it is possible to find a third（number）proportional to them．

So $A$ and $B$ are either prime to one another，or not． And if they are prime to one another then it has（already） been show that it is impossible to find a third（number） proportional to them［Prop．9．16］．

And so let $A$ and $B$ not be prime to one another．And let $B$ make $C$（by）multiplying itself．So $A$ either mea－ sures，or does not measure，$C$ ．Let it first of all measure （C）according to $D$ ．Thus，$A$ has made $C$（by）multiply－

 àvá入oүov тробnúpŋtal ò $\Delta$.










## $\left.{ }^{1}\right)^{\prime}$.





B


E







 тpòs $\alpha \lambda \lambda$ ǹ $\lambda$ ous عíбív.












ing $D$. But, in fact, $B$ has also made $C$ (by) multiplying itself. Thus, the (number created) from (multiplying) $A$, $D$ is equal to the (square) on $B$. Thus, as $A$ is to $B$, (so) $B$ (is) to $D$ [Prop. 7.19]. Thus, a third number has been found proportional to $A, B$, (namely) $D$.

And so let $A$ not measure $C$. I say that it is impossible to find a third number proportional to $A, B$. For, if possible, let it have been found, (and let it be) $D$. Thus, the (number created) from (multiplying) $A, D$ is equal to the (square) on $B$ [Prop. 7.19]. And the (square) on $B$ is $C$. Thus, the (number created) from (multiplying) $A$, $D$ is equal to $C$. Hence, $A$ has made $C$ (by) multiplying $D$. Thus, $A$ measures $C$ according to $D$. But ( $A$ ) was, in fact, also assumed (to be) not measuring ( $C$ ). The very thing (is) absurd. Thus, it is not possible to find a third number proportional to $A, B$ when $A$ does not measure $C$. (Which is) the very thing it was required to show.

## Proposition $19^{\dagger}$

For three given numbers, to investigate when it is possible to find a fourth (number) proportional to them.


Let $A, B, C$ be the three given numbers. And let it be required to investigate when it is possible to find a fourth (number) proportional to them.

In fact, $(A, B, C)$ are either not continuously proportional and the outermost of them are prime to one another, or are continuously proportional and the outermost of them are not prime to one another, or are neither continuously proportional nor are the outermost of them prime to one another, or are continuously proportional and the outermost of them are prime to one another.

In fact, if $A, B, C$ are continuously proportional, and the outermost of them, $A$ and $C$, are prime to one another, (then) it has (already) been shown that it is impossible to find a fourth number proportional to them [Prop. 9.17]. So let $A, B, C$ not be continuously proportional, (with) the outermost of them again being prime to one another. I say that, in this case, it is also impossible to find a fourth (number) proportional to them. For, if possible, let it have been found, (and let it be) $D$. Hence, it will be that as $A$ (is) to $B$, (so) $C$ (is) to $D$. And let it be contrived that as $B$ (is) to $C$, (so) $D$ (is) to $E$. And since






























as $A$ is to $B$, (so) $C$ (is) to $D$, and as $B$ (is) to $C$, (so) $D$ (is) to $E$, thus, via equality, as $A$ (is) to $C$, (so) $C$ (is) to $E$ [Prop. 7.14]. And $A$ and $C$ (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least (numbers) measure those numbers having the same ratio as them (the same number of times), the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, $A$ measures $C$, (as) the leading (measuring) the leading. And it also measures itself. Thus, $A$ measures $A$ and $C$, which are prime to one another. The very thing is impossible. Thus, it is not possible to find a fourth (number) proportional to $A, B$, $C$.

And so let $A, B, C$ again be continuously proportional, and let $A$ and $C$ not be prime to one another. I say that it is possible to find a fourth (number) proportional to them. For let $B$ make $D$ (by) multiplying $C$. Thus, $A$ either measures or does not measure $D$. Let it, first of all, measure ( $D$ ) according to $E$. Thus, $A$ has made $D$ (by) multiplying $E$. But, in fact, $B$ has also made $D$ (by) multiplying $C$. Thus, the (number created) from (multiplying) $A, E$ is equal to the (number created) from (multiplying) $B, C$. Thus, proportionally, as $A$ [is] to $B$, (so) $C$ (is) to $E$ [Prop. 7.19]. Thus, a fourth (number) proportional to $A, B, C$ has been found, (namely) $E$.

And so let $A$ not measure $D$. I say that it is impossible to find a fourth number proportional to $A, B, C$. For, if possible, let it have been found, (and let it be) $E$. Thus, the (number created) from (multiplying) $A, E$ is equal to the (number created) from (multiplying) $B, C$. But, the (number created) from (multiplying) $B, C$ is $D$. And thus the (number created) from (multiplying) $A, E$ is equal to $D$. Thus, $A$ has made $D$ (by) multiplying $E$. Thus, $A$ measures $D$ according to $E$. Hence, $A$ measures $D$. But, it also does not measure ( $D$ ). The very thing (is) absurd. Thus, it is not possible to find a fourth number proportional to $A, B, C$ when $A$ does not measure $D$. And so (let) $A, B, C$ (be) neither continuously proportional, nor (let) the outermost of them (be) prime to one another. And let $B$ make $D$ (by) multiplying $C$. So, similarly, it can be show that if $A$ measures $D$ then it is possible to find a fourth (number) proportional to ( $A, B, C$ ), and impossible if $(A)$ does not measure $(D)$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ The proof of this proposition is incorrect. There are, in fact, only two cases. Either $A, B, C$ are continuously proportional, with $A$ and $C$ prime to one another, or not. In the first case, it is impossible to find a fourth proportional number. In the second case, it is possible to find a fourth proportional number provided that $A$ measures $B$ times $C$. Of the four cases considered by Euclid, the proof given in the second case is incorrect, since it only demonstrates that if $A: B:: C: D$ then a number $E$ cannot be found such that $B: C:: D: E$. The proofs given in the other three
cases are correct.

## $x^{\prime}$.

 $\pi \lambda \dot{\gamma} \vartheta \frac{0}{}$






 $\Gamma$.







 $\pi \lambda \dot{\eta} \vartheta$ ous $\tau \widetilde{\omega} \nu \mathrm{A}, \mathrm{B}, \Gamma$ oi $\mathrm{A}, \mathrm{B}, \Gamma, \mathrm{H} \cdot$ o้ $\pi \varepsilon \rho$ है $\delta \varepsilon \iota ~ \delta \varepsilon i ̈ \zeta \alpha l . ~$

## $x \alpha^{\prime}$.

 äptiós ह̀ $\sigma \tau \tau$.








## Proposition 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.


Let $A, B, C$ be the assigned prime numbers. I say that the (set of all) primes numbers is more numerous than $A$, $B, C$.

For let the least number measured by $A, B, C$ have been taken, and let it be $D E$ [Prop. 7.36]. And let the unit $D F$ have been added to $D E$. So $E F$ is either prime, or not. Let it, first of all, be prime. Thus, the (set of) prime numbers $A, B, C, E F$, (which is) more numerous than $A, B, C$, has been found.

And so let $E F$ not be prime. Thus, it is measured by some prime number [Prop. 7.31]. Let it be measured by the prime (number) $G$. I say that $G$ is not the same as any of $A, B, C$. For, if possible, let it be (the same). And $A, B, C$ (all) measure $D E$. Thus, $G$ will also measure $D E$. And it also measures $E F$. (So) $G$ will also measure the remainder, unit $D F$, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, $G$ is not the same as one of $A, B, C$. And it was assumed (to be) prime. Thus, the (set of) prime numbers $A, B, C, G$, (which is) more numerous than the assigned multitude (of prime numbers), $A, B, C$, has been found. (Which is) the very thing it was required to show.

## Proposition 21

If any multitude whatsoever of even numbers is added together then the whole is even.


For let any multitude whatsoever of even numbers, $A B, B C, C D, D E$, lie together. I say that the whole, $A E$, is even.

For since everyone of $A B, B C, C D, D E$ is even, it has a half part [Def. 7.6]. And hence the whole $A E$ has a half part. And an even number is one (which can be) divided in half [Def. 7.6]. Thus, $A E$ is even. (Which is)

## $\chi \beta^{\prime}$.





 ó AE äptıós દ̇бтıv.






$$
x \gamma^{\prime}
$$




 $\pi \lambda \tilde{\eta} \vartheta \circ \varsigma \pi \varepsilon \rho เ \sigma \sigma o ̀ v$ है $\sigma \tau \omega$, oi $\mathrm{AB}, В \Gamma, \Gamma \Delta \cdot \lambda \varepsilon ́ \gamma \omega$, ǒ $\tau \iota$ кגi ő $\lambda \circ \varsigma$ ó A $\Delta \pi \varepsilon p ı \sigma \sigma o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota \nu . ~$




$\chi \delta^{\prime}$.
 а̋ртıos है $\sigma \tau \alpha$.

 őtı ó 入oเлòs ó ГА «̈ptıós ह̀ $\sigma \tau \iota$.



the very thing it was required to show.

## Proposition 22

If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.


For let any even multitude whatsoever of odd numbers, $A B, B C, C D, D E$, lie together. I say that the whole, $A E$, is even.

For since everyone of $A B, B C, C D, D E$ is odd then, a unit being subtracted from each, everyone of the remainders will be (made) even [Def. 7.7]. And hence the sum of them will be even [Prop. 9.21]. And the multitude of the units is even. Thus, the whole $A E$ is also even [Prop. 9.21]. (Which is) the very thing it was required to show.

## Proposition 23

If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.


For let any multitude whatsoever of odd numbers, $A B, B C, C D$, lie together, and let the multitude of them be odd. I say that the whole, $A D$, is also odd.

For let the unit $D E$ have been subtracted from $C D$. The remainder $C E$ is thus even [Def. 7.7]. And $C A$ is also even [Prop. 9.22]. Thus, the whole $A E$ is also even [Prop. 9.21]. And $D E$ is a unit. Thus, $A D$ is odd [Def. 7.7]. (Which is) the very thing it was required to show.

## Proposition 24

If an even (number) is subtracted from an(other) even number then the remainder will be even.


For let the even (number) $B C$ have been subtracted from the even number $A B$. I say that the remainder $C A$ is even.

For since $A B$ is even, it has a half part [Def. 7.6]. So, for the same (reasons), $B C$ also has a half part. And hence the remainder [ $C A$ has a half part]. [Thus,] $A C$ is even. (Which is) the very thing it was required to show.

## $\chi \varepsilon^{\prime}$.

 $\pi \varepsilon \rho เ \sigma \sigma o ̀ \varsigma ~$ ह̈ $\sigma \tau \alpha$.








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\chiq}
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 äptıos eैठ $\sigma \alpha \mathrm{L}$.





 бєї̧al.

$$
x \zeta^{\prime} .
$$

 $\pi \varepsilon \rho เ \sigma \sigma o ̀ s$ है $\sigma \tau \alpha$.

'A





$$
x n^{\prime} .
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 tiva, ó $\gamma \varepsilon v o ́ \mu \varepsilon v o s ~ a ̈ p t i o s ~ ह ै \sigma \tau \alpha l . ~$

## Proposition 25

If an odd (number) is subtracted from an even number then the remainder will be odd.


For let the odd (number) $B C$ have been subtracted from the even number $A B$. I say that the remainder $C A$ is odd.

For let the unit $C D$ have been subtracted from $B C$. $D B$ is thus even [Def. 7.7]. And $A B$ is also even. And thus the remainder $A D$ is even [Prop. 9.24]. And $C D$ is a unit. Thus, $C A$ is odd [Def. 7.7]. (Which is) the very thing it was required to show.

## Proposition 26

If an odd (number) is subtracted from an odd number then the remainder will be even.


For let the odd (number) $B C$ have been subtracted from the odd (number) $A B$. I say that the remainder $C A$ is even.

For since $A B$ is odd, let the unit $B D$ have been subtracted (from it). Thus, the remainder $A D$ is even [Def. 7.7]. So, for the same (reasons), $C D$ is also even. And hence the remainder $C A$ is even [Prop. 9.24]. (Which is) the very thing it was required to show.

## Proposition 27

If an even (number) is subtracted from an odd number then the remainder will be odd.


For let the even (number) $B C$ have been subtracted from the odd (number) $A B$. I say that the remainder $C A$ is odd.
[For] let the unit $A D$ have been subtracted (from $A B$ ). $D B$ is thus even [Def. 7.7]. And $B C$ is also even. Thus, the remainder $C D$ is also even [Prop. 9.24]. $C A$ (is) thus odd [Def. 7.7]. (Which is) the very thing it was required to show.

## Proposition 28

If an odd number makes some (number by) multiplying an even (number) then the created (number) will be even.








$\chi \vartheta^{\prime}$.



$\Gamma$
Пعрıббòs үàp àpıقuòs ó A $\pi \varepsilon p ı \sigma \sigma o ̀ v ~ \tau o ̀ v ~ B ~ \pi o \lambda \lambda \alpha-$ $\pi \lambda \alpha \sigma \iota \alpha ́ \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ \Gamma \pi o เ \varepsilon i ́ t \omega \cdot \lambda \varepsilon ́ \gamma \omega$, öтı ó $\Gamma \pi \varepsilon p เ \sigma \sigma o ́ s ~ \varepsilon ̇ \sigma \tau \iota \nu . ~$






## $\lambda^{\prime}$.

 ทั $\mu เ \sigma \cup \nu$ बu่тои̃ $\mu \varepsilon \tau \rho ท ̆ \sigma \varepsilon เ . ~$


B
$\Gamma$ $\qquad$


 $\Gamma \cdot \lambda \varepsilon ́ \gamma \omega$, öтı ò $\Gamma$ оủx है $\sigma \tau \iota ~ \pi \varepsilon p ı \sigma \sigma o ́ \varsigma . ~ \varepsilon i ̂ ~ \gamma a ̀ p ~ \delta u v \alpha \tau o ́ v, ~ ह ै \sigma \tau \omega . ~$
 $\pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma ~ \tau o ̀ \nu ~ B ~ \pi \varepsilon \pi o i ́ \eta \chi \varepsilon \nu$. ò B аैp $\alpha$ $\sigma \cup ́ \gamma \varkappa \varepsilon เ \tau \alpha \iota ~ \varepsilon ̇ \chi ~$



For let the odd number $A$ make $C$ (by) multiplying the even (number) $B$. I say that $C$ is even.

For since $A$ has made $C$ (by) multiplying $B, C$ is thus composed out of so many (magnitudes) equal to $B$, as many as (there) are units in $A$ [Def. 7.15]. And $B$ is even. Thus, $C$ is composed out of even (numbers). And if any multitude whatsoever of even numbers is added together then the whole is even [Prop. 9.21]. Thus, $C$ is even. (Which is) the very thing it was required to show.

## Proposition 29

If an odd number makes some (number by) multiplying an odd (number) then the created (number) will be odd.


For let the odd number $A$ make $C$ (by) multiplying the odd (number) $B$. I say that $C$ is odd.

For since $A$ has made $C$ (by) multiplying $B, C$ is thus composed out of so many (magnitudes) equal to $B$, as many as (there) are units in $A$ [Def. 7.15]. And each of $A, B$ is odd. Thus, $C$ is composed out of odd (numbers), (and) the multitude of them is odd. Hence $C$ is odd [Prop. 9.23]. (Which is) the very thing it was required to show.

## Proposition 30

If an odd number measures an even number then it will also measure (one) half of it.


For let the odd number $A$ measure the even (number) $B$. I say that ( $A$ ) will also measure (one) half of $(B)$.

For since $A$ measures $B$, let it measure it according to $C$. I say that $C$ is not odd. For, if possible, let it be (odd). And since $A$ measures $B$ according to $C, A$ has thus made $B$ (by) multiplying $C$. Thus, $B$ is composed out of odd numbers, (and) the multitude of them is odd. $B$ is thus




$\lambda \alpha^{\prime}$.
 $\pi \rho o ̀ \varsigma ~ \tau o ̀ v ~ \delta \iota \pi \lambda \alpha \sigma i ́ v \alpha \alpha$ 人ủ兀oũ $\pi \rho \widetilde{\omega} \tau 0 \varsigma$ है $\sigma \tau \alpha$.














$$
\lambda \beta^{\prime} .
$$




$\Gamma$

## $\Delta$


 a̋ptıoí عíal $\mu$ óvov.



 $\mu \circ \nu \alpha ́ \delta \alpha$ ò A $\pi \rho \widetilde{\omega} \tau o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota \nu, ~ o ́ ~ \mu \varepsilon ́ \gamma \iota \sigma \tau o \varsigma ~ \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma, \Delta \dot{\delta}$
odd [Prop. 9.23]. The very thing (is) absurd. For ( $B$ ) was assumed (to be) even. Thus, $C$ is not odd. Thus, $C$ is even. Hence, $A$ measures $B$ an even number of times. So, on account of this, ( $A$ ) will also measure (one) half of $(B)$. (Which is) the very thing it was required to show.

## Proposition 31

If an odd number is prime to some number then it will also be prime to its double.


For let the odd number $A$ be prime to some number $B$. And let $C$ be double $B$. I say that $A$ is [also] prime to $C$.

For if [ $A$ and $C$ ] are not prime (to one another) then some number will measure them. Let it measure (them), and let it be $D$. And $A$ is odd. Thus, $D$ (is) also odd. And since $D$, which is odd, measures $C$, and $C$ is even, [ $D$ ] will thus also measure half of $C$ [Prop. 9.30]. And $B$ is half of $C$. Thus, $D$ measures $B$. And it also measures $A$. Thus, $D$ measures (both) $A$ and $B$, (despite) them being prime to one another. The very thing is impossible. Thus, $A$ is not unprime to $C$. Thus, $A$ and $C$ are prime to one another. (Which is) the very thing it was required to show.

## Proposition 32

Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even (number) only.


For let any multitude of numbers whatsoever, $B, C$, $D$, have been (continually) doubled, (starting) from the dyad $A$. I say that $B, C, D$ are even-times-even (numbers) only.

In fact, (it is) clear that each [of $B, C, D$ ] is an even-times-even (number). For it is doubled from a dyad [Def. 7.8]. I also say that (they are even-times-even numbers) only. For let a unit be laid down. Therefore, since





## $\lambda \gamma^{\prime}$.




$$
A \longmapsto
$$

 ò A àpтıáxıs $\pi \varepsilon p ı \sigma \sigma o ́ s ~ \varepsilon ̇ \sigma \tau ı ~ \mu o ́ v o v . ~$








## $\lambda \delta^{\prime}$.


 ג̉pтıáxıs $\pi \varepsilon p ı \sigma \sigma o ́ s . ~$














any multitude of numbers whatsoever are continuously proportional, starting from a unit, and the (number) $A$ after the unit is prime, the greatest of $A, B, C, D$, (namely) $D$, will not be measured by any other (numbers) except $A, B, C$ [Prop. 9.13]. And each of $A, B, C$ is even. Thus, $D$ is an even-time-even (number) only [Def. 7.8]. So, similarly, we can show that each of $B, C$ is [also] an even-time-even (number) only. (Which is) the very thing it was required to show.

## Proposition 33

If a number has an odd half then it is an even-timeodd (number) only.

$$
A \longmapsto
$$

For let the number $A$ have an odd half. I say that $A$ is an even-times-odd (number) only.

In fact, (it is) clear that $(A)$ is an even-times-odd (number). For its half, being odd, measures it an even number of times [Def. 7.9]. So I also say that (it is an even-times-odd number) only. For if $A$ is also an even-times-even (number) then it will be measured by an even (number) according to an even number [Def. 7.8]. Hence, its half will also be measured by an even number, (despite) being odd. The very thing is absurd. Thus, $A$ is an even-times-odd (number) only. (Which is) the very thing it was required to show.

## Proposition 34

If a number is neither (one) of the (numbers) doubled from a dyad, nor has an odd half, then it is (both) an even-times-even and an even-times-odd (number).


For let the number $A$ neither be (one) of the (numbers) doubled from a dyad, nor let it have an odd half. I say that $A$ is (both) an even-times-even and an even-times-odd (number).

In fact, (it is) clear that $A$ is an even-times-even (number) [Def. 7.8]. For it does not have an odd half. So I say that it is also an even-times-odd (number). For if we cut $A$ in half, and (then cut) its half in half, and we do this continually, then we will arrive at some odd number which will measure $A$ according to an even number. For if not, we will arrive at a dyad, and $A$ will be (one) of the (numbers) doubled from a dyad. The very opposite thing (was) assumed. Hence, $A$ is an even-times-odd (number) [Def. 7.9]. And it was also shown (to be) an even-times-even (number). Thus, $A$ is (both) an even-times-even and an even-times-odd (number). (Which is)

## $\lambda \varepsilon^{\prime}$.


 $\pi \rho \omega ́ \tau \omega$, , है $\sigma \tau \alpha \iota \dot{\omega} \varsigma \dot{\eta} \tau o u ̃ ~ \delta \varepsilon \cup \tau \varepsilon ́ \rho o v ~ ن ́ \pi \varepsilon \rho o \chi \grave{\eta} \pi \rho o ̀ s ~ t o ̀ v ~ \pi \rho \widetilde{\omega} \tau o \nu$,
 $\pi \alpha ́ v \tau \alpha \varsigma$.




 toùs $\mathrm{A}, \mathrm{B} \mathrm{\Gamma}, \Delta$.


 ò EZ трòs tòv $\Delta$, oüt $\omega \varsigma$ ò $\Delta$ тpòs tòv $\mathrm{B} \mathrm{\Gamma}$ кal ò $\mathrm{B} \mathrm{\Gamma}$ тpòs





 ápa ís ò $\mathrm{K} \Theta$ тpòs tòv $\mathrm{Z} \Theta$, oüt


 $\dot{\omega} \varsigma \dot{\eta}$ тоũ $\delta \varepsilon \cup \tau \varepsilon ́ \rho o \cup ~ \cup ́ \pi \varepsilon \rho o \chi \grave{\eta} \pi \rho o ̀ \varsigma ~ \tau o ̀ v ~ \pi \rho \tilde{\omega} \tau o \nu, ~ o u ̛ \tau \omega \varsigma ~ \dot{\eta} \tau о \tilde{u}$
 ठє亢̃! $\alpha$.
the very thing it was required to show.

## Proposition $35^{\dagger}$

If there is any multitude whatsoever of continually proportional numbers, and (numbers) equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those (numbers) before it.


Let $A, B C, D, E F$ be any multitude whatsoever of continuously proportional numbers, beginning from the least $A$. And let $B G$ and $F H$, each equal to $A$, have been subtracted from $B C$ and $E F$ (respectively). I say that as $G C$ is to $A$, so $E H$ is to $A, B C, D$.

For let $F K$ be made equal to $B C$, and $F L$ to $D$. And since $F K$ is equal to $B C$, of which $F H$ is equal to $B G$, the remainder $H K$ is thus equal to the remainder $G C$. And since as $E F$ is to $D$, so $D$ (is) to $B C$, and $B C$ to $A$ [Prop. 7.13], and $D$ (is) equal to $F L$, and $B C$ to $F K$, and $A$ to $F H$, thus as $E F$ is to $F L$, so $L F$ (is) to $F K$, and $F K$ to $F H$. By separation, as $E L$ (is) to $L F$, so $L K$ (is) to $F K$, and $K H$ to $F H$ [Props. 7.11, 7.13]. And thus as one of the leading (numbers) is to one of the following, so (the sum of) all of the leading (numbers is) to (the sum of) all of the following [Prop. 7.12]. Thus, as $K H$ is to $F H$, so $E L, L K, K H$ (are) to $L F, F K, H F$. And $K H$ (is) equal to $C G$, and $F H$ to $A$, and $L F, F K, H F$ to $D, B C, A$. Thus, as $C G$ is to $A$, so $E H$ (is) to $D$, $B C, A$. Thus, as the excess of the second (number) is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition allows us to sum a geometric series of the form $a$, $a r, a r^{2}, a r^{3}, \cdots a r^{n-1}$. According to Euclid, the sum $S_{n}$ satisfies $(a r-a) / a=\left(a r^{n}-a\right) / S_{n}$. Hence, $S_{n}=a\left(r^{n}-1\right) /(r-1)$.

$$
\lambda \varepsilon^{\prime}
$$





## Proposition $36^{\dagger}$

If any multitude whatsoever of numbers is set out continuously in a double proportion, (starting) from a unit, until the whole sum added together becomes prime, and



 ó $\mathrm{E}, \chi \alpha \grave{\imath}$ ó E тòv $\Delta \pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \alpha \varsigma$ tòv $\mathrm{ZH} \pi о เ \varepsilon i ́ \tau \omega$. $\lambda \varepsilon ́ \gamma \omega$,


"Oбol үáp عíซเv oi $\mathrm{A}, \mathrm{B}, \Gamma, \Delta \tau \widetilde{̣} \pi \lambda \eta \dot{\eta} \vartheta \varepsilon$, тoбoũ tol $\dot{\alpha} \pi o ̀$








 ठ̀̀ $\dot{\alpha} \pi o ̀ ~ \tau o u ̃ ~ \delta \varepsilon u \tau \varepsilon ́ p o u ~ \tau o u ̃ ~ \Theta K ~ x \alpha i ̀ ~ \tau o u ̃ ~ \varepsilon ̇ \sigma \chi \alpha ́ \tau o u ~ \tau o u ̃ ~ Z H ~ \tau \widetilde{̣}$


 äpa ìs ó NK rpòs tòv E , oứt $\omega$ s ó $\Xi \mathrm{H}$ тpòs toùs $\mathrm{M}, \Lambda$,




㐅$\lambda \lambda$ о $\cup \mu \varepsilon \tau \rho \eta \vartheta \dot{\eta} \sigma \varepsilon \tau \alpha \iota \pi \alpha \rho \varepsilon ̀ \xi \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma, \Delta, \mathrm{E}, \Theta \mathrm{K}, \Lambda, \mathrm{M}$
 O , xai ó $\mathrm{O} \mu \eta \delta \varepsilon v i \tau \widetilde{\omega} \nu \mathrm{~A}, \mathrm{~B}, \Gamma, \Delta, \mathrm{E}, \Theta \mathrm{K}, \Lambda, \mathrm{M}$ है $\sigma \tau \omega \dot{\delta}$

the sum multiplied into the last (number) makes some (number), then the (number so) created will be perfect.

For let any multitude of numbers, $A, B, C, D$, be set out (continuouly) in a double proportion, until the whole sum added together is made prime. And let $E$ be equal to the sum. And let $E$ make $F G$ (by) multiplying $D$. I say that $F G$ is a perfect (number).


For as many as is the multitude of $A, B, C, D$, let so many (numbers), $E, H K, L, M$, have been taken in a double proportion, (starting) from $E$. Thus, via equality, as $A$ is to $D$, so $E$ (is) to $M$ [Prop. 7.14]. Thus, the (number created) from (multiplying) $E, D$ is equal to the (number created) from (multiplying) $A, M$. And $F G$ is the (number created) from (multiplying) $E, D$. Thus, $F G$ is also the (number created) from (multiplying) $A$, $M$ [Prop. 7.19]. Thus, $A$ has made $F G$ (by) multiplying $M$. Thus, $M$ measures $F G$ according to the units in $A$. And $A$ is a dyad. Thus, $F G$ is double $M$. And $M, L$, $H K, E$ are also continuously double one another. Thus, $E, H K, L, M, F G$ are continuously proportional in a double proportion. So let $H N$ and $F O$, each equal to the first (number) $E$, have been subtracted from the second (number) $H K$ and the last $F G$ (respectively). Thus, as the excess of the second number is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it [Prop. 9.35]. Thus, as $N K$ is to $E$, so $O G$ (is) to $M, L, K H, E$. And $N K$ is equal to $E$. And thus $O G$ is equal to $M, L, H K, E$. And $F O$ is also equal to $E$, and $E$ to $A, B, C, D$, and a unit. Thus, the whole of $F G$ is equal to $E, H K, L, M$, and $A, B, C, D$, and a unit. And it is measured by them. I also say that $F G$ will be
















 árò тoũ E oi $\mathrm{E}, \Theta \mathrm{K}, \Lambda$. xaí घíбь oi $\mathrm{E}, \Theta \mathrm{K}, \Lambda$ тoĩऽ $\mathrm{B}, \Gamma, \Delta$










 ठєīŋ $\alpha$.
measured by no other (numbers) except $A, B, C, D, E$, $H K, L, M$, and a unit. For, if possible, let some (number) $P$ measure $F G$, and let $P$ not be the same as any of $A, B, C, D, E, H K, L, M$. And as many times as $P$ measures $F G$, so many units let there be in $Q$. Thus, $Q$ has made $F G$ (by) multiplying $P$. But, in fact, $E$ has also made $F G$ (by) multiplying $D$. Thus, as $E$ is to $Q$, so $P$ (is) to $D$ [Prop. 7.19]. And since $A, B, C, D$ are continually proportional, (starting) from a unit, $D$ will thus not be measured by any other numbers except $A, B, C$ [Prop. 9.13]. And $P$ was assumed not (to be) the same as any of $A, B, C$. Thus, $P$ does not measure $D$. But, as $P$ (is) to $D$, so $E$ (is) to $Q$. Thus, $E$ does not measure $Q$ either [Def. 7.20]. And $E$ is a prime (number). And every prime number [is] prime to every (number) which it does not measure [Prop. 7.29]. Thus, $E$ and $Q$ are prime to one another. And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. And as $E$ is to $Q$, (so) $P$ (is) to $D$. Thus, $E$ measures $P$ the same number of times as $Q$ (measures) $D$. And $D$ is not measured by any other (numbers) except $A, B, C$. Thus, $Q$ is the same as one of $A, B, C$. Let it be the same as $B$. And as many as is the multitude of $B, C, D$, let so many (of the set out numbers) have been taken, (starting) from $E$, (namely) $E, H K, L$. And $E, H K, L$ are in the same ratio as $B, C, D$. Thus, via equality, as $B$ (is) to $D$, (so) $E$ (is) to $L$ [Prop. 7.14]. Thus, the (number created) from (multiplying) $B, L$ is equal to the (number created) from multiplying $D, E$ [Prop. 7.19]. But, the (number created) from (multiplying) $D, E$ is equal to the (number created) from (multiplying) $Q, P$. Thus, the (number created) from (multiplying) $Q, P$ is equal to the (number created) from (multiplying) $B, L$. Thus, as $Q$ is to $B$, (so) $L$ (is) to $P$ [Prop. 7.19]. And $Q$ is the same as $B$. Thus, $L$ is also the same as $P$. The very thing (is) impossible. For $P$ was assumed not (to be) the same as any of the (numbers) set out. Thus, $F G$ cannot be measured by any number except $A, B, C, D, E, H K, L$, $M$, and a unit. And $F G$ was shown (to be) equal to (the sum of) $A, B, C, D, E, H K, L, M$, and a unit. And a perfect number is one which is equal to (the sum of) its own parts [Def. 7.22]. Thus, $F G$ is a perfect (number). (Which is) the very thing it was required to show.

[^20]
## ELEMENTS BOOK 10

## Incommensurable Magnitudes $\ddagger$

[^21]${ }^{*} \mathrm{Opol}$.

 $\gamma \varepsilon v \varepsilon ́ \sigma \vartheta \alpha \downarrow$.


 $\mu \varepsilon ́ т \rho о \nu ~ \gamma \varepsilon \nu \varepsilon ́ \sigma \vartheta \alpha \downarrow$.




 $\dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho о \iota ~ \alpha ॅ \lambda о \gamma о \iota ~ \varkappa \alpha \lambda \varepsilon і \sigma \vartheta \vartheta \omega \sigma \alpha \nu$.






## Definitions

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable. ${ }^{\dagger}$
2. (Two) straight-lines are commensurable in square ${ }^{\ddagger}$ when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them. ${ }^{\S}$
3. These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straight-line-those (incommensurable) in length only, and those also (commensurable or incommensurable) in square. ${ }^{\top}$ Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.*
4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their squareroots ${ }^{\$}$ (also be called) irrational-the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).

[^22]$$
\alpha^{\prime}
$$






## Proposition $1^{\dagger}$

If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will


 $\mu \varepsilon \curlyvee \varepsilon ิ \vartheta$ оus.








"Eбт $\omega \sigma \alpha \nu$ oưv גi AK, $\mathrm{K} \Theta, ~ \Theta В ~ \delta ı \alpha ı p \varepsilon ́ \sigma \varepsilon ı \varsigma ~ i \sigma o \pi \lambda \eta \vartheta \varepsilon i ̃ \varsigma ~$










 ג̀ $\propto<1 \rho 0 u ́ \mu \varepsilon v \alpha$.
be less than the lesser laid out magnitude.
Let $A B$ and $C$ be two unequal magnitudes, of which (let) $A B$ (be) the greater. I say that if (a part) greater than half is subtracted from $A B$, and (if a part) greater than half (is subtracted) from the remainder, and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the magnitude $C$.


For $C$, when multiplied (by some number), will sometimes be greater than $A B$ [Def. 5.4]. Let it have been (so) multiplied. And let $D E$ be (both) a multiple of $C$, and greater than $A B$. And let $D E$ have been divided into the (divisions) $D F, F G, G E$, equal to $C$. And let $B H$, (which is) greater than half, have been subtracted from $A B$. And (let) $H K$, (which is) greater than half, (have been subtracted) from $A H$. And let this happen continually, until the divisions in $A B$ become equal in number to the divisions in $D E$.

Therefore, let the divisions (in $A B$ ) be $A K, K H, H B$, being equal in number to $D F, F G, G E$. And since $D E$ is greater than $A B$, and $E G$, (which is) less than half, has been subtracted from $D E$, and $B H$, (which is) greater than half, from $A B$, the remainder $G D$ is thus greater than the remainder $H A$. And since $G D$ is greater than $H A$, and the half $G F$ has been subtracted from $G D$, and $H K$, (which is) greater than half, from $H A$, the remainder $D F$ is thus greater than the remainder $A K$. And $D F$ (is) equal to $C . C$ is thus also greater than $A K$. Thus, $A K$ (is) less than $C$.

Thus, the magnitude $A K$, which is less than the lesser laid out magnitude $C$, is left over from the magnitude $A B$. (Which is) the very thing it was required to show. (The theorem) can similarly be proved even if the (parts) subtracted are halves.
${ }^{\dagger}$ This theorem is the basis of the so-called method of exhaustion, and is generally attributed to Eudoxus of Cnidus.

$$
\beta^{\prime} .
$$


 $\mu \eta \delta$ ह́лотє ж $\alpha \tau \alpha \mu \varepsilon \tau \rho \tilde{n}$ тò $\pi \rho o ̀ ~ \varepsilon ́ \alpha \cup \tau о u ̃, ~ \alpha ̇ \sigma u ́ \mu \mu \varepsilon \tau \rho \alpha ~ ह ै \sigma \tau \alpha l ~ \tau \grave{~}$ $\mu \varepsilon \gamma \varepsilon ́ \vartheta \eta$.




## Proposition 2

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.

For, $A B$ and $C D$ being two unequal magnitudes, and $A B$ (being) the lesser, let the remainder never measure
 $\Gamma \Delta \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta$.


## $E \longmapsto$

$\stackrel{\square}{\vdash} \quad 1 \quad \Delta$
Ei $\gamma \alpha ́ p ~ \varepsilon ̇ \sigma \tau l ~ \sigma u ́ \mu \mu \varepsilon \tau p \alpha, ~ \mu \varepsilon \tau р ท ́ \sigma \varepsilon \iota ~ \tau l ~ \alpha u ̉ \tau \alpha ̀ ~ \mu \varepsilon ́ \gamma \varepsilon \vartheta ั o \varsigma . ~ \mu \varepsilon-~$










 व̈p $\alpha$ ह̇ $\sigma \grave{\iota} \tau \alpha \dot{\alpha} A B, \Gamma \Delta \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta$.

${ }^{\dagger}$ The fact that this will eventually occur is guaranteed by Prop. 10.1.

$$
\gamma^{\prime}
$$

$\Delta u ́ o \mu \varepsilon \gamma \varepsilon \vartheta \widetilde{\omega} \nu ~ \sigma \cup \mu \mu \varepsilon ́ \tau \rho \omega \nu ~ \delta о \vartheta \varepsilon ́ v \tau \omega \nu ~ \tau o ̀ ~ \mu \varepsilon ́ \gamma เ \sigma \tau o \nu ~ \alpha u ̉ \tau \widetilde{\omega} \nu$



## $\mathrm{H} \longmapsto$


 $\mu$ и́троข عúpeĩ̀.


 үàp тоũ $\mathrm{AB} \mu \varepsilon \gamma \varepsilon ́ ध \vartheta o u s ~ t o ̀ ~ A B ~ o u ́ ~ \mu \varepsilon \tau р \grave{\sigma \varepsilon ı . ~}$




the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater. I say that the magnitudes $A B$ and $C D$ are incommensurable.

$\begin{array}{lc}\text { C } & \text { F } \\ \text { For if they are commensurable then some magnitude }\end{array}$ will measure them (both). If possible, let it (so) measure (them), and let it be $E$. And let $A B$ leave $C F$ less than itself (in) measuring $F D$, and let $C F$ leave $A G$ less than itself (in) measuring $B G$, and let this happen continually, until some magnitude which is less than $E$ is left. Let (this) have occurred, ${ }^{\dagger}$ and let $A G$, (which is) less than $E$, have been left. Therefore, since $E$ measures $A B$, but $A B$ measures $D F, E$ will thus also measure $F D$. And it also measures the whole (of) $C D$. Thus, it will also measure the remainder $C F$. But, $C F$ measures $B G$. Thus, $E$ also measures $B G$. And it also measures the whole (of) $A B$. Thus, it will also measure the remainder $A G$, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude cannot measure (both) the magnitudes $A B$ and $C D$. Thus, the magnitudes $A B$ and $C D$ are incommensurable [Def. 10.1].

Thus, if . . . of two unequal magnitudes, and so on ....

## Proposition 3

To find the greatest common measure of two given commensurable magnitudes.


Let $A B$ and $C D$ be the two given magnitudes, of which (let) $A B$ (be) the lesser. So, it is required to find the greatest common measure of $A B$ and $C D$.

For the magnitude $A B$ either measures, or (does) not (measure), $C D$. Therefore, if it measures ( $C D$ ), and (since) it also measures itself, $A B$ is thus a common measure of $A B$ and $C D$. And (it is) clear that (it is) also (the) greatest. For a (magnitude) greater than magnitude $A B$ cannot measure $A B$.

So let $A B$ not measure $C D$. And continually subtracting in turn the lesser (magnitude) from the greater, the

















$\Delta u ́ o$ äp $\alpha \mu \varepsilon \varepsilon є \vartheta \widetilde{\omega} \nu ~ \sigma \cup \mu \mu \varepsilon ́ \tau \rho \omega \nu ~ \delta о \vartheta \varepsilon ́ v \tau \omega \nu ~ \tau \widetilde{\omega} \nu \mathrm{AB}, Г \Delta$


## Пópıбца.



$\delta^{\prime}$.
 $\alpha \cup ๋ \tau \widetilde{\omega} \nu$ Koเvòv $\mu$ ย́tpov ยúpeĩ.






remaining (magnitude) will (at) some time measure the (magnitude) before it, on account of $A B$ and $C D$ not being incommensurable [Prop. 10.2]. And let $A B$ leave $E C$ less than itself (in) measuring $E D$, and let $E C$ leave $A F$ less than itself (in) measuring $F B$, and let $A F$ measure $C E$.

Therefore, since $A F$ measures $C E$, but $C E$ measures $F B, A F$ will thus also measure $F B$. And it also measures itself. Thus, $A F$ will also measure the whole (of) $A B$. But, $A B$ measures $D E$. Thus, $A F$ will also measure $E D$. And it also measures $C E$. Thus, it also measures the whole of $C D$. Thus, $A F$ is a common measure of $A B$ and $C D$. So I say that (it is) also (the) greatest (common measure). For, if not, there will be some magnitude, greater than $A F$, which will measure (both) $A B$ and $C D$. Let it be $G$. Therefore, since $G$ measures $A B$, but $A B$ measures $E D, G$ will thus also measure $E D$. And it also measures the whole of $C D$. Thus, $G$ will also measure the remainder $C E$. But $C E$ measures $F B$. Thus, $G$ will also measure $F B$. And it also measures the whole (of) $A B$. And (so) it will measure the remainder $A F$, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude greater than $A F$ cannot measure (both) $A B$ and $C D$. Thus, $A F$ is the greatest common measure of $A B$ and $C D$.

Thus, the greatest common measure of two given commensurable magnitudes, $A B$ and $C D$, has been found. (Which is) the very thing it was required to show.

## Corollary

So (it is) clear, from this, that if a magnitude measures two magnitudes then it will also measure their greatest common measure.

## Proposition 4

To find the greatest common measure of three given commensurable magnitudes.


Let $A, B, C$ be the three given commensurable magnitudes. So it is required to find the greatest common measure of $A, B, C$.

For let the greatest common measure of the two (magnitudes) $A$ and $B$ have been taken [Prop. 10.3], and let it






















 $\mu \grave{n} \mu \varepsilon \tau \rho \tilde{n}$ tò $\Delta$ tò $\Gamma$, घ̇ $\alpha \nu$ ठè $\mu \varepsilon \tau \rho \tilde{n}, \alpha u$ tò tò $\Delta$.



## Пópıбиа.





be $D$. So $D$ either measures, or [does] not [measure], $C$. Let it, first of all, measure ( $C$ ). Therefore, since $D$ measures $C$, and it also measures $A$ and $B, D$ thus measures $A, B, C$. Thus, $D$ is a common measure of $A, B, C$. And (it is) clear that (it is) also (the) greatest (common measure). For no magnitude larger than $D$ measures (both) $A$ and $B$.

So let $D$ not measure $C$. I say, first, that $C$ and $D$ are commensurable. For if $A, B, C$ are commensurable then some magnitude will measure them which will clearly also measure $A$ and $B$. Hence, it will also measure $D$, the greatest common measure of $A$ and $B$ [Prop. 10.3 corr.]. And it also measures $C$. Hence, the aforementioned magnitude will measure (both) $C$ and $D$. Thus, $C$ and $D$ are commensurable [Def. 10.1]. Therefore, let their greatest common measure have been taken [Prop. 10.3], and let it be $E$. Therefore, since $E$ measures $D$, but $D$ measures (both) $A$ and $B, E$ will thus also measure $A$ and $B$. And it also measures $C$. Thus, $E$ measures $A, B, C$. Thus, $E$ is a common measure of $A, B, C$. So I say that (it is) also (the) greatest (common measure). For, if possible, let $F$ be some magnitude greater than $E$, and let it measure $A$, $B, C$. And since $F$ measures $A, B, C$, it will thus also measure $A$ and $B$, and will (thus) measure the greatest common measure of $A$ and $B$ [Prop. 10.3 corr.]. And $D$ is the greatest common measure of $A$ and $B$. Thus, $F$ measures $D$. And it also measures $C$. Thus, $F$ measures (both) $C$ and $D$. Thus, $F$ will also measure the greatest common measure of $C$ and $D$ [Prop. 10.3 corr.]. And it is $E$. Thus, $F$ will measure $E$, the greater (measuring) the lesser. The very thing is impossible. Thus, some [magnitude] greater than the magnitude $E$ cannot measure $A$, $B, C$. Thus, if $D$ does not measure $C$ then $E$ is the greatest common measure of $A, B, C$. And if it does measure $(C)$ then $D$ itself (is the greatest common measure).

Thus, the greatest common measure of three given commensurable magnitudes has been found. [(Which is) the very thing it was required to show.]

## Corollary

So (it is) clear, from this, that if a magnitude measures three magnitudes then it will also measure their greatest common measure.

So, similarly, the greatest common measure of more (magnitudes) can also be taken, and the (above) corollary will go forward. (Which is) the very thing it was required to show.

## $\varepsilon^{\prime}$.

T $\dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho \alpha ~ \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta ~ \pi \rho o ̀ s ~ \alpha " \lambda \lambda \eta \lambda \alpha ~ \lambda o ́ \gamma o v ~ ह ै \chi \varepsilon เ, ~ o ̀ v ~$




'Eлєì $\gamma \dot{\alpha} p ~ \sigma u ́ \mu \mu \varepsilon \tau \rho \alpha ́ ~ \varepsilon ̇ \sigma \tau \iota ~ \tau \grave{\alpha} \mathrm{~A}, \mathrm{~B}, \mu \varepsilon \tau \rho \eta ́ \sigma \varepsilon \iota ~ \tau \iota ~ \alpha u ̉ \tau \grave{\alpha}$













 тpòs tòv E .



## Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.


Let $A$ and $B$ be commensurable magnitudes. I say that $A$ has to $B$ the ratio which (some) number (has) to (some) number.

For if $A$ and $B$ are commensurable (magnitudes) then some magnitude will measure them. Let it (so) measure (them), and let it be $C$. And as many times as $C$ measures $A$, so many units let there be in $D$. And as many times as $C$ measures $B$, so many units let there be in $E$.

Therefore, since $C$ measures $A$ according to the units in $D$, and a unit also measures $D$ according to the units in it, a unit thus measures the number $D$ as many times as the magnitude $C$ (measures) $A$. Thus, as $C$ is to $A$, so a unit (is) to $D$ [Def. 7.20]. ${ }^{\dagger}$ Thus, inversely, as $A$ (is) to $C$, so $D$ (is) to a unit [Prop. 5.7 corr.]. Again, since $C$ measures $B$ according to the units in $E$, and a unit also measures $E$ according to the units in it, a unit thus measures $E$ the same number of times that $C$ (measures) $B$. Thus, as $C$ is to $B$, so a unit (is) to $E$ [Def. 7.20]. And it was also shown that as $A$ (is) to $C$, so $D$ (is) to a unit. Thus, via equality, as $A$ is to $B$, so the number $D$ (is) to the (number) $E$ [Prop. 5.22].

Thus, the commensurable magnitudes $A$ and $B$ have to one another the ratio which the number $D$ (has) to the number $E$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ There is a slight logical gap here, since Def. 7.20 applies to four numbers, rather than two number and two magnitudes.

$$
\varsigma^{\prime}
$$

 $\pi \rho o ̀ \varsigma ~ \alpha ̉ \rho เ \vartheta \mu o ́ v, ~ \sigma u ́ \mu \mu \varepsilon \tau \rho \alpha ~ \varepsilon ै \sigma \tau \alpha l ~ \tau \grave{\alpha} \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta$.


 $\tau \dot{\alpha} \mathrm{A}, \mathrm{B} \mu \varepsilon \gamma \varepsilon ́ \vartheta \eta$.


## Proposition 6

If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.


For let the two magnitudes $A$ and $B$ have to one another the ratio which the number $D$ (has) to the number $E$. I say that the magnitudes $A$ and $B$ are commensurable.

 бuүหєíб७બ tò Z.

 $\Delta$, tò aủtò $\mu \varepsilon ́ p o s ~ \varepsilon ̇ \sigma \tau i ̀ ~ x \alpha i ̀ ~ \tau o ̀ ~ \Gamma ~ \tau o u ̃ ~ A \cdot ~ ह ै \sigma \tau ル ~ \alpha ̈ p \alpha ~ ஸ ́ s ~ t o ̀ ~ \Gamma ~$













 B.


## По́рьб $\alpha$.







 ónoícs $\dot{\alpha} \nu \alpha \gamma \rho \alpha \varphi o ́ \mu \varepsilon \nu o v . ~ \dot{\alpha} \lambda \lambda{ }^{\circ} \omega \varsigma \dot{\eta} \mathrm{A} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{Z}$, oüt $\omega \varsigma$




## $\zeta$

 ג́рเ७นòऽ тро̀ऽ $\dot{\alpha} \rho เ \vartheta \mu o ́ v . ~$



For, as many units as there are in $D$, let $A$ have been divided into so many equal (divisions). And let $C$ be equal to one of them. And as many units as there are in $E$, let $F$ be the sum of so many magnitudes equal to $C$.

Therefore, since as many units as there are in $D$, so many magnitudes equal to $C$ are also in $A$, therefore whichever part a unit is of $D, C$ is also the same part of $A$. Thus, as $C$ is to $A$, so a unit (is) to $D$ [Def. 7.20]. And a unit measures the number $D$. Thus, $C$ also measures $A$. And since as $C$ is to $A$, so a unit (is) to the [number] $D$, thus, inversely, as $A$ (is) to $C$, so the number $D$ (is) to a unit [Prop. 5.7 corr.]. Again, since as many units as there are in $E$, so many (magnitudes) equal to $C$ are also in $F$, thus as $C$ is to $F$, so a unit (is) to the [number] $E$ [Def. 7.20]. And it was also shown that as $A$ (is) to $C$, so $D$ (is) to a unit. Thus, via equality, as $A$ is to $F$, so $D$ (is) to $E$ [Prop. 5.22]. But, as $D$ (is) to $E$, so $A$ is to $B$. And thus as $A$ (is) to $B$, so (it) also is to $F$ [Prop. 5.11]. Thus, $A$ has the same ratio to each of $B$ and $F$. Thus, $B$ is equal to $F$ [Prop. 5.9]. And $C$ measures $F$. Thus, it also measures $B$. But, in fact, (it) also (measures) $A$. Thus, $C$ measures (both) $A$ and $B$. Thus, $A$ is commensurable with $B$ [Def. 10.1].

Thus, if two magnitudes . . . to one another, and so on ....

## Corollary

So it is clear, from this, that if there are two numbers, like $D$ and $E$, and a straight-line, like $A$, then it is possible to contrive that as the number $D$ (is) to the number $E$, so the straight-line (is) to (another) straight-line (i.e., $F$ ). And if the mean proportion, (say) $B$, is taken of $A$ and $F$, then as $A$ is to $F$, so the (square) on $A$ (will be) to the (square) on $B$. That is to say, as the first (is) to the third, so the (figure) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. But, as $A$ (is) to $F$, so the number $D$ is to the number $E$. Thus, it has also been contrived that as the number $D$ (is) to the number $E$, so the (figure) on the straight-line $A$ (is) to the (similar figure) on the straight-line $B$. (Which is) the very thing it was required to show.

## Proposition 7

Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number.

Let $A$ and $B$ be incommensurable magnitudes. I say that $A$ does not have to $B$ the ratio which (some) number (has) to (some) number.




 x $\alpha \grave{\imath} \tau \dot{\alpha} \dot{\varepsilon} \xi \tilde{n} \varsigma$ ．

$$
\eta^{\prime} .
$$

 $\pi \rho o ̀ s ~ \alpha \dot{\alpha} เ \vartheta \mu o ́ v, \dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho \alpha$ हैбт $\alpha \iota \tau \dot{\alpha} \mu \varepsilon \gamma \varepsilon ́ v \eta$ ．

$\mathrm{B} \longmapsto$

 B $\mu \varepsilon \gamma \varepsilon ́ \vartheta \vartheta \eta$ ．

Eî $\gamma \dot{\alpha} \rho$ है $\sigma \tau \alpha \iota \sigma u ́ \mu \mu \varepsilon \tau \rho \alpha$ ，тò $\mathrm{A} \pi \rho o ̀ s ~ \tau o ̀ ~ B ~ \lambda o ́ \gamma o v ~ \varepsilon ́ \xi \varepsilon ı, ~ o ̊ v ~$
 A，B $\mu \varepsilon \gamma \varepsilon ์ \nexists \eta$ ．


## $\vartheta^{\prime}$ ．

Tม̀ $\alpha \pi o ̀ ~ \tau \widetilde{\omega} \nu \mu n ́ x \varepsilon \iota ~ \sigma \cup \mu \mu \varepsilon ́ \tau \rho \omega \nu ~ \varepsilon u ̉ v \varepsilon เ \widetilde{\omega} \nu ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \alpha$












 ápıひ朱．


$$
\mathrm{B} \longmapsto
$$

For if $A$ has to $B$ the ratio which（some）number（has） to（some）number then $A$ will be commensurable with $B$ ［Prop．10．6］．But it is not．Thus，$A$ does not have to $B$ the ratio which（some）number（has）to（some）number．

Thus，incommensurable numbers do not have to one another，and so on ．．．

## Proposition 8

If two magnitudes do not have to one another the ra－ tio which（some）number（has）to（some）number then the magnitudes will be incommensurable．


For let the two magnitudes $A$ and $B$ not have to one another the ratio which（some）number（has）to（some） number．I say that the magnitudes $A$ and $B$ are incom－ mensurable．

For if they are commensurable，$A$ will have to $B$ the ratio which（some）number（has）to（some）number ［Prop．10．5］．But it does not have（such a ratio）．Thus， the magnitudes $A$ and $B$ are incommensurable．

Thus，if two magnitudes ．．．to one another，and so on ．．．．

## Proposition 9

Squares on straight－lines（which are）commensurable in length have to one another the ratio which（some） square number（has）to（some）square number．And squares having to one another the ratio which（some） square number（has）to（some）square number will also have sides（which are）commensurable in length．But squares on straight－lines（which are）incommensurable in length do not have to one another the ratio which （some）square number（has）to（some）square number． And squares not having to one another the ratio which （some）square number（has）to（some）square number will not have sides（which are）commensurable in length either．


For let $A$ and $B$ be（straight－lines which are）commen－ surable in length．I say that the square on $A$ has to the square on $B$ the ratio which（some）square number（has） to（some）square number．

















 $\mu \dot{\eta}$ кєь.













 àpıখuóv.








 $\tau \tilde{n}$ B $\mu \dot{n} x \varepsilon$.


 $\dot{\eta} \mathrm{A} \tau \tilde{n} \mathrm{~B} \mu \dot{\eta} \chi \varepsilon ⿺$.


For since $A$ is commensurable in length with $B, A$ thus has to $B$ the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (that) which $C$ (has) to $D$. Therefore, since as $A$ is to $B$, so $C$ (is) to $D$. But the (ratio) of the square on $A$ to the square on $B$ is the square of the ratio of $A$ to $B$. For similar figures are in the squared ratio of (their) corresponding sides [Prop. 6.20 corr.]. And the (ratio) of the square on $C$ to the square on $D$ is the square of the ratio of the [number] $C$ to the [number] $D$. For there exits one number in mean proportion to two square numbers, and (one) square (number) has to the (other) square [number] a squared ratio with respect to (that) the side (of the former has) to the side (of the latter) [Prop. 8.11]. And, thus, as the square on $A$ is to the square on $B$, so the square [number] on the (number) $C$ (is) to the square [number] on the [number] $D .^{\dagger}$

And so let the square on $A$ be to the (square) on $B$ as the square (number) on $C$ (is) to the [square] (number) on $D$. I say that $A$ is commensurable in length with $B$.

For since as the square on $A$ is to the [square] on $B$, so the square (number) on $C$ (is) to the [square] (number) on $D$. But, the ratio of the square on $A$ to the (square) on $B$ is the square of the (ratio) of $A$ to $B$ [Prop. 6.20 corr.]. And the (ratio) of the square [number] on the [number] $C$ to the square [number] on the [number] $D$ is the square of the ratio of the [number] $C$ to the [number] $D$ [Prop. 8.11]. Thus, as $A$ is to $B$, so the [number] $C$ also (is) to the [number] $D . A$, thus, has to $B$ the ratio which the number $C$ has to the number $D$. Thus, $A$ is commensurable in length with $B$ [Prop. 10.6]. $\ddagger$

And so let $A$ be incommensurable in length with $B$. I say that the square on $A$ does not have to the [square] on $B$ the ratio which (some) square number (has) to (some) square number.

For if the square on $A$ has to the [square] on $B$ the ratio which (some) square number (has) to (some) square number then $A$ will be commensurable (in length) with $B$. But it is not. Thus, the square on $A$ does not have to the [square] on the $B$ the ratio which (some) square number (has) to (some) square number.

So, again, let the square on $A$ not have to the [square] on $B$ the ratio which (some) square number (has) to (some) square number. I say that $A$ is incommensurable in length with $B$.

For if $A$ is commensurable (in length) with $B$ then the (square) on $A$ will have to the (square) on $B$ the ratio which (some) square number (has) to (some) square number. But it does not have (such a ratio). Thus, $A$ is not commensurable in length with $B$.

Thus, (squares) on (straight-lines which are) com-
mensurable in length, and so on ....

## По́pıбиа.


 $\mu \dot{\gamma}$ кє.

## Corollary

And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straightlines commensurable) in square (are) not always also (commensurable) in length.
${ }^{\dagger}$ There is an unstated assumption here that if $\alpha: \beta:: \gamma: \delta$ then $\alpha^{2}: \beta^{2}:: \gamma^{2}: \delta^{2}$.
${ }^{\ddagger}$ There is an unstated assumption here that if $\alpha^{2}: \beta^{2}:: \gamma^{2}: \delta^{2}$ then $\alpha: \beta:: \gamma: \delta$.
$\therefore$.




 ठuváueı.
















Tǹ $\alpha \not \rho \alpha ~ \pi \rho о \tau \varepsilon \vartheta \varepsilon i \sigma \emptyset ~ \varepsilon u ̉ v \varepsilon i ́ \alpha ~ \tau \tilde{n}$ A $\pi \rho о \sigma \varepsilon u ́ p \eta \nu \tau \alpha l ~ \delta u ́ o ~$



## Proposition $10^{\dagger}$

To find two straight-lines incommensurable with a given straight-line, the one (incommensurable) in length only, the other also (incommensurable) in square.


Let $A$ be the given straight-line. So it is required to find two straight-lines incommensurable with $A$, the one (incommensurable) in length only, the other also (incommensurable) in square.

For let two numbers, $B$ and $C$, not having to one another the ratio which (some) square number (has) to (some) square number - that is to say, not (being) similar plane (numbers) -have been taken. And let it be contrived that as $B$ (is) to $C$, so the square on $A$ (is) to the square on $D$. For we learned (how to do this) [Prop. 10.6 corr.]. Thus, the (square) on $A$ (is) commensurable with the (square) on $D$ [Prop. 10.6]. And since $B$ does not have to $C$ the ratio which (some) square number (has) to (some) square number, the (square) on $A$ thus does not have to the (square) on $D$ the ratio which (some) square number (has) to (some) square number either. Thus, $A$ is incommensurable in length with $D$ [Prop. 10.9]. Let the (straight-line) $E$ (which is) in mean proportion to $A$ and $D$ have been taken [Prop. 6.13]. Thus, as $A$ is to $D$, so the square on $A$ (is) to the (square) on $E$ [Def. 5.9]. And $A$ is incommensurable in length with $D$. Thus, the square on $A$ is also incommensurble with the square on $E$ [Prop. 10.11]. Thus, $A$ is incommensurable in square with $E$.
${ }^{\dagger}$ This whole proposition is regarded by Heiberg as an interpolation into the original text.

## $\alpha^{\prime}$.








 है〒тац.



 tò $\Gamma \uparrow \widetilde{\varphi} \Delta$.








$$
{ }^{\prime} \beta^{\prime} .
$$

 бט́циєтра.


'Етєi $\gamma \alpha ̀ \rho ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ t o ̀ ~ A ~ \tau \widetilde{̣ ~} \Gamma$, tò A äp $\alpha$ трòs







## Proposition 11

If four magnitudes are proportional, and the first is commensurable with the second, then the third will also be commensurable with the fourth. And if the first is incommensurable with the second, then the third will also be incommensurable with the fourth.


Let $A, B, C, D$ be four proportional magnitudes, (such that) as $A$ (is) to $B$, so $C$ (is) to $D$. And let $A$ be commensurable with $B$. I say that $C$ will also be commensurable with $D$.

For since $A$ is commensurable with $B, A$ thus has to $B$ the ratio which (some) number (has) to (some) number [Prop. 10.5]. And as $A$ is to $B$, so $C$ (is) to $D$. Thus, $C$ also has to $D$ the ratio which (some) number (has) to (some) number. Thus, $C$ is commensurable with $D$ [Prop. 10.6].

And so let $A$ be incommensurable with $B$. I say that $C$ will also be incommensurable with $D$. For since $A$ is incommensurable with $B, A$ thus does not have to $B$ the ratio which (some) number (has) to (some) number [Prop. 10.7]. And as $A$ is to $B$, so $C$ (is) to $D$. Thus, $C$ does not have to $D$ the ratio which (some) number (has) to (some) number either. Thus, $C$ is incommensurable with $D$ [Prop. 10.8].

Thus, if four magnitudes, and so on ....

## Proposition 12

(Magnitudes) commensurable with the same magnitude are also commensurable with one another.

For let $A$ and $B$ each be commensurable with $C$. I say that $A$ is also commensurable with $B$.

For since $A$ is commensurable with $C, A$ thus has to $C$ the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which $D$ (has) to $E$. Again, since $C$ is commensurable with $B$, $C$ thus has to $B$ the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which $F$ (has) to $G$. And for any multitude whatsoever
 tòv Z тpòs tòv H , oưt $\omega$ s tòv $\mathrm{K} \pi \rho o ̀ s ~ \tau o ̀ v ~ \Lambda . ~$














## ${ }^{\prime} \gamma^{\prime}$.


 ov ยै $\sigma \tau \alpha$.


 x $\alpha$ tò $\lambda о \iota \pi o ̀ v ~ t o ̀ ~ B ~ \tau \widetilde{̣} \Gamma ~ \Gamma \alpha ́ \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ́ \sigma \tau เ \nu . ~$

Ei $\gamma \alpha ́ p$ ह̇ $\sigma \tau \iota \sigma u ́ \mu \mu \varepsilon \tau \rho o \nu$ tò $\mathrm{B} \tau \widetilde{\varphi} \Gamma, \dot{\alpha} \lambda \lambda \alpha \dot{\alpha} x \alpha i$ tò $\mathrm{A} \tau \widetilde{\varphi}$




$\Lambda \tilde{n} \mu \mu \alpha$.


of given ratios-(namely,) those which $D$ has to $E$, and $F$ to $G$-let the numbers $H, K, L$ (which are) continuously (proportional) in the(se) given ratios have been taken [Prop. 8.4]. Hence, as $D$ is to $E$, so $H$ (is) to $K$, and as $F$ (is) to $G$, so $K$ (is) to $L$.


Therefore, since as $A$ is to $C$, so $D$ (is) to $E$, but as $D$ (is) to $E$, so $H$ (is) to $K$, thus also as $A$ is to $C$, so $H$ (is) to $K$ [Prop. 5.11]. Again, since as $C$ is to $B$, so $F$ (is) to $G$, but as $F$ (is) to $G$, [so] $K$ (is) to $L$, thus also as $C$ (is) to $B$, so $K$ (is) to $L$ [Prop. 5.11]. And also as $A$ is to $C$, so $H$ (is) to $K$. Thus, via equality, as $A$ is to $B$, so $H$ (is) to $L$ [Prop. 5.22]. Thus, $A$ has to $B$ the ratio which the number $H$ (has) to the number $L$. Thus, $A$ is commensurable with $B$ [Prop. 10.6].

Thus, (magnitudes) commensurable with the same magnitude are also commensurable with one another. (Which is) the very thing it was required to show.

## Proposition 13

If two magnitudes are commensurable, and one of them is incommensurable with some magnitude, then the remaining (magnitude) will also be incommensurable with it.


Let $A$ and $B$ be two commensurable magnitudes, and let one of them, $A$, be incommensurable with some other (magnitude), $C$. I say that the remaining (magnitude), $B$, is also incommensurable with $C$.

For if $B$ is commensurable with $C$, but $A$ is also commensurable with $B, A$ is thus also commensurable with $C$ [Prop. 10.12]. But, (it is) also incommensurable (with $C$ ). The very thing (is) impossible. Thus, $B$ is not commensurable with $C$. Thus, (it is) incommensurable.

Thus, if two magnitudes are commensurable, and so on ....

## Lemma

For two given unequal straight-lines, to find by (the square on) which (straight-line) the square on the greater


 $\tau \tilde{n} s \Gamma$.











(straight-line is) larger than (the square on) the lesser. ${ }^{\dagger}$


Let $A B$ and $C$ be the two given unequal straight-lines, and let $A B$ be the greater of them. So it is required to find by (the square on) which (straight-line) the square on $A B$ (is) greater than (the square on) $C$.

Let the semi-circle $A D B$ have been described on $A B$. And let $A D$, equal to $C$, have been inserted into it [Prop. 4.1]. And let $D B$ have been joined. So (it is) clear that the angle $A D B$ is a right-angle [Prop. 3.31], and that the square on $A B$ (is) greater than (the square on) $A D$-that is to say, (the square on) $C$-by (the square on) $D B$ [Prop. 1.47].

And, similarly, the square-root of (the sum of the squares on) two given straight-lines is also found likeso.

Let $A D$ and $D B$ be the two given straight-lines. And let it be necessary to find the square-root of (the sum of the squares on) them. For let them have been laid down such as to encompass a right-angle-(namely), that (angle encompassed) by $A D$ and $D B$. And let $A B$ have been joined. (It is) again clear that $A B$ is the square-root of (the sum of the squares on) $A D$ and $D B$ [Prop. 1.47]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ That is, if $\alpha$ and $\beta$ are the lengths of two given straight-lines, with $\alpha$ being greater than $\beta$, to find a straight-line of length $\gamma$ such that $\alpha^{2}=\beta^{2}+\gamma^{2}$. Similarly, we can also find $\gamma$ such that $\gamma^{2}=\alpha^{2}+\beta^{2}$.

$$
\delta^{\prime}
$$



 $\sigma \cup \mu \mu \varepsilon ́ \tau \rho \circ \cup$ غ́ $\alpha \cup \tau \tilde{n}[\mu \dot{\eta} x \varepsilon \iota]$. x $\alpha i$ ह̇ $\alpha \nu \dot{\eta} \pi \rho \omega ́ \tau \eta ~ \tau \tilde{\eta} \varsigma ~ \delta \varepsilon \cup \tau \varepsilon ́ \rho \alpha \varsigma$

 $\dot{\varepsilon} \alpha \cup \tau \tilde{n}[\mu \dot{n} \chi \varepsilon \iota]$.
"Eбт $\omega \sigma \alpha \nu$ тé $\sigma \sigma \alpha \rho \varepsilon \varsigma ~ \varepsilon u ̉ v \varepsilon i ̃ \alpha l ~ \alpha ̀ v \alpha ́ \lambda o \gamma o v ~ \alpha i ~ A, ~ В, ~ Г, ~ \Delta, ~$






## Proposition 14

If four straight-lines are proportional, and the square on the first is greater than (the square on) the second by the (square) on (some straight-line) commensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) commensurable [in length] with the third. And if the square on the first is greater than (the square on) the second by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the third.

Let $A, B, C, D$ be four proportional straight-lines, (such that) as $A$ (is) to $B$, so $C$ (is) to $D$. And let the square on $A$ be greater than (the square on) $B$ by the













 $\dot{\eta} \mathrm{A} \tau \tilde{n} \mathrm{E}, \alpha \dot{\alpha} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho o ́ s ~ \varepsilon ̇ \sigma \tau \iota ~ \varkappa \alpha i ̀ ~ \dot{\eta} \Gamma \tau \tilde{\eta} \mathrm{Z}$.


$$
\iota \varepsilon^{\prime}
$$






(square) on $E$, and let the square on $C$ be greater than (the square on) $D$ by the (square) on $F$. I say that $A$ is either commensurable (in length) with $E$, and $C$ is also commensurable with $F$, or $A$ is incommensurable (in length) with $E$, and $C$ is also incommensurable with $F$.


For since as $A$ is to $B$, so $C$ (is) to $D$, thus as the (square) on $A$ is to the (square) on $B$, so the (square) on $C$ (is) to the (square) on $D$ [Prop. 6.22]. But the (sum of the squares) on $E$ and $B$ is equal to the (square) on $A$, and the (sum of the squares) on $D$ and $F$ is equal to the (square) on $C$. Thus, as the (sum of the squares) on $E$ and $B$ is to the (square) on $B$, so the (sum of the squares) on $D$ and $F$ (is) to the (square) on $D$. Thus, via separation, as the (square) on $E$ is to the (square) on $B$, so the (square) on $F$ (is) to the (square) on $D$ [Prop. 5.17]. Thus, also, as $E$ is to $B$, so $F$ (is) to $D$ [Prop. 6.22]. Thus, inversely, as $B$ is to $E$, so $D$ (is) to $F$ [Prop. 5.7 corr.]. But, as $A$ is to $B$, so $C$ also (is) to $D$. Thus, via equality, as $A$ is to $E$, so $C$ (is) to $F$ [Prop. 5.22]. Therefore, $A$ is either commensurable (in length) with $E$, and $C$ is also commensurable with $F$, or $A$ is incommensurable (in length) with $E$, and $C$ is also incommensurable with $F$ [Prop. 10.11].

Thus, if, and so on ....

## Proposition 15

If two commensurable magnitudes are added together then the whole will also be commensurable with each of them. And if the whole is commensurable with one of them then the original magnitudes will also be commensurable (with one another).

For let the two commensurable magnitudes $A B$ and $B C$ be laid down together. I say that the whole $A C$ is also commensurable with each of $A B$ and $B C$.


## $\Delta \longmapsto$




 tò $\mathrm{A} \Gamma$ غ́x $\alpha \tau \varepsilon ́ \rho \biguplus ~ \tau \widetilde{\omega} \nu \mathrm{AB}, ~ В \Gamma$.
 ж $\grave{\imath} \tau \grave{\alpha} \mathrm{AB}, ~ В \Gamma ~ \sigma u ́ \mu \mu \varepsilon \tau р \alpha ́ \alpha ~ \varepsilon ̇ \sigma \tau \iota . ~$



 $\tau \dot{\alpha} \mathrm{AB}, \mathrm{B} \Gamma$.


## $l \xi^{\prime}$.






## $\Delta \longmapsto$

 $\lambda \varepsilon ́ \gamma \omega$, öтı x $\alpha$ ö ö̀ov tò $\mathrm{A} \Gamma$ ह́x $\alpha \tau \varepsilon ́ \rho \varphi ~ \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \Gamma \dot{\alpha} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho o ́ v$ ย̇бтเน.







 ВГ $\dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau \iota \nu . ~$








For since $A B$ and $B C$ are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be $D$. Therefore, since $D$ measures (both) $A B$ and $B C$, it will also measure the whole $A C$. And it also measures $A B$ and $B C$. Thus, $D$ measures $A B, B C$, and $A C$. Thus, $A C$ is commensurable with each of $A B$ and $B C$ [Def. 10.1].

And so let $A C$ be commensurable with $A B$. I say that $A B$ and $B C$ are also commensurable.

For since $A C$ and $A B$ are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be $D$. Therefore, since $D$ measures (both) $C A$ and $A B$, it will thus also measure the remainder $B C$. And it also measures $A B$. Thus, $D$ will measure (both) $A B$ and $B C$. Thus, $A B$ and $B C$ are commensurable [Def. 10.1].

Thus, if two magnitudes, and so on ....

## Proposition 16

If two incommensurable magnitudes are added together then the whole will also be incommensurable with each of them. And if the whole is incommensurable with one of them then the original magnitudes will also be incommensurable (with one another).


## D

For let the two incommensurable magnitudes $A B$ and $B C$ be laid down together. I say that that the whole $A C$ is also incommensurable with each of $A B$ and $B C$.

For if $C A$ and $A B$ are not incommensurable then some magnitude will measure [them]. If possible, let it (so) measure (them), and let it be $D$. Therefore, since $D$ measures (both) $C A$ and $A B$, it will thus also measure the remainder $B C$. And it also measures $A B$. Thus, $D$ measures (both) $A B$ and $B C$. Thus, $A B$ and $B C$ are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) $C A$ and $A B$. Thus, $C A$ and $A B$ are incommensurable [Def. 10.1]. So, similarly, we can show that $A C$ and $C B$ are also incommensurable. Thus, $A C$ is incommensurable with each of $A B$ and $B C$.

And so let $A C$ be incommensurable with one of $A B$ and $B C$. So let it, first of all, be incommensurable with

 ध̇б $\sigma \grave{\imath} \tau \alpha ̀ \mathrm{AB}, \mathrm{B} \mathrm{\Gamma}$.


## А $\tilde{\mu} \mu \alpha$.












${ }^{\dagger}$ Note that this lemma only applies to rectangular parallelograms.

$$
\zeta^{\prime}
$$











$A B$. I say that $A B$ and $B C$ are also incommensurable. For if they are commensurable then some magnitude will measure them. Let it (so) measure (them), and let it be $D$. Therefore, since $D$ measures (both) $A B$ and $B C$, it will thus also measure the whole $A C$. And it also measures $A B$. Thus, $D$ measures (both) $C A$ and $A B$. Thus, $C A$ and $A B$ are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) $A B$ and $B C$. Thus, $A B$ and $B C$ are incommensurable [Def. 10.1].

Thus, if two. . . magnitudes, and so on ....

## Lemma

If a parallelogram, ${ }^{\dagger}$ falling short by a square figure, is applied to some straight-line then the applied (parallelogram) is equal (in area) to the (rectangle contained) by the pieces of the straight-line created via the application (of the parallelogram).


For let the parallelogram $A D$, falling short by the square figure $D B$, have been applied to the straight-line $A B$. I say that $A D$ is equal to the (rectangle contained) by $A C$ and $C B$.

And it is immediately obvious. For since $D B$ is a square, $D C$ is equal to $C B$. And $A D$ is the (rectangle contained) by $A C$ and $C D$-that is to say, by $A C$ and $C B$.

Thus, if . . . to some straight-line, and so on ....

## Proposition $17{ }^{\dagger}$

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable in length, then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater. And if the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the











 $\tau \widetilde{\omega} \nu \mathrm{B} \Delta, \Delta \Gamma \mu \varepsilon \tau \grave{\alpha} \tau \circ \tilde{u} \tau \varepsilon \tau \rho \alpha \pi \lambda \alpha \sigma$ íou тoũ $\alpha \pi o ̀ ~ \tau \tilde{\eta} s \Delta \mathrm{E}$ 亿ैбov




 $\alpha \dot{\alpha} o ̀ ~ \tau n ̃ ऽ ~ B \Gamma ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o v . ~ \delta \iota \pi \lambda \alpha \sigma i ́ \omega \nu ~ \gamma \alpha ́ p ~ \varepsilon ̇ \sigma \tau \iota ~ \pi \alpha ́ \alpha \lambda \iota ~ \dot{\eta}$ ВГ













 $\mu$ и́кєь.


greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) commensurable in length.

Let $A$ and $B C$ be two unequal straight-lines, of which (let) $B C$ (be) the greater. And let a (rectangle) equal to the fourth part of the (square) on the lesser, $A$-that is, (equal) to the (square) on half of $A$-falling short by a square figure, have been applied to $B C$. And let it be the (rectangle contained) by $B D$ and $D C$ [see previous lemma]. And let $B D$ be commensurable in length with $D C$. I say that that the square on $B C$ is greater than the (square on) $A$ by (the square on some straight-line) commensurable (in length) with ( $B C$ ).


For let $B C$ have been cut in half at the point $E$ [Prop. 1.10]. And let $E F$ be made equal to $D E$ [Prop. 1.3]. Thus, the remainder $D C$ is equal to $B F$. And since the straight-line $B C$ has been cut into equal (pieces) at $E$, and into unequal (pieces) at $D$, the rectangle contained by $B D$ and $D C$, plus the square on $E D$, is thus equal to the square on $E C$ [Prop. 2.5]. (The same) also (for) the quadruples. Thus, four times the (rectangle contained) by $B D$ and $D C$, plus the quadruple of the (square) on $D E$, is equal to four times the square on $E C$. But, the square on $A$ is equal to the quadruple of the (rectangle contained) by $B D$ and $D C$, and the square on $D F$ is equal to the quadruple of the (square) on $D E$. For $D F$ is double $D E$. And the square on $B C$ is equal to the quadruple of the (square) on $E C$. For, again, $B C$ is double $C E$. Thus, the (sum of the) squares on $A$ and $D F$ is equal to the square on $B C$. Hence, the (square) on $B C$ is greater than the (square) on $A$ by the (square) on $D F$. Thus, $B C$ is greater in square than $A$ by $D F$. It must also be shown that $B C$ is commensurable (in length) with $D F$. For since $B D$ is commensurable in length with $D C, B C$ is thus also commensurable in length with $C D$ [Prop. 10.15]. But, $C D$ is commensurable in length with $C D$ plus $B F$. For $C D$ is equal to $B F$ [Prop. 10.6]. Thus, $B C$ is also commensurable in length with $B F$ plus $C D$ [Prop. 10.12]. Hence, $B C$ is also commensurable in length with the remainder $F D$ [Prop. 10.15]. Thus, the square on $B C$ is greater than (the square on) $A$ by the (square) on (some straight-line) commensurable (in length) with ( $B C$ ).








And so let the square on $B C$ be greater than the (square on) $A$ by the (square) on (some straight-line) commensurable (in length) with $(B C)$. And let a (rectangle) equal to the fourth (part) of the (square) on $A$, falling short by a square figure, have been applied to $B C$. And let it be the (rectangle contained) by $B D$ and $D C$. It must be shown that $B D$ is commensurable in length with $D C$.

For, similarly, by the same construction, we can show that the square on $B C$ is greater than the (square on) $A$ by the (square) on $F D$. And the square on $B C$ is greater than the (square on) $A$ by the (square) on (some straightline) commensurable (in length) with ( $B C$ ). Thus, $B C$ is commensurable in length with $F D$. Hence, $B C$ is also commensurable in length with the remaining sum of $B F$ and $D C$ [Prop. 10.15]. But, the sum of $B F$ and $D C$ is commensurable [in length] with $D C$ [Prop. 10.6]. Hence, $B C$ is also commensurable in length with $C D$ [Prop. 10.12]. Thus, via separation, $B D$ is also commensurable in length with $D C$ [Prop. 10.15].

Thus, if there are two unequal straight-lines, and so on ....
${ }^{\dagger}$ This proposition states that if $\alpha x-x^{2}=\beta^{2} / 4$ (where $\alpha=B C, x=D C$, and $\beta=A$ ) then $\alpha$ and $\sqrt{\alpha^{2}-\beta^{2}}$ are commensurable when $\alpha-x$ are $x$ are commensurable, and vice versa.

$$
i \eta^{\prime}
$$













 ג̇бицце́трои غ̀ $\alpha \cup \tau \tilde{n}$.

## Proposition $18^{\dagger}$

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable [in length], then the square on the greater will be larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater. And if the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) incommensurable [in length].

Let $A$ and $B C$ be two unequal straight-lines, of which (let) $B C$ (be) the greater. And let a (rectangle) equal to the fourth [part] of the (square) on the lesser, $A$, falling short by a square figure, have been applied to $B C$. And let it be the (rectangle contained) by $B D C$. And let $B D$ be incommensurable in length with $D C$. I say that that the square on $B C$ is greater than the (square on) $A$ by the (square) on (some straight-line) incommensurable (in length) with ( $B C$ ).









 ג̇兀ò $\alpha \sigma \cup \mu \mu \varepsilon ́ т \rho о \cup ~ غ ́ \alpha \cup \tau \tilde{n}$.



 $\mu \eta^{\prime}$ кє．






 $\mathrm{B} \Delta \tau \tilde{n} \Delta \Gamma$ वं $\sigma \cup ́ \mu \mu \varepsilon \tau \rho o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota ~ \mu \eta ́ \chi \varepsilon เ . ~$



For，similarly，by the same construction as before，we can show that the square on $B C$ is greater than the （square on）$A$ by the（square）on $F D$ ．［Therefore］it must be shown that $B C$ is incommensurable in length with $D F$ ．For since $B D$ is incommensurable in length with $D C, B C$ is thus also incommensurable in length with $C D$［Prop．10．16］．But，$D C$ is commensurable（in length）with the sum of $B F$ and $D C$［Prop．10．6］．And， thus，$B C$ is incommensurable（in length）with the sum of $B F$ and $D C$［Prop．10．13］．Hence，$B C$ is also incommen－ surable in length with the remainder $F D$［Prop．10．16］． And the square on $B C$ is greater than the（square on） $A$ by the（square）on $F D$ ．Thus，the square on $B C$ is greater than the（square on）$A$ by the（square）on（some straight－line）incommensurable（in length）with（ $B C$ ）．

So，again，let the square on $B C$ be greater than the （square on）$A$ by the（square）on（some straight－line）in－ commensurable（in length）with $(B C)$ ．And let a（rect－ angle）equal to the fourth［part］of the（square）on $A$ ， falling short by a square figure，have been applied to $B C$ ． And let it be the（rectangle contained）by $B D$ and $D C$ ． It must be shown that $B D$ is incommensurable in length with $D C$ ．

For，similarly，by the same construction，we can show that the square on $B C$ is greater than the（square）on $A$ by the（square）on $F D$ ．But，the square on $B C$ is greater than the（square）on $A$ by the（square）on（some straight－line）incommensurable（in length）with（ $B C$ ）． Thus，$B C$ is incommensurable in length with $F D$ ．Hence， $B C$ is also incommensurable（in length）with the re－ maining sum of $B F$ and $D C$［Prop．10．16］．But，the sum of $B F$ and $D C$ is commensurable in length with $D C$［Prop．10．6］．Thus，$B C$ is also incommensurable in length with $D C$［Prop．10．13］．Hence，via separa－ tion，$B D$ is also incommensurable in length with $D C$ ［Prop．10．16］．

Thus，if there are two ．．．straight－lines，and so on ．．．．
$\dagger$ This proposition states that if $\alpha x-x^{2}=\beta^{2} / 4$（where $\alpha=B C, x=D C$ ，and $\beta=A$ ）then $\alpha$ and $\sqrt{\alpha^{2}-\beta^{2}}$ are incommensurable when $\alpha-x$ are $x$ are incommensurable，and vice versa．

## เ̛＇．$^{\prime}$ ．



${ }^{〔} \Upsilon \pi o ̀ ~ \gamma \grave{\alpha} \rho \dot{\rho} \eta \tau \widetilde{\omega} \nu \mu \eta \eta^{\prime} x \varepsilon \iota \sigma u \mu \mu \varepsilon ́ \tau \rho \omega \nu \varepsilon u ̉ \vartheta \varepsilon เ \widetilde{\omega} \nu \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma}$

## Proposition 19

The rectangle contained by rational straight－lines （which are）commensurable in length is rational．

For let the rectangle $A C$ have been enclosed by the
 $А \Gamma$.









## $x^{\prime}$.

 ж人ì $\sigma \dot{\mu} \mu \mu \varepsilon \tau \rho o \nu \tau \tilde{n}, \pi \alpha \rho^{\prime} \hat{\eta} \nu \pi \alpha \rho \alpha ́ x \varepsilon เ \tau \alpha \iota, \mu \dot{\eta} x \varepsilon \iota$.






rational straight-lines $A B$ and $B C$ (which are) commensurable in length. I say that $A C$ is rational.


For let the square $A D$ have been described on $A B$. $A D$ is thus rational [Def. 10.4]. And since $A B$ is commensurable in length with $B C$, and $A B$ is equal to $B D$, $B D$ is thus commensurable in length with $B C$. And as $B D$ is to $B C$, so $D A$ (is) to $A C$ [Prop. 6.1]. Thus, $D A$ is commensurable with $A C$ [Prop. 10.11]. And $D A$ (is) rational. Thus, $A C$ is also rational [Def. 10.4]. Thus, the . . . by rational straight-lines ... commensurable, and so on ....

## Proposition 20

If a rational (area) is applied to a rational (straightline) then it produces as breadth a (straight-line which is) rational, and commensurable in length with the (straightline) to which it is applied.


For let the rational (area) $A C$ have been applied to the rational (straight-line) $A B$, producing the (straight-line) $B C$ as breadth. I say that $B C$ is rational, and commensurable in length with $B A$.

For let the square $A D$ have been described on $A B$.



 $\mathrm{AB} \boldsymbol{\mu} \boldsymbol{\eta} \boldsymbol{\varepsilon}$.

$\chi \alpha^{\prime}$.






 $\delta \varepsilon ̀ \mu \varepsilon ́ \sigma \eta$.


 ठè $\dot{\eta} \mathrm{AB} \tau \tilde{n} \mathrm{~B} \Delta$, $\alpha \sigma \cup ́ \mu \mu \varepsilon \tau \rho о \varsigma ~ \alpha ́ p \alpha ~ \varepsilon ̇ \sigma \tau i ̀ ~ x \alpha i ̀ ~ \dot{\eta} \Delta \mathrm{~B} \tau \tilde{n} \mathrm{~B} \mathrm{\Gamma}$





${ }^{\dagger}$ Thus, a medial straight-line has a length expressible as $k^{1 / 4}$.
$A D$ is thus rational [Def. 10.4]. And $A C$ (is) also rational. $D A$ is thus commensurable with $A C$. And as $D A$ is to $A C$, so $D B$ (is) to $B C$ [Prop. 6.1]. Thus, $D B$ is also commensurable (in length) with $B C$ [Prop. 10.11]. And $D B$ (is) equal to $B A$. Thus, $A B$ (is) also commensurable (in length) with $B C$. And $A B$ is rational. Thus, $B C$ is also rational, and commensurable in length with $A B$ [Def. 10.3].

Thus, if a rational (area) is applied to a rational (straight-line), and so on ....

## Proposition 21

The rectangle contained by rational straight-lines (which are) commensurable in square only is irrational, and its square-root is irrational-let it be called medial. ${ }^{\dagger}$


For let the rectangle $A C$ be contained by the rational straight-lines $A B$ and $B C$ (which are) commensurable in square only. I say that $A C$ is irrational, and its squareroot is irrational-let it be called medial.

For let the square $A D$ have been described on $A B$. $A D$ is thus rational [Def. 10.4]. And since $A B$ is incommensurable in length with $B C$. For they were assumed to be commensurable in square only. And $A B$ (is) equal to $B D . D B$ is thus also incommensurable in length with $B C$. And as $D B$ is to $B C$, so $A D$ (is) to $A C$ [Prop. 6.1]. Thus, $D A$ [is] incommensurable with $A C$ [Prop. 10.11]. And $D A$ (is) rational. Thus, $A C$ is irrational [Def. 10.4]. Hence, its square-root [that is to say, the square-root of the square equal to it] is also irrational [Def. 10.4]-let it be called medial. (Which is) the very thing it was required to show.
$\Lambda \tilde{n} \mu \mu \alpha$.

 ยu゙vะเడّv.


 ZE, EH.









$$
x \beta^{\prime} .
$$






 $\dot{\alpha} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho о \varsigma \tau \tilde{\eta}$ ГВ $\mu \dot{\eta} \chi \varepsilon ь$.



## Lemma

If there are two straight-lines then as the first is to the second, so the (square) on the first (is) to the (rectangle contained) by the two straight-lines.


Let $F E$ and $E G$ be two straight-lines. I say that as $F E$ is to $E G$, so the (square) on $F E$ (is) to the (rectangle contained) by $F E$ and $E G$.

For let the square $D F$ have been described on $F E$. And let $G D$ have been completed. Therefore, since as $F E$ is to $E G$, so $F D$ (is) to $D G$ [Prop. 6.1], and $F D$ is the (square) on $F E$, and $D G$ the (rectangle contained) by $D E$ and $E G$-that is to say, the (rectangle contained) by $F E$ and $E G$-thus as $F E$ is to $E G$, so the (square) on $F E$ (is) to the (rectangle contained) by $F E$ and $E G$. And also, similarly, as the (rectangle contained) by $G E$ and $E F$ is to the (square on) $E F$-that is to say, as $G D$ (is) to $F D$-so $G E$ (is) to $E F$. (Which is) the very thing it was required to show.

## Proposition 22

The square on a medial (straight-line), being applied to a rational (straight-line), produces as breadth a (straight-line which is) rational, and incommensurable in length with the (straight-line) to which it is applied.


Let $A$ be a medial (straight-line), and $C B$ a rational (straight-line), and let the rectangular area $B D$, equal to the (square) on $A$, have been applied to $B C$, producing $C D$ as breadth. I say that $C D$ is rational, and incommensurable in length with $C B$.

For since $A$ is medial, the square on it is equal to a














 $\tau \widetilde{\varphi}$ ठ乇̀ ú $\pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{ZE}, \mathrm{EH} \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau ו ~ \tau o ̀ ~ u ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu \Delta \Gamma$,





† Literally, "rational".

$$
x \gamma^{\prime}
$$


${ }^{3} \mathrm{E} \sigma \tau \omega \mu \varepsilon ́ \sigma \eta \dot{\eta} \mathrm{~A}$, x $\alpha \grave{\imath} \tau \tilde{\eta} \mathrm{A} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho \circ \varsigma$ है $\sigma \tau \omega \dot{\eta} \mathrm{B}$ • $\lambda \varepsilon ́ \gamma \omega$, o้тા xai $\dot{\eta} \mathrm{B} \mu \varepsilon ́ \sigma \eta$ ह̇ $\sigma \tau i ́ v$.





 $\sigma u ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau l ~ x \alpha i ̀ ~ \tau o ̀ ~ \alpha ̇ \pi o ̀ ~ \tau n ̃ s ~ A ~ \tau \widetilde{̣}$ árò $\tau \tilde{\eta} s \mathrm{~B}$. $\alpha \lambda \lambda \lambda \grave{\alpha}$



(rectangular) area contained by rational (straight-lines which are) commensurable in square only [Prop. 10.21]. Let the square on $(A)$ be equal to $G F$. And the square on $(A)$ is also equal to $B D$. Thus, $B D$ is equal to $G F$. And $(B D)$ is also equiangular with $(G F)$. And for equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, proportionally, as $B C$ is to $E G$, so $E F$ (is) to $C D$. And, also, as the (square) on $B C$ is to the (square) on $E G$, so the (square) on $E F$ (is) to the (square) on $C D$ [Prop. 6.22]. And the (square) on $C B$ is commensurable with the (square) on $E G$. For they are each rational. Thus, the (square) on $E F$ is also commensurable with the (square) on $C D$ [Prop. 10.11]. And the (square) on $E F$ is rational. Thus, the (square) on $C D$ is also rational [Def. 10.4]. Thus, $C D$ is rational. And since $E F$ is incommensurable in length with $E G$. For they are commensurable in square only. And as $E F$ (is) to $E G$, so the (square) on $E F$ (is) to the (rectangle contained) by $F E$ and $E G$ [see previous lemma]. The (square) on $E F$ [is] thus incommensurable with the (rectangle contained) by $F E$ and $E G$ [Prop. 10.11]. But, the (square) on $C D$ is commensurable with the (square) on $E F$. For they are rational in square. And the (rectangle contained) by $D C$ and $C B$ is commensurable with the (rectangle contained) by $F E$ and $E G$. For they are (both) equal to the (square) on $A$. Thus, the (square) on $C D$ is also incommensurable with the (rectangle contained) by $D C$ and $C B$ [Prop. 10.13]. And as the (square) on $C D$ (is) to the (rectangle contained) by $D C$ and $C B$, so $D C$ is to $C B$ [see previous lemma]. Thus, $D C$ is incommensurable in length with $C B$ [Prop. 10.11]. Thus, $C D$ is rational, and incommensurable in length with $C B$. (Which is) the very thing it was required to show.

## Proposition 23

A (straight-line) commensurable with a medial (straightline) is medial.

Let $A$ be a medial (straight-line), and let $B$ be commensurable with $A$. I say that $B$ is also a medial (staightline).

Let the rational (straight-line) $C D$ be set out, and let the rectangular area $C E$, equal to the (square) on $A$, have been applied to $C D$, producing $E D$ as width. $E D$ is thus rational, and incommensurable in length with $C D$ [Prop. 10.22]. And let the rectangular area $C F$, equal to the (square) on $B$, have been applied to $C D$, producing $D F$ as width. Therefore, since $A$ is commensurable with $B$, the (square) on $A$ is also commensurable with



 $\mu o ́ v o \nu ~ \sigma \cup \mu \mu \varepsilon ́ \tau \rho \omega \nu ~ \delta u v \alpha \mu \varepsilon ́ v \eta ~ \mu \varepsilon ́ \sigma \eta ~ \varepsilon ̇ \sigma \tau i ́ v . ~ \dot{\eta} \alpha \nprec \alpha ~ \tau o ̀ ~ U ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu$
 $\Delta \mathrm{Z} \dot{\eta} \mathrm{B} \cdot \mu \varepsilon ́ \sigma \eta$ a̋ $\rho \alpha$ ह̇ $\sigma \tau i \nu \dot{\eta} \mathrm{~B}$.


По́рьбцд.
 ov $\mu$ モ́бov ย̇எтív.
the (square) on $B$. But, $E C$ is equal to the (square) on $A$, and $C F$ is equal to the (square) on $B$. Thus, $E C$ is commensurable with $C F$. And as $E C$ is to $C F$, so $E D$ (is) to $D F$ [Prop. 6.1]. Thus, $E D$ is commensurable in length with $D F$ [Prop. 10.11]. And $E D$ is rational, and incommensurable in length with $C D . D F$ is thus also rational [Def. 10.3], and incommensurable in length with $D C$ [Prop. 10.13]. Thus, $C D$ and $D F$ are rational, and commensurable in square only. And the square-root of a (rectangle contained) by rational (straight-lines which are) commensurable in square only is medial [Prop. 10.21]. Thus, the square-root of the (rectangle contained) by $C D$ and $D F$ is medial. And the square on $B$ is equal to the (rectangle contained) by $C D$ and $D F$. Thus, $B$ is a medial (straight-line).


## Corollary

And (it is) clear, from this, that an (area) commensurable with a medial area ${ }^{\dagger}$ is medial.
${ }^{\dagger}$ A medial area is equal to the square on some medial straight-line. Hence, a medial area is expressible as $k^{1 / 2}$.

## $\chi \delta^{\prime}$.



${ }^{'} \Upsilon \pi o ̀ ~ \gamma \grave{\alpha} \rho \mu \varepsilon ́ \sigma \omega \nu ~ \mu ウ ́ x \varepsilon \iota ~ \sigma u \mu \mu \varepsilon ́ \tau \rho \omega \nu ~ \varepsilon u ̉ \vartheta \varepsilon เ \widetilde{\omega} \nu \tau \widetilde{\omega} \nu \mathrm{AB}, ~ В \Gamma$ $\pi \varepsilon \rho เ \varepsilon \chi \varepsilon ́ \sigma \vartheta \omega$ ỏp७oүávıov tò $A \Gamma \cdot \lambda \varepsilon ́ \gamma \omega$, o้тו tò $A \Gamma \mu \varepsilon ́ \sigma o v$ ย่ $\sigma \tau i ้$.






## Proposition 24

A rectangle contained by medial straight-lines (which are) commensurable in length is medial.

For let the rectangle $A C$ be contained by the medial straight-lines $A B$ and $B C$ (which are) commensurable in length. I say that $A C$ is medial.

For let the square $A D$ have been described on $A B$. $A D$ is thus medial [see previous footnote]. And since $A B$ is commensurable in length with $B C$, and $A B$ (is) equal to $B D, D B$ is thus also commensurable in length with $B C$. Hence, $D A$ is also commensurable with $A C$ [Props. 6.1, 10.11]. And $D A$ (is) medial. Thus, $A C$ (is) also medial [Prop. 10.23 corr.]. (Which is) the very thing it was required to show.


Tò Úлò $\mu \varepsilon ́ \sigma \omega \nu ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u \mu \mu \varepsilon ́ т \rho \omega \nu ~ \varepsilon u ̉ v \varepsilon เ \widetilde{\omega} \nu \pi \varepsilon-$









 óp७oү́́viov $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu$ tò $\mathrm{MK} \pi \lambda \alpha ́ \tau o s ~ \pi о เ o u ̃ \nu ~ \tau \eta ̀ \nu ~$
 $\beta \varepsilon \beta \lambda \eta \dot{\eta} \vartheta \omega$ tò $\mathrm{N} \Lambda \pi \lambda \alpha ́ \tau o \varsigma ~ \pi o เ o u ̃ \nu ~ \tau \grave{\eta} \nu \mathrm{~K} \Lambda \cdot \dot{\varepsilon} \pi^{\prime} \varepsilon \cup \cup \vartheta \varepsilon i ́ \alpha s ~ \alpha ̈ p \alpha$











Proposition 25
The rectangle contained by medial straight-lines (which are) commensurable in square only is either rational or medial.


For let the rectangle $A C$ be contained by the medial straight-lines $A B$ and $B C$ (which are) commensurable in square only. I say that $A C$ is either rational or medial.

For let the squares $A D$ and $B E$ have been described on (the straight-lines) $A B$ and $B C$ (respectively). $A D$ and $B E$ are thus each medial. And let the rational (straight-line) $F G$ be laid out. And let the rectangular parallelogram $G H$, equal to $A D$, have been applied to $F G$, producing $F H$ as breadth. And let the rectangular parallelogram $M K$, equal to $A C$, have been applied to $H M$, producing $H K$ as breadth. And, finally, let $N L$, equal to $B E$, have similarly been applied to $K N$, producing $K L$ as breadth. Thus, $F H, H K$, and $K L$ are in a straight-line. Therefore, since $A D$ and $B E$ are each medial, and $A D$ is equal to $G H$, and $B E$ to $N L, G H$ and $N L$ (are) thus each also medial. And they are applied to the rational (straight-line) $F G . F H$ and $K L$ are thus each rational, and incommensurable in length with $F G$ [Prop. 10.22]. And since $A D$ is commensurable with $B E, G H$ is thus also commensurable with $N L$. And as

غ̇л















Tò äpa ن́兀ò $\mu \varepsilon ́ \sigma \omega \nu$ ठuvá $\mu \varepsilon \iota ~ \mu o ́ v o v ~ \sigma \cup \mu \mu \varepsilon ́ \tau p \omega \nu, ~ x \alpha i ̀ ~ \tau \alpha ̀ ~$ $\varepsilon \xi \tilde{\eta} \varsigma$.
$\chi q^{\prime}$.
Méбov $\mu$ ह́бou oủX ن́ $\pi \varepsilon \rho \varepsilon ́ \chi \varepsilon ા ~ ค ̀ \eta \tau \widetilde{̣ ̆ . ~}$


Eỉ үàp $\delta u v \alpha \tau o ́ v, ~ \mu \varepsilon ́ \sigma o v ~ \tau o ̀ ~ A B ~ \mu \varepsilon ́ \sigma o u ~ \tau o u ̃ ~ A \Gamma ~ u ́ \pi \varepsilon p \varepsilon \chi \varepsilon ́ \tau \omega ~$









$G H$ is to $N L$, so $F H$ (is) to $K L$ [Prop. 6.1]. Thus, $F H$ is commensurable in length with $K L$ [Prop. 10.11]. Thus, $F H$ and $K L$ are rational (straight-lines which are) commensurable in length. Thus, the (rectangle contained) by $F H$ and $K L$ is rational [Prop. 10.19]. And since $D B$ is equal to $B A$, and $O B$ to $B C$, thus as $D B$ is to $B C$, so $A B$ (is) to $B O$. But, as $D B$ (is) to $B C$, so $D A$ (is) to $A C$ [Props. 6.1]. And as $A B$ (is) to $B O$, so $A C$ (is) to $C O$ [Prop. 6.1]. Thus, as $D A$ is to $A C$, so $A C$ (is) to $C O$. And $A D$ is equal to $G H$, and $A C$ to $M K$, and $C O$ to $N L$. Thus, as $G H$ is to $M K$, so $M K$ (is) to $N L$. Thus, also, as $F H$ is to $H K$, so $H K$ (is) to $K L$ [Props. 6.1, 5.11]. Thus, the (rectangle contained) by $F H$ and $K L$ is equal to the (square) on $H K$ [Prop. 6.17]. And the (rectangle contained) by $F H$ and $K L$ (is) rational. Thus, the (square) on $H K$ is also rational. Thus, $H K$ is rational. And if it is commensurable in length with $F G$ then $H N$ is rational [Prop. 10.19]. And if it is incommensurable in length with $F G$ then $K H$ and $H M$ are rational (straight-lines which are) commensurable in square only: thus, $H N$ is medial [Prop. 10.21]. Thus, $H N$ is either rational or medial. And $H N$ (is) equal to $A C$. Thus, $A C$ is either rational or medial.

Thus, the . . . by medial straight-lines (which are) commensurable in square only, and so on ....

## Proposition 26

A medial (area) does not exceed a medial (area) by a rational (area). ${ }^{\dagger}$


For, if possible, let the medial (area) $A B$ exceed the medial (area) $A C$ by the rational (area) $D B$. And let the rational (straight-line) $E F$ be laid down. And let the rectangular parallelogram $F H$, equal to $A B$, have been applied to to $E F$, producing $E H$ as breadth. And let $F G$, equal to $A C$, have been cut off (from $F H$ ). Thus, the remainder $B D$ is equal to the remainder $K H$. And $D B$ is rational. Thus, $K H$ is also rational. Therefore, since $A B$ and $A C$ are each medial, and $A B$ is equal to $F H$, and $A C$ to $F G, F H$ and $F G$ are thus each also medial.














 ӧтєр દ̇எтi้ $\dot{\alpha} \delta u ́ v \alpha \tau o v$.

${ }^{\dagger}$ In other words, $\sqrt{k}-\sqrt{k^{\prime}} \neq k^{\prime \prime}$.

$$
x \zeta^{\prime} .
$$

 рเモไoúซa૬.


 $\dot{\omega} \varsigma \dot{\eta} \mathrm{A} \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \mathrm{~B}$, oü $\tau \omega \varsigma \dot{\eta} \Gamma \pi \rho o ̀ s \tau \grave{\eta} \nu \Delta$.

And they are applied to the rational (straight-line) $E F$. Thus, $H E$ and $E G$ are each rational, and incommensurable in length with $E F$ [Prop. 10.22]. And since $D B$ is rational, and is equal to $K H, K H$ is thus also rational. And $(K H)$ is applied to the rational (straight-line) $E F . G H$ is thus rational, and commensurable in length with $E F$ [Prop. 10.20]. But, $E G$ is also rational, and incommensurable in length with $E F$. Thus, $E G$ is incommensurable in length with $G H$ [Prop. 10.13]. And as $E G$ is to $G H$, so the (square) on $E G$ (is) to the (rectangle contained) by $E G$ and $G H$ [Prop. 10.13 lem.]. Thus, the (square) on $E G$ is incommensurable with the (rectangle contained) by $E G$ and $G H$ [Prop. 10.11]. But, the (sum of the) squares on $E G$ and $G H$ is commensurable with the (square) on $E G$. For ( $E G$ and $G H$ are) both rational. And twice the (rectangle contained) by $E G$ and $G H$ is commensurable with the (rectangle contained) by $E G$ and $G H$ [Prop. 10.6]. For (the former) is double the latter. Thus, the (sum of the squares) on $E G$ and $G H$ is incommensurable with twice the (rectangle contained) by $E G$ and $G H$ [Prop. 10.13]. And thus the sum of the (squares) on $E G$ and $G H$ plus twice the (rectangle contained) by $E G$ and $G H$, that is the (square) on $E H$ [Prop. 2.4], is incommensurable with the (sum of the squares) on $E G$ and $G H$ [Prop. 10.16]. And the (sum of the squares) on $E G$ and $G H$ (is) rational. Thus, the (square) on $E H$ is irrational [Def. 10.4]. Thus, $E H$ is irrational [Def. 10.4]. But, (it is) also rational. The very thing is impossible.

Thus, a medial (area) does not exceed a medial (area) by a rational (area). (Which is) the very thing it was required to show.

## Proposition 27

To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.


Let the two rational (straight-lines) $A$ and $B$, (which are) commensurable in square only, be laid down. And let $C$-the mean proportional (straight-line) to $A$ and $B$ -




 $\Gamma \cdot \mu \varepsilon ́ \sigma \eta$ äp $\alpha$ xì $\dot{\eta} \Delta$ ．$\alpha i \Gamma, \Delta$ äp $\alpha \mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i \sigma i ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~$

 ध̇のtiv $\dot{\omega} \varsigma \dot{\eta} \mathrm{A} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma, \dot{\eta} \mathrm{~B} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Delta$ ．$\alpha \lambda \lambda^{\circ} \dot{\omega} \varsigma \dot{\eta} \mathrm{A}$


 ن́лò 七 $\widetilde{\omega} \nu \Gamma, \Delta$ ．


have been taken［Prop．6．13］．And let it be contrived that as $A$（is）to $B$ ，so $C$（is）to $D$［Prop．6．12］．

And since the rational（straight－lines）$A$ and $B$ are commensurable in square only，the（rectangle con－ tained）by $A$ and $B$－that is to say，the（square）on $C$ ［Prop．6．17］－is thus medial［Prop 10．21］．Thus，$C$ is medial［Prop．10．21］．And since as $A$ is to $B$ ，［so］$C$（is） to $D$ ，and $A$ and $B$［are］commensurable in square only， $C$ and $D$ are thus also commensurable in square only ［Prop．10．11］．And $C$ is medial．Thus，$D$ is also medial ［Prop．10．23］．Thus，$C$ and $D$ are medial（straight－lines which are）commensurable in square only．I say that they also contain a rational（area）．For since as $A$ is to $B$ ，so $C$（is）to $D$ ，thus，alternately，as $A$ is to $C$ ，so $B$（is）to $D$［Prop．5．16］．But，as $A$（is）to $C$ ，（so）$C$（is）to $B$ ． And thus as $C$（is）to $B$ ，so $B$（is）to $D$［Prop．5．11］． Thus，the（rectangle contained）by $C$ and $D$ is equal to the（square）on $B$［Prop．6．17］．And the（square）on $B$ （is）rational．Thus，the（rectangle contained）by $C$ and $D$［is］also rational．

Thus，（two）medial（straight－lines，$C$ and $D$ ），con－ taining a rational（area），（which are）commensurable in square only，have been found．${ }^{\dagger}$（Which is）the very thing it was required to show．
${ }^{\dagger} C$ and $D$ have lengths $k^{1 / 4}$ and $k^{3 / 4}$ times that of $A$ ，respectively，where the length of $B$ is $k^{1 / 2}$ times that of $A$ ．

$$
x r^{\prime}
$$

 pıモðoúб人ऽ．








 E．גi $\Delta$ ， E «้p $\alpha \mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i \sigma i ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o \nu ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l . ~ \lambda \varepsilon ́ \gamma \omega ~$
 $\tau \grave{\eta} \nu \Gamma, \dot{\eta} \Delta \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{E}, \dot{\varepsilon} v \alpha \lambda \lambda \lambda \dot{\alpha} \xi \ddot{\alpha} \rho \alpha \dot{\omega} \omega \dot{\eta} \mathrm{~B} \pi \rho o ̀ s \operatorname{\tau ì\nu } \Delta, \dot{\eta}$






## Proposition 28

To find（two）medial（straight－lines），containing a me－ dial（area），（which are）commensurable in square only．


Let the［three］rational（straight－lines）$A, B$ ，and $C$ ， （which are）commensurable in square only，be laid down． And let，$D$ ，the mean proportional（straight－line）to $A$ and $B$ ，have been taken［Prop．6．13］．And let it be con－ trived that as $B$（is）to $C$ ，（so）$D$（is）to $E$［Prop．6．12］．

Since the rational（straight－lines）$A$ and $B$ are com－ mensurable in square only，the（rectangle contained）by $A$ and $B$－that is to say，the（square）on $D$［Prop．6．17］－ is medial［Prop．10．21］．Thus，$D$（is）medial［Prop．10．21］． And since $B$ and $C$ are commensurable in square only， and as $B$ is to $C$ ，（so）$D$（is）to $E, D$ and $E$ are thus com－ mensurable in square only［Prop．10．11］．And $D$（is）me－ dial．$E$（is）thus also medial［Prop．10．23］．Thus，$D$ and $E$ are medial（straight－lines which are）commensurable in square only．So，I say that they also enclose a medial （area）．For since as $B$ is to $C$ ，（so）$D$（is）to $E$ ，thus，

alternately, as $B$ (is) to $D$, (so) $C$ (is) to $E$ [Prop. 5.16]. And as $B$ (is) to $D$, (so) $D$ (is) to $A$. And thus as $D$ (is) to $A$, (so) $C$ (is) to $E$. Thus, the (rectangle contained) by $A$ and $C$ is equal to the (rectangle contained) by $D$ and $E$ [Prop. 6.16]. And the (rectangle contained) by $A$ and $C$ is medial [Prop. 10.21]. Thus, the (rectangle contained) by $D$ and $E$ (is) also medial.

Thus, (two) medial (straight-lines, $D$ and $E$ ), containing a medial (area), (which are) commensurable in square only, have been found. (Which is) the very thing it was required to show.
${ }^{\dagger} D$ and $E$ have lengths $k^{1 / 4}$ and $k^{1 / 2} / k^{1 / 4}$ times that of $A$, respectively, where the lengths of $B$ and $C$ are $k^{1 / 2}$ and $k^{1 / 2}$ times that of $A$, respectively.

## $\Lambda \tilde{\eta} \mu \mu \alpha \alpha^{\prime}$.















 тoเoũซl 七òv $\alpha \pi o ̀ ~ \tau o u ̃ ~ B \Delta ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega v o v . ~$








## Lemma I

To find two square numbers such that the sum of them is also square.


Let the two numbers $A B$ and $B C$ be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder $A C$ is thus even. Let $A C$ have been cut in half at $D$. And let $A B$ and $B C$ also be either similar plane (numbers), or square (numbers) -which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) $A B$ and $B C$, plus the square on $C D$, is equal to the square on $B D$ [Prop. 2.6]. And the (number created) from (multiplying) $A B$ and $B C$ is square-inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found-(namely,) the (number created) from (multiplying) $A B$ and $B C$, and the (square) on $C D$-which, (when) added (together), make the square on $B D$.

And (it is) clear that two square (numbers) have again been found-(namely,) the (square) on $B D$, and the (square) on $C D$-such that their difference-(namely,) the (rectangle) contained by $A B$ and $B C$-is square whenever $A B$ and $B C$ are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) have been found-(namely,) the (square) on $B D$, and the (square) on $D C$-between which the difference-(namely,) the (rectangle) contained by $A B$ and $B C$-is not square. (Which is) the very thing it was required to show.

## $\Lambda \tilde{n} \mu \mu \alpha \beta^{\prime}$.







































## Lemma II

To find two square numbers such that the sum of them is not square.


For let the (number created) from (multiplying) $A B$ and $B C$, as we said, be square. And (let) $C A$ (be) even. And let $C A$ have been cut in half at $D$. So it is clear that the square (number created) from (multiplying) $A B$ and $B C$, plus the square on $C D$, is equal to the square on $B D$ [see previous lemma]. Let the unit $D E$ have been subtracted (from $B D$ ). Thus, the (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$, is less than the square on $B D$. I say, therefore, that the square (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$, is not square.

For if it is square, it is either equal to the (square) on $B E$, or less than the (square) on $B E$, but cannot any more be greater (than the square on $B E$ ), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$, be equal to the (square) on $B E$. And let $G A$ be double the unit $D E$. Therefore, since the whole of $A C$ is double the whole of $C D$, of which $A G$ is double $D E$, the remain$\operatorname{der} G C$ is thus double the remainder $E C$. Thus, $G C$ has been cut in half at $E$. Thus, the (number created) from (multiplying) $G B$ and $B C$, plus the (square) on $C E$, is equal to the square on $B E$ [Prop. 2.6]. But, the (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$, was also assumed (to be) equal to the square on $B E$. Thus, the (number created) from (multiplying) $G B$ and $B C$, plus the (square) on $C E$, is equal to the (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$. And subtracting the (square) on $C E$ from both, $A B$ is inferred (to be) equal to $G B$. The very thing is absurd. Thus, the (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$, is not equal to the (square) on $B E$. So I say that (it is) not less than the (square) on $B E$ either. For, if possible, let it be equal to the (square) on $B F$. And (let) $H A$ (be) double $D F$. And it can again be inferred that $H C$ (is) double $C F$. Hence, $C H$ has also been cut in half at $F$. And, on account of this, the (number created) from (multiplying) $H B$ and $B C$, plus the (square) on $F C$, becomes equal to the (square) on $B F$ [Prop. 2.6]. And the (number created) from (multiplying) $A B$ and $B C$, plus the (square) on $C E$, was also assumed (to be) equal to the (square) on $B F$. Hence, the (number created) from (multiplying) $H B$ and $B C$, plus the (square) on $C F$, will also be equal to the (number created) from (multiplying) $A B$ and $B C$,

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\chi \vartheta^{\prime} .
$$

Eúpeĩ $\delta u ́ o$ ค̊ $\eta \tau \alpha ̀ s ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u \mu \mu \varepsilon ́ \tau p o u s, ~ \omega ̈ \sigma \tau \varepsilon ~ \tau \eta ̀ v ~$
 $\dot{\varepsilon} \alpha \cup \tau \tilde{n} \mu \dot{\eta} x \varepsilon \iota$ ．





 غ̇лを弓ょú $\chi \vartheta \omega \dot{\eta} \mathrm{ZB}$ ．



















plus the（square）on $C E$ ．The very thing is absurd．Thus， the（number created）from（multiplying）$A B$ and $B C$ ， plus the（square）on $C E$ ，is not equal to less than the （square）on $B E$ ．And it was shown that（is it）not equal to the（square）on $B E$ either．Thus，the（number created） from（multiplying）$A B$ and $B C$ ，plus the square on $C E$ ， is not square．（Which is）the very thing it was required to show．

## Proposition 29

To find two rational（straight－lines which are）com－ mensurable in square only，such that the square on the greater is larger than the（square on the）lesser by the （square）on（some straight－line which is）commensurable in length with the greater．


For let some rational（straight－line）$A B$ be laid down， and two square numbers，$C D$ and $D E$ ，such that the dif－ ference between them，$C E$ ，is not square［Prop． 10.28 lem．I］．And let the semi－circle $A F B$ have been drawn on $A B$ ．And let it be contrived that as $D C$（is）to $C E$ ，so the square on $B A$（is）to the square on $A F$［Prop． 10.6 corr．］． And let $F B$ have been joined．
［Therefore，］since as the（square）on $B A$ is to the （square）on $A F$ ，so $D C$（is）to $C E$ ，the（square）on $B A$ thus has to the（square）on $A F$ the ratio which the number $D C$（has）to the number $C E$ ．Thus，the （square）on $B A$ is commensurable with the（square）on $A F$［Prop．10．6］．And the（square）on $A B$（is）rational ［Def．10．4］．Thus，the（square）on $A F$（is）also ratio－ nal．Thus，$A F$（is）also rational．And since $D C$ does not have to $C E$ the ratio which（some）square num－ ber（has）to（some）square number，the（square）on $B A$ thus does not have to the（square）on $A F$ the ra－ tio which（some）square number has to（some）square number either．Thus，$A B$ is incommensurable in length with $A F$［Prop．10．9］．Thus，the rational（straight－lines） $B A$ and $A F$ are commensurable in square only．And since as $D C$［is］to $C E$ ，so the（square）on $B A$（is）to the（square）on $A F$ ，thus，via conversion，as $C D$（is） to $D E$ ，so the（square）on $A B$（is）to the（square）on
$\varepsilon \alpha \cup \tau \tilde{n}$.


 ठદǐ̌ $\alpha$.
$B F$ [Props. 5.19 corr., $3.31,1.47$ ]. And $C D$ has to $D E$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $A B$ also has to the (square) on $B F$ the ratio which (some) square number has to (some) square number. $A B$ is thus commensurable in length with $B F$ [Prop. 10.9]. And the (square) on $A B$ is equal to the (sum of the squares) on $A F$ and $F B$ [Prop. 1.47]. Thus, the square on $A B$ is greater than (the square on) $A F$ by (the square on) $B F$, (which is) commensurable (in length) with $(A B)$.

Thus, two rational (straight-lines), $B A$ and $A F$, commensurable in square only, have been found such that the square on the greater, $A B$, is larger than (the square on) the lesser, $A F$, by the (square) on $B F$, (which is) commensurable in length with $(A B) .^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger} B A$ and $A F$ have lengths 1 and $\sqrt{1-k^{2}}$ times that of $A B$, respectively, where $k=\sqrt{D E / C D}$.

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\lambda^{\prime}
$$


 غ́ $\cup \tau \tilde{n} \mu \dot{\eta} x \varepsilon ા$.




 BA rpòs tò $\alpha \pi o ̀ ~ \tau n ̃ s ~ A Z, ~ x \alpha \grave{~ \varepsilon ̇ \pi \varepsilon \zeta \varepsilon u ́ \chi \vartheta \omega ~ \dot{\eta} Z B . ~}$







 $\dot{\alpha} \sigma u ́ \mu \mu \varepsilon \tau \rho о \varsigma ~ \alpha ้ p \alpha ~ \varepsilon ̇ \sigma \tau i \nu ~ \dot{\eta} \mathrm{AB}$ т $\tilde{n} \mathrm{BZ} \mu \dot{\eta} x \varepsilon เ$. x $\alpha \grave{\iota}$ ठúv $\alpha \tau \alpha \iota \dot{\eta}$


## Proposition 30

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.


Let the rational (straight-line) $A B$ be laid out, and the two square numbers, $C E$ and $E D$, such that the sum of them, $C D$, is not square [Prop. 10.28 lem. II]. And let the semi-circle $A F B$ have been drawn on $A B$. And let it be contrived that as $D C$ (is) to $C E$, so the (square) on $B A$ (is) to the (square) on $A F$ [Prop. 10.6 corr]. And let $F B$ have been joined.

So, similarly to the (proposition) before this, we can show that $B A$ and $A F$ are rational (straight-lines which are) commensurable in square only. And since as $D C$ is to $C E$, so the (square) on $B A$ (is) to the (square) on $A F$, thus, via conversion, as $C D$ (is) to $D E$, so the (square) on $A B$ (is) to the (square) on $B F$ [Props. 5.19 corr., 3.31, 1.47]. And $C D$ does not have to $D E$ the ratio which (some) square number (has) to (some) square number.




Thus, the (square) on $A B$ does not have to the (square) on $B F$ the ratio which (some) square number has to (some) square number either. Thus, $A B$ is incommensurable in length with $B F$ [Prop. 10.9]. And the square on $A B$ is greater than the (square on) $A F$ by the (square) on $F B$ [Prop. 1.47], (which is) incommensurable (in length) with ( $A B$ ).

Thus, $A B$ and $A F$ are rational (straight-lines which are) commensurable in square only, and the square on $A B$ is greater than (the square on) $A F$ by the (square) on $F B$, (which is) incommensurable (in length) with $(A B) .^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger} A B$ and $A F$ have lengths 1 and $1 / \sqrt{1+k^{2}}$ times that of $A B$, respectively, where $k=\sqrt{D E / C E}$.

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\lambda \alpha^{\prime}
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 $\dot{\eta} \mathrm{A} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{~B}$, oưt $\tau \omega \varsigma \dot{\eta} \Gamma$ трòs t $\grave{\nu} \nu \Delta$. $\sigma u ́ \mu \mu \varepsilon \tau \rho o s ~ \delta \grave{\varepsilon} \dot{\eta} \mathrm{~A}$







## Proposition 31

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.


Let two rational (straight-lines), $A$ and $B$, commensurable in square only, be laid out, such that the square on the greater $A$ is larger than the (square on the) lesser $B$ by the (square) on (some straight-line) commensurable in length with ( $A$ ) [Prop. 10.29]. And let the (square) on $C$ be equal to the (rectangle contained) by $A$ and $B$. And the (rectangle contained by) $A$ and $B$ (is) medial [Prop. 10.21]. Thus, the (square) on $C$ (is) also medial. Thus, $C$ (is) also medial [Prop. 10.21]. And let the (rectangle contained) by $C$ and $D$ be equal to the (square) on $B$. And the (square) on $B$ (is) rational. Thus, the (rectangle contained) by $C$ and $D$ (is) also rational. And since as $A$ is to $B$, so the (rectangle contained) by $A$ and $B$ (is) to the (square) on $B$ [Prop. 10.21 lem.], but the (square) on $C$ is equal to the (rectangle contained) by $A$ and $B$, and the (rectangle contained) by $C$ and $D$ to the (square) on $B$, thus as $A$ (is) to $B$, so the (square) on $C$ (is) to the (rectangle contained) by $C$ and $D$. And as the (square) on $C$ (is) to the (rectangle contained) by




$C$ and $D$, so $C$ (is) to $D$ [Prop. 10.21 lem.]. And thus as $A$ (is) to $B$, so $C$ (is) to $D$. And $A$ is commensurable in square only with $B$. Thus, $C$ (is) also commensurable in square only with $D$ [Prop. 10.11]. And $C$ is medial. Thus, $D$ (is) also medial [Prop. 10.23]. And since as $A$ is to $B$, (so) $C$ (is) to $D$, and the square on $A$ is greater than (the square on) $B$ by the (square) on (some straight-line) commensurable (in length) with ( $A$ ), the square on $C$ is thus also greater than (the square on) $D$ by the (square) on (some straight-line) commensurable (in length) with (C) [Prop. 10.14].

Thus, two medial (straight-lines), $C$ and $D$, commensurable in square only, (and) containing a rational (area), have been found. And the square on $C$ is greater than (the square on) $D$ by the (square) on (some straight-line) commensurable in length with ( $C$ ). ${ }^{\dagger}$

So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with $C$ ), provided that the square on $A$ is greater than (the square on $B$ ) by the (square) on (some straight-line) incommensurable (in length) with ( $A$ ) [Prop. 10.30].
${ }^{\dagger} C$ and $D$ have lengths $\left(1-k^{2}\right)^{1 / 4}$ and $\left(1-k^{2}\right)^{3 / 4}$ times that of $A$, respectively, where $k$ is defined in the footnote to Prop. 10.29.
$\ddagger C$ and $D$ would have lengths $1 /\left(1+k^{2}\right)^{1 / 4}$ and $1 /\left(1+k^{2}\right)^{3 / 4}$ times that of $A$, respectively, where $k$ is defined in the footnote to Prop. 10.30.

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\lambda \beta^{\prime} .
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## Proposition 32

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.


Let three rational (straight-lines), $A, B$ and $C$, commensurable in square only, be laid out such that the square on $A$ is greater than (the square on $C$ ) by the (square) on (some straight-line) commensurable (in length) with ( $A$ ) [Prop. 10.29]. And let the (square) on $D$ be equal to the (rectangle contained) by $A$ and $B$. Thus, the (square) on $D$ (is) medial. Thus, $D$ is also medial [Prop. 10.21]. And let the (rectangle contained) by $D$ and $E$ be equal to the (rectangle contained) by $B$ and $C$. And since as the (rectangle contained) by $A$ and $B$ is to the (rectangle contained) by $B$ and $C$, so $A$ (is) to $C$ [Prop. 10.21 lem.], but the (square) on $D$ is equal to the (rectangle contained) by $A$ and $B$, and the (rectangle












contained) by $D$ and $E$ to the (rectangle contained) by $B$ and $C$, thus as $A$ is to $C$, so the (square) on $D$ (is) to the (rectangle contained) by $D$ and $E$. And as the (square) on $D$ (is) to the (rectangle contained) by $D$ and $E$, so $D$ (is) to $E$ [Prop. 10.21 lem.]. And thus as $A$ (is) to $C$, so $D$ (is) to $E$. And $A$ (is) commensurable in square [only] with $C$. Thus, $D$ (is) also commensurable in square only with $E$ [Prop. 10.11]. And $D$ (is) medial. Thus, $E$ (is) also medial [Prop. 10.23]. And since as $A$ is to $C$, (so) $D$ (is) to $E$, and the square on $A$ is greater than (the square on) $C$ by the (square) on (some straight-line) commensurable (in length) with $(A)$, the square on $D$ will thus also be greater than (the square on) $E$ by the (square) on (some straight-line) commensurable (in length) with ( $D$ ) [Prop. 10.14]. So, I also say that the (rectangle contained) by $D$ and $E$ is medial. For since the (rectangle contained) by $B$ and $C$ is equal to the (rectangle contained) by $D$ and $E$, and the (rectangle contained) by $B$ and $C$ (is) medial [for $B$ and $C$ are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by $D$ and $E$ (is) thus also medial.

Thus, two medial (straight-lines), $D$ and $E$, commensurable in square only, (and) containing a medial (area), have been found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater. ${ }^{\dagger}$.

So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on $A$ is greater than (the square on) $C$ by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30]. $\ddagger$
${ }^{\dagger} D$ and $E$ have lengths $k^{1 / 4}$ and $k^{1 / 4} \sqrt{1-k^{2}}$ times that of $A$, respectively, where the length of $B$ is $k^{\prime 1 / 2}$ times that of $A$, and $k$ is defined in the footnote to Prop. 10.29.
$\ddagger D$ and $E$ would have lengths $k^{\prime 1 / 4}$ and $k^{\prime 1 / 4} / \sqrt{1+k^{2}}$ times that of $A$, respectively, where the length of $B$ is $k^{\prime 1 / 2}$ times that of $A$, and $k$ is defined in the footnote to Prop. 10.30.

## $\Lambda \tilde{\eta} \mu \mu \alpha$.






 $\tau \tilde{\eta} \mathrm{BA}$.

## Lemma

Let $A B C$ be a right-angled triangle having the (angle) $A$ a right-angle. And let the perpendicular $A D$ have been drawn. I say that the (rectangle contained) by $C B D$ is equal to the (square) on $B A$, and the (rectangle contained) by $B C D$ (is) equal to the (square) on $C A$, and the (rectangle contained) by $B D$ and $D C$ (is) equal to the (square) on $A D$, and, further, the (rectangle contained) by $B C$ and $A D$ [is] equal to the (rectangle contained) by $B A$ and $A C$.







 $\tau \tilde{\eta} \mathrm{A} \cdot$.











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\lambda \gamma^{\prime}
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 $\delta^{\prime} \dot{\cup} \pi^{\prime} \alpha \cup ๋ \tau \widetilde{\omega} \nu \mu \varepsilon ́ \sigma o v$.








And, first of all, (let us prove) that the (rectangle contained) by $C B D$ [is] equal to the (square) on $B A$.


For since $A D$ has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, $A B D$ and $A D C$ are thus triangles (which are) similar to the whole, $A B C$, and to one another [Prop. 6.8]. And since triangle $A B C$ is similar to triangle $A B D$, thus as $C B$ is to $B A$, so $B A$ (is) to $B D$ [Prop. 6.4]. Thus, the (rectangle contained) by $C B D$ is equal to the (square) on $A B$ [Prop. 6.17].

So, for the same (reasons), the (rectangle contained) by $B C D$ is also equal to the (square) on $A C$.

And since if a (straight-line) is drawn from the rightangle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as $B D$ is to $D A$, so $A D$ (is) to $D C$. Thus, the (rectangle contained) by $B D$ and $D C$ is equal to the (square) on $D A$ [Prop. 6.17].

I also say that the (rectangle contained) by $B C$ and $A D$ is equal to the (rectangle contained) by $B A$ and $A C$. For since, as we said, $A B C$ is similar to $A B D$, thus as $B C$ is to $C A$, so $B A$ (is) to $A D$ [Prop. 6.4]. Thus, the (rectangle contained) by $B C$ and $A D$ is equal to the (rectangle contained) by $B A$ and $A C$ [Prop. 6.16]. (Which is) the very thing it was required to show.

## Proposition 33

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Let the two rational (straight-lines) $A B$ and $B C$, (which are) commensurable in square only, be laid out such that the square on the greater, $A B$, is larger than (the square on) the lesser, $B C$, by the (square) on (some straight-line which is) incommensurable (in length) with $(A B)$ [Prop. 10.30]. And let $B C$ have been cut in half at $D$. And let a parallelogram equal to the (square) on ei-





















 $\tau \widetilde{\omega} \nu \dot{\alpha} \pi^{\prime} \alpha \cup \cup \tau \widetilde{\omega} \nu \tau \varepsilon \tau \rho \alpha \gamma \dot{\omega} \nu \omega \nu$.



ther of $B D$ or $D C$, (and) falling short by a square figure, have been applied to $A B$ [Prop. 6.28], and let it be the (rectangle contained) by $A E B$. And let the semi-circle $A F B$ have been drawn on $A B$. And let $E F$ have been drawn at right-angles to $A B$. And let $A F$ and $F B$ have been joined.


And since $A B$ and $B C$ are [two] unequal straightlines, and the square on $A B$ is greater than (the square on) $B C$ by the (square) on (some straight-line which is) incommensurable (in length) with $(A B)$. And a parallelogram, equal to one quarter of the (square) on $B C$ that is to say, (equal) to the (square) on half of it-(and) falling short by a square figure, has been applied to $A B$, and makes the (rectangle contained) by $A E B . A E$ is thus incommensurable (in length) with $E B$ [Prop. 10.18]. And as $A E$ is to $E B$, so the (rectangle contained) by $B A$ and $A E$ (is) to the (rectangle contained) by $A B$ and $B E$. And the (rectangle contained) by $B A$ and $A E$ (is) equal to the (square) on $A F$, and the (rectangle contained) by $A B$ and $B E$ to the (square) on $B F$ [Prop. 10.32 lem.]. The (square) on $A F$ is thus incommensurable with the (square) on $F B$ [Prop. 10.11]. Thus, $A F$ and $F B$ are incommensurable in square. And since $A B$ is rational, the (square) on $A B$ is also rational. Hence, the sum of the (squares) on $A F$ and $F B$ is also rational [Prop. 1.47]. And, again, since the (rectangle contained) by $A E$ and $E B$ is equal to the (square) on $E F$, and the (rectangle contained) by $A E$ and $E B$ was assumed (to be) equal to the (square) on $B D, F E$ is thus equal to $B D$. Thus, $B C$ is double $F E$. And hence the (rectangle contained) by $A B$ and $B C$ is commensurable with the (rectangle contained) by $A B$ and $E F$ [Prop. 10.6]. And the (rectangle contained) by $A B$ and $B C$ (is) medial [Prop. 10.21]. Thus, the (rectangle contained) by $A B$ and $E F$ (is) also medial [Prop. 10.23 corr.]. And the (rectangle contained) by $A B$ and $E F$ (is) equal to the (rectangle contained) by $A F$ and $F B$ [Prop. 10.32 lem.]. Thus, the (rectangle contained) by $A F$ and $F B$ (is) also medial. And the sum of the squares on them was also shown (to be) rational.

Thus, the two straight-lines, $A F$ and $F B$, (which are) incommensurable in square, have been found, making the sum of the squares on them rational, and the (rectangle contained) by them medial. (Which is) the very thing it was required to show.
$\dagger A F$ and $F B$ have lengths $\sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}$ and $\sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}$ times that of $A B$, respectively, where $k$ is defined in the footnote to Prop. 10.30.

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\lambda \delta^{\prime}
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## Proposition 34

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.


Let the two medial (straight-lines) $A B$ and $B C$, (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on $A B$ is greater than (the square on) $B C$ by the (square) on (some straight-line) incommensurable (in length) with ( $A B$ ) [Prop. 10.31]. And let the semi-circle $A D B$ have been drawn on $A B$. And let $B C$ have been cut in half at $E$. And let a (rectangular) parallelogram equal to the (square) on $B E$, (and) falling short by a square figure, have been applied to $A B$, (and let it be) the (rectangle contained by) $A F B$ [Prop. 6.28]. Thus, $A F$ [is] incommensurable in length with $F B$ [Prop. 10.18]. And let $F D$ have been drawn from $F$ at right-angles to $A B$. And let $A D$ and $D B$ have been joined.

Since $A F$ is incommensurable (in length) with $F B$, the (rectangle contained) by $B A$ and $A F$ is thus also incommensurable with the (rectangle contained) by $A B$ and $B F$ [Prop. 10.11]. And the (rectangle contained) by $B A$ and $A F$ (is) equal to the (square) on $A D$, and the (rectangle contained) by $A B$ and $B F$ to the (square) on $D B$ [Prop. 10.32 lem.]. Thus, the (square) on $A D$ is also incommensurable with the (square) on $D B$. And since the (square) on $A B$ is medial, the sum of the (squares) on $A D$ and $D B$ (is) thus also medial [Props. 3.31, 1.47]. And since $B C$ is double $D F$ [see previous proposition], the (rectangle contained) by $A B$ and $B C$ (is) thus also double the (rectangle contained) by $A B$ and $F D$. And the (rectangle contained) by $A B$ and $B C$ (is) rational. Thus, the (rectangle contained) by $A B$ and $F D$ (is) also rational [Prop. 10.6, Def. 10.4]. And the (rectangle contained) by $A B$ and $F D$ (is) equal to the (rectangle contained) by $A D$ and $D B$ [Prop. 10.32 lem.]. And hence the (rectangle contained) by $A D$ and $D B$ is rational.

Thus, two straight-lines, $A D$ and $D B$, (which are) incommensurable in square, have been found, making the sum of the squares on them medial, and the (rectangle
contained) by them rational. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger} A D$ and $D B$ have lengths $\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}+k\right] /\left[2\left(1+k^{2}\right)\right]}$ and $\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}-k\right] /\left[2\left(1+k^{2}\right)\right]}$ times that of $A B$, respectively, where $k$ is defined in the footnote to Prop. 10.29.

$$
\lambda \varepsilon^{\prime} .
$$



 $\tau \widetilde{\omega} \nu \dot{\alpha} \pi^{\prime} \alpha \dot{\tau} \tau \widetilde{\omega} \nu \tau \varepsilon \tau \rho \alpha \gamma \omega ้ \nu \omega$.




 ó $\mu$ оíus.














 $\dot{\varepsilon} \chi \tau \widetilde{\omega} \nu \dot{\alpha} \pi \grave{o} \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B} \tau \widetilde{\mu} \dot{\text { ínò }} \tau \widetilde{\omega} \nu \mathrm{A} \Delta, \Delta \mathrm{B}$.





## Proposition 35

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.


Let the two medial (straight-lines) $A B$ and $B C$, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on $A B$ is greater than (the square on) $B C$ by the (square) on (some straight-line) incommensurable (in length) with ( $A B$ ) [Prop. 10.32]. And let the semi-circle $A D B$ have been drawn on $A B$. And let the remainder (of the figure) be generated similarly to the above (proposition).

And since $A F$ is incommensurable in length with $F B$ [Prop. 10.18], $A D$ is also incommensurable in square with $D B$ [Prop. 10.11]. And since the (square) on $A B$ is medial, the sum of the (squares) on $A D$ and $D B$ (is) thus also medial [Props. 3.31, 1.47]. And since the (rectangle contained) by $A F$ and $F B$ is equal to the (square) on each of $B E$ and $D F, B E$ is thus equal to $D F$. Thus, $B C$ (is) double $F D$. And hence the (rectangle contained) by $A B$ and $B C$ is double the (rectangle) contained by $A B$ and $F D$. And the (rectangle contained) by $A B$ and $B C$ (is) medial. Thus, the (rectangle contained) by $A B$ and $F D$ (is) also medial. And it is equal to the (rectangle contained) by $A D$ and $D B$ [Prop. 10.32 lem.]. Thus, the (rectangle contained) by $A D$ and $D B$ (is) also medial. And since $A B$ is incommensurable in length with $B C$, and $C B$ (is) commensurable (in length) with $B E$, $A B$ (is) thus also incommensurable in length with $B E$ [Prop. 10.13]. And hence the (square) on $A B$ is also incommensurable with the (rectangle contained) by $A B$ and $B E$ [Prop. 10.11]. But the (sum of the squares) on $A D$ and $D B$ is equal to the (square) on $A B$ [Prop. 1.47]. And the (rectangle contained) by $A B$ and $F D$-that is to say, the (rectangle contained) by $A D$ and $D B$-is equal to the (rectangle contained) by $A B$ and $B E$. Thus, the
${ }^{\dagger} A D$ and $D B$ have lengths $k^{\prime 1 / 4} \sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}$ and $k^{\prime 1 / 4} \sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}$ times that of $A B$, respectively, where $k$ and $k^{\prime}$ are defined in the footnote to Prop. 10.32.
sum of the (squares) on $A D$ and $D B$ is incommensurable with the (rectangle contained) by $A D$ and $D B$.

Thus, two straight-lines, $A D$ and $D B$, (which are) incommensurable in square, have been found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them. ${ }^{\dagger}$ (Which is) the very thing it was required to show. defied in the foot

$$
\lambda \epsilon^{\prime}
$$

















 ठعї̆ $\alpha$.
 .

## Proposition 36

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is irrational-let it be called a binomial (straight-line). ${ }^{\dagger}$


For let the two rational (straight-lines), $A B$ and $B C$, (which are) commensurable in square only, be laid down together. I say that the whole (straight-line), $A C$, is irrational. For since $A B$ is incommensurable in length with $B C$-for they are commensurable in square only-and as $A B$ (is) to $B C$, so the (rectangle contained) by $A B C$ (is) to the (square) on $B C$, the (rectangle contained) by $A B$ and $B C$ is thus incommensurable with the (square) on $B C$ [Prop. 10.11]. But, twice the (rectangle contained) by $A B$ and $B C$ is commensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.6]. And (the sum of) the (squares) on $A B$ and $B C$ is commensurable with the (square) on $B C$-for the rational (straight-lines) $A B$ and $B C$ are commensurable in square only [Prop. 10.15]. Thus, twice the (rectangle contained) by $A B$ and $B C$ is incommensurable with (the sum of) the (squares) on $A B$ and $B C$ [Prop. 10.13]. And, via composition, twice the (rectangle contained) by $A B$ and $B C$, plus (the sum of) the (squares) on $A B$ and $B C$-that is to say, the (square) on $A C$ [Prop. 2.4]-is incommensurable with the sum of the (squares) on $A B$ and $B C$ [Prop. 10.16]. And the sum of the (squares) on $A B$ and $B C$ (is) rational. Thus, the (square) on $A C$ [is] irrational [Def. 10.4]. Hence, $A C$ is also irrational [Def. 10.4]-let it be called a binomial (straight-line). ${ }^{\ddagger}$ (Which is) the very thing it was required to show.

[^23](see Prop. 10.73), are the positive roots of the quartic $x^{4}-2(1+k) x^{2}+(1-k)^{2}=0$.
$$
\lambda \zeta^{\prime} .
$$
’Eàv $\delta u ́ o ~ \mu \varepsilon ́ \sigma \alpha l ~ \delta u v a ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau р о ا ~ \sigma u \nu \tau \varepsilon \vartheta \overparen{\omega ั \sigma \iota ~}$
 ठúo $\mu \varepsilon ́ \sigma \omega \nu \pi \rho \omega ́ \tau \eta$.

$\Sigma u \gamma \varkappa \varepsilon i \sigma \vartheta \omega \sigma \alpha \nu \gamma \grave{\alpha} \rho$ ठúo $\mu \varepsilon ́ \sigma \alpha l ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l ~$
 ย่ $\sigma \tau \iota$.



 $\tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma}$. p̀ntòv dè tò ú $\pi o ̀ \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma} \cdot$ ن́ $\pi o ́ x \varepsilon เ \nu \tau \alpha l ~ \gamma \grave{\alpha} \rho$




## Proposition 37

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational-let it be called a first bimedial (straight-line). ${ }^{\dagger}$


For let the two medial (straight-lines), $A B$ and $B C$, commensurable in square only, (and) containing a rational (area), be laid down together. I say that the whole (straight-line), $A C$, is irrational.

For since $A B$ is incommensurable in length with $B C$, (the sum of) the (squares) on $A B$ and $B C$ is thus also incommensurable with twice the (rectangle contained) by $A B$ and $B C$ [see previous proposition]. And, via composition, (the sum of) the (squares) on $A B$ and $B C$, plus twice the (rectangle contained) by $A B$ and $B C$ that is, the (square) on $A C$ [Prop. 2.4]-is incommensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.16]. And the (rectangle contained) by $A B$ and $B C$ (is) rational-for $A B$ and $B C$ were assumed to enclose a rational (area). Thus, the (square) on $A C$ (is) irrational. Thus, $A C$ (is) irrational [Def. 10.4]-let it be called a first bimedial (straight-line). ${ }^{\ddagger}$ (Which is) the very thing it was required to show.
† Literally, "first from two medials".
$\ddagger$ Thus, a first bimedial straight-line has a length expressible as $k^{1 / 4}+k^{3 / 4}$. The first bimedial and the corresponding first apotome of a medial, whose length is expressible as $k^{1 / 4}-k^{3 / 4}$ (see Prop. 10.74), are the positive roots of the quartic $x^{4}-2 \sqrt{k}(1+k) x^{2}+k(1-k)^{2}=0$.

$$
\lambda \eta^{\prime}
$$

’Eàv ठúo $\mu \varepsilon ́ \sigma \alpha l ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau p o l ~ \sigma u \nu \tau \varepsilon \vartheta \widetilde{\omega} \sigma \iota ~$
 ठúo $\mu \varepsilon ́ \sigma \omega \nu$ ठuєтépa.

$\Sigma u \gamma \varkappa \varepsilon i \sigma \vartheta \omega \sigma \alpha \nu \gamma \grave{\alpha} \rho$ ठúo $\mu \varepsilon ́ \sigma \alpha l ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o ь ~$


## Proposition 38

If two medial (straight-lines), commensurable in square only, which contain a medial (area), are added together then the whole (straight-line) is irrational-let it be called a second bimedial (straight-line).


For let the two medial (straight-lines), $A B$ and $B C$, commensurable in square only, (and) containing a medial

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(area), be laid down together [Prop. 10.28]. I say that $A C$ is irrational.

For let the rational (straight-line) $D E$ be laid down, and let (the rectangle) $D F$, equal to the (square) on $A C$, have been applied to $D E$, making $D G$ as breadth [Prop. 1.44]. And since the (square) on $A C$ is equal to (the sum of) the (squares) on $A B$ and $B C$, plus twice the (rectangle contained) by $A B$ and $B C$ [Prop. 2.4], so let (the rectangle) $E H$, equal to (the sum of) the squares on $A B$ and $B C$, have been applied to $D E$. The remainder $H F$ is thus equal to twice the (rectangle contained) by $A B$ and $B C$. And since $A B$ and $B C$ are each medial, (the sum of) the squares on $A B$ and $B C$ is thus also medial. ${ }^{\ddagger}$ And twice the (rectangle contained) by $A B$ and $B C$ was also assumed (to be) medial. And $E H$ is equal to (the sum of) the squares on $A B$ and $B C$, and $F H$ (is) equal to twice the (rectangle contained) by $A B$ and $B C$. Thus, $E H$ and $H F$ (are) each medial. And they were applied to the rational (straight-line) $D E$. Thus, $D H$ and $H G$ are each rational, and incommensurable in length with $D E$ [Prop. 10.22]. Therefore, since $A B$ is incommensurable in length with $B C$, and as $A B$ is to $B C$, so the (square) on $A B$ (is) to the (rectangle contained) by $A B$ and $B C$ [Prop. 10.21 lem.], the (square) on $A B$ is thus incommensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.11]. But, the sum of the squares on $A B$ and $B C$ is commensurable with the (square) on $A B$ [Prop. 10.15], and twice the (rectangle contained) by $A B$ and $B C$ is commensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.6]. Thus, the sum of the (squares) on $A B$ and $B C$ is incommensurable with twice the (rectangle contained) by $A B$ and $B C$ [Prop. 10.13]. But, $E H$ is equal to (the sum of) the squares on $A B$ and $B C$, and $H F$ is equal to twice the (rectangle) contained by $A B$ and $B C$. Thus, $E H$ is incommensurable with $H F$. Hence, $D H$ is also incommensurable in length with $H G$ [Props. 6.1, 10.11]. Thus, $D H$ and $H G$ are rational (straight-lines which are) commensurable in square only. Hence, $D G$ is irrational [Prop. 10.36]. And $D E$ (is) rational. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20]. The area $D F$ is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4]. And $A C$ is the square-root of $D F$. $A C$ is thus irrational-let it be called a second bimedial (straight-line). ${ }^{\S}$ (Which is) the very thing it was required to show.

[^24]$\left[\left(k-k^{\prime}\right)^{2} / k\right]=0$.

## $\lambda \vartheta^{\prime}$.



 б乇̀ $\mu \varepsilon$ í $\omega \nu$.


 АГ.








 ठєi̧̋al.

## Proposition 39

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together then the whole straight-line is irrational-let it be called a major (straight-line).

$$
\begin{array}{lll}
\mathrm{A} & \mathrm{~B} & \mathrm{C}
\end{array}
$$

For let the two straight-lines, $A B$ and $B C$, incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33]. I say that $A C$ is irrational.

For since the (rectangle contained) by $A B$ and $B C$ is medial, twice the (rectangle contained) by $A B$ and $B C$ is [thus] also medial [Props. 10.6, 10.23 corr.]. And the sum of the (squares) on $A B$ and $B C$ (is) rational. Thus, twice the (rectangle contained) by $A B$ and $B C$ is incommensurable with the sum of the (squares) on $A B$ and $B C$ [Def. 10.4]. Hence, (the sum of) the squares on $A B$ and $B C$, plus twice the (rectangle contained) by $A B$ and $B C$-that is, the (square) on $A C$ [Prop. 2.4]-is also incommensurable with the sum of the (squares) on $A B$ and $B C$ [Prop. 10.16] [and the sum of the (squares) on $A B$ and $B C$ (is) rational]. Thus, the (square) on $A C$ is irrational. Hence, $A C$ is also irrational [Def. 10.4]-let it be called a major (straight-line). ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ Thus, a major straight-line has a length expressible as $\sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}+\sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}$. The major and the corresponding minor, whose length is expressible as $\sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}-\sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}$ (see Prop. 10.76), are the positive roots of the quartic $x^{4}-2 x^{2}+k^{2} /\left(1+k^{2}\right)=0$.

$$
\mu^{\prime}
$$







 AГ.




## Proposition 40

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational, are added together then the whole straight-line is irrational-let it be called the square-root of a rational plus a medial (area).


For let the two straight-lines, $A B$ and $B C$, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34]. I say that $A C$ is irrational.

For since the sum of the (squares) on $A B$ and $B C$ is medial, and twice the (rectangle contained) by $A B$ and




$B C$ (is) rational, the sum of the (squares) on $A B$ and $B C$ is thus incommensurable with twice the (rectangle contained) by $A B$ and $B C$. Hence, the (square) on $A C$ is also incommensurable with twice the (rectangle contained) by $A B$ and $B C$ [Prop. 10.16]. And twice the (rectangle contained) by $A B$ and $B C$ (is) rational. The (square) on $A C$ (is) thus irrational. Thus, $A C$ (is) irrational [Def. 10.4]-let it be called the square-root of a rational plus a medial (area). ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ Thus, the square-root of a rational plus a medial (area) has a length expressible as $\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}+k\right] /\left[2\left(1+k^{2}\right)\right]}+\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}-k\right] /\left[2\left(1+k^{2}\right)\right]}$. This and the corresponding irrational with a minus sign, whose length is expressible as $\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}+k\right] /\left[2\left(1+k^{2}\right)\right]}-\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}-k\right] /\left[2\left(1+k^{2}\right)\right]}$ (see Prop. 10.77), are the positive roots of the quartic $x^{4}-\left(2 / \sqrt{1+k^{2}}\right) x^{2}+k^{2} /\left(1+k^{2}\right)^{2}=0$.

$$
\mu \alpha^{\prime}
$$








 ย̇எтเน.











## Proposition 41

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational-let it be called the square-root of (the sum of) two medial (areas).


For let the two straight-lines, $A B$ and $B C$, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35]. I say that $A C$ is irrational.

Let the rational (straight-line) $D E$ be laid out, and let (the rectangle) $D F$, equal to (the sum of) the (squares) on $A B$ and $B C$, and (the rectangle) $G H$, equal to twice the (rectangle contained) by $A B$ and $B C$, have been applied to $D E$. Thus, the whole of $D H$ is equal to the square on $A C$ [Prop. 2.4]. And since the sum of the (squares) on $A B$ and $B C$ is medial, and is equal to $D F$, $D F$ is thus also medial. And it is applied to the rational (straight-line) $D E$. Thus, $D G$ is rational, and incommen-






surable in length with $D E$ [Prop. 10.22]. So, for the same (reasons), $G K$ is also rational, and incommensurable in length with $G F$-that is to say, $D E$. And since (the sum of) the (squares) on $A B$ and $B C$ is incommensurable with twice the (rectangle contained) by $A B$ and $B C$, $D F$ is incommensurable with $G H$. Hence, $D G$ is also incommensurable (in length) with $G K$ [Props. 6.1, 10.11]. And they are rational. Thus, $D G$ and $G K$ are rational (straight-lines which are) commensurable in square only. Thus, $D K$ is irrational, and that (straight-line which is) called binomial [Prop. 10.36]. And $D E$ (is) rational. Thus, $D H$ is irrational, and its square-root is irrational [Def. 10.4]. And $A C$ (is) the square-root of $H D$. Thus, $A C$ is irrational-let it be called the square-root of (the sum of) two medial (areas). ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ Thus, the square-root of (the sum of) two medial (areas) has a length expressible as $k^{\prime 1 / 4}\left(\sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}+\sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}\right)$. This and the corresponding irrational with a minus sign, whose length is expressible as $k^{\prime 1 / 4}\left(\sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}-\sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}\right)$ (see Prop. 10.78), are the positive roots of the quartic $x^{4}-2 k^{\prime 1 / 2} x^{2}+k^{\prime} k^{2} /\left(1+k^{2}\right)=0$.

## $\Lambda \tilde{n} \mu \mu \alpha$.







 $\tau \widetilde{\omega} \mathrm{A} \Delta, \Delta \mathrm{B}$.







 ن́ $\pi o ̀ ~ \tau \tilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B} \mu \varepsilon \tau \dot{\alpha}$ toũ $\dot{\alpha} \pi o ̀ ~ \tau \tilde{\eta} \varsigma \Delta \mathrm{E} \cdot \widetilde{\omega} \nu$ tò $\alpha \pi o ̀ ~ \tau \eta ̃ \varsigma ~$







## Lemma

We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.


Let the straight-line $A B$ be laid out, and let the whole (straight-line) have been cut into unequal parts at each of the (points) $C$ and $D$. And let $A C$ be assumed (to be) greater than $D B$. I say that (the sum of) the (squares) on $A C$ and $C B$ is greater than (the sum of) the (squares) on $A D$ and $D B$.

For let $A B$ have been cut in half at $E$. And since $A C$ is greater than $D B$, let $D C$ have been subtracted from both. Thus, the remainder $A D$ is greater than the remainder $C B$. And $A E$ (is) equal to $E B$. Thus, $D E$ (is) less than $E C$. Thus, points $C$ and $D$ are not equally far from the point of bisection. And since the (rectangle contained) by $A C$ and $C B$, plus the (square) on $E C$, is equal to the (square) on $E B$ [Prop. 2.5], but, moreover, the (rectangle contained) by $A D$ and $D B$, plus the (square) on $D E$, is also equal to the (square) on $E B$ [Prop. 2.5], the (rectangle contained) by $A C$ and $C B$, plus the (square) on $E C$, is thus equal to the (rectangle contained) by $A D$ and $D B$, plus the (square) on $D E$. And, of these, the (square) on $D E$ is less than the (square) on $E C$. And, thus, the
remaining (rectangle contained) by $A C$ and $C B$ is less than the (rectangle contained) by $A D$ and $D B$. And, hence, twice the (rectangle contained) by $A C$ and $C B$ is less than twice the (rectangle contained) by $A D$ and $D B$. And thus the remaining sum of the (squares) on $A C$ and $C B$ is greater than the sum of the (squares) on $A D$ and $D B .^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ Since, $A C^{2}+C B^{2}+2 A C C B=A D^{2}+D B^{2}+2 A D D B=A B^{2}$.

$$
\mu \beta^{\prime} .
$$

 عis t̀̀ óvó $\mu \alpha \tau \alpha$.








 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu ~ \Gamma \mathrm{~B}$, oữ $\omega$ s $\dot{\eta} \mathrm{B} \Delta$ тpòs $\operatorname{\tau \eta } \nu \Delta \mathrm{A}$, xal हैन $\tau \alpha l \dot{\eta} \mathrm{AB}$















## Proposition 42

A binomial (straight-line) can be divided into its (component) terms at one point only. ${ }^{\dagger}$


Let $A B$ be a binomial (straight-line) which has been divided into its (component) terms at $C . A C$ and $C B$ are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. I say that $A B$ cannot be divided at another point into two rational (straight-lines which are) commensurable in square only.

For, if possible, let it also have been divided at $D$, such that $A D$ and $D B$ are also rational (straight-lines which are) commensurable in square only. So, (it is) clear that $A C$ is not the same as $D B$. For, if possible, let it be (the same). So, $A D$ will also be the same as $C B$. And as $A C$ will be to $C B$, so $B D$ (will be) to $D A$. And $A B$ will (thus) also be divided at $D$ in the same (manner) as the division at $C$. The very opposite was assumed. Thus, $A C$ is not the same as $D B$. So, on account of this, points $C$ and $D$ are not equally far from the point of bisection. Thus, by whatever (amount the sum of) the (squares) on $A C$ and $C B$ differs from (the sum of) the (squares) on $A D$ and $D B$, twice the (rectangle contained) by $A D$ and $D B$ also differs from twice the (rectangle contained) by $A C$ and $C B$ by this (same amount)—on account of both (the sum of) the (squares) on $A C$ and $C B$, plus twice the (rectangle contained) by $A C$ and $C B$, and (the sum of) the (squares) on $A D$ and $D B$, plus twice the (rectangle contained) by $A D$ and $D B$, being equal to the (square) on $A B$ [Prop. 2.4]. But, (the sum of) the (squares) on $A C$ and $C B$ differs from (the sum of) the (squares) on $A D$ and $D B$ by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by $A D$ and $D B$ also differs from twice the (rectangle contained) by $A C$ and $C B$ by a rational (area, despite both) being medial (areas) [Prop. 10.21]. The very thing is absurd. For a medial (area) cannot exceed a medial (area) by a rational (area) [Prop. 10.26].
${ }^{\dagger}$ In other words, $k+k^{\prime 1 / 2}=k^{\prime \prime}+k^{\prime \prime \prime 1 / 2}$ has only one solution: i.e., $k^{\prime \prime}=k$ and $k^{\prime \prime \prime}=k^{\prime}$. Likewise, $k^{1 / 2}+k^{\prime 1 / 2}=k^{\prime \prime 1 / 2}+k^{\prime \prime \prime 1 / 2}$ has only one solution: i.e., $k^{\prime \prime}=k$ and $k^{\prime \prime \prime}=k^{\prime}$ (or, equivalently, $k^{\prime \prime}=k^{\prime}$ and $k^{\prime \prime \prime}=k$ ).

$$
\mu \gamma^{\prime} .
$$





 ठเаlрعі̃тац.












## Proposition 43

A first bimedial (straight-line) can be divided (into its component terms) at one point only. ${ }^{\dagger}$


Let $A B$ be a first bimedial (straight-line) which has been divided at $C$, such that $A C$ and $C B$ are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37]. I say that $A B$ cannot be (so) divided at another point.

For, if possible, let it also have been divided at $D$, such that $A D$ and $D B$ are also medial (straight-lines), commensurable in square only, (and) containing a rational (area). Since, therefore, by whatever (amount) twice the (rectangle contained) by $A D$ and $D B$ differs from twice the (rectangle contained) by $A C$ and $C B$, (the sum of) the (squares) on $A C$ and $C B$ differs from (the sum of) the (squares) on $A D$ and $D B$ by this (same amount) [Prop. 10.41 lem.]. And twice the (rectangle contained) by $A D$ and $D B$ differs from twice the (rectangle contained) by $A C$ and $C B$ by a rational (area). For (they are) both rational (areas). (The sum of) the (squares) on $A C$ and $C B$ thus differs from (the sum of) the (squares) on $A D$ and $D B$ by a rational (area, despite both) being medial (areas). The very thing is absurd [Prop. 10.26].

Thus, a first bimedial (straight-line) cannot be divided into its (component) terms at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, $k^{1 / 4}+k^{3 / 4}=k^{1 / 4}+k^{3 / 4}$ has only one solution: i.e., $k^{\prime}=k$.
$\mu \delta^{\prime}$.
'H ėx ठúo $\mu \varepsilon ́ \sigma \omega \nu ~ \delta \varepsilon \cup \tau \varepsilon ́ p \alpha ~ x \alpha \vartheta ’ ~ ह ै \nu ~ \mu o ́ v o v ~ \sigma \eta \mu \varepsilon i ̃ o \nu ~$

 $\Gamma$, ๕̋ $\sigma \tau \varepsilon \tau \grave{\alpha} \varsigma ~ А Г, ~ Г В ~ \mu \varepsilon ́ \sigma \alpha s ~ \varepsilon i v \alpha a l ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u \mu \mu \varepsilon ́ \tau p o u s ~$




## Proposition 44

A second bimedial (straight-line) can be divided (into its component terms) at one point only. ${ }^{\dagger}$

Let $A B$ be a second bimedial (straight-line) which has been divided at $C$, so that $A C$ and $B C$ are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38]. So, (it is) clear that $C$ is not (located) at the point of bisection, since ( $A C$ and $B C$ ) are not commensurable in length. I say that $A B$ cannot be (so) divided at another point.


 $\tau \grave{\eta} \nu \mathrm{A} \Gamma \cdot \delta \tilde{\eta} \lambda o \nu \delta \dot{\eta}$, ǒ $\tau \iota \nprec \alpha i ~ \tau \dot{\alpha} \alpha \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B}, \dot{\omega} \varsigma$ ह́ $\pi \alpha ́ \nu \omega$













 $\dot{\eta} \mathrm{A} \Gamma \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \mathrm{~B}$, oưt $\omega \varsigma$ tò $\alpha \pi o ̀ ~ \tau \tilde{\eta} \varsigma ~ А \Gamma ~ \pi \rho o ̀ s ~ \tau o ̀ ~ U ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu$

 АГ, ГВ• ठuvá $\mu \varepsilon \iota ~ \gamma \alpha ́ p ~ \varepsilon i \sigma ı ~ \sigma u ́ \mu \mu \varepsilon \tau p o l ~ \alpha i ~ А Г, ~ Г В . ~ \tau \widetilde{̣} ~ \delta e ̀ ~ i ́ \pi o ̀ ~$












 $\tau \widetilde{\omega} \nu \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B}$. $\dot{\alpha} \lambda \lambda \alpha$ $\tau \dot{\alpha} \alpha \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B} \mu \varepsilon i \zeta o v \alpha ́$






For, if possible, let it also have been (so) divided at $D$, so that $A C$ is not the same as $D B$, but $A C$ (is), by hypothesis, greater. So, (it is) clear that (the sum of) the (squares) on $A D$ and $D B$ is also less than (the sum of) the (squares) on $A C$ and $C B$, as we showed above [Prop. 10.41 lem.]. And $A D$ and $D B$ are medial (straight-lines), commensurable in square only, (and) containing a medial (area). And let the rational (straightline) $E F$ be laid down. And let the rectangular parallelogram $E K$, equal to the (square) on $A B$, have been applied to $E F$. And let $E G$, equal to (the sum of) the (squares) on $A C$ and $C B$, have been cut off (from $E K$ ). Thus, the remainder, $H K$, is equal to twice the (rectangle contained) by $A C$ and $C B$ [Prop. 2.4]. So, again, let $E L$, equal to (the sum of) the (squares) on $A D$ and $D B$-which was shown (to be) less than (the sum of) the (squares) on $A C$ and $C B$-have been cut off (from $E K$ ). And, thus, the remainder, $M K$, (is) equal to twice the (rectangle contained) by $A D$ and $D B$. And since (the sum of) the (squares) on $A C$ and $C B$ is medial, $E G$ (is) thus [also] medial. And it is applied to the rational (straight-line) $E F$. Thus, $E H$ is rational, and incommensurable in length with $E F$ [Prop. 10.22]. So, for the same (reasons), $H N$ is also rational, and incommensurable in length with $E F$. And since $A C$ and $C B$ are medial (straight-lines which are) commensurable in square only, $A C$ is thus incommensurable in length with $C B$. And as $A C$ (is) to $C B$, so the (square) on $A C$ (is) to the (rectangle contained) by $A C$ and $C B$ [Prop. 10.21 lem.]. Thus, the (square) on $A C$ is incommensurable with the (rectangle contained) by $A C$ and $C B$ [Prop. 10.11]. But, (the sum of) the (squares) on $A C$ and $C B$ is commensurable with the (square) on $A C$. For, $A C$ and $C B$ are commensurable in square [Prop. 10.15]. And twice the (rectangle contained) by $A C$ and $C B$ is commensurable with the (rectangle contained) by $A C$ and $C B$ [Prop. 10.6]. And thus (the sum of) the squares on $A C$ and $C B$ is incommensurable with twice the (rectangle contained) by $A C$ and $C B$ [Prop. 10.13]. But, $E G$ is equal to (the sum of) the (squares) on $A C$ and $C B$, and $H K$ equal to twice the (rectangle contained) by $A C$ and $C B$. Thus, $E G$ is incommensurable with $H K$. Hence, $E H$ is also incom-
mensurable in length with $H N$ [Props. 6.1, 10.11]. And (they are) rational (straight-lines). Thus, $E H$ and $H N$ are rational (straight-lines which are) commensurable in square only. And if two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is that irrational called binomial [Prop. 10.36]. Thus, EN is a binomial (straightline) which has been divided (into its component terms) at $H$. So, according to the same (reasoning), $E M$ and $M N$ can be shown (to be) rational (straight-lines which are) commensurable in square only. And $E N$ will (thus) be a binomial (straight-line) which has been divided (into its component terms) at the different (points) $H$ and $M$ (which is absurd [Prop. 10.42]). And $E H$ is not the same as $M N$, since (the sum of) the (squares) on $A C$ and $C B$ is greater than (the sum of) the (squares) on $A D$ and $D B$. But, (the sum of) the (squares) on $A D$ and $D B$ is greater than twice the (rectangle contained) by $A D$ and $D B$ [Prop. 10.59 lem.]. Thus, (the sum of) the (squares) on $A C$ and $C B$-that is to say, $E G$-is also much greater than twice the (rectangle contained) by $A D$ and $D B$ that is to say, $M K$. Hence, $E H$ is also greater than $M N$ [Prop. 6.1]. Thus, $E H$ is not the same as $M N$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, $k^{1 / 4}+k^{\prime 1 / 2} / k^{1 / 4}=k^{\prime \prime 1 / 4}+k^{\prime \prime \prime 1 / 2} / k^{\prime \prime 1 / 4}$ has only one solution: i.e., $k^{\prime \prime}=k$ and $k^{\prime \prime \prime}=k^{\prime}$.
$\mu \varepsilon^{\prime}$





 oủ ठıаıрعі̃таı.




 $\Delta \mathrm{B}$ тoũ $\delta i \varsigma ~ \cup ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu ~ А Г, ~ Г В, ~ \alpha ̀ \lambda \lambda \dot{\alpha} \tau \grave{\alpha} \alpha \pi o ̀ ~ \tau \widetilde{\omega} \nu ~ А Г, ~ Г В ~$






## Proposition 45

A major (straight-line) can only be divided (into its component terms) at the same point. ${ }^{\dagger}$


Let $A B$ be a major (straight-line) which has been divided at $C$, so that $A C$ and $C B$ are incommensurable in square, making the sum of the squares on $A C$ and $C B$ rational, and the (rectangle contained) by $A C$ and $C D$ medial [Prop. 10.39]. I say that $A B$ cannot be (so) divided at another point.

For, if possible, let it also have been divided at $D$, such that $A D$ and $D B$ are also incommensurable in square, making the sum of the (squares) on $A D$ and $D B$ rational, and the (rectangle contained) by them medial. And since, by whatever (amount the sum of) the (squares) on $A C$ and $C B$ differs from (the sum of) the (squares) on $A D$ and $D B$, twice the (rectangle contained) by $A D$ and $D B$ also differs from twice the (rectangle contained) by $A C$ and $C B$ by this (same amount). But, (the sum of) the (squares) on $A C$ and $C B$ exceeds (the sum of) the (squares) on $A D$ and $D B$ by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle
${ }^{\dagger}$ In other words, $\sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}+\sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}=\sqrt{\left[1+k^{\prime} /\left(1+k^{\prime 2}\right)^{1 / 2}\right] / 2}+\sqrt{\left[1-k^{\prime} /\left(1+k^{\prime 2}\right)^{1 / 2}\right] / 2}$ has only one solution: i.e., $k^{\prime}=k$.

$$
\mu \tau^{\prime}
$$

 ঠıаюреїта.




 бпиеі̃ov oú ठıаıреїтац.

Eî $\gamma \dot{\rho}$ ठ











## Proposition 46

The square-root of a rational plus a medial (area) can be divided (into its component terms) at one point only. ${ }^{\dagger}$


Let $A B$ be the square-root of a rational plus a medial (area) which has been divided at $C$, so that $A C$ and $C B$ are incommensurable in square, making the sum of the (squares) on $A C$ and $C B$ medial, and twice the (rectangle contained) by $A C$ and $C B$ rational [Prop. 10.40]. I say that $A B$ cannot be (so) divided at another point.

For, if possible, let it also have been divided at $D$, so that $A D$ and $D B$ are also incommensurable in square, making the sum of the (squares) on $A D$ and $D B$ medial, and twice the (rectangle contained) by $A D$ and $D B$ rational. Therefore, since by whatever (amount) twice the (rectangle contained) by $A C$ and $C B$ differs from twice the (rectangle contained) by $A D$ and $D B$, (the sum of) the (squares) on $A D$ and $D B$ also differs from (the sum of) the (squares) on $A C$ and $C B$ by this (same amount). And twice the (rectangle contained) by $A C$ and $C B$ exceeds twice the (rectangle contained) by $A D$ and $D B$ by a rational (area). (The sum of) the (squares) on $A D$ and $D B$ thus also exceeds (the sum of) the (squares) on $A C$ and $C B$ by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, the square-root of a rational plus a medial (area) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one point (only). (Which is) the very thing it was required to show.
$\dagger$ In other words, $\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}+k\right] /\left[2\left(1+k^{2}\right)\right]}+\sqrt{\left[\left(1+k^{2}\right)^{1 / 2}-k\right] /\left[2\left(1+k^{2}\right)\right]}=\sqrt{\left[\left(1+k^{\prime 2}\right)^{1 / 2}+k^{\prime}\right] /\left[2\left(1+k^{\prime 2}\right)\right]}$ $+\sqrt{\left[\left(1+k^{\prime 2}\right)^{1 / 2}-k^{\prime}\right] /\left[2\left(1+k^{\prime 2}\right)\right]}$ has only one solution: i.e., $k^{\prime}=k$.

$$
\mu \zeta^{\prime} .
$$



## Proposition 47

The square-root of (the sum of) two medial (areas) can be divided (into its component terms) at one point only. ${ }^{\dagger}$































Let $A B$ be [the square-root of (the sum of) two medial (areas)] which has been divided at $C$, such that $A C$ and $C B$ are incommensurable in square, making the sum of the (squares) on $A C$ and $C B$ medial, and the (rectangle contained) by $A C$ and $C B$ medial, and, moreover, incommensurable with the sum of the (squares) on ( $A C$ and $C B$ ) [Prop. 10.41]. I say that $A B$ cannot be divided at another point fulfilling the prescribed (conditions).

For, if possible, let it have been divided at $D$, such that $A C$ is again manifestly not the same as $D B$, but $A C$ (is), by hypothesis, greater. And let the rational (straight-line) $E F$ be laid down. And let $E G$, equal to (the sum of) the (squares) on $A C$ and $C B$, and $H K$, equal to twice the (rectangle contained) by $A C$ and $C B$, have been applied to $E F$. Thus, the whole of $E K$ is equal to the square on $A B$ [Prop. 2.4]. So, again, let $E L$, equal to (the sum of) the (squares) on $A D$ and $D B$, have been applied to $E F$. Thus, the remainder-twice the (rectangle contained) by $A D$ and $D B$-is equal to the remainder, $M K$. And since the sum of the (squares) on $A C$ and $C B$ was assumed (to be) medial, $E G$ is also medial. And it is applied to the rational (straight-line) $E F . H E$ is thus rational, and incommensurable in length with $E F$ [Prop. 10.22]. So, for the same (reasons), $H N$ is also rational, and incommensurable in length with $E F$. And since the sum of the (squares) on $A C$ and $C B$ is incommensurable with twice the (rectangle contained) by $A C$ and $C B, E G$ is thus also incommensurable with $G N$. Hence, $E H$ is also incommensurable with $H N$ [Props. 6.1, 10.11]. And they are (both) rational (straight-lines). Thus, $E H$ and $H N$ are rational (straight-lines which are) commensurable in square only. Thus, $E N$ is a binomial (straightline) which has been divided (into its component terms) at $H$ [Prop. 10.36]. So, similarly, we can show that it has also been (so) divided at $M$. And $E H$ is not the same as $M N$. Thus, a binomial (straight-line) has been divided (into its component terms) at different points. The very thing is absurd [Prop. 10.42]. Thus, the square-root of (the sum of) two medial (areas) cannot be divided (into
its component terms) at different points. Thus, it can be (so) divided at one [point] only.
${ }^{\dagger}$ In other words, $k^{\prime 1 / 4} \sqrt{\left[1+k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}+k^{\prime 1 / 4} \sqrt{\left[1-k /\left(1+k^{2}\right)^{1 / 2}\right] / 2}=k^{\prime \prime \prime 1 / 4} \sqrt{\left[1+k^{\prime \prime} /\left(1+k^{\prime 2}\right)^{1 / 2}\right] / 2}$
$+k^{\prime \prime \prime 1 / 4} \sqrt{\left[1-k^{\prime \prime} /\left(1+k^{\prime \prime 2}\right)^{1 / 2}\right] / 2}$ has only one solution: i.e., $k^{\prime \prime}=k$ and $k^{\prime \prime \prime}=k^{\prime}$.
"Opot $\delta$ ع́útepot.
















## $\mu \eta^{\prime}$.



 ๕̀ $\chi \varepsilon เ \nu, ~ o ̈ \nu ~ \tau \varepsilon \tau р \alpha ́ \gamma \omega \nu о \varsigma ~ \alpha ̉ p ı \vartheta \mu o ̀ s ~ \pi р o ̀ s ~ \tau \varepsilon \tau р \alpha ́ \gamma \omega \nu о \nu ~ \alpha ̉ p เ \vartheta \mu o ́ v, ~$









## Definitions II

5. Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line).
6. And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line).
7. And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line).
8. So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line).
9. And if the lesser (term is commensurable), a fifth (binomial straight-line).
10. And if neither (term is commensurable), a sixth (binomial straight-line).

## Proposition 48

To find a first binomial (straight-line).
Let two numbers $A C$ and $C B$ be laid down such that their sum $A B$ has to $B C$ the ratio which (some) square number (has) to (some) square number, and does not have to $C A$ the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let some rational (straight-line) $D$ be laid down. And let $E F$ be commensurable in length with $D . E F$ is thus also rational [Def. 10.3]. And let it have been contrived that as the number $B A$ (is) to $A C$, so the (square) on $E F$ (is) to the (square) on $F G$ [Prop. 10.6 corr.]. And $A B$ has to $A C$ the ratio which (some) number (has) to (some) num-

 $\pi \rho o ̀ s ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o v ~ \alpha ́ p ı \vartheta \mu o ́ v, ~ o u ̉ \delta e ̀ ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau n ̃ s ~ E Z ~ \alpha ́ p \alpha ~ \pi \rho o ̀ s ~$












 tò $\alpha \pi o ̀ ~ \tau n ̃ s ~ E Z ~ \alpha ै p \alpha ~ \pi p o ̀ s ~ t o ̀ ~ \alpha ́ \pi o ̀ ~ \tau n ̃ s ~ \Theta ~ \lambda o ́ \gamma o v ~ ह ै \chi ~ \chi \varepsilon ا, ~ o ̋ v ~$


 $\sigma \cup ́ \mu \mu \varepsilon \tau \rho о \varsigma \dot{\eta} \mathrm{EZ} \tau \tilde{n} \Delta \mu \dot{\eta} \varkappa \varepsilon \iota$.
 ठєĭ̧al.
ber. Thus, the (square) on $E F$ also has to the (square) on $F G$ the ratio which (some) number (has) to (some) number. Hence, the (square) on $E F$ is commensurable with the (square) on $F G$ [Prop. 10.6]. And $E F$ is rational. Thus, $F G$ (is) also rational. And since $B A$ does not have to $A C$ the ratio which (some) square number (has) to (some) square number, thus the (square) on $E F$ does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, $E F$ is incommensurable in length with $F G$ [Prop 10.9]. $E F$ and $F G$ are thus rational (straight-lines which are) commensurable in square only. Thus, $E G$ is a binomial (straight-line) [Prop. 10.36]. I say that (it is) also a first (binomial straight-line).


For since as the number $B A$ is to $A C$, so the (square) on $E F$ (is) to the (square) on $F G$, and $B A$ (is) greater than $A C$, the (square) on $E F$ (is) thus also greater than the (square) on $F G$ [Prop. 5.14]. Therefore, let (the sum of) the (squares) on $F G$ and $H$ be equal to the (square) on $E F$. And since as $B A$ is to $A C$, so the (square) on $E F$ (is) to the (square) on $F G$, thus, via conversion, as $A B$ is to $B C$, so the (square) on $E F$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. And $A B$ has to $B C$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $E F$ also has to the (square) on $H$ the ratio which (some) square number (has) to (some) square number. Thus, $E F$ is commensurable in length with $H$ [Prop. 10.9]. Thus, the square on $E F$ is greater than (the square on) $F G$ by the (square) on (some straight-line) commensurable (in length) with $(E F)$. And $E F$ and $F G$ are rational (straight-lines). And $E F$ (is) commensurable in length with $D$.

Thus, $E G$ is a first binomial (straight-line) [Def. 10.5]. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the rational straight-line has unit length then the length of a first binomial straight-line is $k+k \sqrt{1-k^{\prime 2}}$. This, and the first apotome, whose length is $k-k \sqrt{1-k^{\prime 2}}$ [Prop. 10.85], are the roots of $x^{2}-2 k x+k^{2} k^{\prime 2}=0$.

$\mu \vartheta^{\prime}$.<br>

## Proposition 49

To find a second binomial (straight-line).

’Еथx
 है $\chi \varepsilon เ \nu, ~ o ̈ \nu ~ \tau \varepsilon \tau р \alpha ́ \gamma \omega \nu о \varsigma ~ \alpha ́ p ı \vartheta \mu o ̀ s ~ \pi \rho o ̀ s ~ \tau \varepsilon \tau р \alpha ́ \gamma \omega \nu о \nu ~ \alpha ̉ р \imath \vartheta \mu o ́ v, ~$
















 $\pi \rho o ̀ s ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau \tilde{n} \varsigma ~ \Theta . ~ \dot{\alpha} \lambda \lambda \lambda^{\prime}$ ó $\mathrm{AB} \pi \rho o ̀ s ~ \tau o ̀ v ~ В \Gamma ~ \lambda o ́ \gamma o v ~ ह ै \chi \varepsilon ı, ~ o ̋ v ~$







 ठعї̆ $\alpha$.


Let the two numbers $A C$ and $C B$ be laid down such that their sum $A B$ has to $B C$ the ratio which (some) square number (has) to (some) square number, and does not have to $A C$ the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) $D$ be laid down. And let $E F$ be commensurable in length with $D$. $E F$ is thus a rational (straight-line). So, let it also have been contrived that as the number $C A$ (is) to $A B$, so the (square) on $E F$ (is) to the (square) on $F G$ [Prop. 10.6 corr.]. Thus, the (square) on $E F$ is commensurable with the (square) on $F G$ [Prop. 10.6]. Thus, $F G$ is also a rational (straightline). And since the number $C A$ does not have to $A B$ the ratio which (some) square number (has) to (some) square number, the (square) on $E F$ does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, $E F$ is incommensurable in length with $F G$ [Prop. 10.9]. $E F$ and $F G$ are thus rational (straight-lines which are) commensurable in square only. Thus, $E G$ is a binomial (straightline) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

For since, inversely, as the number $B A$ is to $A C$, so the (square) on $G F$ (is) to the (square) on $F E$ [Prop. 5.7 corr.], and $B A$ (is) greater than $A C$, the (square) on $G F$ (is) thus [also] greater than the (square) on $F E$ [Prop. 5.14]. Let (the sum of) the (squares) on $E F$ and $H$ be equal to the (square) on $G F$. Thus, via conversion, as $A B$ is to $B C$, so the (square) on $F G$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. But, $A B$ has to $B C$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $F G$ also has to the (square) on $H$ the ratio which (some) square number (has) to (some) square number. Thus, $F G$ is commensurable in length with $H$ [Prop. 10.9]. Hence, the square on $F G$ is greater than (the square on) $F E$ by the (square) on (some straight-line) commensurable in length with $(F G)$. And $F G$ and $F E$ are rational (straight-lines which are) commensurable in square only. And the lesser term $E F$ is commensurable in length with the rational (straightline) $D$ (previously) laid down.

Thus, $E G$ is a second binomial (straight-line) [Def. 10.6]. ${ }^{\dagger}$ (Which is) the very thing it was required to show.

[^25]whose length is $k / \sqrt{1-k^{\prime 2}}-k$ [Prop. 10.86], are the roots of $x^{2}-\left(2 k / \sqrt{1-k^{\prime 2}}\right) x+k^{2}\left[k^{\prime 2} /\left(1-k^{\prime 2}\right)\right]=0$.

## $v^{\prime}$.























 $\pi \rho o ̀ s ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu о \nu ~ \alpha ́ p \imath \vartheta \mu o ́ v ~ \alpha ́ \sigma u ́ \mu \mu \varepsilon \tau \rho о \varsigma ~ \alpha ̈ p \alpha ~ \varepsilon ̇ \sigma \tau i \nu ~ \dot{\eta} \mathrm{ZH}$

 x $\alpha$ ì трít $\eta$.










 $\dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{H} \Theta, \mathrm{K} \cdot \dot{\alpha} \nu \alpha \sigma \tau \rho \varepsilon ́ \psi \alpha \nu \tau \iota \not ้ p \alpha[\hat{\varepsilon} \sigma \tau i \nu]$ $\dot{\omega} \varsigma$ ó $\mathrm{AB} \pi \rho o ̀ s$



Proposition 50
To find a third binomial (straight-line).


Let the two numbers $A C$ and $C B$ be laid down such that their sum $A B$ has to $B C$ the ratio which (some) square number (has) to (some) square number, and does not have to $A C$ the ratio which (some) square number (has) to (some) square number. And let some other nonsquare number $D$ also be laid down, and let it not have to each of $B A$ and $A C$ the ratio which (some) square number (has) to (some) square number. And let some rational straight-line $E$ be laid down, and let it have been contrived that as $D$ (is) to $A B$, so the (square) on $E$ (is) to the (square) on $F G$ [Prop. 10.6 corr.]. Thus, the (square) on $E$ is commensurable with the (square) on $F G$ [Prop. 10.6]. And $E$ is a rational (straight-line). Thus, $F G$ is also a rational (straight-line). And since $D$ does not have to $A B$ the ratio which (some) square number has to (some) square number, the (square) on $E$ does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. $E$ is thus incommensurable in length with $F G$ [Prop. 10.9]. So, again, let it have been contrived that as the number $B A$ (is) to $A C$, so the (square) on $F G$ (is) to the (square) on $G H$ [Prop. 10.6 corr.]. Thus, the (square) on $F G$ is commensurable with the (square) on $G H$ [Prop. 10.6]. And $F G$ (is) a rational (straight-line). Thus, $G H$ (is) also a rational (straight-line). And since $B A$ does not have to $A C$ the ratio which (some) square number (has) to (some) square number, the (square) on $F G$ does not have to the (square) on $H G$ the ratio which (some) square number (has) to (some) square number either. Thus, $F G$ is incommensurable in length with $G H$ [Prop. 10.9]. $F G$ and $G H$ are thus rational (straightlines which are) commensurable in square only. Thus, $F H$ is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a third (binomial straight-line).

For since as $D$ is to $A B$, so the (square) on $E$ (is) to the (square) on $F G$, and as $B A$ (is) to $A C$, so the (square) on $F G$ (is) to the (square) on $G H$, thus, via equality, as $D$ (is) to $A C$, so the (square) on $E$ (is) to the (square) on $G H$ [Prop. 5.22]. And $D$ does not







have to $A C$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $E$ does not have to the (square) on $G H$ the ratio which (some) square number (has) to (some) square number either. Thus, $E$ is incommensurable in length with $G H$ [Prop. 10.9]. And since as $B A$ is to $A C$, so the (square) on $F G$ (is) to the (square) on $G H$, the (square) on $F G$ (is) thus greater than the (square) on $G H$ [Prop. 5.14]. Therefore, let (the sum of) the (squares) on $G H$ and $K$ be equal to the (square) on $F G$. Thus, via conversion, as $A B$ [is] to $B C$, so the (square) on $F G$ (is) to the (square) on $K$ [Prop. 5.19 corr.]. And $A B$ has to $B C$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $F G$ also has to the (square) on $K$ the ratio which (some) square number (has) to (some) square number. Thus, $F G$ [is] commensurable in length with $K$ [Prop. 10.9]. Thus, the square on $F G$ is greater than (the square on) $G H$ by the (square) on (some straight-line) commensurable (in length) with ( $F G$ ). And $F G$ and $G H$ are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with $E$.

Thus, $F H$ is a third binomial (straight-line) [Def. 10.7]. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the rational straight-line has unit length then the length of a third binomial straight-line is $k^{1 / 2}\left(1+\sqrt{1-k^{\prime 2}}\right)$. This, and the third apotome, whose length is $k^{1 / 2}\left(1-\sqrt{1-k^{\prime 2}}\right)$ [Prop. 10.87], are the roots of $x^{2}-2 k^{1 / 2} x+k k^{\prime 2}=0$.



 ย̇xxยíб७










## Proposition 51

To find a fourth binomial (straight-line).


Let the two numbers $A C$ and $C B$ be laid down such that $A B$ does not have to $B C$, or to $A C$ either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) $D$ be laid down. And let $E F$ be commensurable in length with $D$. Thus, $E F$ is also a rational (straight-line). And let it have been contrived that as the number $B A$ (is) to $A C$, so the (square) on $E F$ (is) to the (square) on $F G$ [Prop. 10.6 corr.]. Thus, the (square) on $E F$ is commensurable with the (square) on $F G$ [Prop. 10.6]. Thus, $F G$ is also a rational (straightline). And since $B A$ does not have to $A C$ the ratio which (some) square number (has) to (some) square number,

O้тl $x \alpha i ̀ \tau \varepsilon \tau \alpha ́ p \tau \eta$.












 ठعǐ̌ $\alpha$.
the (square) on $E F$ does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, $E F$ is incommensurable in length with $F G$ [Prop. 10.9]. Thus, $E F$ and $F G$ are rational (straight-lines which are) commensurable in square only. Hence, $E G$ is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fourth (binomial straight-line).

For since as $B A$ is to $A C$, so the (square) on $E F$ (is) to the (square) on $F G$ [and $B A$ (is) greater than $A C$ ], the (square) on $E F$ (is) thus greater than the (square) on $F G$ [Prop. 5.14]. Therefore, let (the sum of) the squares on $F G$ and $H$ be equal to the (square) on $E F$. Thus, via conversion, as the number $A B$ (is) to $B C$, so the (square) on $E F$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. And $A B$ does not have to $B C$ the ratio which (some) square numbber (has) to (some) square number. Thus, the (square) on $E F$ does not have to the (square) on $H$ the ratio which (some) square number (has) to (some) square number either. Thus, $E F$ is incommensurable in length with $H$ [Prop. 10.9]. Thus, the square on $E F$ is greater than (the square on) $G F$ by the (square) on (some straight-line) incommensurable (in length) with ( $E F$ ). And $E F$ and $F G$ are rational (straight-lines which are) commensurable in square only. And $E F$ is commensurable in length with $D$.

Thus, $E G$ is a fourth binomial (straight-line) [Def. 10.8]. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the rational straight-line has unit length then the length of a fourth binomial straight-line is $k\left(1+1 / \sqrt{1+k^{\prime}}\right)$. This, and the fourth apotome, whose length is $k\left(1-1 / \sqrt{1+k^{\prime}}\right)$ [Prop. 10.88], are the roots of $x^{2}-2 k x+k^{2} k^{\prime} /\left(1+k^{\prime}\right)=0$.

$$
\nu \beta^{\prime}
$$













## Proposition 52

To find a fifth binomial straight-line.


Let the two numbers $A C$ and $C B$ be laid down such that $A B$ does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.]. And let some rational straight-line $D$ be laid down. And let $E F$ be commensurable [in length] with $D$. Thus, $E F$ (is) a rational (straightline). And let it have been contrived that as $C A$ (is) to $A B$, so the (square) on $E F$ (is) to the (square) on $F G$ [Prop. 10.6 corr.]. And $C A$ does not have to $A B$ the ra-













 بóvov $\sigma u ́ \mu \mu \varepsilon \tau \rho o l, x \alpha i ~ \tau o ̀ ~ E Z ~ ह ै \lambda \alpha \tau \tau о \nu ~ o ̋ v o \mu \alpha ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~$

 ठعǐ̌ $\alpha$.
tio which (some) square number (has) to (some) square number. Thus, the (square) on $E F$ does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, EF and $F G$ are rational (straight-lines which are) commensurable in square only [Prop. 10.9]. Thus, $E G$ is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For since as $C A$ is to $A B$, so the (square) on $E F$ (is) to the (square) on $F G$, inversely, as $B A$ (is) to $A C$, so the (square) on $F G$ (is) to the (square) on $F E$ [Prop. 5.7 corr.]. Thus, the (square) on $G F$ (is) greater than the (square) on $F E$ [Prop. 5.14]. Therefore, let (the sum of) the (squares) on $E F$ and $H$ be equal to the (square) on $G F$. Thus, via conversion, as the number $A B$ is to $B C$, so the (square) on $G F$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. And $A B$ does not have to $B C$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $F G$ does not have to the (square) on $H$ the ratio which (some) square number (has) to (some) square number either. Thus, $F G$ is incommensurable in length with $H$ [Prop. 10.9]. Hence, the square on $F G$ is greater than (the square on) $F E$ by the (square) on (some straight-line) incommensurable (in length) with ( $F G$ ). And $G F$ and $F E$ are rational (straight-lines which are) commensurable in square only. And the lesser term $E F$ is commensurable in length with the rational (straight-line previously) laid down, $D$.

Thus, $E G$ is a fifth binomial (straight-line). ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the rational straight-line has unit length then the length of a fifth binomial straight-line is $k\left(\sqrt{1+k^{\prime}}+1\right)$. This, and the fifth apotome, whose length is $k\left(\sqrt{1+k^{\prime}}-1\right)$ [Prop. 10.89], are the roots of $x^{2}-2 k \sqrt{1+k^{\prime}} x+k^{2} k^{\prime}=0$.










## Proposition 53

To find a sixth binomial (straight-line).


Let the two numbers $A C$ and $C B$ be laid down such that $A B$ does not have to each of them the ratio which (some) square number (has) to (some) square number. And let $D$ also be another number, which is not square, and does not have to each of $B A$ and $A C$ the ratio which (some) square number (has) to (some) square number either [Prop. 10.28 lem. I]. And let some rational straightline $E$ be laid down. And let it have been contrived that





 tñs ZH тpòs tò $\alpha \pi o ̀ ~ \tau n ̃ s ~ H \Theta . ~ \sigma u ́ \mu \mu \varepsilon \tau p o v ~ \alpha ̈ p \alpha ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau n ̃ ̧ ~ Z H ~$






 оัтા xal ยैxтท.





 oủठè tò $\alpha \pi o ̀ ~ \tau \eta ̃ s ~ E ~ \alpha ̈ p \alpha ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau n ̃ s ~ H \Theta ~ \lambda o ́ \gamma o v ~$










 ג́pı७
 $\alpha i \mathrm{ZH}, \mathrm{H} \Theta$ ṕntaì $\delta u v \alpha ́ \mu \varepsilon \iota ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l, ~ \chi \alpha i ̀ ~ o u ̉ \delta \varepsilon \tau \varepsilon ́ p \alpha ~$


as $D$ (is) to $A B$, so the (square) on $E$ (is) to the (square) on $F G$ [Prop. 10.6 corr.]. Thus, the (square) on $E$ (is) commensurable with the (square) on $F G$ [Prop. 10.6]. And $E$ is rational. Thus, $F G$ (is) also rational. And since $D$ does not have to $A B$ the ratio which (some) square number (has) to (some) square number, the (square) on $E$ thus does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, $E$ (is) incommensurable in length with $F G$ [Prop. 10.9]. So, again, let it have be contrived that as $B A$ (is) to $A C$, so the (square) on $F G$ (is) to the (square) on $G H$ [Prop. 10.6 corr.]. The (square) on $F G$ (is) thus commensurable with the (square) on $H G$ [Prop. 10.6]. The (square) on $H G$ (is) thus rational. Thus, $H G$ (is) rational. And since $B A$ does not have to $A C$ the ratio which (some) square number (has) to (some) square number, the (square) on $F G$ does not have to the (square) on $G H$ the ratio which (some) square number (has) to (some) square number either. Thus, $F G$ is incommensurable in length with $G H$ [Prop. 10.9]. Thus, $F G$ and $G H$ are rational (straight-lines which are) commensurable in square only. Thus, $F H$ is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a sixth (binomial straight-line).

For since as $D$ is to $A B$, so the (square) on $E$ (is) to the (square) on $F G$, and also as $B A$ is to $A C$, so the (square) on $F G$ (is) to the (square) on $G H$, thus, via equality, as $D$ is to $A C$, so the (square) on $E$ (is) to the (square) on $G H$ [Prop. 5.22]. And $D$ does not have to $A C$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $E$ does not have to the (square) on $G H$ the ratio which (some) square number (has) to (some) square number either. $E$ is thus incommensurable in length with $G H$ [Prop. 10.9]. And ( $E$ ) was also shown (to be) incommensurable (in length) with $F G$. Thus, $F G$ and $G H$ are each incommensurable in length with $E$. And since as $B A$ is to $A C$, so the (square) on $F G$ (is) to the (square) on $G H$, the (square) on $F G$ (is) thus greater than the (square) on $G H$ [Prop. 5.14]. Therefore, let (the sum of) the (squares) on $G H$ and $K$ be equal to the (square) on $F G$. Thus, via conversion, as $A B$ (is) to $B C$, so the (square) on $F G$ (is) to the (square) on $K$ [Prop. 5.19 corr.]. And $A B$ does not have to $B C$ the ratio which (some) square number (has) to (some) square number. Hence, the (square) on $F G$ does not have to the (square) on $K$ the ratio which (some) square number (has) to (some) square number either. Thus, $F G$ is incommensurable in length with $K$ [Prop. 10.9]. The square on $F G$ is thus greater than (the square on) $G H$ by the (square) on (some straight-line which is) incom-
mensurable (in length) with ( $F G$ ). And $F G$ and $G H$ are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with the rational (straight-line) $E$ (previously) laid down.

Thus, $F H$ is a sixth binomial (straight-line) [Def. 10.10]. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the rational straight-line has unit length then the length of a sixth binomial straight-line is $\sqrt{k}+\sqrt{k^{\prime}}$. This, and the sixth apotome, whose length is $\sqrt{k}-\sqrt{k^{\prime}}$ [Prop. 10.90], are the roots of $x^{2}-2 \sqrt{k} x+\left(k-k^{\prime}\right)=0$.

## $\Lambda \tilde{\mu} \mu \mu \alpha$.


 $\mathrm{ZB} \tau \tilde{n} \mathrm{BH}$. x $\alpha \grave{\imath} \sigma \cup \mu \pi \varepsilon \pi \lambda n \rho \omega ́ \sigma \vartheta \omega$ tò $\mathrm{A} \Gamma \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o v$. $\lambda \varepsilon ́ \gamma \omega$, ơтı тєтра́ү $\omega$ vóv ह̇ $\sigma \tau \iota ~ \tau o ̀ ~ A \Gamma, ~ x \alpha \grave{~ o ̛ \tau \iota ~ \tau \widetilde{\omega} \nu ~} \mathrm{AB}, \mathrm{B} \Gamma$ $\mu \varepsilon ́ \sigma o v ~ \dot{\alpha} v \alpha ́ \lambda o \gamma o ́ v ~ \varepsilon ̇ \sigma \tau l ~ \tau o ̀ ~ \Delta H, ~ x \alpha l ~ ह ै \tau \iota ~ \tau \widetilde{\omega} \nu ~ А Г, ~ Г В ~ \mu \varepsilon ́ \sigma o v ~$













 $\Delta \Gamma$.








## Lemma

Let $A B$ and $B C$ be two squares, and let them be laid down such that $D B$ is straight-on to $B E . F B$ is, thus, also straight-on to $B G$. And let the parallelogram $A C$ have been completed. I say that $A C$ is a square, and that $D G$ is the mean proportional to $A B$ and $B C$, and, moreover, $D C$ is the mean proportional to $A C$ and $C B$.


For since $D B$ is equal to $B F$, and $B E$ to $B G$, the whole of $D E$ is thus equal to the whole of $F G$. But $D E$ is equal to each of $A H$ and $K C$, and $F G$ is equal to each of $A K$ and $H C$ [Prop. 1.34]. Thus, $A H$ and $K C$ are also equal to $A K$ and $H C$, respectively. Thus, the parallelogram $A C$ is equilateral. And (it is) also right-angled. Thus, $A C$ is a square.

And since as $F B$ is to $B G$, so $D B$ (is) to $B E$, but as $F B$ (is) to $B G$, so $A B$ (is) to $D G$, and as $D B$ (is) to $B E$, so $D G$ (is) to $B C$ [Prop. 6.1], thus also as $A B$ (is) to $D G$, so $D G$ (is) to $B C$ [Prop. 5.11]. Thus, $D G$ is the mean proportional to $A B$ and $B C$.

So I also say that $D C$ [is] the mean proportional to $A C$ and $C B$.

For since as $A D$ is to $D K$, so $K G$ (is) to $G C$. For [they are] respectively equal. And, via composition, as $A K$ (is) to $K D$, so $K C$ (is) to $C G$ [Prop. 5.18]. But as $A K$ (is) to $K D$, so $A C$ (is) to $C D$, and as $K C$ (is) to $C G$, so $D C$ (is) to $C B$ [Prop. 6.1]. Thus, also, as $A C$ (is) to $D C$, so $D C$ (is) to $B C$ [Prop. 5.11]. Thus, $D C$ is the mean proportional to $A C$ and $C B$. Which (is the very thing) it

## $v \delta^{\prime}$.


 $\dot{\eta} x \alpha \lambda о \cup \mu \varepsilon ́ v \eta$ モ̇x ठ́́o o̊vo $\mu \dot{\alpha} \tau \omega \nu$.



 ỏvoú́t $\omega \nu$.




 $\dot{\rho} \eta \tau \tilde{n} \tau \tilde{n} \mathrm{AB} \mu \dot{\eta} x \varepsilon \iota$. $\tau \varepsilon \tau \mu \dot{\eta} \sigma \vartheta \omega$ ò̀ $\dot{\eta} \mathrm{E} \Delta$ ठí $\alpha \alpha \alpha \tau \dot{\alpha}$ tò Z






 $\Gamma \Delta \pi \alpha p \alpha ́ \lambda \lambda \eta \lambda o l ~ \alpha i ~ H \Theta, ~ E K, ~ Z \Lambda \cdot ~ \chi \alpha i ~ \tau \widetilde{\omega} ~ \mu \varepsilon ̀ v ~ A \Theta ~ \pi \alpha p \alpha \lambda \lambda \eta-~$


 $\pi \varepsilon \pi \lambda \eta \rho \omega ́ \sigma \vartheta \omega$ тò $\Sigma \Pi \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu$. тєтра́ү $\omega \nu$ оv $\alpha \rho \alpha$

 EH• xaì ćs $\alpha$ 人p $\alpha$ tò $\mathrm{A} \Theta$ тpòs $\mathrm{E} \Lambda$, tò $\mathrm{E} \Lambda$ тpòs $\mathrm{KH} \cdot \tau \widetilde{\omega} \nu$

 $\mu \varepsilon ́ \sigma o \nu ~ \alpha ̉ \nu \alpha ́ \lambda o \gamma o ́ v ~ \varepsilon ̇ \sigma \tau l ~ \tau o ̀ ~ E \Lambda . ~ ह ै \sigma \tau l ~ \delta غ ̀ ~ \tau \widetilde{\omega} \nu ~ \alpha u ̉ \tau \widetilde{\omega} \nu \tau \widetilde{\omega} \nu \Sigma N$,





’Eлعì $\gamma \dot{\alpha} \rho \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ s ~ \varepsilon ̇ \sigma \tau ו \nu ~ \dot{\eta} A H ~ \tau \tilde{n} H E, ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ s ~ \varepsilon ̇ \sigma \tau \tau ~$

was prescribed to show.

## Proposition 54

If an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called binomial. ${ }^{\dagger}$


For let the area $A C$ be contained by the rational (straight-line) $A B$ and by the first binomial (straightline) $A D$. I say that square-root of area $A C$ is the irrational (straight-line which is) called binomial.

For since $A D$ is a first binomial (straight-line), let it have been divided into its (component) terms at $E$, and let $A E$ be the greater term. So, (it is) clear that $A E$ and $E D$ are rational (straight-lines which are) commensurable in square only, and that the square on $A E$ is greater than (the square on) $E D$ by the (square) on (some straight-line) commensurable (in length) with ( $A E$ ), and that $A E$ is commensurable (in length) with the rational (straight-line) $A B$ (first) laid out [Def. 10.5]. So, let $E D$ have been cut in half at point $F$. And since the square on $A E$ is greater than (the square on) $E D$ by the (square) on (some straight-line) commensurable (in length) with ( $A E$ ), thus if a (rectangle) equal to the fourth part of the (square) on the lesser (term) -that is to say, the (square) on $E F$-falling short by a square figure, is applied to the greater (term) $A E$, then it divides it into (terms which are) commensurable (in length) [Prop 10.17]. Therefore, let the (rectangle contained) by $A G$ and $G E$, equal to the (square) on $E F$, have been applied to $A E . A G$ is thus commensurable in length with $E G$. And let $G H, E K$, and $F L$ have been drawn from (points) $G, E$, and $F$ (respectively), parallel to either of $A B$ or $C D$. And let the square $S N$, equal to the parallelogram $A H$, have been constructed, and (the square) $N Q$, equal to (the parallelogram) $G K$ [Prop. 2.14]. And let $M N$ be laid down so as to be straight-on to $N O . R N$ is thus also straight-on to $N P$. And let the parallelogram $S Q$ have been completed. $S Q$ is thus a square [Prop. 10.53 lem.]. And since the (rectangle contained) by $A G$ and $G E$ is equal to the (square) on $E F$, thus as $A G$ is to $E F$, so $F E$ (is) to $E G$ [Prop. 6.17]. And thus as $A H$ (is) to $E L$, (so) $E L$ (is)





 $\mathrm{AE} \tau \tilde{n} \mathrm{E} \Delta \mu \dot{\eta} x \varepsilon \iota, \alpha \dot{\alpha} \lambda^{\circ} \dot{\eta} \mu \varepsilon ̀ \nu \mathrm{AE} \tau \tilde{n} \mathrm{AH}$ ह́ $\sigma \tau \iota \sigma \cup ́ \mu \mu \varepsilon \tau \rho \circ \varsigma$,


 $\tau \widetilde{\varphi} \mathrm{MP} \dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau \tau \nu . ~ \dot{\alpha} \lambda \lambda{ }^{\circ} \dot{\omega} \varsigma ~ \tau o ̀ ~ \Sigma N ~ \pi \rho o ̀ s ~ M P, ~ \dot{\eta} \mathrm{ON}$



 цóvov oú $\mu \mu \varepsilon \tau \rho o l$.
${ }^{`} \mathrm{H} \mathrm{M} \Xi$ äp $\alpha$ ह̉x $\delta u ́ o ~ o ̉ v o \mu \alpha ́ \tau \omega \nu ~ \varepsilon ̇ \sigma \tau i ̀ ~ x \alpha i ̀ ~ \delta u ́ v \alpha \tau \alpha l ~ t o ̀ ~ A \Gamma . ~$

to $K G$ [Prop. 6.1]. Thus, $E L$ is the mean proportional to $A H$ and $G K$. But, $A H$ is equal to $S N$, and $G K$ (is) equal to $N Q . E L$ is thus the mean proportional to $S N$ and $N Q$. And $M R$ is also the mean proportional to the same(namely), $S N$ and $N Q$ [Prop. 10.53 lem.]. $E L$ is thus equal to $M R$. Hence, it is also equal to $P O$ [Prop. 1.43]. And $A H$ plus $G K$ is equal to $S N$ plus $N Q$. Thus, the whole of $A C$ is equal to the whole of $S Q$-that is to say, to the square on $M O$. Thus, $M O$ (is) the square-root of (area) $A C$. I say that $M O$ is a binomial (straight-line).

For since $A G$ is commensurable (in length) with $G E$, $A E$ is also commensurable (in length) with each of $A G$ and $G E$ [Prop. 10.15]. And $A E$ was also assumed (to be) commensurable (in length) with $A B$. Thus, $A G$ and $G E$ are also commensurable (in length) with $A B$ [Prop. 10.12]. And $A B$ is rational. $A G$ and $G E$ are thus each also rational. Thus, $A H$ and $G K$ are each rational (areas), and $A H$ is commensurable with $G K$ [Prop. 10.19]. But, $A H$ is equal to $S N$, and $G K$ to $N Q$. $S N$ and $N Q$-that is to say, the (squares) on $M N$ and $N O$ (respectively) -are thus also rational and commensurable. And since $A E$ is incommensurable in length with $E D$, but $A E$ is commensurable (in length) with $A G$, and $D E$ (is) commensurable (in length) with $E F$, $A G$ (is) thus also incommensurable (in length) with $E F$ [Prop. 10.13]. Hence, $A H$ is also incommensurable with $E L$ [Props. 6.1, 10.11]. But, $A H$ is equal to $S N$, and $E L$ to $M R$. Thus, $S N$ is also incommensurable with $M R$. But, as $S N$ (is) to $M R$, (so) $P N$ (is) to $N R$ [Prop. 6.1]. $P N$ is thus incommensurable (in length) with $N R$ [Prop. 10.11]. And $P N$ (is) equal to $M N$, and $N R$ to $N O$. Thus, $M N$ is incommensurable (in length) with $N O$. And the (square) on $M N$ is commensurable with the (square) on $N O$, and each (is) rational. $M N$ and $N O$ are thus rational (straight-lines which are) commensurable in square only.

Thus, $M O$ is (both) a binomial (straight-line) [Prop. 10.36], and the square-root of $A C$. (Which is) the very thing it was required to show.

[^26]$$
v \varepsilon^{\prime}
$$

 $\dot{\eta} x \alpha \lambda o \cup \mu \varepsilon ́ v \eta$ モ̇x ठט́o $\mu \varepsilon ́ \sigma \omega \nu \pi \rho \omega \dot{\epsilon} \tau \eta$.

## Proposition 55

If an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called first bimedial. ${ }^{\dagger}$







 $\sigma \cup \mu \mu \varepsilon ́ \tau \rho о \cup ~ \varepsilon ́ \alpha \cup \tau \tilde{n}, ~ x \alpha i ̀ ~ \tau o ̀ ~ ह ै \lambda \alpha \tau \tau о \nu ~ o ̋ v o \mu \alpha ~ \dot{\eta} \mathrm{E} \Delta$ бú $\mu \mu \varepsilon \tau \rho o ́ v$


 $\tau \tilde{n}$ HE $\mu \dot{n} x \varepsilon \iota . ~ \chi \alpha \grave{~} \delta \iota \dot{\alpha} \tau \widetilde{\omega} \nu \mathrm{H}, \mathrm{E}, \mathrm{Z} \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o \iota ~ \ddot{\eta} \chi \vartheta \omega \sigma \alpha \nu$ $\tau \alpha i ̈ \varsigma ~ A B, \Gamma \Delta$ 人i $\mathrm{H} \Theta, \mathrm{EK}, \mathrm{Z} \Lambda$, $x \alpha \grave{\imath} \tau \widetilde{\varphi} \mu \varepsilon ̀ \nu \mathrm{~A} \Theta \pi \alpha p \alpha \lambda \lambda \eta-$









 $\dot{\varepsilon} x \alpha \tau \varepsilon ́ \rho \alpha \tau \widetilde{\omega} \nu \mathrm{AH}$, HE. $\dot{\alpha} \lambda \lambda \dot{\alpha} \dot{\eta} \mathrm{AE} \dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho o s \tau \tilde{n} \mathrm{AB} \mu \dot{\eta} x \varepsilon เ \cdot$


 NП $\mu \varepsilon ́ \sigma o v ~ \varepsilon ̇ \sigma \tau i ́ v . ~ x \alpha i ̀ ~ \alpha i ~ M N, ~ N \Xi ~ \alpha ̈ p \alpha ~ \mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i ̄ \sigma i ́ v . ~ x \alpha i ̀ ~$



 $\alpha \lambda^{\lambda} \dot{\eta} \mu \varepsilon ̀ \nu \mathrm{AE} \sigma u ̛ \mu \mu \varepsilon \tau \rho o ́ s ~ \varepsilon ̇ \sigma \tau \iota ~ \tau \tilde{n} \mathrm{AH}, \dot{\eta}$ ठ̇̀ $\mathrm{E} \Delta \tau \tilde{n} \mathrm{EZ}$












For let the area $A B C D$ be contained by the rational (straight-line) $A B$ and by the second binomial (straightline) $A D$. I say that the square-root of area $A C$ is a first bimedial (straight-line).

For since $A D$ is a second binomial (straight-line), let it have been divided into its (component) terms at $E$, such that $A E$ is the greater term. Thus, $A E$ and $E D$ are rational (straight-lines which are) commensurable in square only, and the square on $A E$ is greater than (the square on) $E D$ by the (square) on (some straight-line) commensurable (in length) with $(A E)$, and the lesser term $E D$ is commensurable in length with $A B$ [Def. 10.6]. Let $E D$ have been cut in half at $F$. And let the (rectangle contained) by $A G E$, equal to the (square) on $E F$, have been applied to $A E$, falling short by a square figure. $A G$ (is) thus commensurable in length with $G E$ [Prop. 10.17]. And let $G H, E K$, and $F L$ have been drawn through (points) $G, E$, and $F$ (respectively), parallel to $A B$ and $C D$. And let the square $S N$, equal to the parallelogram $A H$, have been constructed, and the square $N Q$, equal to $G K$. And let $M N$ be laid down so as to be straight-on to $N O$. Thus, $R N$ [is] also straight-on to $N P$. And let the square $S Q$ have been completed. So, (it is) clear from what has been previously demonstrated [Prop. 10.53 lem.] that $M R$ is the mean proportional to $S N$ and $N Q$, and (is) equal to $E L$, and that $M O$ is the square-root of the area $A C$. So, we must show that $M O$ is a first bimedial (straight-line).

Since $A E$ is incommensurable in length with $E D$, and $E D$ (is) commensurable (in length) with $A B$, $A E$ (is) thus incommensurable (in length) with $A B$ [Prop. 10.13]. And since $A G$ is commensurable (in length) with $E G, A E$ is also commensurable (in length) with each of $A G$ and $G E$ [Prop. 10.15]. But, $A E$ is incommensurable in length with $A B$. Thus, $A G$ and $G E$ are also (both) incommensurable (in length) with $A B$ [Prop. 10.13]. Thus, $B A, A G$, and ( $B A$, and) $G E$ are (pairs of) rational (straight-lines which are) commensurable in square only. And, hence, each of $A H$ and $G K$ is a medial (area) [Prop. 10.21]. Hence, each of $S N$ and $N Q$ is also a medial (area). Thus, $M N$ and $N O$ are medial (straight-lines). And since $A G$ (is) commensurable in length with $G E, A H$ is also commensurable
$\pi \varepsilon \rho เ \varepsilon ́ \chi$ OU $\pi \rho \omega ́$ тท.

with $G K$-that is to say, $S N$ with $N Q$-that is to say, the (square) on $M N$ with the (square) on $N O$ [hence, $M N$ and $N O$ are commensurable in square] [Props. 6.1, 10.11]. And since $A E$ is incommensurable in length with $E D$, but $A E$ is commensurable (in length) with $A G$, and $E D$ commensurable (in length) with $E F, A G$ (is) thus incommensurable (in length) with $E F$ [Prop. 10.13]. Hence, $A H$ is also incommensurable with $E L$-that is to say, $S N$ with $M R$-that is to say, $P N$ with $N R$-that is to say, $M N$ is incommensurable in length with $N O$ [Props. 6.1, 10.11]. But $M N$ and $N O$ have also been shown to be medial (straight-lines) which are commensurable in square. Thus, $M N$ and $N O$ are medial (straightlines which are) commensurable in square only. So, I say that they also contain a rational (area). For since $D E$ was assumed (to be) commensurable (in length) with each of $A B$ and $E F, E F$ (is) thus also commensurable with $E K$ [Prop. 10.12]. And they (are) each rational. Thus, $E L-$ that is to say, $M R$-(is) rational [Prop. 10.19]. And $M R$ is the (rectangle contained) by $M N O$. And if two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together, then the whole is (that) irrational (straight-line which is) called first bimedial [Prop. 10.37].

Thus, $M O$ is a first bimedial (straight-line). (Which is) the very thing it was required to show.
$\dagger$ If the rational straight-line has unit length then this proposition states that the square-root of a second binomial straight-line is a first bimedial straight-line: i.e., a second binomial straight-line has a length $k / \sqrt{1-k^{\prime 2}}+k$ whose square-root can be written $\rho\left(k^{\prime \prime 1 / 4}+k^{\prime \prime 3 / 4}\right)$, where $\rho=\sqrt{(k / 2)\left(1+k^{\prime}\right) /\left(1-k^{\prime}\right)}$ and $k^{\prime \prime}=\left(1-k^{\prime}\right) /\left(1+k^{\prime}\right)$. This is the length of a first bimedial straight-line (see Prop. 10.37), since $\rho$ is rational.
$\nu q^{\prime}$.







 $\mu \varepsilon ́ \sigma \omega \nu$ ठெvtépa.


## Proposition 56

If an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called second bimedial. ${ }^{\dagger}$


For let the area $A B C D$ be contained by the rational (straight-line) $A B$ and by the third binomial (straightline) $A D$, which has been divided into its (component) terms at $E$, of which $A E$ is the greater. I say that the square-root of area $A C$ is the irrational (straight-line which is) called second bimedial.






 ठєutépa.








For let the same construction be made as previously. And since $A D$ is a third binomial (straight-line), $A E$ and $E D$ are thus rational (straight-lines which are) commensurable in square only, and the square on $A E$ is greater than (the square on) $E D$ by the (square) on (some straight-line) commensurable (in length) with ( $A E$ ), and neither of $A E$ and $E D$ [is] commensurable in length with $A B$ [Def. 10.7]. So, similarly to that which has been previously demonstrated, we can show that $M O$ is the square-root of area $A C$, and $M N$ and $N O$ are medial (straight-lines which are) commensurable in square only. Hence, $M O$ is bimedial. So, we must show that (it is) also second (bimedial).
[And] since $D E$ is incommensurable in length with $A B$-that is to say, with $E K$-and $D E$ (is) commensurable (in length) with $E F, E F$ is thus incommensurable in length with $E K$ [Prop. 10.13]. And they are (both) rational (straight-lines). Thus, $F E$ and $E K$ are rational (straight-lines which are) commensurable in square only. $E L$-that is to say, $M R$-[is] thus medial [Prop. 10.21]. And it is contained by $M N O$. Thus, the (rectangle contained) by $M N O$ is medial.

Thus, $M O$ is a second bimedial (straight-line) [Prop. 10.38]. (Which is) the very thing it was required to show.
$\dagger$ If the rational straight-line has unit length then this proposition states that the square-root of a third binomial straight-line is a second bimedial straight-line: i.e., a third binomial straight-line has a length $k^{1 / 2}\left(1+\sqrt{1-k^{\prime 2}}\right)$ whose square-root can be written $\rho\left(k^{1 / 4}+k^{\prime \prime 1 / 2} / k^{1 / 4}\right)$, where $\rho=\sqrt{\left(1+k^{\prime}\right) / 2}$ and $k^{\prime \prime}=k\left(1-k^{\prime}\right) /\left(1+k^{\prime}\right)$. This is the length of a second bimedial straight-line (see Prop. 10.38), since $\rho$ is rational.

$$
\nu \zeta^{\prime} .
$$


 $\dot{\eta} \chi \alpha \lambda о \cup \mu \varepsilon ́ v \eta \mu \varepsilon i \zeta \omega \nu$.










## Proposition 57

If an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called major. ${ }^{\dagger}$


For let the area $A C$ be contained by the rational (straight-line) $A B$ and the fourth binomial (straight-line) $A D$, which has been divided into its (component) terms at $E$, of which let $A E$ be the greater. I say that the squareroot of $A C$ is the irrational (straight-line which is) called major.

For since $A D$ is a fourth binomial (straight-line), $A E$ and $E D$ are thus rational (straight-lines which are) com-

 $\dot{\eta} \mathrm{AH} \tau \tilde{n} \mathrm{HE} \mu \dot{\eta} x \varepsilon \iota$. $\ddot{\eta} \chi \vartheta \omega \sigma \alpha \nu \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o \iota ~ \tau \tilde{n} \mathrm{AB} \alpha i \mathrm{H} \Theta$,









 $\tau \tilde{n} \mathrm{EZ}, \alpha \dot{\alpha} \cup^{\mu} \mu \mu \varepsilon \tau \rho o s ~ \alpha ้ p \alpha \dot{\eta}$ EZ $\tau \tilde{n}$ EK $\mu \dot{\eta} x \varepsilon \iota$. $\alpha i \mathrm{EK}$, EZ






 б̀̀ $\mu \varepsilon$ í弓 $\omega \nu$.


mensurable in square only, and the square on $A E$ is greater than (the square on) $E D$ by the (square) on (some straight-line) incommensurable (in length) with ( $A E$ ), and $A E$ [is] commensurable in length with $A B$ [Def. 10.8]. Let $D E$ have been cut in half at $F$, and let the parallelogram (contained by) $A G$ and $G E$, equal to the (square) on $E F$, (and falling short by a square figure) have been applied to $A E . A G$ is thus incommensurable in length with $G E$ [Prop. 10.18]. Let $G H, E K$, and $F L$ have been drawn parallel to $A B$, and let the rest (of the construction) have been made the same as the (proposition) before this. So, it is clear that $M O$ is the square-root of area $A C$. So, we must show that $M O$ is the irrational (straight-line which is) called major.

Since $A G$ is incommensurable in length with $E G, A H$ is also incommensurable with $G K$-that is to say, $S N$ with $N Q$ [Props. 6.1, 10.11]. Thus, $M N$ and $N O$ are incommensurable in square. And since $A E$ is commensurable in length with $A B, A K$ is rational [Prop. 10.19]. And it is equal to the (sum of the squares) on $M N$ and $N O$. Thus, the sum of the (squares) on $M N$ and $N O$ [is] also rational. And since $D E$ [is] incommensurable in length with $A B$ [Prop. 10.13]-that is to say, with $E K$-but $D E$ is commensurable (in length) with $E F, E F$ (is) thus incommensurable in length with $E K$ [Prop. 10.13]. Thus, $E K$ and $E F$ are rational (straightlines which are) commensurable in square only. $L E-$ that is to say, $M R$-(is) thus medial [Prop. 10.21]. And it is contained by $M N$ and $N O$. The (rectangle contained) by $M N$ and $N O$ is thus medial. And the [sum] of the (squares) on $M N$ and $N O$ (is) rational, and $M N$ and $N O$ are incommensurable in square. And if two straightlines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together, then the whole is the irrational (straight-line which is) called major [Prop. 10.39].

Thus, $M O$ is the irrational (straight-line which is) called major. And (it is) the square-root of area $A C$. (Which is) the very thing it was required to show.

[^27]




 بavepòv $\delta \dot{\eta}$, ơтı $\dot{\eta}$ тò $А \Gamma ~ \chi \omega p i ́ o v ~ \delta u v a \mu \varepsilon ́ v \eta ~ \varepsilon ̇ \sigma \tau i v ~ \dot{\eta} \mathrm{M} \Xi$.





 $\dot{\alpha} \lambda \lambda \dot{\alpha} \dot{\eta} \mathrm{AE} \tau \tilde{n} \mathrm{E} \Delta$ ह̇ $\sigma \tau \omega \nu \dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho \circ \varsigma \varsigma^{\circ}$ к人i $\dot{\eta} \mathrm{AB}$ 人́p $\alpha \tau \tilde{n}$












For let the area $A C$ be contained by the rational (straight-line) $A B$ and the fifth binomial (straight-line) $A D$, which has been divided into its (component) terms at $E$, such that $A E$ is the greater term. [So] I say that the square-root of area $A C$ is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).


For let the same construction be made as that shown previously. So, (it is) clear that $M O$ is the square-root of area $A C$. So, we must show that $M O$ is the square-root of a rational plus a medial (area).

For since $A G$ is incommensurable (in length) with $G E$ [Prop. 10.18], $A H$ is thus also incommensurable with $H E$-that is to say, the (square) on $M N$ with the (square) on $N O$ [Props. 6.1, 10.11]. Thus, $M N$ and $N O$ are incommensurable in square. And since $A D$ is a fifth binomial (straight-line), and $E D$ [is] its lesser segment, $E D$ (is) thus commensurable in length with $A B$ [Def. 10.9]. But, $A E$ is incommensurable (in length) with $E D$. Thus, $A B$ is also incommensurable in length with $A E$ [ $B A$ and $A E$ are rational (straight-lines which are) commensurable in square only] [Prop. 10.13]. Thus, $A K$-that is to say, the sum of the (squares) on $M N$ and $N O$-is medial [Prop. 10.21]. And since $D E$ is commensurable in length with $A B$-that is to say, with $E K$-but, $D E$ is commensurable (in length) with $E F$, $E F$ is thus also commensurable (in length) with $E K$ [Prop. 10.12]. And $E K$ (is) rational. Thus, $E L$-that is to say, $M R$-that is to say, the (rectangle contained) by $M N O$-(is) also rational [Prop. 10.19]. $M N$ and $N O$ are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

Thus, $M O$ is the square-root of a rational plus a medial (area) [Prop. 10.40]. And (it is) the square-root of area $A C$. (Which is) the very thing it was required to show.

[^28]the square root of a rational plus a medial area (see Prop. 10.40), since $\rho$ is rational.
$v v^{\prime}$.






















 वं б́йцвтроь.



## Proposition 59

If an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of (the sum of) two medial (areas). ${ }^{\dagger}$


For let the area $A B C D$ be contained by the rational (straight-line) $A B$ and the sixth binomial (straight-line) $A D$, which has been divided into its (component) terms at $E$, such that $A E$ is the greater term. So, I say that the square-root of $A C$ is the square-root of (the sum of) two medial (areas).
[For] let the same construction be made as that shown previously. So, (it is) clear that $M O$ is the square-root of $A C$, and that $M N$ is incommensurable in square with $N O$. And since $E A$ is incommensurable in length with $A B$ [Def. 10.10], $E A$ and $A B$ are thus rational (straightlines which are) commensurable in square only. Thus, $A K$-that is to say, the sum of the (squares) on $M N$ and $N O$-is medial [Prop. 10.21]. Again, since $E D$ is incommensurable in length with $A B$ [Def. 10.10], $F E$ is thus also incommensurable (in length) with $E K$ [Prop. 10.13]. Thus, $F E$ and $E K$ are rational (straightlines which are) commensurable in square only. Thus, $E L$-that is to say, $M R$-that is to say, the (rectangle contained) by $M N O$-is medial [Prop. 10.21]. And since $A E$ is incommensurable (in length) with $E F, A K$ is also incommensurable with $E L$ [Props. 6.1, 10.11]. But, $A K$ is the sum of the (squares) on $M N$ and $N O$, and $E L$ is the (rectangle contained) by $M N O$. Thus, the sum of the (squares) on $M N O$ is incommensurable with the (rectangle contained) by $M N O$. And each of them is medial. And $M N$ and $N O$ are incommensurable in square.

Thus, $M O$ is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. And (it is) the square-root of $A C$. (Which is) the very thing it was required to show.

[^29]
## $\Lambda \tilde{n} \mu \mu \alpha$.

















## $\xi '$.














## Lemma

If a straight-line is cut unequally then (the sum of) the squares on the unequal (parts) is greater than twice the rectangle contained by the unequal (parts).


Let $A B$ be a straight-line, and let it have been cut unequally at $C$, and let $A C$ be greater (than $C B$ ). I say that (the sum of) the (squares) on $A C$ and $C B$ is greater than twice the (rectangle contained) by $A C$ and $C B$.

For let $A B$ have been cut in half at $D$. Therefore, since a straight-line has been cut into equal (parts) at $D$, and into unequal (parts) at $C$, the (rectangle contained) by $A C$ and $C B$, plus the (square) on $C D$, is thus equal to the (square) on $A D$ [Prop. 2.5]. Hence, the (rectangle contained) by $A C$ and $C B$ is less than the (square) on $A D$. Thus, twice the (rectangle contained) by $A C$ and $C B$ is less than double the (square) on $A D$. But, (the sum of) the (squares) on $A C$ and $C B$ [is] double (the sum of) the (squares) on $A D$ and $D C$ [Prop. 2.9]. Thus, (the sum of) the (squares) on $A C$ and $C B$ is greater than twice the (rectangle contained) by $A C$ and $C B$. (Which is) the very thing it was required to show.

## Proposition 60

The square on a binomial (straight-line) applied to a rational (straight-line) produces as breadth a first binomial (straight-line). ${ }^{\dagger}$


Let $A B$ be a binomial (straight-line), having been divided into its (component) terms at $C$, such that $A C$ is the greater term. And let the rational (straight-line) $D E$ be laid down. And let the (rectangle) $D E F G$, equal to the (square) on $A B$, have been applied to $D E$, producing $D G$ as breadth. I say that $D G$ is a first binomial (straightline).

For let $D H$, equal to the (square) on $A C$, and $K L$, equal to the (square) on $B C$, have been applied to $D E$.






























 $\tau \tilde{\eta} \varsigma \dot{\varepsilon} \lambda \alpha \dot{\alpha} \sigma \sigma o v o s ~ \mu \varepsilon і ̈ \zeta o v ~ \delta u ́ v \alpha \tau \alpha l ~ \tau \widetilde{̣}$ 秋ò $\sigma \cup \mu \mu \varepsilon ́ \tau \rho o u ~ \varepsilon ́ \alpha u \tau \tilde{n} \cdot \dot{\eta}$



 ठєֹ̧̈ $\alpha$.

Thus, the remaining twice the (rectangle contained) by $A C$ and $C B$ is equal to $M F$ [Prop. 2.4]. Let $M G$ have been cut in half at $N$, and let $N O$ have been drawn parallel [to each of $M L$ and $G F$ ]. $M O$ and $N F$ are thus each equal to once the (rectangle contained) by $A C B$. And since $A B$ is a binomial (straight-line), having been divided into its (component) terms at $C, A C$ and $C B$ are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Thus, the (squares) on $A C$ and $C B$ are rational, and commensurable with one another. And hence the sum of the (squares) on $A C$ and $C B$ (is rational) [Prop. 10.15], and is equal to $D L$. Thus, $D L$ is rational. And it is applied to the rational (straightline) $D E . D M$ is thus rational, and commensurable in length with $D E$ [Prop. 10.20]. Again, since $A C$ and $C B$ are rational (straight-lines which are) commensurable in square only, twice the (rectangle contained) by $A C$ and $C B$-that is to say, $M F$-is thus medial [Prop. 10.21]. And it is applied to the rational (straight-line) $M L . M G$ is thus also rational, and incommensurable in length with $M L$-that is to say, with $D E$ [Prop. 10.22]. And $M D$ is also rational, and commensurable in length with $D E$. Thus, $D M$ is incommensurable in length with $M G$ [Prop. 10.13]. And they are rational. $D M$ and $M G$ are thus rational (straight-lines which are) commensurable in square only. Thus, $D G$ is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a first (binomial straight-line).

Since the (rectangle contained) by $A C B$ is the mean proportional to the squares on $A C$ and $C B$ [Prop. 10.53 lem.], $M O$ is thus also the mean proportional to $D H$ and $K L$. Thus, as $D H$ is to $M O$, so $M O$ (is) to $K L$-that is to say, as $D K$ (is) to $M N$, (so) $M N$ (is) to $M K$ [Prop. 6.1]. Thus, the (rectangle contained) by $D K$ and $K M$ is equal to the (square) on $M N$ [Prop. 6.17]. And since the (square) on $A C$ is commensurable with the (square) on $C B, D H$ is also commensurable with $K L$. Hence, $D K$ is also commensurable with $K M$ [Props. 6.1, 10.11]. And since (the sum of) the squares on $A C$ and $C B$ is greater than twice the (rectangle contained) by $A C$ and $C B$ [Prop. 10.59 lem.], $D L$ (is) thus also greater than $M F$. Hence, $D M$ is also greater than $M G$ [Props. 6.1, 5.14]. And the (rectangle contained) by $D K$ and $K M$ is equal to the (square) on $M N$-that is to say, to one quarter the (square) on $M G$. And $D K$ (is) commensurable (in length) with $K M$. And if there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable (in length), then the square on the greater is larger
than (the square on) the lesser by the (square) on (some straight-line) commensurable (in length) with the greater [Prop. 10.17]. Thus, the square on $D M$ is greater than (the square on) $M G$ by the (square) on (some straightline) commensurable (in length) with ( $D M$ ). And $D M$ and $M G$ are rational. And $D M$, which is the greater term, is commensurable in length with the (previously) laid down rational (straight-line) $D E$.

Thus, $D G$ is a first binomial (straight-line) [Def. 10.5]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, the square of a binomial is a first binomial. See Prop. 10.54.
$\xi \alpha^{\prime}$.



 $\mu \varepsilon ́ \sigma \alpha s ~ x \alpha \tau \dot{\alpha}$ тò $\Gamma$, $\widetilde{\omega} \nu \mu \varepsilon i \zeta \omega \nu \dot{\eta} А \Gamma$, x $\alpha \grave{~ \varepsilon ̇ x x \varepsilon ا ́ \sigma \vartheta \omega ~ p ̀ \eta \tau \grave{\eta} \dot{\eta}}$




 $\Gamma$, גi $А Г, Г В$ äpa $\mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i \sigma i ̀ ~ \delta u v \alpha ́ \mu ц ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon т \rho o ь ~$

 $\rho \alpha \beta \varepsilon ́ \beta \lambda \eta \tau \alpha l \cdot \rho \eta \tau \grave{\eta} \alpha \rho \alpha$ ह̇ $\sigma \tau i \nu \dot{\eta} \mathrm{M} \Delta$ x $\alpha \grave{\iota} \dot{\alpha} \sigma \cup ́ \mu \mu \varepsilon \tau \rho о \varsigma ~ \tau \tilde{n} \Delta \mathrm{E}$





 ठєutép $\alpha$.



 $\dot{\eta} \Delta \mathrm{K} \tau \tilde{n} \mathrm{KM} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota \nu . ~ x \alpha i ́ ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ ن ́ ~ u ̀ o ̀ ~ \tau \widetilde{\omega} \nu \Delta \mathrm{KM}$


## Proposition 61

The square on a first bimedial (straight-line) applied to a rational (straight-line) produces as breadth a second binomial (straight-line). ${ }^{\dagger}$


Let $A B$ be a first bimedial (straight-line) having been divided into its (component) medial (straight-lines) at $C$, of which $A C$ (is) the greater. And let the rational (straight-line) $D E$ be laid down. And let the parallelogram $D F$, equal to the (square) on $A B$, have been applied to $D E$, producing $D G$ as breadth. I say that $D G$ is a second binomial (straight-line).

For let the same construction have been made as in the (proposition) before this. And since $A B$ is a first bimedial (straight-line), having been divided at $C$, $A C$ and $C B$ are thus medial (straight-lines) commensurable in square only, and containing a rational (area) [Prop. 10.37]. Hence, the (squares) on $A C$ and $C B$ are also medial [Prop. 10.21]. Thus, $D L$ is medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) $D E . M D$ is thus rational, and incommensurable in length with $D E$ [Prop. 10.22]. Again, since twice the (rectangle contained) by $A C$ and $C B$ is rational, $M F$ is also rational. And it is applied to the rational (straight-line) $M L$. Thus, $M G$ [is] also rational, and commensurable in length with $M L$-that is to say, with $D E$ [Prop. 10.20]. $D M$ is thus incommensurable in length with $M G$ [Prop. 10.13]. And they are rational. $D M$ and $M G$ are thus rational, and commensu-
 $\mu \eta^{\prime}$ кь.

rable in square only. $D G$ is thus a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

For since (the sum of) the squares on $A C$ and $C B$ is greater than twice the (rectangle contained) by $A C$ and $C B$ [Prop. 10.59], $D L$ (is) thus also greater than $M F$. Hence, $D M$ (is) also (greater) than $M G$ [Prop. 6.1]. And since the (square) on $A C$ is commensurable with the (square) on $C B, D H$ is also commensurable with $K L$. Hence, $D K$ is also commensurable (in length) with $K M$ [Props. 6.1, 10.11]. And the (rectangle contained) by $D K M$ is equal to the (square) on $M N$. Thus, the square on $D M$ is greater than (the square on) $M G$ by the (square) on (some straight-line) commensurable (in length) with ( $D M$ ) [Prop. 10.17]. And $M G$ is commensurable in length with $D E$.

Thus, $D G$ is a second binomial (straight-line) [Def. 10.6].
${ }^{\dagger}$ In other words, the square of a first bimedial is a second binomial. See Prop. 10.55.

$$
\xi \beta^{\prime} .
$$

 $\beta \alpha \lambda \lambda o ́ \mu \varepsilon v o \nu ~ \pi \lambda \alpha ́ \tau о \varsigma ~ \pi о เ \varepsilon і ̃ ~ \tau \eta ̀ \nu ~ ย ̉ x ~ \delta u ́ o ~ o ̉ v o \mu \alpha ́ \tau \omega \nu ~ \tau р i ́ t \eta \nu . ~$


















## Proposition 62

The square on a second bimedial (straight-line) applied to a rational (straight-line) produces as breadth a third binomial (straight-line). ${ }^{\dagger}$


Let $A B$ be a second bimedial (straight-line) having been divided into its (component) medial (straight-lines) at $C$, such that $A C$ is the greater segment. And let $D E$ be some rational (straight-line). And let the parallelogram $D F$, equal to the (square) on $A B$, have been applied to $D E$, producing $D G$ as breadth. I say that $D G$ is a third binomial (straight-line).

Let the same construction be made as that shown previously. And since $A B$ is a second bimedial (straightline), having been divided at $C, A C$ and $C B$ are thus medial (straight-lines) commensurable in square only, and containing a medial (area) [Prop. 10.38]. Hence, the sum of the (squares) on $A C$ and $C B$ is also medial [Props. 10.15, 10.23 corr.]. And it is equal to $D L$. Thus, $D L$ (is) also medial. And it is applied to the rational (straight-line) $D E . \quad M D$ is thus also rational, and in-

 ГВ т $\Delta \Lambda \tau \widetilde{\varphi} \mathrm{MZ} \cdot \stackrel{\omega}{\sigma} \sigma \varepsilon$ каi $\dot{\eta} \Delta \mathrm{M} \tau \widetilde{\varphi} \mathrm{MH} \dot{\alpha} \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ s ~ \varepsilon ̇ \sigma \tau \iota . ~ \chi \alpha i ́ ~$
 őt каі трítท.



 $\Delta \mathrm{M}, \mathrm{MH} \sigma \cup ́ \mu \mu \varepsilon \tau \rho o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota ~ \tau \tilde{n} \Delta \mathrm{E} \mu \eta^{\prime} x \varepsilon \iota$.

commensurable in length with $D E$ [Prop. 10.22]. So, for the same (reasons), $M G$ is also rational, and incommensurable in length with $M L$-that is to say, with $D E$. Thus, $D M$ and $M G$ are each rational, and incommensurable in length with $D E$. And since $A C$ is incommensurable in length with $C B$, and as $A C$ (is) to $C B$, so the (square) on $A C$ (is) to the (rectangle contained) by $A C B$ [Prop. 10.21 lem.], the (square) on $A C$ (is) also incommensurable with the (rectangle contained) by $A C B$ [Prop. 10.11]. And hence the sum of the (squares) on $A C$ and $C B$ is incommensurable with twice the (rectangle contained) by $A C B$-that is to say, $D L$ with $M F$ [Props. 10.12, 10.13]. Hence, $D M$ is also incommensurable (in length) with $M G$ [Props. 6.1, 10.11]. And they are rational. $D G$ is thus a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a third (binomial straight-line).

So, similarly to the previous (propositions), we can conclude that $D M$ is greater than $M G$, and $D K$ (is) commensurable (in length) with $K M$. And the (rectangle contained) by $D K M$ is equal to the (square) on $M N$. Thus, the square on $D M$ is greater than (the square on) $M G$ by the (square) on (some straight-line) commensurable (in length) with ( $D M$ ) [Prop. 10.17]. And neither of $D M$ and $M G$ is commensurable in length with $D E$.

Thus, $D G$ is a third binomial (straight-line) [Def. 10.7]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, the square of a second bimedial is a third binomial. See Prop. 10.56.

$$
\xi \gamma^{\prime} .
$$

Tò $\alpha \pi o ̀ ~ \tau \tilde{\eta} s ~ \mu \varepsilon i \zeta o v o s ~ \pi \alpha p \alpha ̀ ~ \rho ́ \eta \tau \grave{\nu} \nu ~ \pi \alpha \rho \alpha \beta \alpha \lambda \lambda o ́ \mu \varepsilon v o v ~$



 $\pi \alpha p \alpha ̀ ~ \tau \grave{\nu} \nu \mathrm{E} \pi \alpha \rho \alpha \beta \varepsilon \beta \lambda \eta \dot{\sigma} \vartheta \omega$ тò $\Delta \mathrm{Z} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu$
 غ̇бтì $\tau \varepsilon \tau \alpha ́ p \tau \eta$.



## Proposition 63

The square on a major (straight-line) applied to a rational (straight-line) produces as breadth a fourth binomial (straight-line). ${ }^{\dagger}$


Let $A B$ be a major (straight-line) having been divided at $C$, such that $A C$ is greater than $C B$, and (let) $D E$ (be) a rational (straight-line). And let the parallelogram $D F$, equal to the (square) on $A B$, have been applied to $D E$, producing $D G$ as breadth. I say that $D G$ is a fourth binomial (straight-line).

Let the same construction be made as that shown pre-





















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 ठєi̧̋ $\alpha$.
viously. And since $A B$ is a major (straight-line), having been divided at $C, A C$ and $C B$ are incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. Therefore, since the sum of the (squares) on $A C$ and $C B$ is rational, $D L$ is thus rational. Thus, $D M$ (is) also rational, and commensurable in length with $D E$ [Prop. 10.20]. Again, since twice the (rectangle contained) by $A C$ and $C B$-that is to say, $M F$-is medial, and is (applied to) the rational (straight-line) $M L, M G$ is thus also rational, and incommensurable in length with $D E$ [Prop. 10.22]. $D M$ is thus also incommensurable in length with $M G$ [Prop. 10.13]. $D M$ and $M G$ are thus rational (straight-lines which are) commensurable in square only. Thus, $D G$ is a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a fourth (binomial straight-line).

So, similarly to the previous (propositions), we can show that $D M$ is greater than $M G$, and that the (rectangle contained) by $D K M$ is equal to the (square) on $M N$. Therefore, since the (square) on $A C$ is incommensurable with the (square) on $C B, D H$ is also incommensurable with $K L$. Hence, $D K$ is also incommensurable with $K M$ [Props. 6.1, 10.11]. And if there are two unequal straight-lines, and a parallelogram equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable (in length), then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) incommensurable in length with the greater [Prop. 10.18]. Thus, the square on $D M$ is greater than (the square on) $M G$ by the (square) on (some straight-line) incommensurable (in length) with ( $D M$ ). And $D M$ and $M G$ are rational (straight-lines which are) commensurable in square only. And $D M$ is commensurable (in length) with the (previously) laid down rational (straight-line) $D E$.

Thus, $D G$ is a fourth binomial (straight-line) [Def. 10.8]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, the square of a major is a fourth binomial. See Prop. 10.57.

## $\xi \delta^{\prime}$.





 $\Delta \mathrm{E} \pi \alpha \rho \alpha \beta \varepsilon \beta \lambda \dot{\eta} \sigma \vartheta \omega$ тò $\Delta \mathrm{Z} \pi \lambda \alpha ́ \tau o s ~ \pi o เ o u ̃ \nu ~ \tau \eta ̀ \nu ~ \Delta H \cdot \lambda \varepsilon ́ \gamma \omega$,


## Proposition 64

The square on the square-root of a rational plus a medial (area) applied to a rational (straight-line) produces as breadth a fifth binomial (straight-line). ${ }^{\dagger}$

Let $A B$ be the square-root of a rational plus a medial (area) having been divided into its (component) straightlines at $C$, such that $A C$ is greater. And let the rational (straight-line) $D E$ be laid down. And let the (parallelogram) $D F$, equal to the (square) on $A B$, have been ap-


| $\vdash$ | $\quad$ | $\quad$ |
| :--- | :--- | :--- |







 tò ठís ítò $\tau \widetilde{\omega} \nu \mathrm{A}$,








 ठєĭ̧al.
plied to $D E$, producing $D G$ as breadth. I say that $D G$ is a fifth binomial straight-line.


Let the same construction be made as in the (propositions) before this. Therefore, since $A B$ is the square-root of a rational plus a medial (area), having been divided at $C, A C$ and $C B$ are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. Therefore, since the sum of the (squares) on $A C$ and $C B$ is medial, $D L$ is thus medial. Hence, $D M$ is rational and incommensurable in length with $D E$ [Prop. 10.22]. Again, since twice the (rectangle contained) by $A C B-$ that is to say, $M F$-is rational, $M G$ (is) thus rational and commensurable (in length) with $D E$ [Prop. 10.20]. $D M$ (is) thus incommensurable (in length) with $M G$ [Prop. 10.13]. Thus, $D M$ and $M G$ are rational (straightlines which are) commensurable in square only. Thus, $D G$ is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For, similarly (to the previous propositions), it can be shown that the (rectangle contained) by $D K M$ is equal to the (square) on $M N$, and $D K$ (is) incommensurable in length with $K M$. Thus, the square on $D M$ is greater than (the square on) $M G$ by the (square) on (some straight-line) incommensurable (in length) with ( $D M$ ) [Prop. 10.18]. And $D M$ and $M G$ are [rational] (straight-lines which are) commensurable in square only, and the lesser $M G$ is commensurable in length with $D E$.

Thus, $D G$ is a fifth binomial (straight-line) [Def. 10.9]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, the square of the square-root of a rational plus medial is a fifth binomial. See Prop. 10.58.

## $\xi \varepsilon^{\prime}$.

Tò árò tñs ठúo $\mu \varepsilon ́ \sigma \alpha ~ \delta u v \alpha \mu \varepsilon ́ v \eta s ~ \pi \alpha p \alpha ̀ ~ \rho ீ \eta t \eta ̀ \nu ~ \pi \alpha p \alpha-~$






## Proposition 65

The square on the square-root of (the sum of) two medial (areas) applied to a rational (straight-line) produces as breadth a sixth binomial (straight-line). ${ }^{\dagger}$

Let $A B$ be the square-root of (the sum of) two medial (areas), having been divided at $C$. And let $D E$ be a rational (straight-line). And let the (parallelogram) $D F$, equal to the (square) on $A B$, have been applied to $D E$,









 $\sigma \cup \gamma x \varepsilon \dot{\prime} \mu \varepsilon \nu \frac{}{}$


 x $\alpha$ ย゙ $์$ тท.





 ठєĩ̌ $\alpha$.
producing $D G$ as breadth. I say that $D G$ is a sixth binomial (straight-line).


For let the same construction be made as in the previous (propositions). And since $A B$ is the square-root of (the sum of) two medial (areas), having been divided at $C, A C$ and $C B$ are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.41]. Hence, according to what has been previously demonstrated, $D L$ and $M F$ are each medial. And they are applied to the rational (straight-line) $D E$. Thus, $D M$ and $M G$ are each rational, and incommensurable in length with $D E$ [Prop. 10.22]. And since the sum of the (squares) on $A C$ and $C B$ is incommensurable with twice the (rectangle contained) by $A C$ and $C B, D L$ is thus incommensurable with $M F$. Thus, $D M$ (is) also incommensurable (in length) with $M G$ [Props. 6.1, 10.11]. $D M$ and $M G$ are thus rational (straight-lines which are) commensurable in square only. Thus, $D G$ is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a sixth (binomial straight-line).

So, similarly (to the previous propositions), we can again show that the (rectangle contained) by $D K M$ is equal to the (square) on $M N$, and that $D K$ is incommensurable in length with $K M$. And so, for the same (reasons), the square on $D M$ is greater than (the square on) $M G$ by the (square) on (some straight-line) incommensurable in length with ( $D M$ ) [Prop. 10.18]. And neither of $D M$ and $M G$ is commensurable in length with the (previously) laid down rational (straight-line) $D E$.

Thus, $D G$ is a sixth binomial (straight-line) [Def. 10.10]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ In other words, the square of the square-root of two medials is a sixth binomial. See Prop. 10.59.

$$
\xi \varepsilon^{\prime}
$$





## Proposition 66

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order.





 $\dot{\eta} \mathrm{AB} \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ \Gamma \Delta$, oưt $\omega \varsigma \dot{\eta} \mathrm{AE} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu ~ \Gamma Z . ~ \chi \alpha i ̀ ~ \lambda o l \pi \dot{\eta}$
































Let $A B$ be a binomial (straight-line), and let $C D$ be commensurable in length with $A B$. I say that $C D$ is a binomial (straight-line), and (is) the same in order as $A B$.


For since $A B$ is a binomial (straight-line), let it have been divided into its (component) terms at $E$, and let $A E$ be the greater term. $A E$ and $E B$ are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Let it have been contrived that as $A B$ (is) to $C D$, so $A E$ (is) to $C F$ [Prop. 6.12]. Thus, the remainder $E B$ is also to the remainder $F D$, as $A B$ (is) to $C D$ [Props. 6.16, 5.19 corr.]. And $A B$ (is) commensurable in length with $C D$. Thus, $A E$ is also commensurable (in length) with $C F$, and $E B$ with $F D$ [Prop. 10.11]. And $A E$ and $E B$ are rational. Thus, $C F$ and $F D$ are also rational. And as $A E$ is to $C F$, (so) $E B$ (is) to $F D$ [Prop. 5.11]. Thus, alternately, as $A E$ is to $E B$, (so) $C F$ (is) to $F D$ [Prop. 5.16]. And $A E$ and $E B$ [are] commensurable in square only. Thus, $C F$ and $F D$ are also commensurable in square only [Prop. 10.11]. And they are rational. $C D$ is thus a binomial (straight-line) [Prop. 10.36]. So, I say that it is the same in order as $A B$.

For the square on $A E$ is greater than (the square on) $E B$ by the (square) on (some straight-line) either commensurable or incommensurable (in length) with ( $A E$ ). Therefore, if the square on $A E$ is greater than (the square on) $E B$ by the (square) on (some straight-line) commensurable (in length) with $(A E)$ then the square on $C F$ will also be greater than (the square on) $F D$ by the (square) on (some straight-line) commensurable (in length) with ( $C F$ ) [Prop. 10.14]. And if $A E$ is commensurable (in length) with (some previously) laid down rational (straight-line) then $C F$ will also be commensurable (in length) with it [Prop. 10.12]. And, on account of this, $A B$ and $C D$ are each first binomial (straightlines) [Def. 10.5] -that is to say, the same in order. And if $E B$ is commensurable (in length) with the (previously) laid down rational (straight-line) then $F D$ is also commensurable (in length) with it [Prop. 10.12], and, again, on account of this, $(C D)$ will be the same in order as $A B$. For each of them will be second binomial (straightlines) [Def. 10.6]. And if neither of $A E$ and $E B$ is commensurable (in length) with the (previously) laid down rational (straight-line) then neither of $C F$ and $F D$ will be commensurable (in length) with it [Prop. 10.13], and each (of $A B$ and $C D$ ) is a third (binomial straight-line)

## $\xi \zeta^{\prime}$.

 $\mu \varepsilon ́ \sigma \omega \nu$ ह̇எтì $\varkappa \alpha \grave{\imath} \tau \tilde{n} \tau \alpha ́ \xi \varepsilon ા ~ \dot{\eta} \alpha u ̉ \tau ท ́$.

"E $\sigma \tau \omega$ ह̇x ठúo $\mu \varepsilon ́ \sigma \omega \nu \dot{\eta} \mathrm{AB}$, x $\alpha$ 亿 $\tau \tilde{n} \mathrm{AB} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho \circ \varsigma$ है $\sigma \tau \omega$
 $\dot{\eta} \alpha u ̋ n \eta \tau \tilde{n} \mathrm{AB}$.






 ஸ́s $\dot{\eta}$ AE трòs $\mathrm{EB}, \dot{\eta}$ ГZ $\pi \rho o ̀ s ~ Z \Delta, ~ \alpha i ~ \delta e ̀ ~ A E, ~ E B ~ \delta u v \alpha ́ \mu \varepsilon ı ~$






[Def. 10.7]. And if the square on $A E$ is greater than (the square on) $E B$ by the (square) on (some straightline) incommensurable (in length) with ( $A E$ ) then the square on $C F$ is also greater than (the square on) $F D$ by the (square) on (some straight-line) incommensurable (in length) with ( $C F$ ) [Prop. 10.14]. And if $A E$ is commensurable (in length) with the (previously) laid down rational (straight-line) then $C F$ is also commensurable (in length) with it [Prop. 10.12], and each (of $A B$ and $C D$ ) is a fourth (binomial straight-line) [Def. 10.8]. And if $E B$ (is commensurable in length with the previously laid down rational straight-line) then $F D$ (is) also (commensurable in length with it), and each (of $A B$ and $C D$ ) will be a fifth (binomial straight-line) [Def. 10.9]. And if neither of $A E$ and $E B$ (is commensurable in length with the previously laid down rational straight-line) then also neither of $C F$ and $F D$ is commensurable (in length) with the laid down rational (straight-line), and each (of $A B$ and $C D$ ) will be a sixth (binomial straight-line) [Def. 10.10].

Hence, a (straight-line) commensurable in length with a binomial (straight-line) is a binomial (straightline), and the same in order. (Which is) the very thing it was required to show.

## Proposition 67

A (straight-line) commensurable in length with a bimedial (straight-line) is itself also bimedial, and the same in order.


Let $A B$ be a bimedial (straight-line), and let $C D$ be commensurable in length with $A B$. I say that $C D$ is bimedial, and the same in order as $A B$.

For since $A B$ is a bimedial (straight-line), let it have been divided into its (component) medial (straight-lines) at $E$. Thus, $A E$ and $E B$ are medial (straight-lines which are) commensurable in square only [Props. 10.37, 10.38]. And let it have been contrived that as $A B$ (is) to $C D$, (so) $A E$ (is) to $C F$ [Prop. 6.12]. And thus as the remainder $E B$ is to the remainder $F D$, so $A B$ (is) to $C D$ [Props. 5.19 corr., 6.16]. And $A B$ (is) commensurable in length with $C D$. Thus, $A E$ and $E B$ are also commensurable (in length) with $C F$ and $F D$, respectively [Prop. 10.11]. And $A E$ and $E B$ (are) medial. Thus, $C F$ and $F D$ (are) also medial [Prop. 10.23]. And since as $A E$ is to $E B$, (so) $C F$ (is) to $F D$, and $A E$ and $E B$ are commensurable in square only, $C F$ and $F D$ are [thus]

AE $\pi \rho o ̀ s ~ \tau o ̀ ~ \alpha ̉ \pi o ̀ ~ \tau \tilde{n} \varsigma ~ \Gamma Z, ~ o u ̛ \tau \omega \varsigma ~ \tau o ̀ ~ U ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AEB} \pi \rho o ̀ \varsigma ~ \tau o ̀ ~$



 $\mu \varepsilon ́ \sigma o \nu, \mu \varepsilon ́ \sigma o \nu, \chi \alpha i ́ ~ \varepsilon ̇ \sigma \tau \tau \nu$ ย̇x $\kappa \tau \varepsilon ́ p \alpha ~ \delta \varepsilon \cup \tau \varepsilon ́ p \alpha . ~$



＂E $\sigma \tau \omega \mu \varepsilon$ i丂 $\omega \nu \dot{\eta} \mathrm{AB}$ ，$\alpha \alpha i ̀ \tau \tilde{n} \mathrm{AB} \sigma \dot{\prime} \mu \mu \varepsilon \tau \rho o s$ है $\sigma \tau \omega \dot{\eta} \Gamma \Delta$ ． $\lambda \varepsilon ́ \gamma \omega$ ，öть $\dot{\eta} \Gamma \Delta \mu \varepsilon i \zeta \omega \nu$ ह̇ $\sigma \tau i ⿱ 亠 䒑$.
















also commensurable in square only［Prop．10．11］．And they were also shown（to be）medial．Thus，$C D$ is a bi－ medial（straight－line）．So，I say that it is also the same in order as $A B$ ．

For since as $A E$ is to $E B$ ，（so）$C F$（is）to $F D$ ，thus also as the（square）on $A E$（is）to the（rectangle con－ tained）by $A E B$ ，so the（square）on $C F$（is）to the（rect－ angle contained）by $C F D$［Prop． 10.21 lem．］．Alter－ nately，as the（square）on $A E$（is）to the（square）on $C F$ ，so the（rectangle contained）by $A E B$（is）to the （rectangle contained）by $C F D$［Prop．5．16］．And the （square）on $A E$（is）commensurable with the（square） on $C F$ ．Thus，the（rectangle contained）by $A E B$（is） also commensurable with the（rectangle contained）by $C F D$［Prop．10．11］．Therefore，either the（rectangle contained）by $A E B$ is rational，and the（rectangle con－ tained）by $C F D$ is rational［and，on account of this， （ $A E$ and $C D$ ）are first bimedial（straight－lines）］，or（the rectangle contained by $A E B$ is）medial，and（the rect－ angle containe by $C F D$ is）medial，and（ $A B$ and $C D$ ） are each second（bimedial straight－lines）［Props．10．23， 10．37，10．38］．

And，on account of this，$C D$ will be the same in order as $A B$ ．（Which is）the very thing it was required to show．

## Proposition 68

A（straight－line）commensurable（in length）with a major（straight－line）is itself also major．


Let $A B$ be a major（straight－line），and let $C D$ be com－ mensurable（in length）with $A B$ ．I say that $C D$ is a major （straight－line）．

Let $A B$ have been divided（into its component terms） at $E . A E$ and $E B$ are thus incommensurable in square， making the sum of the squares on them rational，and the （rectangle contained）by them medial［Prop．10．39］．And let（the）same（things）have been contrived as in the pre－ vious（propositions）．And since as $A B$ is to $C D$ ，so $A E$ （is）to $C F$ and $E B$ to $F D$ ，thus also as $A E$（is）to $C F$ ， so $E B$（is）to $F D$［Prop．5．11］．And $A B$（is）commen－ surable（in length）with $C D$ ．Thus，$A E$ and $E B$（are） also commensurable（in length）with $C F$ and $F D$ ，re－ spectively［Prop．10．11］．And since as $A E$ is to $C F$ ，so $E B$（is）to $F D$ ，also，alternately，as $A E$（is）to $E B$ ，so $C F$（is）to $F D$［Prop．5．16］，and thus，via composition， as $A B$ is to $B E$ ，so $C D$（is）to $D F$［Prop．5．18］．And thus as the（square）on $A B$（is）to the（square）on $B E$ ，so the
 $\Gamma \Delta$, oú $\tau \omega \varsigma ~ \tau \grave{\alpha} \alpha \dot{\alpha} \pi \grave{\tau} \tau \widetilde{\omega} \nu \mathrm{AE}, \mathrm{EB} \pi \rho o ̀ \varsigma \tau \dot{\alpha} \alpha \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \Gamma \mathrm{Z}, \mathrm{Z} \Delta$.








 ж $\alpha \lambda о \cup \mu \varepsilon ́ v \eta ~ \mu \varepsilon i \zeta \omega \nu$.
 ठعǐ̌ $\alpha$.

$$
\xi \vartheta^{\prime}
$$





 $\mu \varepsilon ́ \sigma o \nu ~ \delta u v \alpha \mu \varepsilon ́ v \eta ~ \varepsilon ̇ \sigma \tau i ́ v . ~$









(square) on $C D$ (is) to the (square) on $D F$ [Prop. 6.20]. So, similarly, we can also show that as the (square) on $A B$ (is) to the (square) on $A E$, so the (square) on $C D$ (is) to the (square) on $C F$. And thus as the (square) on $A B$ (is) to (the sum of) the (squares) on $A E$ and $E B$, so the (square) on $C D$ (is) to (the sum of) the (squares) on $C F$ and $F D$. And thus, alternately, as the (square) on $A B$ is to the (square) on $C D$, so (the sum of) the (squares) on $A E$ and $E B$ (is) to (the sum of) the (squares) on $C F$ and $F D$ [Prop. 5.16]. And the (square) on $A B$ (is) commensurable with the (square) on $C D$. Thus, (the sum of) the (squares) on $A E$ and $E B$ (is) also commensurable with (the sum of) the (squares) on $C F$ and $F D$ [Prop. 10.11]. And the (squares) on $A E$ and $E B$ (added) together are rational. The (squares) on $C F$ and $F D$ (added) together (are) thus also rational. So, similarly, twice the (rectangle contained) by $A E$ and $E B$ is also commensurable with twice the (rectangle contained) by $C F$ and $F D$. And twice the (rectangle contained) by $A E$ and $E B$ is medial. Therefore, twice the (rectangle contained) by $C F$ and $F D$ (is) also medial [Prop. 10.23 corr.]. $C F$ and $F D$ are thus (straight-lines which are) incommensurable in square [Prop 10.13], simultaneously making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. The whole, $C D$, is thus that irrational (straight-line) called major [Prop. 10.39].

Thus, a (straight-line) commensurable (in length) with a major (straight-line) is major. (Which is) the very thing it was required to show.

## Proposition 69

A (straight-line) commensurable (in length) with the square-root of a rational plus a medial (area) is [itself also] the square-root of a rational plus a medial (area).


Let $A B$ be the square-root of a rational plus a medial (area), and let $C D$ be commensurable (in length) with $A B$. We must show that $C D$ is also the square-root of a rational plus a medial (area).

Let $A B$ have been divided into its (component) straight-lines at $E . A E$ and $E B$ are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that $C F$ and $F D$ are also incommensurable in square, and that the sum of the (squares) on $A E$ and
$\mathrm{Z} \Delta$ pintóv.
 סєī̧al.

$o^{\prime}$.<br> ėठтív.
















 $\tau \widetilde{\sim}$ ГZ, Z $\Delta$.


## $o \alpha^{\prime}$.




$E B$ (is) commensurable with the sum of the (squares) on $C F$ and $F D$, and the (rectangle contained) by $A E$ and $E B$ with the (rectangle contained) by $C F$ and $F D$. And hence the sum of the squares on $C F$ and $F D$ is medial, and the (rectangle contained) by $C F$ and $F D$ (is) rational.

Thus, $C D$ is the square-root of a rational plus a medial (area) [Prop. 10.40]. (Which is) the very thing it was required to show.

## Proposition 70

A (straight-line) commensurable (in length) with the square-root of (the sum of) two medial (areas) is (itself also) the square-root of (the sum of) two medial (areas).


Let $A B$ be the square-root of (the sum of) two medial (areas), and (let) $C D$ (be) commensurable (in length) with $A B$. We must show that $C D$ is also the square-root of (the sum of) two medial (areas).

For since $A B$ is the square-root of (the sum of) two medial (areas), let it have been divided into its (component) straight-lines at $E$. Thus, $A E$ and $E B$ are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on $A E$ and $E B$ incommensurable with the (rectangle) contained by $A E$ and $E B$ [Prop. 10.41]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that $C F$ and $F D$ are also incommensurable in square, and (that) the sum of the (squares) on $A E$ and $E B$ (is) commensurable with the sum of the (squares) on $C F$ and $F D$, and the (rectangle contained) by $A E$ and $E B$ with the (rectangle contained) by $C F$ and $F D$. Hence, the sum of the squares on $C F$ and $F D$ is also medial, and the (rectangle contained) by $C F$ and $F D$ (is) medial, and, moreover, the sum of the squares on $C F$ and $F D$ (is) incommensurable with the (rectangle contained) by $C F$ and $F D$.

Thus, $C D$ is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. (Which is) the very thing it was required to show.

## Proposition 71

When a rational and a medial (area) are added together, four irrational (straight-lines) arise (as the squareroots of the total area)-either a binomial, or a first bi-







 ӨI $\pi \lambda \alpha ́ \tau o s ~ \pi o เ o u ̃ \nu ~ \tau \grave{\nu} \nu ~ \Theta K . ~ x \alpha i ̀ ~ \varepsilon ́ \pi \varepsilon ो ~ p ̣ \eta \tau o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ A B ~ x \alpha i ́ ~$
 тウ̀̀ $\mathrm{EZ} \pi \alpha \rho \alpha \beta \dot{\beta} \beta \lambda \eta \tau \alpha l \pi \lambda \alpha ́ \tau o s ~ \pi o เ o u ̃ \nu ~ \tau \grave{\eta} \nu \mathrm{E} \Theta \cdot \dot{\eta} \mathrm{E} \Theta$ 㒸 $\rho \alpha$

























medial, or a major, or the square-root of a rational plus a medial (area).

Let $A B$ be a rational (area), and $C D$ a medial (area). I say that the square-root of area $A D$ is either binomial, or first bimedial, or major, or the square-root of a rational plus a medial (area).


For $A B$ is either greater or less than $C D$. Let it, first of all, be greater. And let the rational (straight-line) $E F$ be laid down. And let (the rectangle) $E G$, equal to $A B$, have been applied to $E F$, producing $E H$ as breadth. And let (the recatangle) $H I$, equal to $D C$, have been applied to $E F$, producing $H K$ as breadth. And since $A B$ is rational, and is equal to $E G, E G$ is thus also rational. And it has been applied to the [rational] (straight-line) $E F$, producing $E H$ as breadth. $E H$ is thus rational, and commensurable in length with $E F$ [Prop. 10.20]. Again, since $C D$ is medial, and is equal to $H I, H I$ is thus also medial. And it is applied to the rational (straight-line) $E F$, producing $H K$ as breadth. $H K$ is thus rational, and incommensurable in length with $E F$ [Prop. 10.22]. And since $C D$ is medial, and $A B$ rational, $A B$ is thus incommensurable with $C D$. Hence, $E G$ is also incommensurable with $H I$. And as $E G$ (is) to $H I$, so $E H$ is to $H K$ [Prop. 6.1]. Thus, $E H$ is also incommensurable in length with $H K$ [Prop. 10.11]. And they are both rational. Thus, $E H$ and $H K$ are rational (straight-lines which are) commensurable in square only. $E K$ is thus a binomial (straight-line), having been divided (into its component terms) at $H$ [Prop. 10.36]. And since $A B$ is greater than $C D$, and $A B$ (is) equal to $E G$, and $C D$ to $H I, E G$ (is) thus also greater than $H I$. Thus, $E H$ is also greater than $H K$ [Prop. 5.14]. Therefore, the square on $E H$ is greater than (the square on) $H K$ either by the (square) on (some straight-line) commensurable in length with $(E H)$, or by the (square) on (some straightline) incommensurable (in length with $E H$ ). Let it, first of all, be greater by the (square) on (some straight-line) commensurable (in length with $E H$ ). And the greater
 $\chi \alpha \lambda о \cup \mu \varepsilon ́ v \eta \mu \varepsilon i \zeta \omega \nu$. $\dot{\eta}$ äp $\alpha$ тò EI $\chi \omega \rho i ́ o \nu ~ \delta u v \alpha \mu \varepsilon ́ v \eta ~ \mu \varepsilon i \zeta \omega \nu$

' $\mathrm{A} \lambda \lambda \alpha$ 人 $\delta \dot{\eta}$ है $\sigma \tau \omega$ है $\lambda \alpha \sigma \sigma o v$ тò AB тoũ $\Gamma \Delta$ ' $x \alpha i$ tò EH

 $\sigma \cup \mu \mu \varepsilon ́ \tau \rho о \cup ~ \dot{\varepsilon} \alpha \cup \tau \tilde{n} \tilde{\eta} \tau \widetilde{\uparrow} \alpha \dot{\alpha} \pi o ̀ ~ \alpha ́ \sigma u \mu \mu \varepsilon ́ \tau \rho o u . ~ \delta u v \alpha ́ \sigma \vartheta \omega ~ \pi \rho o ́ \tau \varepsilon \rho o v ~$















 ү'ү

(of the two components of $E K$ ) $H E$ is commensurable (in length) with the (previously) laid down (straightline) $E F . E K$ is thus a first binomial (straight-line) [Def. 10.5]. And $E F$ (is) rational. And if an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is a binomial (straight-line) [Prop. 10.54]. Thus, the square-root of $E I$ is a binomial (straight-line). Hence the squareroot of $A D$ is also a binomial (straight-line). And, so, let the square on $E H$ be greater than (the square on) $H K$ by the (square) on (some straight-line) incommensurable (in length) with $(E H)$. And the greater (of the two components of $E K) E H$ is commensurable in length with the (previously) laid down rational (straight-line) $E F$. Thus, $E K$ is a fourth binomial (straight-line) [Def. 10.8]. And $E F$ (is) rational. And if an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line) called major [Prop. 10.57]. Thus, the square-root of area $E I$ is a major (straight-line). Hence, the square-root of $A D$ is also major.

And so, let $A B$ be less than $C D$. Thus, $E G$ is also less than $H I$. Hence, $E H$ is also less than $H K$ [Props. 6.1, 5.14]. And the square on $H K$ is greater than (the square on) $E H$ either by the (square) on (some straightline) commensurable (in length) with ( $H K$ ), or by the (square) on (some straight-line) incommensurable (in length) with ( $H K$ ). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with ( $H K$ ). And the lesser (of the two components of $E K$ ) $E H$ is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, $E K$ is a second binomial (straight-line) [Def. 10.6]. And $E F$ (is) rational. And if an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is a first bimedial (straightline) [Prop. 10.55]. Thus, the square-root of area $E I$ is a first bimedial (straight-line). Hence, the square-root of $A D$ is also a first bimedial (straight-line). And so, let the square on $H K$ be greater than (the square on) $H E$ by the (square) on (some straight-line) incommensurable (in length) with ( $H K$ ). And the lesser (of the two components of $E K$ ) $E H$ is commensurable (in length) with the (previously) laid down rational (straight-line) $E F$. Thus, $E K$ is a fifth binomial (straight-line) [Def. 10.9]. And $E F$ (is) rational. And if an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the square-root of a rational plus a medial (area) [Prop. 10.58]. Thus, the square-root of area $E I$ is the square-root of a rational plus a medial (area). Hence, the square-root of area $A D$ is also the

$$
\xi \beta^{\prime} .
$$






 દ̇のтì $\delta \varepsilon u \tau \varepsilon ́ \rho \alpha ~ \grave{\eta}$ ठúo $\mu \varepsilon ́ \sigma \alpha ~ \delta u v \alpha \mu \varepsilon ́ v \eta$.



















 EI, тOUtモ́ $\sigma \tau \iota ~ \tau o ̀ ~ A \Delta, ~ \delta u v \alpha \mu \varepsilon ́ v \eta ~ \varepsilon ̉ x ~ \delta u ́ o ~ \mu \varepsilon ́ \sigma \omega \nu ~ \varepsilon ̉ \sigma \tau i ̀ ~ \delta \varepsilon u \tau \varepsilon ́ p \alpha . ~$
square-root of a rational plus a medial (area).
Thus, when a rational and a medial area are added together, four irrational (straight-lines) arise (as the squareroots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area). (Which is) the very thing it was required to show.

## Proposition 72

When two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area) -either a second bimedial, or the squareroot of (the sum of) two medial (areas).


For let the two medial (areas) $A B$ and $C D$, (which are) incommensurable with one another, have been added together. I say that the square-root of area $A D$ is either a second bimedial, or the square-root of (the sum of) two medial (areas).

For $A B$ is either greater than or less than $C D$. By chance, let $A B$, first of all, be greater than $C D$. And let the rational (straight-line) $E F$ be laid down. And let $E G$, equal to $A B$, have been applied to $E F$, producing $E H$ as breadth, and $H I$, equal to $C D$, producing $H K$ as breadth. And since $A B$ and $C D$ are each medial, $E G$ and $H I$ (are) thus also each medial. And they are applied to the rational straight-line $F E$, producing $E H$ and $H K$ (respectively) as breadth. Thus, $E H$ and $H K$ are each rational (straight-lines which are) incommensurable in length with $E F$ [Prop. 10.22]. And since $A B$ is incommensurable with $C D$, and $A B$ is equal to $E G$, and $C D$ to $H I, E G$ is thus also incommensurable with $H I$. And as $E G$ (is) to $H I$, so $E H$ is to $H K$ [Prop. 6.1]. $E H$ is thus incommensurable in length with $H K$ [Prop. 10.11]. Thus, $E H$ and $H K$ are rational (straight-lines which are) commensurable in square only. $E K$ is thus a binomial (straight-line) [Prop. 10.36]. And the square on $E H$ is greater than (the square on) $H K$ either by the (square)





 ย̇бтív.




 ठט́o $\mu$ ह́ $\sigma \alpha$ ठuvaцévŋ.












 $\pi \varepsilon ́ \mu \pi \tau \eta \nu$. Tò $\delta \varepsilon ̀ ~ \alpha ̉ \pi o ̀ ~ \tau \eta ̃ ऽ ~ \delta u ́ o ~ \mu \varepsilon ́ \sigma \alpha ~ \delta u v \alpha \mu \varepsilon ́ v \eta s ~ \pi \alpha p \alpha ̀ ~ \rho ́ \eta \tau \eta ̀ \nu ~$ $\pi \alpha p \alpha \beta \alpha \lambda \lambda o ́ \mu \varepsilon \nu \circ \nu \pi \lambda \alpha ́ \tau o \varsigma \pi о เ \varepsilon і ̃ ~ \tau \grave{\eta} \nu$ ह̉x ठúo ỏvo $\mu \alpha ́ \tau \omega \nu$ हैx $\tau \eta \nu$.


 $\dot{\alpha} \lambda \lambda \dot{n} \lambda \omega \nu$.
on (some straight-line) commensurable (in length) with ( $E H$ ), or by the (square) on (some straight-line) incommensurable (in length with $E H$ ). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with $(E H)$. And neither of $E H$ or $H K$ is commensurable in length with the (previously) laid down rational (straight-line) $E F$. Thus, $E K$ is a third binomial (straight-line) [Def. 10.7]. And $E F$ (is) rational. And if an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is a second bimedial (straight-line) [Prop. 10.56]. Thus, the square-root of $E I$-that is to say, of $A D$ is a second bimedial. And so, let the square on $E H$ be greater than (the square) on $H K$ by the (square) on (some straight-line) incommensurable in length with $(E H)$. And $E H$ and $H K$ are each incommensurable in length with $E F$. Thus, $E K$ is a sixth binomial (straightline) [Def. 10.10]. And if an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the square-root of (the sum of) two medial (areas) [Prop. 10.59]. Hence, the square-root of area $A D$ is also the square-root of (the sum of) two medial (areas).
[So, similarly, we can show that, even if $A B$ is less than $C D$, the square-root of area $A D$ is either a second bimedial or the square-root of (the sum of) two medial (areas).]

Thus, when two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the squareroots of the total area) -either a second bimedial, or the square-root of (the sum of) two medial (areas).

A binomial (straight-line), and the (other) irrational (straight-lines) after it, are neither the same as a medial (straight-line) nor (the same) as one another. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straightline which is) also incommensurable in length with (the straight-line) to which it is applied [Prop. 10.22]. And the (square) on a binomial (straight-line), applied to a rational (straight-line), produces as breadth a first binomial [Prop. 10.60]. And the (square) on a first bimedial (straight-line), applied to a rational (straight-line), produces as breadth a second binomial [Prop. 10.61]. And the (square) on a second bimedial (straight-line), applied to a rational (straight-line), produces as breadth a third binomial [Prop. 10.62]. And the (square) on a major (straight-line), applied to a rational (straight-line), produces as breadth a fourth binomial [Prop. 10.63]. And the (square) on the square-root of a rational plus a medial

$$
o \gamma^{\prime}
$$

















${ }^{\dagger}$ See footnote to Prop. 10.36.

$$
o \delta^{\prime}
$$




(area), applied to a rational (straight-line), produces as breadth a fifth binomial [Prop. 10.64]. And the (square) on the square-root of (the sum of) two medial (areas), applied to a rational (straight-line), produces as breadth a sixth binomial [Prop. 10.65]. And the aforementioned breadths differ from the first (breadth), and from one another-from the first, because it is rational-and from one another, because they are not the same in order. Hence, the (previously mentioned) irrational (straightlines) themselves also differ from one another.

## Proposition 73

If a rational (straight-line), which is commensurable in square only with the whole, is subtracted from a(nother) rational (straight-line) then the remainder is an irrational (straight-line). Let it be called an apotome.


For let the rational (straight-line) $B C$, which commensurable in square only with the whole, have been subtracted from the rational (straight-line) $A B$. I say that the remainder $A C$ is that irrational (straight-line) called an apotome.

For since $A B$ is incommensurable in length with $B C$, and as $A B$ is to $B C$, so the (square) on $A B$ (is) to the (rectangle contained) by $A B$ and $B C$ [Prop. 10.21 lem.], the (square) on $A B$ is thus incommensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.11]. But, the (sum of the) squares on $A B$ and $B C$ is commensurable with the (square) on $A B$ [Prop. 10.15], and twice the (rectangle contained) by $A B$ and $B C$ is commensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.6]. And, inasmuch as the (sum of the squares) on $A B$ and $B C$ is equal to twice the (rectangle contained) by $A B$ and $B C$ plus the (square) on $C A$ [Prop. 2.7], the (sum of the squares) on $A B$ and $B C$ is thus also incommensurable with the remaining (square) on $A C$ [Props. 10.13, 10.16]. And the (sum of the squares) on $A B$ and $B C$ is rational. $A C$ is thus an irrational (straight-line) [Def. 10.4]. And let it be called an apotome. ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 74

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a rational (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational

 $\mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o s ~ o u ̛ \sigma \alpha ~ \tau \tilde{n} A B, \mu \varepsilon \tau \dot{\alpha}$ ठè $\tau \tilde{\eta} \varsigma \mathrm{AB}$ ṕntòv $\pi o \circ o u ̃ \sigma \alpha$ tò ن́ $\pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma} \cdot \lambda \hat{\varepsilon} \gamma \omega$, ö $\tau \iota \dot{\eta} \lambda o \iota \pi \dot{\eta} \dot{\eta} \mathrm{~A} \Gamma$










[^30]$$
o \varepsilon^{\prime} .
$$



'A

 $\dot{\eta} \lambda o \iota \pi \grave{\eta} \dot{\eta} \mathrm{~A} \Gamma$ 敞 бєu七ép $\alpha$.
(straight-line). Let it be called a first apotome of a medial (straight-line).


For let the medial (straight-line) $B C$, which is commensurable in square only with $A B$, and which makes with $A B$ the rational (rectangle contained) by $A B$ and $B C$, have been subtracted from the medial (straight-line) $A B$ [Prop. 10.27]. I say that the remainder $A C$ is an irrational (straight-line). Let it be called the first apotome of a medial (straight-line).

For since $A B$ and $B C$ are medial (straight-lines), the (sum of the squares) on $A B$ and $B C$ is also medial. And twice the (rectangle contained) by $A B$ and $B C$ (is) rational. The (sum of the squares) on $A B$ and $B C$ (is) thus incommensurable with twice the (rectangle contained) by $A B$ and $B C$. Thus, twice the (rectangle contained) by $A B$ and $B C$ is also incommensurable with the remaining (square) on $A C$ [Prop. 2.7], since if the whole is incommensurable with one of the (constituent magnitudes) then the original magnitudes will also be incommensurable (with one another) [Prop. 10.16]. And twice the (rectangle contained) by $A B$ and $B C$ (is) rational. Thus, the (square) on $A C$ is irrational. Thus, $A C$ is an irrational (straight-line) [Def. 10.4]. Let it be called a first apotome of a medial (straight-line). ${ }^{\dagger}$

## Proposition 75

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a medial (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).

For let the medial (straight-line) $C B$, which is commensurable in square only with the whole, $A B$, and which contains with the whole, $A B$, the medial (rectangle contained) by $A B$ and $B C$, have been subtracted from the medial (straight-line) $A B$ [Prop. 10.28]. I say that the remainder $A C$ is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).


 тท̀̀ $\Delta \mathrm{H}, \tau \widetilde{̣}$ ठ











 $\tau \widetilde{\uparrow} \mu \varepsilon ̀ \nu \alpha \dot{\alpha} \pi o ̀ ~ \tau n ̃ \varsigma ~ A B ~ \sigma u ̛ ́ \mu \mu \varepsilon \tau p \alpha ́ ~ \varepsilon ̉ \sigma \tau \iota ~ \tau \alpha ̀ ~ \alpha ̇ \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma}, \tau \widetilde{\widetilde{\omega}}$














For let the rational (straight-line) $D I$ be laid down. And let $D E$, equal to the (sum of the squares) on $A B$ and $B C$, have been applied to $D I$, producing $D G$ as breadth. And let $D H$, equal to twice the (rectangle contained) by $A B$ and $B C$, have been applied to $D I$, producing $D F$ as breadth. The remainder $F E$ is thus equal to the (square) on $A C$ [Prop. 2.7]. And since the (squares) on $A B$ and $B C$ are medial and commensurable (with one another), $D E$ (is) thus also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straightline) $D I$, producing $D G$ as breadth. Thus, $D G$ is rational, and incommensurable in length with $D I$ [Prop. 10.22]. Again, since the (rectangle contained) by $A B$ and $B C$ is medial, twice the (rectangle contained) by $A B$ and $B C$ is thus also medial [Prop. 10.23 corr.]. And it is equal to $D H$. Thus, $D H$ is also medial. And it has been applied to the rational (straight-line) $D I$, producing $D F$ as breadth. $D F$ is thus rational, and incommensurable in length with $D I$ [Prop. 10.22]. And since $A B$ and $B C$ are commensurable in square only, $A B$ is thus incommensurable in length with $B C$. Thus, the square on $A B$ (is) also incommensurable with the (rectangle contained) by $A B$ and $B C$ [Props. 10.21 lem., 10.11]. But, the (sum of the squares) on $A B$ and $B C$ is commensurable with the (square) on $A B$ [Prop. 10.15], and twice the (rectangle contained) by $A B$ and $B C$ is commensurable with the (rectangle contained) by $A B$ and $B C$ [Prop. 10.6]. Thus, twice the (rectangle contained) by $A B$ and $B C$ is incommensurable with the (sum of the squares) on $A B$ and $B C$ [Prop. 10.13]. And $D E$ is equal to the (sum of the squares) on $A B$ and $B C$, and $D H$ to twice the (rectangle contained) by $A B$ and $B C$. Thus, $D E$ [is] incommensurable with $D H$. And as $D E$ (is) to $D H$, so $G D$ (is) to $D F$ [Prop. 6.1]. Thus, $G D$ is incommensurable with $D F$ [Prop. 10.11]. And they are both rational (straight-lines). Thus, $G D$ and $D F$ are rational (straight-lines which are) commensurable in square only. Thus, $F G$ is an apotome [Prop. 10.73]. And $D I$ (is) rational. And the (area) contained by a rational and an irrational (straight-line) is irrational [Prop. 10.20], and its square-root is irrational.
${ }^{\dagger}$ See footnote to Prop. 10.38.

$$
O \xi^{\prime} .
$$
















${ }^{\dagger}$ See footnote to Prop. 10.39.

$$
o \zeta^{\prime}
$$












And $A C$ is the square-root of $F E$. Thus, $A C$ is an irrational (straight-line) [Def. 10.4]. And let it be called the second apotome of a medial (straight-line). ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 76

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the (squares) on them (added) together rational, and the (rectangle contained) by them medial, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called a minor (straight-line).


For let the straight-line $B C$, which is incommensurable in square with the whole, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line $A B$ [Prop. 10.33]. I say that the remainder $A C$ is that irrational (straight-line) called minor.

For since the sum of the squares on $A B$ and $B C$ is rational, and twice the (rectangle contained) by $A B$ and $B C$ (is) medial, the (sum of the squares) on $A B$ and $B C$ is thus incommensurable with twice the (rectangle contained) by $A B$ and $B C$. And, via conversion, the (sum of the squares) on $A B$ and $B C$ is incommensurable with the remaining (square) on $A C$ [Props. 2.7, 10.16]. And the (sum of the squares) on $A B$ and $B C$ (is) rational. The (square) on $A C$ (is) thus irrational. Thus, $A C$ (is) an irrational (straight-line) [Def. 10.4]. Let it be called a minor (straight-line). ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 77

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called that which makes with a rational (area) a medial whole.


For let the straight-line $B C$, which is incommensurable in square with $A B$, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line $A B$ [Prop. 10.34]. I say that the remainder $A C$ is the







${ }^{\dagger}$ See footnote to Prop. 10.40.

$$
o \eta^{\prime}
$$










 $\mu \varepsilon ́ \sigma o \cup \mu \varepsilon ́ \sigma o \nu ~ \tau o ̀ ~ o ̈ \lambda o \nu ~ \pi o เ o v ̃ \sigma \alpha . ~$










aforementioned irrational (straight-line).
For since the sum of the squares on $A B$ and $B C$ is medial, and twice the (rectangle contained) by $A B$ and $B C$ rational, the (sum of the squares) on $A B$ and $B C$ is thus incommensurable with twice the (rectangle contained) by $A B$ and $B C$. Thus, the remaining (square) on $A C$ is also incommensurable with twice the (rectangle contained) by $A B$ and $B C$ [Props. 2.7, 10.16]. And twice the (rectangle contained) by $A B$ and $B C$ is rational. Thus, the (square) on $A C$ is irrational. Thus, $A C$ is an irrational (straight-line) [Def. 10.4]. And let it be called that which makes with a rational (area) a medial whole. ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 78

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them, is subtracted from a(nother) straight-line then the remainder is an irrational (straightline). Let it be called that which makes with a medial (area) a medial whole.


For let the straight-line $B C$, which is incommensurable in square $A B$, and fulfils the (other) prescribed (conditions), have been subtracted from the (straightline) $A B$ [Prop. 10.35]. I say that the remainder $A C$ is the irrational (straight-line) called that which makes with a medial (area) a medial whole.

For let the rational (straight-line) $D I$ be laid down. And let $D E$, equal to the (sum of the squares) on $A B$ and $B C$, have been applied to $D I$, producing $D G$ as breadth. And let $D H$, equal to twice the (rectangle contained) by $A B$ and $B C$, have been subtracted (from $D E$ ) [producing $D F$ as breadth]. Thus, the remainder $F E$ is equal to the (square) on $A C$ [Prop. 2.7]. Hence, $A C$ is the square-root of $F E$. And since the sum of the squares on












${ }^{\dagger}$ See footnote to Prop. 10.41.
$O \vartheta^{\prime}$.





 oŨ $\sigma \alpha \tau \tilde{n}$ ö̀ñ.











 ठuvá $\mu \varepsilon \iota \mu$ н́vov $\sigma u ́ \mu \mu \varepsilon \tau \rho о \varsigma ~ o u ̃ \sigma \alpha ~ \tau \tilde{n}$ ö̀n.

$A B$ and $B C$ is medial, and is equal to $D E, D E$ [is] thus medial. And it is applied to the rational (straight-line) $D I$, producing $D G$ as breadth. Thus, $D G$ is rational, and incommensurable in length with $D I$ [Prop 10.22]. Again, since twice the (rectangle contained) by $A B$ and $B C$ is medial, and is equal to $D H, D H$ is thus medial. And it is applied to the rational (straight-line) $D I$, producing $D F$ as breadth. Thus, $D F$ is also rational, and incommensurable in length with $D I$ [Prop. 10.22]. And since the (sum of the squares) on $A B$ and $B C$ is incommensurable with twice the (rectangle contained) by $A B$ and $B C, D E$ (is) also incommensurable with $D H$. And as $D E$ (is) to $D H$, so $D G$ also is to $D F$ [Prop. 6.1]. Thus, $D G$ (is) incommensurable (in length) with $D F$ [Prop. 10.11]. And they are both rational. Thus, $G D$ and $D F$ are rational (straight-lines which are) commensurable in square only. Thus, $F G$ is an apotome [Prop. 10.73]. And $F H$ (is) rational. And the [rectangle] contained by a rational (straight-line) and an apotome is irrational [Prop. 10.20], and its square-root is irrational. And $A C$ is the squareroot of $F E$. Thus, $A C$ is irrational. Let it be called that which makes with a medial (area) a medial whole. ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 79

[Only] one rational straight-line, which is commensurable in square only with the whole, can be attached to an apotome. ${ }^{\dagger}$


Let $A B$ be an apotome, with $B C$ (so) attached to it. $A C$ and $C B$ are thus rational (straight-lines which are) commensurable in square only [Prop. 10.73]. I say that another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to $A B$.

For, if possible, let $B D$ be (so) attached (to $A B$ ). Thus, $A D$ and $D B$ are also rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And since by whatever (area) the (sum of the squares) on $A D$ and $D B$ exceeds twice the (rectangle contained) by $A D$ and $D B$, the (sum of the squares) on $A C$ and $C B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by this (same area). For both exceed by the same (area)(namely), the (square) on $A B$ [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on $A D$ and $D B$ exceeds the (sum of the squares) on $A C$ and $C B$, twice the (rectangle contained) by $A D$ and $D B$ [also] exceeds twice the (rectangle contained) by $A C$ and

$C B$ by this (same area). And the (sum of the squares) on $A D$ and $D B$ exceeds the (sum of the squares) on $A C$ and $C B$ by a rational (area). For both (are) rational (areas). Thus, twice the (rectangle contained) by $A D$ and $D B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.21], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26]. Thus, another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to $A B$.

Thus, only one rational (straight-line), which is commensurable in square only with the whole, can be attached to an apotome. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to Prop. 10.42, with minus signs instead of plus signs.

$$
\pi^{\prime}
$$

 $\mu \varepsilon ́ \sigma \eta ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o s ~ o u ̛ \sigma \alpha ~ \tau n ̃ ~ o ̈ \lambda \eta, ~ \mu \varepsilon \tau \grave{\alpha} \delta \grave{\varepsilon} \tau \tilde{\eta} \varsigma$ ő $\lambda \eta s$ ค́ntòv $\pi \varepsilon \rho เ \varepsilon ́ \chi o \cup \sigma \alpha$.


 $\mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l ~ ¢ ̣ \eta \tau o ̀ v ~ \pi \varepsilon p l \varepsilon ́ \chi o v \sigma \alpha l ~ t o ̀ ~ u ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu ~ А Г, ~ Г В \cdot ~$

 $\pi \varepsilon p t \varepsilon ́ \chi o \cup \sigma \alpha$.

















## Proposition 80

Only one medial straight-line, which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line). ${ }^{\dagger}$

$$
\mathrm{A} \quad \mathrm{~B} \quad \mathrm{C} D
$$

For let $A B$ be a first apotome of a medial (straightline), and let $B C$ be (so) attached to $A B$. Thus, $A C$ and $C B$ are medial (straight-lines which are) commensurable in square only, containing a rational (area)(namely, that contained) by $A C$ and $C B$ [Prop. 10.74]. I say that a(nother) medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, cannot be attached to $A B$.

For, if possible, let $D B$ also be (so) attached to $A B$. Thus, $A D$ and $D B$ are medial (straight-lines which are) commensurable in square only, containing a rational (area)-(namely, that) contained by $A D$ and $D B$ [Prop. 10.74]. And since by whatever (area) the (sum of the squares) on $A D$ and $D B$ exceeds twice the (rectangle contained) by $A D$ and $D B$, the (sum of the squares) on $A C$ and $C B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by this (same area). For [again] both exceed by the same (area)-(namely), the (square) on $A B$ [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on $A D$ and $D B$ exceeds the (sum of the squares) on $A C$ and $C B$, twice the (rectangle contained) by $A D$ and $D B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by this (same area). And twice the (rectangle contained) by $A D$ and $D B$ exceeds twice
the (rectangle contained) by $A C$ and $C B$ by a rational (area). For both (are) rational (areas). Thus, the (sum of the squares) on $A D$ and $D B$ also exceeds the (sum of the) [squares] on $A C$ and $C B$ by a rational (area). The very thing is impossible. For both are medial (areas) [Props. 10.15, 10.23 corr.], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

Thus, only one medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to Prop. 10.43, with minus signs instead of plus signs.

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\pi \alpha^{\prime}
$$


 oั̀ $\eta \varsigma ~ \mu \varepsilon ́ \sigma o \nu ~ \pi \varepsilon p เ \varepsilon ́ \chi o \cup \sigma \alpha$.


 бú $\mu \mu \varepsilon \tau \rho o \iota \mu \varepsilon ́ \sigma o \nu ~ \pi \varepsilon \rho เ \varepsilon ́ \chi o v \sigma \alpha l ~ \tau o ̀ ~ ن ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu ~ А Г, ~ Г В \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~$

 $\pi \varepsilon \rho เ \varepsilon ́ \chi o \cup \sigma \alpha$.

Eî $\gamma \dot{\alpha} \rho \delta u v \alpha \tau o ́ v, \pi \rho o \sigma \alpha \rho \mu o \zeta \varepsilon ́ \tau \omega \dot{\eta} \mathrm{~B} \Delta \cdot x \alpha \grave{i} \alpha \mathrm{~A} \Delta, \Delta \mathrm{~B}$













## Proposition 81

Only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line). ${ }^{\dagger}$


Let $A B$ be a second apotome of a medial (straightline), with $B C$ (so) attached to $A B$. Thus, $A C$ and $C B$ are medial (straight-lines which are) commensurable in square only, containing a medial (area)-(namely, that contained) by $A C$ and $C B$ [Prop. 10.75]. I say that a(nother) medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, cannot be attached to $A B$.

For, if possible, let $B D$ be (so) attached. Thus, $A D$ and $D B$ are also medial (straight-lines which are) commensurable in square only, containing a medial (area) (namely, that contained) by $A D$ and $D B$ [Prop. 10.75]. And let the rational (straight-line) $E F$ be laid down. And let $E G$, equal to the (sum of the squares) on $A C$ and $C B$, have been applied to $E F$, producing $E M$ as breadth. And let $H G$, equal to twice the (rectangle contained) by $A C$ and $C B$, have been subtracted (from $E G$ ), producing $H M$ as breadth. The remainder $E L$ is thus equal to the (square) on $A B$ [Prop. 2.7]. Hence, $A B$ is the




















 $\tau \tilde{n}$ ö̀ $\eta$. ö $\pi \varepsilon \rho$ ह̇бтiv $\alpha \delta \dot{v} v \alpha \tau o \nu$.



square-root of $E L$. So, again, let $E I$, equal to the (sum of the squares) on $A D$ and $D B$ have been applied to $E F$, producing $E N$ as breadth. And $E L$ is also equal to the square on $A B$. Thus, the remainder $H I$ is equal to twice the (rectangle contained) by $A D$ and $D B$ [Prop. 2.7]. And since $A C$ and $C B$ are (both) medial (straight-lines), the (sum of the squares) on $A C$ and $C B$ is also medial. And it is equal to $E G$. Thus, $E G$ is also medial [Props. $10.15,10.23$ corr.]. And it is applied to the rational (straight-line) $E F$, producing $E M$ as breadth. Thus, $E M$ is rational, and incommensurable in length with $E F$ [Prop. 10.22]. Again, since the (rectangle contained) by $A C$ and $C B$ is medial, twice the (rectangle contained) by $A C$ and $C B$ is also medial [Prop. 10.23 corr.]. And it is equal to $H G$. Thus, $H G$ is also medial. And it is applied to the rational (straight-line) $E F$, producing $H M$ as breadth. Thus, $H M$ is also rational, and incommensurable in length with $E F$ [Prop. 10.22]. And since $A C$ and $C B$ are commensurable in square only, $A C$ is thus incommensurable in length with $C B$. And as $A C$ (is) to $C B$, so the (square) on $A C$ is to the (rectangle contained) by $A C$ and $C B$ [Prop. 10.21 corr.]. Thus, the (square) on $A C$ is incommensurable with the (rectangle contained) by $A C$ and $C B$ [Prop. 10.11]. But, the (sum of the squares) on $A C$ and $C B$ is commensurable with the (square) on $A C$, and twice the (rectangle contained) by $A C$ and $C B$ is commensurable with the (rectangle contained) by $A C$ and $C B$ [Prop. 10.6]. Thus, the (sum of the squares) on $A C$ and $C B$ is incommensurable with twice the (rectangle contained) by $A C$ and $C B$ [Prop. 10.13]. And $E G$ is equal to the (sum of the squares) on $A C$ and $C B$. And $G H$ is equal to twice the (rectangle contained) by $A C$ and $C B$. Thus, $E G$ is incommensurable with $H G$. And as $E G$ (is) to $H G$, so $E M$ is to $H M$ [Prop. 6.1]. Thus, $E M$ is incommensurable in length with $M H$ [Prop. 10.11]. And they are both rational (straight-lines). Thus, $E M$ and $M H$ are rational (straight-lines which are) commensurable in square only. Thus, $E H$ is an apotome [Prop. 10.73], and $H M$ (is) attached to it. So, similarly, we can show that $H N$ (is) also (commensurable in square only with $E N$ and is) attached to $(E H)$. Thus, different straight-lines, which are commensurable in square only with the whole, are attached to an apotome. The very thing is impossible [Prop. 10.79].

Thus, only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to Prop. 10.44, with minus signs instead of plus signs.

$$
\pi \beta^{\prime}
$$



 $\mu \varepsilon ́ \sigma o v$.




 عủvะĩ oủ $\pi \rho \circ \sigma \alpha \rho \mu o ́ \sigma \varepsilon l ~ \tau \alpha ̀ ~ \alpha u ̉ \tau \alpha ̀ ~ \pi o เ o u ̃ \sigma \alpha . ~$

 غ̇ $\tau \varepsilon i ́, \widetilde{\varphi} \dot{\varphi} \dot{u} \pi \varepsilon \rho \varepsilon ́ \chi \varepsilon \iota \tau \grave{\alpha} \alpha \dot{\alpha} \pi \grave{\tau} \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B} \tau \widetilde{\omega} \nu \dot{\alpha} \pi o ̀ \tau \widetilde{\omega} \nu \mathrm{~A} \Gamma, \Gamma \mathrm{~B}$,
 $\tau \widetilde{\omega} \nu \mathrm{A} \Gamma, \Gamma \mathrm{B}, \tau \grave{\alpha} \delta \grave{\varepsilon} \alpha{ }_{\alpha} \tau \grave{o} \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B} \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \alpha \tau \widetilde{\omega} \nu \dot{\alpha} \pi o ̀$


 үáp ह̇бтıレ $\alpha \mu \varphi o ́ \tau \varepsilon \rho \alpha . ~$


 ઠєǐそ $\alpha$.

## Proposition 82

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the (sum of the) squares on them rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).


Let $A B$ be a minor (straight-line), and let $B C$ be (so) attached to $A B$. Thus, $A C$ and $C B$ are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial [Prop. 10.76]. I say that another another straight-line fulfilling the same (conditions) cannot be attached to $A B$.

For, if possible, let $B D$ be (so) attached (to $A B$ ). Thus, $A D$ and $D B$ are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76]. And since by whatever (area) the (sum of the squares) on $A D$ and $D B$ exceeds the (sum of the squares) on $A C$ and $C B$, twice the (rectangle contained) by $A D$ and $D B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by this (same area) [Prop. 2.7]. And the (sum of the) squares on $A D$ and $D B$ exceeds the (sum of the) squares on $A C$ and $C B$ by a rational (area). For both are rational (areas). Thus, twice the (rectangle contained) by $A D$ and $D B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, only one straight-line, which is incommensurable in square with the whole, and (with the whole) makes the squares on them (added) together rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line). (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to Prop. 10.45, with minus signs instead of plus signs.

$$
\pi \gamma^{\prime}
$$

Tñ $\mu \varepsilon \tau \alpha ̀ ~ \rho ீ \eta \tau о u ̃ ~ \mu \varepsilon ́ \sigma o v ~ \tau o ̀ ~ o ̋ \lambda o v ~ \pi o เ o u ́ \sigma n ~ \mu i ́ \alpha ~ \mu o ́ v o v ~ \pi \rho o-~$





## Proposition 83

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, can be attached to that (straight-line) which with a rational (area) makes a medial whole. ${ }^{\dagger}$



 oủ тробариóбєl $\tau \alpha ̀ \alpha u ̉ \tau \alpha ̀ ~ \pi o เ o u ̃ \sigma \alpha . ~$













Let $A B$ be a (straight-line) which with a rational (area) makes a medial whole, and let $B C$ be (so) attached to $A B$. Thus, $A C$ and $C B$ are (straight-lines which are) incommensurable in square, fulfilling the (other) proscribed (conditions) [Prop. 10.77]. I say that another (straight-line) fulfilling the same (conditions) cannot be attached to $A B$.

For, if possible, let $B D$ be (so) attached (to $A B$ ). Thus, $A D$ and $D B$ are also straight-lines (which are) incommensurable in square, fulfilling the (other) prescribed (conditions) [Prop. 10.77]. Therefore, analogously to the (propositions) before this, since by whatever (area) the (sum of the squares) on $A D$ and $D B$ exceeds the (sum of the squares) on $A C$ and $C B$, twice the (rectangle contained) by $A D$ and $D B$ also exceeds twice the (rectangle contained) by $A C$ and $C B$ by this (same area). And twice the (rectangle contained) by $A D$ and $D B$ exceeds twice the (rectangle contained) by $A C$ and $C B$ by a rational (area). For they are (both) rational (areas). Thus, the (sum of the squares) on $A D$ and $D B$ also exceeds the (sum of the squares) on $A C$ and $C B$ by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, another straight-line cannot be attached to $A B$, which is incommensurable in square with the whole, and fulfills the (other) aforementioned (conditions) with the whole. Thus, only one (such straight-line) can be (so) attached. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to Prop. 10.46, with minus signs instead of plus signs.

$$
\pi \delta^{\prime}
$$











## Proposition 84

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the (squares) on them, can be attached to that (straight-line) which with a medial (area) makes a medial whole. ${ }^{\dagger}$

Let $A B$ be a (straight-line) which with a medial (area) makes a medial whole, $B C$ being (so) attached to it. Thus, $A C$ and $C B$ are incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78]. I say that a(nother) (straight-line) fulfilling the aforementioned (conditions) cannot be attached to $A B$.



 $\mathrm{A} \Delta, \Delta \mathrm{B} \mu \varepsilon ́ \sigma o \nu x \alpha \grave{l}$ है $\tau \iota \tau \dot{\alpha} \dot{\alpha} \pi \grave{o} \tau \widetilde{\omega} \nu \mathrm{~A} \Delta, \Delta \mathrm{~B} \dot{\alpha} \sigma \dot{\jmath} \mu \mu \varepsilon \tau \rho \alpha \tau \widetilde{\varphi}$





 $\pi \alpha \rho \alpha ̀ ~ \tau \grave{\eta} \nu \mathrm{EZ} \pi \alpha \rho \alpha \beta \varepsilon \beta \lambda \dot{\eta} \sigma \vartheta \omega$ 七ò $\mathrm{EI} \pi \lambda \alpha ́ \tau o s ~ \pi o เ o u ̃ \nu ~ \tau \eta ̀ \nu ~ E N . ~$

























For, if possible, let $B D$ be (so) attached. Hence, $A D$ and $D B$ are also (straight-lines which are) incommensurable in square, making the squares on $A D$ and $D B$ (added) together medial, and twice the (rectangle contained) by $A D$ and $D B$ medial, and, moreover, the (sum of the squares) on $A D$ and $D B$ incommensurable with twice the (rectangle contained) by $A D$ and $D B$ [Prop. 10.78]. And let the rational (straight-line) $E F$ be laid down. And let $E G$, equal to the (sum of the squares) on $A C$ and $C B$, have been applied to $E F$, producing $E M$ as breadth. And let $H G$, equal to twice the (rectangle contained) by $A C$ and $C B$, have been applied to $E F$, producing $H M$ as breadth. Thus, the remaining (square) on $A B$ is equal to $E L$ [Prop. 2.7]. Thus, $A B$ is the squareroot of $E L$. Again, let $E I$, equal to the (sum of the squares) on $A D$ and $D B$, have been applied to $E F$, producing $E N$ as breadth. And the (square) on $A B$ is also equal to $E L$. Thus, the remaining twice the (rectangle contained) by $A D$ and $D B$ [is] equal to $H I$ [Prop. 2.7]. And since the sum of the (squares) on $A C$ and $C B$ is medial, and is equal to $E G, E G$ is thus also medial. And it is applied to the rational (straight-line) $E F$, producing $E M$ as breadth. $E M$ is thus rational, and incommensurable in length with $E F$ [Prop. 10.22]. Again, since twice the (rectangle contained) by $A C$ and $C B$ is medial, and is equal to $H G, H G$ is thus also medial. And it is applied to the rational (straight-line) $E F$, producing $H M$ as breadth. $H M$ is thus rational, and incommensurable in length with $E F$ [Prop. 10.22]. And since the (sum of the squares) on $A C$ and $C B$ is incommensurable with twice the (rectangle contained) by $A C$ and $C B, E G$ is also incommensurable with $H G$. Thus, $E M$ is also incommensurable in length with $M H$ [Props. 6.1, 10.11]. And they are both rational (straight-lines). Thus, $E M$ and $M H$ are rational (straight-lines which are) commensurable in square only. Thus, $E H$ is an apotome [Prop. 10.73], with $H M$ attached to it. So, similarly, we can show that $E H$ is again an apotome, with $H N$ attached to it. Thus, different rational (straight-lines), which are commensurable in square only with the whole, are attached to an apotome. The very thing was shown
(to be) impossible [Prop. 10.79]. Thus, another straightline cannot be (so) attached to $A B$.

Thus, only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the squares on them (added) together medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with the (rectangle contained) by them, can be attached to $A B$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is equivalent to Prop. 10.47, with minus signs instead of plus signs.

## "Opot tpítol.



 $x \alpha \lambda \varepsilon i ́ \sigma \vartheta \omega$ аं $\pi о \tau о \mu \dot{\eta} \pi \rho \omega ́ \tau \eta$.

 $\tau \widetilde{\varphi} \alpha \dot{\alpha}$ ò $\sigma \cup \mu \mu \varepsilon ́ \tau \rho о \cup \dot{\varepsilon} \alpha \cup \tau \tilde{n}, \chi \alpha \lambda \varepsilon i ́ \sigma \vartheta \omega \alpha \dot{\alpha} \pi о \tau о \mu \grave{\eta} \delta \varepsilon \cup \tau \varepsilon ́ \rho \alpha$.





 $\tau \varepsilon \tau \alpha ́ p \tau \eta$.

เモ'. 'E $\alpha \nu$ ठغ̀ $\dot{\eta} \pi \rho o \sigma \alpha \rho \mu o ́ \zeta o v \sigma \alpha, \pi \varepsilon ́ \mu \pi \tau \eta$.


## Definitions III

11. Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome.
12. And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome.
13. And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome.
14. Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome.
15. And if the attached (straight-line is commensurable), a fifth (apotome).
16. And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

## Proposition 85

To find a first apotome.




 öv тєтра́ $\gamma \omega \nu \circ \varsigma \alpha \dot{\alpha} \rho เ \vartheta \mu o ̀ \varsigma ~ \pi \rho o ̀ \varsigma ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o v ~ \alpha ́ p เ \vartheta \mu o ́ v . ~ \chi \alpha \grave{~} \pi \varepsilon-$
 $\tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \circ \nu \pi \rho o ̀ s ~ t o ̀ ~ \alpha ́ \pi o ̀ ~ t \eta ̀ s ~ Н Г ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega v o v \cdot ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o v ~$
 тñs BH























Let the rational (straight-line) $A$ be laid down. And let $B G$ be commensurable in length with $A . B G$ is thus also a rational (straight-line). And let two square numbers $D E$ and $E F$ be laid down, and let their difference $F D$ be not square [Prop. 10.28 lem. I]. Thus, $E D$ does not have to $D F$ the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as $E D$ (is) to $D F$, so the square on $B G$ (is) to the square on $G C$ [Prop. 10.6. corr.]. Thus, the (square) on $B G$ is commensurable with the (square) on $G C$ [Prop. 10.6]. And the (square) on $B G$ (is) rational. Thus, the (square) on $G C$ (is) also rational. Thus, $G C$ is also rational. And since $E D$ does not have to $D F$ the ratio which (some) square number (has) to (some) square number, the (square) on $B G$ thus does not have to the (square) on $G C$ the ratio which (some) square number (has) to (some) square number either. Thus, $B G$ is incommensurable in length with $G C$ [Prop. 10.9]. And they are both rational (straight-lines). Thus, $B G$ and $G C$ are rational (straight-lines which are) commensurable in square only. Thus, $B C$ is an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

Let the (square) on $H$ be that (area) by which the (square) on $B G$ is greater than the (square) on $G C$ [Prop. 10.13 lem.]. And since as $E D$ is to $F D$, so the (square) on $B G$ (is) to the (square) on $G C$, thus, via conversion, as $D E$ is to $E F$, so the (square) on $G B$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. And $D E$ has to $E F$ the ratio which (some) square-number (has) to (some) square-number. For each is a square (number). Thus, the (square) on $G B$ also has to the (square) on $H$ the ratio which (some) square number (has) to (some) square number. Thus, $B G$ is commensurable in length with $H$ [Prop. 10.9]. And the square on $B G$ is greater than (the square on) $G C$ by the (square) on $H$. Thus, the square on $B G$ is greater than (the square on) $G C$ by the (square) on (some straight-line) commensurable in length with $(B G)$. And the whole, $B G$, is commensurable in length with the (previously) laid down rational (straight-line) $A$. Thus, $B C$ is a first apotome [Def. 10.11].

Thus, the first apotome $B C$ has been found. (Which is) the very thing it was required to find.

## Proposition 86

To find a second apotome.
${ }^{\circ}$ Еxxєíб७
 $\dot{\alpha}$ рı

























Let the rational (straight-line) $A$, and $G C$ (which is) commensurable in length with $A$, be laid down. Thus, $G C$ is a rational (straight-line). And let the two square numbers $D E$ and $E F$ be laid down, and let their difference $D F$ be not square [Prop. 10.28 lem. I]. And let it have been contrived that as $F D$ (is) to $D E$, so the square on $C G$ (is) to the square on $G B$ [Prop. 10.6 corr.]. Thus, the square on $C G$ is commensurable with the square on $G B$ [Prop. 10.6]. And the (square) on $C G$ (is) rational. Thus, the (square) on $G B$ [is] also rational. Thus, $B G$ is a rational (straight-line). And since the square on $G C$ does not have to the (square) on $G B$ the ratio which (some) square number (has) to (some) square number, $C G$ is incommensurable in length with $G B$ [Prop. 10.9]. And they are both rational (straight-lines). Thus, $C G$ and $G B$ are rational (straight-lines which are) commensurable in square only. Thus, $B C$ is an apotome [Prop. 10.73]. So, I say that it is also a second (apotome).


For let the (square) on $H$ be that (area) by which the (square) on $B G$ is greater than the (square) on $G C$ [Prop. 10.13 lem.]. Therefore, since as the (square) on $B G$ is to the (square) on $G C$, so the number $E D$ (is) to the number $D F$, thus, also, via conversion, as the (square) on $B G$ is to the (square) on $H$, so $D E$ (is) to $E F$ [Prop. 5.19 corr.]. And $D E$ and $E F$ are each square (numbers). Thus, the (square) on $B G$ has to the (square) on $H$ the ratio which (some) square number (has) to (some) square number. Thus, $B G$ is commensurable in length with $H$ [Prop. 10.9]. And the square on $B G$ is greater than (the square on) $G C$ by the (square) on $H$. Thus, the square on $B G$ is greater than (the square on) $G C$ by the (square) on (some straight-line) commensurable in length with $(B G)$. And the attachment $C G$ is commensurable (in length) with the (prevously) laid down rational (straight-line) $A$. Thus, $B C$ is a second apotome [Def. 10.12]. ${ }^{\dagger}$

Thus, the second apotome $B C$ has been found. (Which is) the very thing it was required to show.

Proposition 87
To find a third apotome.

$\mathrm{E} \longmapsto$

K


























 $\lambda \varepsilon ́ \gamma \omega$ ón, ötı xai tpítๆ.



















Let the rational (straight-line) $A$ be laid down. And let the three numbers, $E, B C$, and $C D$, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. And let $C B$ have to $B D$ the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as $E$ (is) to $B C$, so the square on $A$ (is) to the square on $F G$, and as $B C$ (is) to $C D$, so the square on $F G$ (is) to the (square) on $G H$ [Prop. 10.6 corr.]. Therefore, since as $E$ is to $B C$, so the square on $A$ (is) to the square on $F G$, the square on $A$ is thus commensurable with the square on $F G$ [Prop. 10.6]. And the square on $A$ (is) rational. Thus, the (square) on $F G$ (is) also rational. Thus, $F G$ is a rational (straight-line). And since $E$ does not have to $B C$ the ratio which (some) square number (has) to (some) square number, the square on $A$ thus does not have to the [square] on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, $A$ is incommensurable in length with $F G$ [Prop. 10.9]. Again, since as $B C$ is to $C D$, so the square on $F G$ is to the (square) on $G H$, the square on $F G$ is thus commensurable with the (square) on $G H$ [Prop. 10.6]. And the (square) on $F G$ (is) rational. Thus, the (square) on $G H$ (is) also rational. Thus, $G H$ is a rational (straightline). And since $B C$ does not have to $C D$ the ratio which (some) square number (has) to (some) square number, the (square) on $F G$ thus does not have to the (square) on $G H$ the ratio which (some) square number (has) to (some) square number either. Thus, $F G$ is incommensurable in length with $G H$ [Prop. 10.9]. And both are rational (straight-lines). $F G$ and $G H$ are thus rational (straight-lines which are) commensurable in square only. Thus, $F H$ is an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since as $E$ is to $B C$, so the square on $A$ (is) to the (square) on $F G$, and as $B C$ (is) to $C D$, so the (square) on $F G$ (is) to the (square) on $H G$, thus, via equality, as $E$ is to $C D$, so the (square) on $A$ (is) to the (square) on $H G$ [Prop. 5.22]. And $E$ does not have to $C D$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $A$ does not have to the (square) on $G H$ the ratio which (some) square number (has) to (some) square number either. $A$ (is) thus incommensurable in length with $G H$ [Prop. 10.9]. Thus, neither of $F G$ and $G H$ is commensurable in length with the





${ }^{\dagger}$ See footnote to Prop. 10.50.




 ג́pıখ








 [ $\lambda \varepsilon ́ \gamma \omega$ б́ń, öт兀 жаi $\tau \varepsilon \tau \alpha ́ p \tau \eta$.]






(previously) laid down rational (straight-line) $A$. Therefore, let the (square) on $K$ be that (area) by which the (square) on $F G$ is greater than the (square) on $G H$ [Prop. 10.13 lem.]. Therefore, since as $B C$ is to $C D$, so the (square) on $F G$ (is) to the (square) on $G H$, thus, via conversion, as $B C$ is to $B D$, so the square on $F G$ (is) to the square on $K$ [Prop. 5.19 corr.]. And $B C$ has to $B D$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $F G$ also has to the (square) on $K$ the ratio which (some) square number (has) to (some) square number. $F G$ is thus commensurable in length with $K$ [Prop. 10.9]. And the square on $F G$ is (thus) greater than (the square on) $G H$ by the (square) on (some straight-line) commensurable (in length) with $(F G)$. And neither of $F G$ and $G H$ is commensurable in length with the (previously) laid down rational (straight-line) $A$. Thus, $F H$ is a third apotome [Def. 10.13].

Thus, the third apotome FH has been found. (Which is) very thing it was required to show.

## Proposition 88

To find a fourth apotome.


Let the rational (straight-line) $A$, and $B G$ (which is) commensurable in length with $A$, be laid down. Thus, $B G$ is also a rational (straight-line). And let the two numbers $D F$ and $F E$ be laid down such that the whole, $D E$, does not have to each of $D F$ and $E F$ the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as $D E$ (is) to $E F$, so the square on $B G$ (is) to the (square) on $G C$ [Prop. 10.6 corr.]. The (square) on $B G$ is thus commensurable with the (square) on $G C$ [Prop. 10.6]. And the (square) on $B G$ (is) rational. Thus, the (square) on $G C$ (is) also rational. Thus, $G C$ (is) a rational (straightline). And since $D E$ does not have to $E F$ the ratio which (some) square number (has) to (some) square number, the (square) on $B G$ thus does not have to the (square) on $G C$ the ratio which (some) square number (has) to (some) square number either. Thus, $B G$ is incommensurable in length with $G C$ [Prop. 10.9]. And they are both rational (straight-lines). Thus, $B G$ and $G C$ are rational (straight-lines which are) commensurable in square only. Thus, $B C$ is an apotome [Prop. 10.73]. [So, I say that (it





 $\tau \circ \mu \dot{\eta}$ ह̇бть $\tau \varepsilon \tau \alpha ́ \rho \tau \eta$.

is) also a fourth (apotome).]
Now, let the (square) on $H$ be that (area) by which the (square) on $B G$ is greater than the (square) on $G C$ [Prop. 10.13 lem.]. Therefore, since as $D E$ is to $E F$, so the (square) on $B G$ (is) to the (square) on $G C$, thus, also, via conversion, as $E D$ is to $D F$, so the (square) on $G B$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. And $E D$ does not have to $D F$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $G B$ does not have to the (square) on $H$ the ratio which (some) square number (has) to (some) square number either. Thus, $B G$ is incommensurable in length with $H$ [Prop. 10.9]. And the square on $B G$ is greater than (the square on) $G C$ by the (square) on $H$. Thus, the square on $B G$ is greater than (the square) on $G C$ by the (square) on (some straight-line) incommensurable (in length) with $(B G)$. And the whole, $B G$, is commensurable in length with the the (previously) laid down rational (straightline) $A$. Thus, $B C$ is a fourth apotome [Def. 10.14]. ${ }^{\dagger}$

Thus, a fourth apotome has been found. (Which is) the very thing it was required to show.

## Proposition 89

To find a fifth apotome.


Let the rational (straight-line) $A$ be laid down, and let $C G$ be commensurable in length with $A$. Thus, $C G$ [is] a rational (straight-line). And let the two numbers $D F$ and $F E$ be laid down such that $D E$ again does not have to each of $D F$ and $F E$ the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as $F E$ (is) to $E D$, so the (square) on $C G$ (is) to the (square) on $G B$. Thus, the (square) on $G B$ (is) also rational [Prop. 10.6]. Thus, $B G$ is also rational. And since as $D E$ is to $E F$, so the (square) on $B G$ (is) to the (square) on $G C$. And $D E$ does not have to $E F$ the ratio which (some) square number (has) to (some) square number. The (square) on $B G$ thus does not have to the (square) on $G C$ the ratio which (some) square number (has) to (some) square number either. Thus, $B G$ is incommensurable in length with $G C$ [Prop. 10.9]. And they are both rational (straight-lines). $B G$ and $G C$ are thus rational (straight-lines which are) commensurable in square only. Thus, $B C$ is an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).





 тñ ย̇x $\pi \varepsilon ́ \mu \pi \tau \eta$.

${ }^{\dagger}$ See footnote to Prop. 10.52.
4.






 $\Gamma \Delta$, oưt $\omega \varsigma$ tò $\dot{\alpha} \pi o ̀ ~ \tau \tilde{\eta} \varsigma \mathrm{ZH} \pi \rho o ̀ s ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau \tilde{n} \varsigma ~ H \Theta . ~$


K







 ó $\mathrm{B} \Gamma \pi \rho o ̀ s ~ \tau o ̀ v ~ \Gamma \Delta, ~ o u ̛ \tau \omega s ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau n ̃ s ~ Z H ~ \pi p o ̀ s ~ t o ̀ ~ \alpha ̉ \pi o ̀ ~ \tau n ̃ s ~$


For, let the (square) on $H$ be that (area) by which the (square) on $B G$ is greater than the (square) on $G C$ [Prop. 10.13 lem.]. Therefore, since as the (square) on $B G$ (is) to the (square) on $G C$, so $D E$ (is) to $E F$, thus, via conversion, as $E D$ is to $D F$, so the (square) on $B G$ (is) to the (square) on $H$ [Prop. 5.19 corr.]. And $E D$ does not have to $D F$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $B G$ does not have to the (square) on $H$ the ratio which (some) square number (has) to (some) square number either. Thus, $B G$ is incommensurable in length with $H$ [Prop. 10.9]. And the square on $B G$ is greater than (the square on) $G C$ by the (square) on $H$. Thus, the square on $G B$ is greater than (the square on) $G C$ by the (square) on (some straight-line) incommensurable in length with $(G B)$. And the attachment $C G$ is commensurable in length with the (previously) laid down rational (straightline) $A$. Thus, $B C$ is a fifth apotome [Def. 10.15]. ${ }^{\dagger}$

Thus, the fifth apotome $B C$ has been found. (Which is) the very thing it was required to show.

## Proposition 90

To find a sixth apotome.
Let the rational (straight-line) $A$, and the three numbers $E, B C$, and $C D$, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. Furthermore, let $C B$ also not have to $B D$ the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as $E$ (is) to $B C$, so the (square) on $A$ (is) to the (square) on $F G$, and as $B C$ (is) to $C D$, so the (square) on $F G$ (is) to the (square) on $G H$ [Prop. 10.6 corr.].


Therefore, since as $E$ is to $B C$, so the (square) on $A$ (is) to the (square) on $F G$, the (square) on $A$ (is) thus commensurable with the (square) on $F G$ [Prop. 10.6]. And the (square) on $A$ (is) rational. Thus, the (square) on $F G$ (is) also rational. Thus, $F G$ is also a rational (straight-line). And since $E$ does not have to $B C$ the ratio which (some) square number (has) to (some) square number, the (square) on $A$ thus does not have to the (square) on $F G$ the ratio which (some) square number (has) to (some) square number either. Thus, $A$ is in-
















 $\mu \eta$ ńx


 tò $\alpha \pi o ̀ ~ \tau n ̃ s ~ K . ~ o ́ ~ \delta e ̀ ~ Г В ~ \pi p o ̀ s ~ t o ̀ v ~ B \Delta ~ \lambda o ́ \gamma o v ~ o u ̛ x ~ ह ै \chi \varepsilon ા, ~ o ̋ v ~$








commensurable in length with $F G$ [Prop. 10.9]. Again, since as $B C$ is to $C D$, so the (square) on $F G$ (is) to the (square) on $G H$, the (square) on $F G$ (is) thus commensurable with the (square) on $G H$ [Prop. 10.6]. And the (square) on $F G$ (is) rational. Thus, the (square) on $G H$ (is) also rational. Thus, $G H$ (is) also rational. And since $B C$ does not have to $C D$ the ratio which (some) square number (has) to (some) square number, the (square) on $F G$ thus does not have to the (square) on $G H$ the ratio which (some) square (number) has to (some) square (number) either. Thus, $F G$ is incommensurable in length with $G H$ [Prop. 10.9]. And both are rational (straightlines). Thus, $F G$ and $G H$ are rational (straight-lines which are) commensurable in square only. Thus, $F H$ is an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since as $E$ is to $B C$, so the (square) on $A$ (is) to the (square) on $F G$, and as $B C$ (is) to $C D$, so the (square) on $F G$ (is) to the (square) on $G H$, thus, via equality, as $E$ is to $C D$, so the (square) on $A$ (is) to the (square) on $G H$ [Prop. 5.22]. And $E$ does not have to $C D$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $A$ does not have to the (square) $G H$ the ratio which (some) square number (has) to (some) square number either. $A$ is thus incommensurable in length with $G H$ [Prop. 10.9]. Thus, neither of $F G$ and $G H$ is commensurable in length with the rational (straight-line) $A$. Therefore, let the (square) on $K$ be that (area) by which the (square) on $F G$ is greater than the (square) on $G H$ [Prop. 10.13 lem.]. Therefore, since as $B C$ is to $C D$, so the (square) on $F G$ (is) to the (square) on $G H$, thus, via conversion, as $C B$ is to $B D$, so the (square) on $F G$ (is) to the (square) on $K$ [Prop. 5.19 corr.]. And $C B$ does not have to $B D$ the ratio which (some) square number (has) to (some) square number. Thus, the (square) on $F G$ does not have to the (square) on $K$ the ratio which (some) square number (has) to (some) square number either. $F G$ is thus incommensurable in length with $K$ [Prop. 10.9]. And the square on $F G$ is greater than (the square on) $G H$ by the (square) on $K$. Thus, the square on $F G$ is greater than (the square on) $G H$ by the (square) on (some straightline) incommensurable in length with $(F G)$. And neither of $F G$ and $G H$ is commensurable in length with the (previously) laid down rational (straight-line) $A$. Thus, $F H$ is a sixth apotome [Def. 10.16].

Thus, the sixth apotome $F H$ has been found. (Which is) the very thing it was required to show.

[^31]
## $4 \alpha^{\prime}$.


















 д̈ $\rho \alpha$ غ் $\chi \alpha \tau \varepsilon ́ \rho \alpha \tau \widetilde{\omega} v \mathrm{AZ}, \mathrm{ZH} \sigma \dot{\mu} \mu \mu \varepsilon \tau \rho o ́ s ~ \varepsilon ̇ \sigma \tau \iota ~ \mu \dot{\eta}\langle\varepsilon \iota . ~ \dot{\alpha} \lambda \lambda \dot{\alpha} \dot{\eta}$























## Proposition 91

If an area is contained by a rational (straight-line) and a first apotome then the square-root of the area is an apotome.

For let the area $A B$ have been contained by the rational (straight-line) $A C$ and the first apotome $A D$. I say that the square-root of area $A B$ is an apotome.


For since $A D$ is a first apotome, let $D G$ be its attachment. Thus, $A G$ and $D G$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And the whole, $A G$, is commensurable (in length) with the (previously) laid down rational (straight-line) $A C$, and the square on $A G$ is greater than (the square on) $G D$ by the (square) on (some straight-line) commensurable in length with $(A G)$ [Def. 10.11]. Thus, if (an area) equal to the fourth part of the (square) on $D G$ is applied to $A G$, falling short by a square figure, then it divides ( $A G$ ) into (parts which are) commensurable (in length) [Prop. 10.17]. Let $D G$ have been cut in half at $E$. And let (an area) equal to the (square) on $E G$ have been applied to $A G$, falling short by a square figure. And let it be the (rectangle contained) by $A F$ and $F G . A F$ is thus commensurable (in length) with $F G$. And let $E H, F I$, and $G K$ have been drawn through points $E, F$, and $G$ (respectively), parallel to $A C$.

And since $A F$ is commensurable in length with $F G$, $A G$ is thus also commensurable in length with each of $A F$ and $F G$ [Prop. 10.15]. But $A G$ is commensurable (in length) with $A C$. Thus, each of $A F$ and $F G$ is also commensurable in length with $A C$ [Prop. 10.12]. And $A C$ is a rational (straight-line). Thus, $A F$ and $F G$ (are) each also rational (straight-lines). Hence, $A I$ and $F K$ are also each rational (areas) [Prop. 10.19]. And since $D E$ is commensurable in length with $E G, D G$ is thus also commensurable in length with each of $D E$ and $E G$ [Prop. 10.15]. And $D G$ (is) rational, and incommensurable in length with $A C . D E$ and $E G$ (are) thus each rational, and incommensurable in length with $A C$ [Prop. 10.13]. Thus, $D H$ and $E K$ are each medial (areas) [Prop. 10.21].

So let the square $L M$, equal to $A I$, be laid down. And let the square $N O$, equal to $F K$, have been sub-


















## ' $\beta^{\prime}$.





tracted (from $L M$ ), having with it the common angle $L P M$. Thus, the squares $L M$ and $N O$ are about the same diagonal [Prop. 6.26]. Let $P R$ be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the rectangle contained by $A F$ and $F G$ is equal to the square $E G$, thus as $A F$ is to $E G$, so $E G$ (is) to $F G$ [Prop. 6.17]. But, as $A F$ (is) to $E G$, so $A I$ (is) to $E K$, and as $E G$ (is) to $F G$, so $E K$ is to $K F$ [Prop. 6.1]. Thus, $E K$ is the mean proportional to $A I$ and $K F$ [Prop. 5.11]. And $M N$ is also the mean proportional to $L M$ and $N O$, as shown before [Prop. 10.53 lem.]. And $A I$ is equal to the square $L M$, and $K F$ to $N O$. Thus, $M N$ is also equal to $E K$. But, $E K$ is equal to $D H$, and $M N$ to $L O$ [Prop. 1.43]. Thus, $D K$ is equal to the gnomon $U V W$ and $N O$. And $A K$ is also equal to (the sum of) the squares $L M$ and $N O$. Thus, the remainder $A B$ is equal to $S T$. And $S T$ is the square on $L N$. Thus, the square on $L N$ is equal to $A B$. Thus, $L N$ is the square-root of $A B$. So, I say that $L N$ is an apotome.

For since $A I$ and $F K$ are each rational (areas), and are equal to $L M$ and $N O$ (respectively), thus $L M$ and $N O$-that is to say, the (squares) on each of $L P$ and $P N$ (respectively) -are also each rational (areas). Thus, $L P$ and $P N$ are also each rational (straight-lines). Again, since $D H$ is a medial (area), and is equal to $L O, L O$ is thus also a medial (area). Therefore, since $L O$ is medial, and $N O$ rational, $L O$ is thus incommensurable with $N O$. And as $L O$ (is) to $N O$, so $L P$ is to $P N$ [Prop. 6.1]. $L P$ is thus incommensurable in length with $P N$ [Prop. 10.11]. And they are both rational (straightlines). Thus, $L P$ and $P N$ are rational (straight-lines which are) commensurable in square only. Thus, $L N$ is an apotome [Prop. 10.73]. And it is the square-root of area $A B$. Thus, the square-root of area $A B$ is an apotome.

Thus, if an area is contained by a rational (straightline), and so on ....

## Proposition 92

If an area is contained by a rational (straight-line) and a second apotome then the square-root of the area is a first apotome of a medial (straight-line).

































 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{ZH} \cdot \dot{\alpha} \lambda \lambda \lambda^{\prime} \dot{\omega} \varsigma \mu \mathrm{E} \nu \dot{\eta} \mathrm{AZ} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{EH}$, oütcus tò AI



















For let the area $A B$ have been contained by the rational (straight-line) $A C$ and the second apotome $A D$. I say that the square-root of area $A B$ is the first apotome of a medial (straight-line).

For let $D G$ be an attachment to $A D$. Thus, $A G$ and $G D$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment $D G$ is commensurable (in length) with the (previously) laid down rational (straight-line) $A C$, and the square on the whole, $A G$, is greater than (the square on) the attachment, $G D$, by the (square) on (some straight-line) commensurable in length with $(A G)$ [Def. 10.12]. Therefore, since the square on $A G$ is greater than (the square on) $G D$ by the (square) on (some straight-line) commensurable (in length) with ( $A G$ ), thus if (an area) equal to the fourth part of the (square) on $G D$ is applied to $A G$, falling short by a square figure, then it divides ( $A G$ ) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let $D G$ have been cut in half at $E$. And let (an area) equal to the (square) on $E G$ have been applied to $A G$, falling short by a square figure. And let it be the (rectangle contained) by $A F$ and $F G$. Thus, $A F$ is commensurable in length with $F G . A G$ is thus also commensurable in length with each of $A F$ and $F G$ [Prop. 10.15]. And $A G$ (is) a rational (straight-line), and incommensurable in length with $A C . A F$ and $F G$ are thus also each rational (straight-lines), and incommensurable in length with $A C$ [Prop. 10.13]. Thus, $A I$ and $F K$ are each medial (areas) [Prop. 10.21]. Again, since $D E$ is commensurable (in length) with $E G$, thus $D G$ is also commensurable (in length) with each of $D E$ and $E G$ [Prop. 10.15]. But, $D G$ is commensurable in length with $A C$ [thus, $D E$ and $E G$ are also each rational, and commensurable in length with $A C]$. Thus, $D H$ and $E K$ are each rational (areas) [Prop. 10.19].

Therefore, let the square $L M$, equal to $A I$, have been constructed. And let $N O$, equal to $F K$, which is about the same angle $L P M$ as $L M$, have been subtracted (from $L M$ ). Thus, the squares $L M$ and $N O$ are about the same diagonal [Prop. 6.26]. Let $P R$ be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since $A I$ and $F K$ are medial (areas), and are equal to the (squares) on $L P$ and $P N$ (respectively), [thus] the (squares) on $L P$ and $P N$ are also medial. Thus, $L P$ and $P N$ are also medial (straight-lines which are) commensurable in square only. ${ }^{\dagger}$ And since the (rectangle contained) by $A F$ and $F G$ is equal to the (square) on $E G$, thus as $A F$ is to $E G$, so $E G$ (is) to $F G$ [Prop. 10.17]. But, as $A F$ (is) to $E G$, so $A I$ (is) to $E K$. And as $E G$ (is) to $F G$, so $E K$ [is] to $F K$ [Prop. 6.1]. Thus, $E K$ is the mean proportional to $A I$



and $F K$ [Prop. 5.11]. And $M N$ is also the mean proportional to the squares $L M$ and $N O$ [Prop. 10.53 lem.]. And $A I$ is equal to $L M$, and $F K$ to $N O$. Thus, $M N$ is also equal to $E K$. But, $D H$ [is] equal to $E K$, and $L O$ equal to $M N$ [Prop. 1.43]. Thus, the whole (of) $D K$ is equal to the gnomon $U V W$ and $N O$. Therefore, since the whole (of) $A K$ is equal to $L M$ and $N O$, of which $D K$ is equal to the gnomon $U V W$ and $N O$, the remainder $A B$ is thus equal to $T S$. And $T S$ is the (square) on $L N$. Thus, the (square) on $L N$ is equal to the area $A B . L N$ is thus the square-root of area $A B$. [So], I say that $L N$ is the first apotome of a medial (straight-line).

For since $E K$ is a rational (area), and is equal to $L O, L O$-that is to say, the (rectangle contained) by $L P$ and $P N$-is thus a rational (area). And $N O$ was shown (to be) a medial (area). Thus, $L O$ is incommensurable with $N O$. And as $L O$ (is) to $N O$, so $L P$ is to $P N$ [Prop. 6.1]. Thus, $L P$ and $P N$ are incommensurable in length [Prop. 10.11]. $L P$ and $P N$ are thus medial (straight-lines which are) commensurable in square only, and which contain a rational (area). Thus, $L N$ is the first apotome of a medial (straight-line) [Prop. 10.74]. And it is the square-root of area $A B$.

Thus, the square root of area $A B$ is the first apotome of a medial (straight-line). (Which is) the very thing it was required to show.
$\dagger$ There is an error in the argument here. It should just say that $L P$ and $P N$ are commensurable in square, rather than in square only, since $L P$ and $P N$ are only shown to be incommensurable in length later on.

$$
4 \gamma^{\prime}
$$







"E $\sigma \tau \omega$ ү $\alpha \rho \tau \tilde{n} \mathrm{~A} \Delta \pi \rho o \sigma \alpha \rho \mu o ́ \zeta o v \sigma \alpha \dot{\eta} \Delta \mathrm{H} \cdot \alpha i \mathrm{AH}, \mathrm{H} \Delta$





## Proposition 93

If an area is contained by a rational (straight-line) and a third apotome then the square-root of the area is a second apotome of a medial (straight-line).


For let the area $A B$ have been contained by the rational (straight-line) $A C$ and the third apotome $A D$. I say that the square-root of area $A B$ is the second apotome of a medial (straight-line).

For let $D G$ be an attachment to $A D$. Thus, $A G$ and $G D$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of $A G$ and $G D$ is commensurable in length with the (previ-





 $\tau \tilde{n} \mathrm{~A} \Gamma \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda$ оı $\alpha \mathrm{E}$ E, ZI, HK• $\sigma u ́ \mu \mu \varepsilon \tau \rho o l ~ \alpha ̈ p \alpha ~ \varepsilon i \sigma i \nu ~ \alpha i ~$




 $\mu \dot{\eta} x \varepsilon \iota, x \alpha i ̀ \dot{\eta} \Delta \mathrm{H} \alpha{ }_{\alpha} \rho \alpha$ غ́x $\alpha \tau \varepsilon ́ \rho \alpha \tau \widetilde{\omega} \nu \Delta \mathrm{E}$, $\mathrm{EH} \sigma \dot{\prime} \mu \mu \varepsilon \tau \rho o ́ s$ ह̇ $\sigma \tau \iota$



 $\mu \dot{\eta} x \varepsilon \iota \dot{\eta} \mathrm{AH} \tau \tilde{n} \mathrm{H} \Delta$. $\alpha \lambda \lambda^{\circ} \dot{\eta} \mu \varepsilon ̀ \nu \mathrm{AH} \tau \tilde{n} \mathrm{AZ} \sigma u ́ \mu \mu \varepsilon \tau \rho o ́ \varsigma ~ \varepsilon ̇ \sigma \tau \iota$

 EK• $\dot{\alpha} \sigma u ́ \mu \mu \varepsilon \tau \rho o \nu \alpha \not p \alpha ~ \varepsilon ̇ \sigma \tau i ̀ ~ \tau o ̀ ~ A I ~ \tau \widetilde{̣}$ EK.




 ஸ́s $\dot{\eta} \mathrm{AZ} \pi \rho o ̀ s ~ \tau \dot{\eta} \nu \mathrm{EH}$, oút $\tau \omega \varsigma \dot{\eta} \mathrm{EH} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{ZH} . \dot{\alpha} \lambda \lambda{ }^{\circ} \dot{\omega} \varsigma$
























ously) laid down rational (straight-line) $A C$, and the square on the whole, $A G$, is greater than (the square on) the attachment, $D G$, by the (square) on (some straightline) commensurable (in length) with $(A G)$ [Def. 10.13]. Therefore, since the square on $A G$ is greater than (the square on) $G D$ by the (square) on (some straight-line) commensurable (in length) with $(A G)$, thus if (an area) equal to the fourth part of the square on $D G$ is applied to $A G$, falling short by a square figure, then it divides ( $A G$ ) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let $D G$ have been cut in half at $E$. And let (an area) equal to the (square) on $E G$ have been applied to $A G$, falling short by a square figure. And let it be the (rectangle contained) by $A F$ and $F G$. And let $E H, F I$, and $G K$ have been drawn through points $E, F$, and $G$ (respectively), parallel to $A C$. Thus, $A F$ and $F G$ are commensurable (in length). $A I$ (is) thus also commensurable with $F K$ [Props. 6.1, 10.11]. And since $A F$ and $F G$ are commensurable in length, $A G$ is thus also commensurable in length with each of $A F$ and $F G$ [Prop. 10.15]. And $A G$ (is) rational, and incommensurable in length with $A C$. Hence, $A F$ and $F G$ (are) also (rational, and incommensurable in length with $A C$ ) [Prop. 10.13]. Thus, $A I$ and $F K$ are each medial (areas) [Prop. 10.21]. Again, since $D E$ is commensurable in length with $E G, D G$ is also commensurable in length with each of $D E$ and $E G$ [Prop. 10.15]. And $G D$ (is) rational, and incommensurable in length with $A C$. Thus, $D E$ and $E G$ (are) each also rational, and incommensurable in length with $A C$ [Prop. 10.13]. $D H$ and $E K$ are thus each medial (areas) [Prop. 10.21]. And since $A G$ and $G D$ are commensurable in square only, $A G$ is thus incommensurable in length with $G D$. But, $A G$ is commensurable in length with $A F$, and $D G$ with $E G$. Thus, $A F$ is incommensurable in length with $E G$ [Prop. 10.13]. And as $A F$ (is) to $E G$, so $A I$ is to $E K$ [Prop. 6.1]. Thus, $A I$ is incommensurable with $E K$ [Prop. 10.11].

Therefore, let the square $L M$, equal to $A I$, have been constructed. And let $N O$, equal to $F K$, which is about the same angle as $L M$, have been subtracted (from $L M$ ). Thus, $L M$ and $N O$ are about the same diagonal [Prop. 6.26]. Let $P R$ be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by $A F$ and $F G$ is equal to the (square) on $E G$, thus as $A F$ is to $E G$, so $E G$ (is) to $F G$ [Prop. 6.17]. But, as $A F$ (is) to $E G$, so $A I$ is to $E K$ [Prop. 6.1]. And as $E G$ (is) to $F G$, so $E K$ is to $F K$ [Prop. 6.1]. And thus as $A I$ (is) to $E K$, so $E K$ (is) to $F K$ [Prop. 5.11]. Thus, $E K$ is the mean proportional to $A I$ and $F K$. And $M N$ is also the mean proportional to the squares $L M$ and $N O$ [Prop. 10.53 lem.]. And $A I$ is
 ठúvatal тò $\mathrm{AB} \chi \omega$ рíov.



## $40^{\circ}$.







equal to $L M$, and $F K$ to $N O$. Thus, $E K$ is also equal to $M N$. But, $M N$ is equal to $L O$, and $E K$ [is] equal to $D H$ [Prop. 1.43]. And thus the whole of $D K$ is equal to the gnomon $U V W$ and $N O$. And $A K$ (is) also equal to $L M$ and $N O$. Thus, the remainder $A B$ is equal to $S T$-that is to say, to the square on $L N$. Thus, $L N$ is the square-root of area $A B$. I say that $L N$ is the second apotome of a medial (straight-line).

For since $A I$ and $F K$ were shown (to be) medial (areas), and are equal to the (squares) on $L P$ and $P N$ (respectively), the (squares) on each of $L P$ and $P N$ (are) thus also medial. Thus, $L P$ and $P N$ (are) each medial (straight-lines). And since $A I$ is commensurable with $F K$ [Props. 6.1, 10.11], the (square) on $L P$ (is) thus also commensurable with the (square) on $P N$. Again, since $A I$ was shown (to be) incommensurable with $E K$, $L M$ is thus also incommensurable with $M N$-that is to say, the (square) on $L P$ with the (rectangle contained) by $L P$ and $P N$. Hence, $L P$ is also incommensurable in length with $P N$ [Props. 6.1, 10.11]. Thus, $L P$ and $P N$ are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a medial (area).

For since $E K$ was shown (to be) a medial (area), and is equal to the (rectangle contained) by $L P$ and $P N$, the (rectangle contained) by $L P$ and $P N$ is thus also medial. Hence, $L P$ and $P N$ are medial (straight-lines which are) commensurable in square only, and which contain a medial (area). Thus, $L N$ is the second apotome of a medial (straight-line) [Prop. 10.75]. And it is the square-root of area $A B$.

Thus, the square-root of area $A B$ is the second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

## Proposition 94

If an area is contained by a rational (straight-line) and a fourth apotome then the square-root of the area is a minor (straight-line).


For let the area $A B$ have been contained by the rational (straight-line) $A C$ and the fourth apotome $A D$. I say that the square-root of area $A B$ is a minor (straight-
















 тò $\mathrm{AI} \tau \widetilde{\square} \mathrm{ZK}$.




























 $\chi$ срíov.


line). For let $D G$ be an attachment to $A D$. Thus, $A G$ and $D G$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and $A G$ is commensurable in length with the (previously) laid down rational (straight-line) $A C$, and the square on the whole, $A G$, is greater than (the square on) the attachment, $D G$, by the square on (some straight-line) incommensurable in length with $(A G)$ [Def. 10.14]. Therefore, since the square on $A G$ is greater than (the square on) $G D$ by the (square) on (some straight-line) incommensurable in length with $(A G)$, thus if (some area), equal to the fourth part of the (square) on $D G$, is applied to $A G$, falling short by a square figure, then it divides ( $A G$ ) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let $D G$ have been cut in half at $E$, and let (some area), equal to the (square) on $E G$, have been applied to $A G$, falling short by a square figure, and let it be the (rectangle contained) by $A F$ and $F G$. Thus, $A F$ is incommensurable in length with $F G$. Therefore, let $E H$, $F I$, and $G K$ have been drawn through $E, F$, and $G$ (respectively), parallel to $A C$ and $B D$. Therefore, since $A G$ is rational, and commensurable in length with $A C$, the whole (area) $A K$ is thus rational [Prop. 10.19]. Again, since $D G$ is incommensurable in length with $A C$, and both are rational (straight-lines), $D K$ is thus a medial (area) [Prop. 10.21]. Again, since $A F$ is incommensurable in length with $F G, A I$ (is) thus also incommensurable with $F K$ [Props. 6.1, 10.11].

Therefore, let the square $L M$, equal to $A I$, have been constructed. And let $N O$, equal to $F K$, (and) about the same angle, $L P M$, have been subtracted (from $L M$ ). Thus, the squares $L M$ and $N O$ are about the same diagonal [Prop. 6.26]. Let $P R$ be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by $A F$ and $F G$ is equal to the (square) on $E G$, thus, proportionally, as $A F$ is to $E G$, so $E G$ (is) to $F G$ [Prop. 6.17]. But, as $A F$ (is) to $E G$, so $A I$ is to $E K$, and as $E G$ (is) to $F G$, so $E K$ is to $F K$ [Prop. 6.1]. Thus, $E K$ is the mean proportional to $A I$ and $F K$ [Prop. 5.11]. And $M N$ is also the mean proportional to the squares $L M$ and $N O$ [Prop. 10.13 lem.], and $A I$ is equal to $L M$, and $F K$ to $N O$. $E K$ is thus also equal to $M N$. But, $D H$ is equal to $E K$, and $L O$ is equal to $M N$ [Prop. 1.43]. Thus, the whole of $D K$ is equal to the gnomon $U V W$ and $N O$. Therefore, since the whole of $A K$ is equal to the (sum of the) squares $L M$ and $N O$, of which $D K$ is equal to the gnomon $U V W$ and the square $N O$, the remainder $A B$ is thus equal to $S T$-that is to say, to the square on $L N$. Thus, $L N$ is the square-root of area $A B$. I say that $L N$ is the irrational (straight-line which is) called minor.

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For since $A K$ is rational, and is equal to the (sum of the) squares $L P$ and $P N$, the sum of the (squares) on $L P$ and $P N$ is thus rational. Again, since $D K$ is medial, and $D K$ is equal to twice the (rectangle contained) by $L P$ and $P N$, thus twice the (rectangle contained) by $L P$ and $P N$ is medial. And since $A I$ was shown (to be) incommensurable with $F K$, the square on $L P$ (is) thus also incommensurable with the square on $P N$. Thus, $L P$ and $P N$ are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. $L N$ is thus the irrational (straight-line) called minor [Prop. 10.76]. And it is the square-root of area $A B$.

Thus, the square-root of area $A B$ is a minor (straightline). (Which is) the very thing it was required to show.

## Proposition 95

If an area is contained by a rational (straight-line) and a fifth apotome then the square-root of the area is that (straight-line) which with a rational (area) makes a medial whole.


For let the area $A B$ have been contained by the rational (straight-line) $A C$ and the fifth apotome $A D$. I say that the square-root of area $A B$ is that (straight-line) which with a rational (area) makes a medial whole.

For let $D G$ be an attachment to $A D$. Thus, $A G$ and $D G$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment $G D$ is commensurable in length the the (previously) laid down rational (straight-line) $A C$, and the square on the whole, $A G$, is greater than (the square on) the attachment, $D G$, by the (square) on (some straight-line) incommensurable (in length) with ( $A G$ ) [Def. 10.15]. Thus, if (some area), equal to the fourth part of the (square) on $D G$, is applied to $A G$, falling short by a square figure, then it divides ( $A G$ ) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let $D G$ have been divided in half at point $E$, and let (some area), equal to the (square) on $E G$, have been applied to $A G$, falling short by a square figure, and let it be the (rectangle contained) by $A F$ and $F G$. Thus, $A F$ is incommensurable in length with $F G$. And since $A G$ is incommensurable
















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 $\mathrm{H} \Delta$ p̀ntaí $\varepsilon i \sigma \iota ~ \delta u v a ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l, ~ x a i ~ o u ̛ \delta \varepsilon \tau e ́ p \alpha ~$
in length with $C A$, and both are rational (straight-lines), $A K$ is thus a medial (area) [Prop. 10.21]. Again, since $D G$ is rational, and commensurable in length with $A C$, $D K$ is a rational (area) [Prop. 10.19].

Therefore, let the square $L M$, equal to $A I$, have been constructed. And let the square $N O$, equal to $F K$, (and) about the same angle, $L P M$, have been subtracted (from $N O$ ). Thus, the squares $L M$ and $N O$ are about the same diagonal [Prop. 6.26]. Let $P R$ be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly (to the previous propositions), we can show that $L N$ is the square-root of area $A B$. I say that $L N$ is that (straight-line) which with a rational (area) makes a medial whole.

For since $A K$ was shown (to be) a medial (area), and is equal to (the sum of) the squares on $L P$ and $P N$, the sum of the (squares) on $L P$ and $P N$ is thus medial. Again, since $D K$ is rational, and is equal to twice the (rectangle contained) by $L P$ and $P N$, (the latter) is also rational. And since $A I$ is incommensurable with $F K$, the (square) on $L P$ is thus also incommensurable with the (square) on $P N$. Thus, $L P$ and $P N$ are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational. Thus, the remainder $L N$ is the irrational (straight-line) called that which with a rational (area) makes a medial whole [Prop. 10.77]. And it is the square-root of area $A B$.

Thus, the square-root of area $A B$ is that (straightline) which with a rational (area) makes a medial whole. (Which is) the very thing it was required to show.

## Proposition 96

If an area is contained by a rational (straight-line) and a sixth apotome then the square-root of the area is that (straight-line) which with a medial (area) makes a medial whole.


For let the area $A B$ have been contained by the rational (straight-line) $A C$ and the sixth apotome $A D$. I say that the square-root of area $A B$ is that (straight-line) which with a medial (area) makes a medial whole.

For let $D G$ be an attachment to $A D$. Thus, $A G$ and













 $\alpha i \mathrm{AH}, \mathrm{H} \Delta$ ठuvá $\mu \varepsilon \iota \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o i ́ ~ \varepsilon i \sigma เ \nu, ~ \alpha ̇ \sigma u ́ \mu \mu \varepsilon \tau \rho o s ~ « ้ p \alpha ~$

 $\mathrm{K} \Delta$.


















 $\mu \varepsilon ́ \sigma o u ~ \mu \varepsilon ́ \sigma o v ~ \tau o ̀ ~ o ̋ \lambda o v ~ \pi o เ o u ̃ \sigma \alpha \cdot ~ \chi \alpha l ̀ ~ \delta u ́ v \alpha \tau \alpha l ~ \tau o ̀ ~ A B ~ \chi \omega p i ́ o v . ~$


$G D$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of them is commensurable in length with the (previously) laid down rational (straight-line) $A C$, and the square on the whole, $A G$, is greater than (the square on) the attachment, $D G$, by the (square) on (some straight-line) incommensurable in length with $(A G)$ [Def. 10.16]. Therefore, since the square on $A G$ is greater than (the square on) $G D$ by the (square) on (some straight-line) incommensurable in length with $(A G)$, thus if (some area), equal to the fourth part of square on $D G$, is applied to $A G$, falling short by a square figure, then it divides $(A G)$ into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let $D G$ have been cut in half at [point] $E$. And let (some area), equal to the (square) on $E G$, have been applied to $A G$, falling short by a square figure. And let it be the (rectangle contained) by $A F$ and $F G$. $A F$ is thus incommensurable in length with $F G$. And as $A F$ (is) to $F G$, so $A I$ is to $F K$ [Prop. 6.1]. Thus, $A I$ is incommensurable with $F K$ [Prop. 10.11]. And since $A G$ and $A C$ are rational (straight-lines which are) commensurable in square only, $A K$ is a medial (area) [Prop. 10.21]. Again, since $A C$ and $D G$ are rational (straight-lines which are) incommensurable in length, $D K$ is also a medial (area) [Prop. 10.21]. Therefore, since $A G$ and $G D$ are commensurable in square only, $A G$ is thus incommensurable in length with $G D$. And as $A G$ (is) to $G D$, so $A K$ is to $K D$ [Prop. 6.1]. Thus, $A K$ is incommensurable with $K D$ [Prop. 10.11].

Therefore, let the square $L M$, equal to $A I$, have been constructed. And let $N O$, equal to $F K$, (and) about the same angle, have been subtracted (from $L M$ ). Thus, the squares $L M$ and $N O$ are about the same diagonal [Prop. 6.26]. Let $P R$ be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly to the above, we can show that $L N$ is the square-root of area $A B$. I say that $L N$ is that (straight-line) which with a medial (area) makes a medial whole.

For since $A K$ was shown (to be) a medial (area), and is equal to the (sum of the) squares on $L P$ and $P N$, the sum of the (squares) on $L P$ and $P N$ is medial. Again, since $D K$ was shown (to be) a medial (area), and is equal to twice the (rectangle contained) by $L P$ and $P N$, twice the (rectangle contained) by $L P$ and $P N$ is also medial. And since $A K$ was shown (to be) incommensurable with $D K$, [thus] the (sum of the) squares on $L P$ and $P N$ is also incommensurable with twice the (rectangle contained) by $L P$ and $P N$. And since $A I$ is incommensurable with $F K$, the (square) on $L P$ (is) thus also incommensurable with the (square) on $P N$. Thus, $L P$ and $P N$ are (straight-lines which are) incommensu-


#### Abstract

$\zeta^{\prime}$. Tò $\alpha \pi o ̀ ~ \alpha \dot{\alpha} \pi o \tau o \mu \tilde{\eta} s ~ \pi \alpha p \alpha ̀ ~ \rho ́ \eta \tau \grave{\eta} \nu \pi \alpha p \alpha \beta \alpha \lambda \lambda o ́ \mu \varepsilon \nu o \nu \pi \lambda \alpha ́ \tau o s$ $\pi о เ \varepsilon \grave{~} \alpha \pi о \tau о \mu \grave{\nu} \nu \pi \rho \omega ́ \tau \eta \nu$.



 AB 亿̛бov $\pi \alpha \rho \alpha ̀ ~ \tau \grave{̀} \nu \Gamma \Delta \pi \alpha p \alpha \beta \varepsilon \beta \lambda \dot{\eta} \sigma \vartheta \omega$ тò $\Gamma \mathrm{E} \pi \lambda \alpha ́ \tau o \varsigma ~ \pi o เ o u ̃ \nu ~$

"E $\sigma \tau \omega$ үàp $\tau \tilde{n} \mathrm{AB} \pi \rho о \sigma \alpha \rho \mu o ́ \zeta o v \sigma \alpha \dot{\eta} \mathrm{BH}$ • ai $\alpha \rho \alpha \mathrm{AH}$,






















rable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by medial, and, furthermore, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them. Thus, $L N$ is the irrational (straight-line) called that which with a medial (area) makes a medial whole [Prop. 10.78]. And it is the square-root of area $A B$.

Thus, the square-root of area $(A B)$ is that (straightline) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

## Proposition 97

The (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.


Let $A B$ be an apotome, and $C D$ a rational (straightline). And let $C E$, equal to the (square) on $A B$, have been applied to $C D$, producing $C F$ as breadth. I say that $C F$ is a first apotome.

For let $B G$ be an attachment to $A B$. Thus, $A G$ and $G B$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let $C H$, equal to the (square) on $A G$, and $K L$, (equal) to the (square) on $B G$, have been applied to $C D$. Thus, the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, of which $C E$ is equal to the (square) on $A B$. The remainder $F L$ is thus equal to twice the (rectangle contained) by $A G$ and $G B$ [Prop. 2.7]. Let $F M$ have been cut in half at point $N$. And let $N O$ have been drawn through $N$, parallel to $C D$. Thus, $F O$ and $L N$ are each equal to the (rectangle contained) by $A G$ and $G B$. And since the (sum of the squares) on $A G$ and $G B$ is rational, and $D M$ is equal to the (sum of the squares) on $A G$ and $G B, D M$ is thus rational. And it has been applied to the rational (straight-line) $C D$, producing $C M$ as breadth. Thus, $C M$ is rational, and commensurable in length with $C D$ [Prop. 10.20]. Again, since twice the (rectangle contained) by $A G$ and $G B$ is medial, and $F L$ (is) equal to twice the (rectangle contained) by $A G$ and $G B, F L$ (is) thus a medial (area). And it is applied to the rational (straight-line) $C D$, producing $F M$ as breadth. $F M$ is
 ठウ́, őtı каі $\pi \rho \omega ́ \tau \eta$.










 tò $\mathrm{K} \Lambda$, oứt $\omega \varsigma \dot{\eta} \Gamma \mathrm{K} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{KM} \cdot \sigma u ́ \mu \mu \varepsilon \tau \rho o \varsigma ~ \alpha ้ p \alpha ~ \varepsilon ̇ \sigma \tau i \nu ~ \dot{\eta}$


 KM, x $\alpha i ́ ~ \varepsilon ̇ \sigma \tau \iota ~ \sigma u ́ \mu \mu \varepsilon \tau p o s ~ \dot{\eta} \Gamma K \tau \tilde{n} \mathrm{KM}, \dot{\eta} \alpha \nsim \alpha$ ГМ $\tau \tilde{\eta} \varsigma \mathrm{MZ}$

 $\dot{\alpha} \pi о \tau о \mu \dot{\prime}$ ध̇ $\sigma \tau \iota \pi \rho \omega ́ \tau \eta$.



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Tò $\alpha \pi o ̀ ~ \mu \varepsilon ́ \sigma \eta s ~ \alpha \dot{\alpha} \pi о \tau o \mu \tilde{\eta} s ~ \pi \rho \omega ́ t \eta s ~ \pi \alpha \rho \alpha ̀ ~ \rho ं \eta t i ̀ \nu ~ \pi \alpha \rho \alpha-$



thus rational, and incommensurable in length with $C D$ [Prop. 10.22]. And since the (sum of the squares) on $A G$ and $G B$ is rational, and twice the (rectangle contained) by $A G$ and $G B$ medial, the (sum of the squares) on $A G$ and $G B$ is thus incommensurable with twice the (rectangle contained) by $A G$ and $G B$. And $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, and $F L$ to twice the (rectangle contained) by $A G$ and $G B . D M$ is thus incommensurable with $F L$. And as $D M$ (is) to $F L$, so $C M$ is to $F M$ [Prop. 6.1]. $C M$ is thus incommensurable in length with $F M$ [Prop. 10.11]. And both are rational (straightlines). Thus, $C M$ and $M F$ are rational (straight-lines which are) commensurable in square only. $C F$ is thus an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

For since the (rectangle contained) by $A G$ and $G B$ is the mean proportional to the (squares) on $A G$ and $G B$ [Prop. 10.21 lem.], and $C H$ is equal to the (square) on $A G$, and $K L$ equal to the (square) on $B G$, and $N L$ to the (rectangle contained) by $A G$ and $G B, N L$ is thus also the mean proportional to $C H$ and $K L$. Thus, as $C H$ is to $N L$, so $N L$ (is) to $K L$. But, as $C H$ (is) to $N L$, so $C K$ is to $N M$, and as $N L$ (is) to $K L$, so $N M$ is to $K M$ [Prop. 6.1]. Thus, the (rectangle contained) by $C K$ and $K M$ is equal to the (square) on $N M$ that is to say, to the fourth part of the (square) on $F M$ [Prop. 6.17]. And since the (square) on $A G$ is commensurable with the (square) on $G B, C H$ [is] also commensurable with $K L$. And as $C H$ (is) to $K L$, so $C K$ (is) to $K M$ [Prop. 6.1]. $C K$ is thus commensurable (in length) with $K M$ [Prop. 10.11]. Therefore, since $C M$ and $M F$ are two unequal straight-lines, and the (rectangle contained) by $C K$ and $K M$, equal to the fourth part of the (square) on $F M$, has been applied to $C M$, falling short by a square figure, and $C K$ is commensurable (in length) with $K M$, the square on $C M$ is thus greater than (the square on) $M F$ by the (square) on (some straight-line) commensurable in length with ( $C M$ ) [Prop. 10.17]. And $C M$ is commensurable in length with the (previously) laid down rational (straight-line) $C D$. Thus, $C F$ is a first apotome [Def. 10.15].

Thus, the (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth. (Which is) the very thing it was required to show.

## Proposition 98

The (square) on a first apotome of a medial (straightline), applied to a rational (straight-line), produces a second apotome as breadth.

Let $A B$ be a first apotome of a medial (straight-line),

 HB $\mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i \sigma i ̀ ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l ~ ค ̣ \eta \tau o ̀ v ~ \pi \varepsilon p l e ́ \chi o u \sigma \alpha l . ~$













 $\mathrm{Z} \Lambda$, oút












and $C D$ a rational (straight-line). And let $C E$, equal to the (square) on $A B$, have been applied to $C D$, producing $C F$ as breadth. I say that $C F$ is a second apotome.

For let $B G$ be an attachment to $A B$. Thus, $A G$ and $G B$ are medial (straight-lines which are) commensurable in square only, containing a rational (area) [Prop. 10.74]. And let $C H$, equal to the (square) on $A G$, have been applied to $C D$, producing $C K$ as breadth, and $K L$, equal to the (square) on $G B$, producing $K M$ as breadth. Thus, the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$. Thus, $C L$ (is) also a medial (area) [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) $C D$, producing $C M$ as breadth. $C M$ is thus rational, and incommensurable in length with $C D$ [Prop. 10.22]. And since $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, of which the (square) on $A B$ is equal to $C E$, the remainder, twice the (rectangle contained) by $A G$ and $G B$, is thus equal to $F L$ [Prop. 2.7]. And twice the (rectangle contained) by $A G$ and $G B$ [is] rational. Thus, $F L$ (is) rational. And it is applied to the rational (straight-line) $F E$, producing $F M$ as breadth. $F M$ is thus also rational, and commensurable in length with $C D$ [Prop. 10.20]. Therefore, since the (sum of the squares) on $A G$ and $G B$-that is to say, $C L$-is medial, and twice the (rectangle contained) by $A G$ and $G B-$ that is to say, $F L$-(is) rational, $C L$ is thus incommensurable with $F L$. And as $C L$ (is) to $F L$, so $C M$ is to $F M$ [Prop. 6.1]. Thus, $C M$ (is) incommensurable in length with $F M$ [Prop. 10.11]. And they are both rational (straight-lines). Thus, $C M$ and $M F$ are rational (straight-lines which are) commensurable in square only. $C F$ is thus an apotome [Prop. 10.73]. So, I say that (it is) also a second (apotome).


For let $F M$ have been cut in half at $N$. And let $N O$ have been drawn through (point) $N$, parallel to $C D$. Thus, $F O$ and $N L$ are each equal to the (rectangle contained) by $A G$ and $G B$. And since the (rectangle contained) by $A G$ and $G B$ is the mean proportional to the squares on $A G$ and $G B$ [Prop. 10.21 lem.], and the (square) on $A G$ is equal to $C H$, and the (rectangle contained) by $A G$ and $G B$ to $N L$, and the (square) on










 غ் $\alpha \cup \tilde{n} \mu \dot{\eta} x \varepsilon \iota . ~ x \alpha i ́ l ~ \varepsilon ̇ \sigma \tau เ \nu ~ \dot{\eta} \pi \rho о \sigma \alpha \rho \mu o ́ \zeta о \cup \sigma \alpha ~ \dot{\eta}$ ZM $\sigma \dot{\mu} \mu \mu \varepsilon \tau \rho о \varsigma ~$
 ठєutépa.

 ठєĭ̧al.

## $4 \vartheta^{\prime}$.

 $\beta \alpha \lambda \lambda o ́ \mu \varepsilon \nu o \nu ~ \pi \lambda \alpha ́ \tau o \varsigma ~ \pi о เ \varepsilon \imath ̃ ~ \alpha ́ \pi о \tau о \mu \grave{\eta} \nu \tau \rho i ́ \tau \eta \nu . ~$




 $\mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i \sigma i ̀ ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l ~ \mu \varepsilon ́ \sigma o v ~ \pi \varepsilon p ı \varepsilon ́ \chi o v \sigma \alpha l . ~ \chi \alpha i ̀ ~$





$B G$ to $K L, N L$ is thus also the mean proportional to $C H$ and $K L$. Thus, as $C H$ is to $N L$, so $N L$ (is) to $K L$ [Prop. 5.11]. But, as $C H$ (is) to $N L$, so $C K$ is to $N M$, and as $N L$ (is) to $K L$, so $N M$ is to $M K$ [Prop. 6.1]. Thus, as $C K$ (is) to $N M$, so $N M$ is to $K M$ [Prop. 5.11]. The (rectangle contained) by $C K$ and $K M$ is thus equal to the (square) on $N M$ [Prop. 6.17]-that is to say, to the fourth part of the (square) on $F M$ [and since the (square) on $A G$ is commensurable with the (square) on $B G, C H$ is also commensurable with $K L$-that is to say, $C K$ with $K M]$. Therefore, since $C M$ and $M F$ are two unequal straight-lines, and the (rectangle contained) by $C K$ and $K M$, equal to the fourth part of the (square) on $M F$, has been applied to the greater $C M$, falling short by a square figure, and divides it into commensurable (parts), the square on $C M$ is thus greater than (the square on) $M F$ by the (square) on (some straight-line) commensurable in length with ( $C M$ ) [Prop. 10.17]. The attachment $F M$ is also commensurable in length with the (previously) laid down rational (straight-line) $C D . C F$ is thus a second apotome [Def. 10.16].

Thus, the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth. (Which is) the very thing it was required to show.

## Proposition 99

The (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.


Let $A B$ be the second apotome of a medial (straightline), and $C D$ a rational (straight-line). And let $C E$, equal to the (square) on $A B$, have been applied to $C D$, producing $C F$ as breadth. I say that $C F$ is a third apotome.

For let $B G$ be an attachment to $A B$. Thus, $A G$ and $G B$ are medial (straight-lines which are) commensurable in square only, containing a medial (area) [Prop. 10.75]. And let $C H$, equal to the (square) on $A G$, have been applied to $C D$, producing $C K$ as breadth. And let $K L$,




















 $\dot{\eta}$ ГZ. $\lambda \varepsilon ́ \gamma \omega$ ón, ötı xail трítŋ.

















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equal to the (square) on $B G$, have been applied to $K H$, producing $K M$ as breadth. Thus, the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$ [and the (sum of the squares) on $A G$ and $G B$ is medial]. $C L$ (is) thus also medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) $C D$, producing $C M$ as breadth. Thus, $C M$ is rational, and incommensurable in length with $C D$ [Prop. 10.22]. And since the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, of which $C E$ is equal to the (square) on $A B$, the remainder $L F$ is thus equal to twice the (rectangle contained) by $A G$ and $G B$ [Prop. 2.7]. Therefore, let $F M$ have been cut in half at point $N$. And let $N O$ have been drawn parallel to $C D$. Thus, $F O$ and $N L$ are each equal to the (rectangle contained) by $A G$ and $G B$. And the (rectangle contained) by $A G$ and $G B$ (is) medial. Thus, $F L$ is also medial. And it is applied to the rational (straight-line) $E F$, producing $F M$ as breadth. $F M$ is thus rational, and incommensurable in length with $C D$ [Prop. 10.22]. And since $A G$ and $G B$ are commensurable in square only, $A G$ [is] thus incommensurable in length with $G B$. Thus, the (square) on $A G$ is also incommensurable with the (rectangle contained) by $A G$ and $G B$ [Props. 6.1, 10.11]. But, the (sum of the squares) on $A G$ and $G B$ is commensurable with the (square) on $A G$, and twice the (rectangle contained) by $A G$ and $G B$ with the (rectangle contained) by $A G$ and $G B$. The (sum of the squares) on $A G$ and $G B$ is thus incommensurable with twice the (rectangle contained) by $A G$ and $G B$ [Prop. 10.13]. But, $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, and $F L$ is equal to twice the (rectangle contained) by $A G$ and $G B$. Thus, $C L$ is incommensurable with $F L$. And as $C L$ (is) to $F L$, so $C M$ is to $F M$ [Prop. 6.1]. $C M$ is thus incommensurable in length with $F M$ [Prop. 10.11]. And they are both rational (straight-lines). Thus, $C M$ and $M F$ are rational (straight-lines which are) commensurable in square only. $C F$ is thus an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since the (square) on $A G$ is commensurable with the (square) on $G B, C H$ (is) thus also commensurable with $K L$. Hence, $C K$ (is) also (commensurable in length) with $K M$ [Props. 6.1, 10.11]. And since the (rectangle contained) by $A G$ and $G B$ is the mean proportional to the (squares) on $A G$ and $G B$ [Prop. 10.21 lem.], and $C H$ is equal to the (square) on $A G$, and $K L$ equal to the (square) on $G B$, and $N L$ equal to the (rectangle contained) by $A G$ and $G B, N L$ is thus also the mean proportional to $C H$ and $K L$. Thus, as $C H$ is to $N L$, so $N L$ (is) to $K L$. But, as $C H$ (is) to $N L$, so $C K$ is to $N M$, and as $N L$ (is) to $K L$, so $N M$ (is) to $K M$ [Prop. 6.1].

$$
\rho^{\prime} \text {. }
$$













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Thus, as $C K$ (is) to $M N$, so $M N$ is to $K M$ [Prop. 5.11]. Thus, the (rectangle contained) by $C K$ and $K M$ is equal to the [(square) on $M N$-that is to say, to the] fourth part of the (square) on $F M$ [Prop. 6.17]. Therefore, since $C M$ and $M F$ are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on $F M$, has been applied to $C M$, falling short by a square figure, and divides it into commensurable (parts), the square on $C M$ is thus greater than (the square on) $M F$ by the (square) on (some straight-line) commensurable (in length) with ( $C M$ ) [Prop. 10.17]. And neither of $C M$ and $M F$ is commensurable in length with the (previously) laid down rational (straight-line) $C D . C F$ is thus a third apotome [Def. 10.13].

Thus, the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth. (Which is) the very thing it was required to show.

## Proposition 100

The (square) on a minor (straight-line), applied to a rational (straight-line), produces a fourth apotome as breadth.

A
B $\quad$ G


Let $A B$ be a minor (straight-line), and $C D$ a rational (straight-line). And let $C E$, equal to the (square) on $A B$, have been applied to the rational (straight-line) $C D$, producing $C F$ as breadth. I say that $C F$ is a fourth apotome.

For let $B G$ be an attachment to $A B$. Thus, $A G$ and $G B$ are incommensurable in square, making the sum of the squares on $A G$ and $G B$ rational, and twice the (rectangle contained) by $A G$ and $G B$ medial [Prop. 10.76]. And let $C H$, equal to the (square) on $A G$, have been applied to $C D$, producing $C K$ as breadth, and $K L$, equal to the (square) on $B G$, producing $K M$ as breadth. Thus, the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$. And the sum of the (squares) on $A G$ and $G B$ is rational. $C L$ is thus also rational. And it is applied to the rational (straight-line) $C D$, producing $C M$ as breadth. Thus, $C M$ (is) also rational, and commensurable in length with $C D$ [Prop. 10.20]. And since the












 غ̇எтì $\dot{\eta}$ ГZ. $\lambda \varepsilon ́ \gamma \omega$ [ $\delta \dot{\eta}]$, ơтı x $\alpha \grave{l} \tau \varepsilon \tau \alpha ́ p \tau \eta . ~$





 $\mu \varepsilon ́ \sigma o \nu ~ \alpha ̉ v \alpha ́ \lambda o \gamma o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ ن ́ \pi o ̀ ~ \tau డ ̃ \nu ~ A H, ~ H B, ~ x \alpha i ́ ~ \varepsilon ̇ \sigma \tau \iota \nu ~ l ̂ \sigma o v ~$













 $\dot{\eta} \alpha \nsim \alpha$ ГZ $\dot{\alpha} \pi о \tau о \mu \dot{n}$ ह̉ $\sigma \tau l ~ \tau \varepsilon \tau \alpha ́ \rho \tau \eta . ~$

whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, of which $C E$ is equal to the (square) on $A B$, the remainder $F L$ is thus equal to twice the (rectangle contained) by $A G$ and $G B$ [Prop. 2.7]. Therefore, let $F M$ have been cut in half at point $N$. And let $N O$ have been drawn through $N$, parallel to either of $C D$ or $M L$. Thus, $F O$ and $N L$ are each equal to the (rectangle contained) by $A G$ and $G B$. And since twice the (rectangle contained) by $A G$ and $G B$ is medial, and is equal to $F L$, $F L$ is thus also medial. And it is applied to the rational (straight-line) $F E$, producing $F M$ as breadth. Thus, $F M$ is rational, and incommensurable in length with $C D$ [Prop. 10.22]. And since the sum of the (squares) on $A G$ and $G B$ is rational, and twice the (rectangle contained) by $A G$ and $G B$ medial, the (sum of the squares) on $A G$ and $G B$ is [thus] incommensurable with twice the (rectangle contained) by $A G$ and $G B$. And $C L$ (is) equal to the (sum of the squares) on $A G$ and $G B$, and $F L$ equal to twice the (rectangle contained) by $A G$ and $G B . C L$ [is] thus incommensurable with $F L$. And as $C L$ (is) to $F L$, so $C M$ is to $M F$ [Prop. 6.1]. $C M$ is thus incommensurable in length with $M F$ [Prop. 10.11]. And both are rational (straight-lines). Thus, $C M$ and $M F$ are rational (straight-lines which are) commensurable in square only. $C F$ is thus an apotome [Prop. 10.73]. [So], I say that (it is) also a fourth (apotome).

For since $A G$ and $G B$ are incommensurable in square, the (square) on $A G$ (is) thus also incommensurable with the (square) on $G B$. And $C H$ is equal to the (square) on $A G$, and $K L$ equal to the (square) on $G B$. Thus, $C H$ is incommensurable with $K L$. And as $C H$ (is) to $K L$, so $C K$ is to $K M$ [Prop. 6.1]. $C K$ is thus incommensurable in length with $K M$ [Prop. 10.11]. And since the (rectangle contained) by $A G$ and $G B$ is the mean proportional to the (squares) on $A G$ and $G B$ [Prop. 10.21 lem.], and the (square) on $A G$ is equal to $C H$, and the (square) on $G B$ to $K L$, and the (rectangle contained) by $A G$ and $G B$ to $N L, N L$ is thus the mean proportional to $C H$ and $K L$. Thus, as $C H$ is to $N L$, so $N L$ (is) to $K L$. But, as $C H$ (is) to $N L$, so $C K$ is to $N M$, and as $N L$ (is) to $K L$, so $N M$ is to $K M$ [Prop. 6.1]. Thus, as $C K$ (is) to $M N$, so $M N$ is to $K M$ [Prop. 5.11]. The (rectangle contained) by $C K$ and $K M$ is thus equal to the (square) on $M N$-that is to say, to the fourth part of the (square) on $F M$ [Prop. 6.17]. Therefore, since $C M$ and $M F$ are two unequal straight-lines, and the (rectangle contained) by $C K$ and $K M$, equal to the fourth part of the (square) on $M F$, has been applied to $C M$, falling short by a square figure, and divides it into incommensurable (parts), the square on $C M$ is thus greater than (the square on) $M F$ by the (square) on (some straight-line) incommensurable

$$
\rho \alpha^{\prime} .
$$






 $\dot{\alpha} \pi о \tau о \mu \grave{n}$ ह̀ $\sigma \tau \iota ~ \pi \varepsilon ́ \mu \pi \tau \eta$.






















 òn, ötı жаі̀ $\pi \varepsilon ́ \mu \pi \tau \eta$.


(in length) with ( $C M$ ) [Prop. 10.18]. And the whole of $C M$ is commensurable in length with the (previously) laid down rational (straight-line) $C D$. Thus, $C F$ is a fourth apotome [Def. 10.14].

Thus, the (square) on a minor, and so on ...

## Proposition 101

The (square) on that (straight-line) which with a rational (area) makes a medial whole, applied to a rational (straight-line), produces a fifth apotome as breadth.


Let $A B$ be that (straight-line) which with a rational (area) makes a medial whole, and $C D$ a rational (straight-line). And let $C E$, equal to the (square) on $A B$, have been applied to $C D$, producing $C F$ as breadth. I say that $C F$ is a fifth apotome.

Let $B G$ be an attachment to $A B$. Thus, the straightlines $A G$ and $G B$ are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational [Prop. 10.77]. And let $C H$, equal to the (square) on $A G$, have been applied to $C D$, and $K L$, equal to the (square) on $G B$. The whole of $C L$ is thus equal to the (sum of the squares) on $A G$ and $G B$. And the sum of the (squares) on $A G$ and $G B$ together is medial. Thus, $C L$ is medial. And it has been applied to the rational (straight-line) $C D$, producing $C M$ as breadth. $C M$ is thus rational, and incommensurable (in length) with $C D$ [Prop. 10.22]. And since the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, of which $C E$ is equal to the (square) on $A B$, the remainder $F L$ is thus equal to twice the (rectangle contained) by $A G$ and $G B$ [Prop. 2.7]. Therefore, let $F M$ have been cut in half at $N$. And let $N O$ have been drawn through $N$, parallel to either of $C D$ or $M L$. Thus, $F O$ and $N L$ are each equal to the (rectangle contained) by $A G$ and $G B$. And since twice the (rectangle contained) by $A G$ and $G B$ is rational, and [is] equal to $F L, F L$ is thus rational. And it is applied to the rational (straight-line) $E F$, producing $F M$ as breadth. Thus, $F M$ is rational, and commensurable in length with $C D$ [Prop. 10.20]. And since $C L$ is medial, and $F L$ rational,













Tò $\alpha$ สò tñs $\mu \varepsilon \tau \grave{\alpha} \mu \varepsilon ́ \sigma o u ~ \mu \varepsilon ́ \sigma o v ~ t o ̀ ~ o ̈ \lambda o v ~ \pi o เ o u ́ \sigma \eta s ~ \pi \alpha p \alpha ̀ ~$


"E $\sigma \tau \omega \dot{\eta} \mu \varepsilon \tau \grave{\alpha} \mu \varepsilon ́ \sigma o u ~ \mu \varepsilon ́ \sigma o v ~ \tau o ̀ ~ o ̈ \lambda o v ~ \pi o เ o u ̃ \sigma \alpha ~ \dot{\eta} \mathrm{AB}, \dot{\rho} \eta \tau \grave{\eta}$





 AH, HB $\mu \varepsilon ́ \sigma o \nu ~ x \alpha i ~ \alpha ̆ \sigma u ́ \mu \mu \varepsilon \tau \rho o \nu ~ \tau \grave{\alpha} \alpha \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AH}, \mathrm{HB} \tau \widetilde{\varphi}$
$C L$ is thus incommensurable with $F L$. And as $C L$ (is) to $F L$, so $C M$ (is) to $M F$ [Prop. 6.1]. $C M$ is thus incommensurable in length with $M F$ [Prop. 10.11]. And both are rational. Thus, $C M$ and $M F$ are rational (straightlines which are) commensurable in square only. $C F$ is thus an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

For, similarly (to the previous propositions), we can show that the (rectangle contained) by $C K M$ is equal to the (square) on $N M$-that is to say, to the fourth part of the (square) on $F M$. And since the (square) on $A G$ is incommensurable with the (square) on $G B$, and the (square) on $A G$ (is) equal to $C H$, and the (square) on $G B$ to $K L, C H$ (is) thus incommensurable with $K L$. And as $C H$ (is) to $K L$, so $C K$ (is) to $K M$ [Prop. 6.1]. Thus, $C K$ (is) incommensurable in length with $K M$ [Prop. 10.11]. Therefore, since $C M$ and $M F$ are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on $F M$, has been applied to $C M$, falling short by a square figure, and divides it into incommensurable (parts), the square on $C M$ is thus greater than (the square on) $M F$ by the (square) on (some straight-line) incommensurable (in length) with ( $C M$ ) [Prop. 10.18]. And the attachment $F M$ is commensurable with the (previously) laid down rational (straightline) $C D$. Thus, $C F$ is a fifth apotome [Def. 10.15]. (Which is) the very thing it was required to show.

## Proposition 102

The (square) on that (straight-line) which with a medial (area) makes a medial whole, applied to a rational (straight-line), produces a sixth apotome as breadth.


Let $A B$ be that (straight-line) which with a medial (area) makes a medial whole, and $C D$ a rational (straight-line). And let $C E$, equal to the (square) on $A B$, have been applied to $C D$, producing $C F$ as breadth. I say that $C F$ is a sixth apotome.

For let $B G$ be an attachment to $A B$. Thus, $A G$ and $G B$ are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle
 $\mu \varepsilon ̀ v ~ \alpha ̇ o ̀ ~ \tau \tilde{\eta} \varsigma ~ A H$ '̛oov tò $\Gamma \Theta \pi \lambda \alpha ́ \tau o s ~ \pi o เ o u ̃ \nu ~ \tau \eta ̀ \nu ~ \Gamma K, ~ \tau \widetilde{̣}$












 $\dot{\eta} \Gamma \mathrm{M}$ трòs $\tau \grave{\eta} \nu \mathrm{MZ}$ • $\alpha \sigma \dot{\prime} \mu \mu \varepsilon \tau \rho \circ \varsigma$ ä $\rho \alpha$ ह̇ $\sigma \tau i \nu \dot{\eta} \Gamma \mathrm{M} \tau \tilde{n} \mathrm{MZ}$











 $\alpha \dot{\alpha} o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AH}, \mathrm{HB} \mu \varepsilon ́ \sigma o \nu ~ \alpha ̀ \nu \alpha ́ \lambda o \gamma o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ u ́ \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AH}$,







contained) by $A G$ and $G B$ medial, and the (sum of the squares) on $A G$ and $G B$ incommensurable with twice the (rectangle contained) by $A G$ and $G B$ [Prop. 10.78]. Therefore, let $C H$, equal to the (square) on $A G$, have been applied to $C D$, producing $C K$ as breadth, and $K L$, equal to the (square) on $B G$. Thus, the whole of $C L$ is equal to the (sum of the squares) on $A G$ and $G B$. $C L$ [is] thus also medial. And it is applied to the rational (straight-line) $C D$, producing $C M$ as breadth. Thus, $C M$ is rational, and incommensurable in length with $C D$ [Prop. 10.22]. Therefore, since $C L$ is equal to the (sum of the squares) on $A G$ and $G B$, of which $C E$ (is) equal to the (square) on $A B$, the remainder $F L$ is thus equal to twice the (rectangle contained) by $A G$ and $G B$ [Prop. 2.7]. And twice the (rectangle contained) by $A G$ and $G B$ (is) medial. Thus, $F L$ is also medial. And it is applied to the rational (straight-line) $F E$, producing $F M$ as breadth. $F M$ is thus rational, and incommensurable in length with $C D$ [Prop. 10.22]. And since the (sum of the squares) on $A G$ and $G B$ is incommensurable with twice the (rectangle contained) by $A G$ and $G B$, and $C L$ equal to the (sum of the squares) on $A G$ and $G B$, and $F L$ equal to twice the (rectangle contained) by $A G$ and $G B, C L$ [is] thus incommensurable with $F L$. And as $C L$ (is) to $F L$, so $C M$ is to $M F$ [Prop. 6.1]. Thus, $C M$ is incommensurable in length with $M F$ [Prop. 10.11]. And they are both rational. Thus, $C M$ and $M F$ are rational (straight-lines which are) commensurable in square only. $C F$ is thus an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since $F L$ is equal to twice the (rectangle contained) by $A G$ and $G B$, let $F M$ have been cut in half at $N$, and let $N O$ have been drawn through $N$, parallel to $C D$. Thus, $F O$ and $N L$ are each equal to the (rectangle contained) by $A G$ and $G B$. And since $A G$ and $G B$ are incommensurable in square, the (square) on $A G$ is thus incommensurable with the (square) on $G B$. But, $C H$ is equal to the (square) on $A G$, and $K L$ is equal to the (square) on $G B$. Thus, $C H$ is incommensurable with $K L$. And as $C H$ (is) to $K L$, so $C K$ is to $K M$ [Prop. 6.1]. Thus, $C K$ is incommensurable (in length) with $K M$ [Prop. 10.11]. And since the (rectangle contained) by $A G$ and $G B$ is the mean proportional to the (squares) on $A G$ and $G B$ [Prop. 10.21 lem.], and $C H$ is equal to the (square) on $A G$, and $K L$ equal to the (square) on $G B$, and $N L$ equal to the (rectangle contained) by $A G$ and $G B, N L$ is thus also the mean proportional to $C H$ and $K L$. Thus, as $C H$ is to $N L$, so $N L$ (is) to $K L$. And for the same (reasons as the preceding propositions), the square on $C M$ is greater than (the square on) $M F$ by the (square) on (some straight-line)
$p \gamma^{\prime}$.
 $\tau \alpha ́ \xi \varepsilon ા \dot{\eta} \alpha \dot{u} \tau \dot{\eta}$.


 rñ AB .









 AB ].












 $\mathrm{Z} \Delta$.


incommensurable (in length) with ( $C M$ ) [Prop. 10.18]. And neither of them is commensurable with the (previously) laid down rational (straight-line) $C D$. Thus, $C F$ is a sixth apotome [Def. 10.16]. (Which is) the very thing it was required to show.

## Proposition 103

A (straight-line) commensurable in length with an apotome is an apotome, and (is) the same in order.


Let $A B$ be an apotome, and let $C D$ be commensurable in length with $A B$. I say that $C D$ is also an apotome, and (is) the same in order as $A B$.

For since $A B$ is an apotome, let $B E$ be an attachment to it. Thus, $A E$ and $E B$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let it have been contrived that the (ratio) of $B E$ to $D F$ is the same as the ratio of $A B$ to $C D$ [Prop. 6.12]. Thus, also, as one is to one, (so) all [are] to all [Prop. 5.12]. And thus as the whole $A E$ is to the whole $C F$, so $A B$ (is) to $C D$. And $A B$ (is) commensurable in length with $C D . A E$ (is) thus also commensurable (in length) with $C F$, and $B E$ with $D F$ [Prop. 10.11]. And $A E$ and $B E$ are rational (straight-lines which are) commensurable in square only. Thus, $C F$ and $F D$ are also rational (straight-lines which are) commensurable in square only [Prop. 10.13]. [ $C D$ is thus an apotome. So, I say that (it is) also the same in order as $A B$.]

Therefore, since as $A E$ is to $C F$, so $B E$ (is) to $D F$, thus, alternately, as $A E$ is to $E B$, so $C F$ (is) to $F D$ [Prop. 5.16]. So, the square on $A E$ is greater than (the square on) $E B$ either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with ( $A E$ ). Therefore, if the (square) on $A E$ is greater than (the square on) $E B$ by the (square) on (some straight-line) commensurable (in length) with ( $A E$ ) then the square on $C F$ will also be greater than (the square on) $F D$ by the (square) on (some straight-line) commensurable (in length) with ( $C F$ ) [Prop. 10.14]. And if $A E$ is commensurable in length with a (previously) laid down rational (straight-line) then so (is) $C F$ [Prop. 10.12], and if $B E$ (is commensurable), so (is) $D F$, and if neither of $A E$ or $E B$ (are commensurable), neither (are) either of $C F$ or $F D$ [Prop. 10.13]. And if the (square) on $A E$ is greater [than (the square on) $E B$ ] by the (square) on (some straight-line) incommensurable (in


 $\tau \alpha \dot{\xi} \varepsilon \iota \dot{\eta} \alpha \cup \dot{\tau} \dot{\eta} \tau \tilde{n} \mathrm{AB}$.


 BE трòs тウ̀̀ $\Delta Z$. $\sigma u ́ \mu \mu \varepsilon \tau \rho o s ~ \alpha ̈ p \alpha ~[\varepsilon ̇ \sigma \tau i] ~ \chi \alpha i ̀ ~ \dot{\eta} \mathrm{AE} \mathrm{\tau} \mathrm{\tilde{} \mathrm{\eta}} \Gamma \mathrm{Z}$, $\dot{\eta}$ ठè $\mathrm{BE} \tau \tilde{\eta} \Delta \mathrm{Z}$. ai $\delta \grave{e ̀ ~} \mathrm{AE}$, $\mathrm{EB} \mu \varepsilon ́ \sigma \alpha l ~ \varepsilon i \sigma i ̀ ~ \delta u v \alpha ́ \mu \varepsilon ı ~ \mu o ́ v o v ~$

 x $\alpha \grave{1} \tau \tilde{n} \tau \alpha \xi \varepsilon ા ~ \varepsilon ̇ \sigma \tau i \nu ~ \dot{\eta} \alpha u ̉ \tau \grave{\eta} \tau \tilde{n} \mathrm{AB}$.
 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{Z} \Delta\left[\dot{\alpha} \lambda \lambda{ }^{\circ} \dot{\omega} \varsigma \mu \bar{\varepsilon} \nu \dot{\eta} \mathrm{AE} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{~EB}\right.$, oü $\tau \omega \varsigma$ tò









 $\mathrm{Z} \Delta$.


length) with ( $A E$ ) then the (square) on $C F$ will also be greater than (the square on) $F D$ by the (square) on (some straight-line) incommensurable (in length) with ( $C F$ ) [Prop. 10.14]. And if $A E$ is commensurable in length with a (previously) laid down rational (straightline), so (is) $C F$ [Prop. 10.12], and if $B E$ (is commensurable), so (is) $D F$, and if neither of $A E$ or $E B$ (are commensurable), neither (are) either of $C F$ or $F D$ [Prop. 10.13].

Thus, $C D$ is an apotome, and (is) the same in order as $A B$ [Defs. 10.11-10.16]. (Which is) the very thing it was required to show.

## Proposition 104

A (straight-line) commensurable (in length) with an apotome of a medial (straight-line) is an apotome of a medial (straight-line), and (is) the same in order.


Let $A B$ be an apotome of a medial (straight-line), and let $C D$ be commensurable in length with $A B$. I say that $C D$ is also an apotome of a medial (straight-line), and (is) the same in order as $A B$.

For since $A B$ is an apotome of a medial (straightline), let $E B$ be an attachment to it. Thus, $A E$ and $E B$ are medial (straight-lines which are) commensurable in square only [Props. 10.74, 10.75]. And let it have been contrived that as $A B$ is to $C D$, so $B E$ (is) to $D F$ [Prop. 6.12]. Thus, $A E$ [is] also commensurable (in length) with $C F$, and $B E$ with $D F$ [Props. 5.12, 10.11]. And $A E$ and $E B$ are medial (straight-lines which are) commensurable in square only. $C F$ and $F D$ are thus also medial (straight-lines which are) commensurable in square only [Props. 10.23, 10.13]. Thus, $C D$ is an apotome of a medial (straight-line) [Props. 10.74, 10.75]. So, I say that it is also the same in order as $A B$.
[For] since as $A E$ is to $E B$, so $C F$ (is) to $F D$ [Props. 5.12, 5.16] [but as $A E$ (is) to $E B$, so the (square) on $A E$ (is) to the (rectangle contained) by $A E$ and $E B$, and as $C F$ (is) to $F D$, so the (square) on $C F$ (is) to the (rectangle contained) by $C F$ and $F D$ ], thus as the (square) on $A E$ is to the (rectangle contained) by $A E$ and $E B$, so the (square) on $C F$ also (is) to the (rectangle contained) by $C F$ and $F D$ [Prop. 10.21 lem.] [and, alternately, as the (square) on $A E$ (is) to the (square) on $C F$, so the (rectangle contained) by $A E$ and $E B$ (is) to the (rectangle contained) by $C F$ and $F D$ ]. And the (square) on $A E$ (is) commensurable with the (square)








 oứt





 tò $\dot{\alpha} \pi \grave{o}$ тñऽ $\mathrm{AE} \pi \rho o ̀ \varsigma ~ \tau o ̀ ~ u ́ \pi o ̀ ~ \tau \omega ̃ \nu ~ A E, ~ E B, ~ o u ̛ ̃ \omega \varsigma ~ \tau o ̀ ~ \alpha ́ \alpha o ̀ ~$





 $\dot{\cup} \pi^{\prime} \alpha u ̋ \tau \widetilde{\omega} \nu \mu \varepsilon ́ \sigma o \nu$.

on $C F$. Thus, the (rectangle contained) by $A E$ and $E B$ is also commensurable with the (rectangle contained) by $C F$ and $F D$ [Props. 5.16, 10.11]. Therefore, either the (rectangle contained) by $A E$ and $E B$ is rational, and the (rectangle contained) by $C F$ and $F D$ will also be rational [Def. 10.4], or the (rectangle contained) by $A E$ and $E B$ [is] medial, and the (rectangle contained) by $C F$ and $F D$ [is] also medial [Prop. 10.23 corr.].

Therefore, $C D$ is the apotome of a medial (straightline), and is the same in order as $A B$ [Props. 10.74, 10.75]. (Which is) the very thing it was required to show.

## Proposition 105

A (straight-line) commensurable (in length) with a minor (straight-line) is a minor (straight-line).


For let $A B$ be a minor (straight-line), and (let) $C D$ (be) commensurable (in length) with $A B$. I say that $C D$ is also a minor (straight-line).

For let the same things have been contrived (as in the former proposition). And since $A E$ and $E B$ are (straight-lines which are) incommensurable in square [Prop. 10.76], $C F$ and $F D$ are thus also (straight-lines which are) incommensurable in square [Prop. 10.13]. Therefore, since as $A E$ is to $E B$, so $C F$ (is) to $F D$ [Props. 5.12, 5.16], thus also as the (square) on $A E$ is to the (square) on $E B$, so the (square) on $C F$ (is) to the (square) on $F D$ [Prop. 6.22]. Thus, via composition, as the (sum of the squares) on $A E$ and $E B$ is to the (square) on $E B$, so the (sum of the squares) on $C F$ and $F D$ (is) to the (square) on $F D$ [Prop. 5.18], [also alternately]. And the (square) on $B E$ is commensurable with the (square) on $D F$ [Prop. 10.104]. The sum of the squares on $A E$ and $E B$ (is) thus also commensurable with the sum of the squares on $C F$ and $F D$ [Prop. 5.16, 10.11]. And the sum of the (squares) on $A E$ and $E B$ is rational [Prop. 10.76]. Thus, the sum of the (squares) on $C F$ and $F D$ is also rational [Def. 10.4]. Again, since as the (square) on $A E$ is to the (rectangle contained) by $A E$ and $E B$, so the (square) on $C F$ (is) to the (rectangle contained) by $C F$ and $F D$ [Prop. 10.21 lem.], and the square on $A E$ (is) commensurable with the square on $C F$, the (rectangle contained) by $A E$ and $E B$ is thus also commensurable with the (rectangle contained) by $C F$ and $F D$. And the (rectangle contained) by $A E$ and $E B$ (is) medial [Prop. 10.76]. Thus, the (rectangle contained) by $C F$ and $F D$ (is) also medial [Prop. 10.23 corr.]. $C F$ and


#### Abstract

$p q^{\prime}$. ${ }^{`} H$ тñ $\mu \varepsilon \tau \alpha ̀ ~ \rho ́ \eta \tau o u ̃ ~ \mu \varepsilon ́ \sigma o v ~ \tau o ̀ ~ o ̈ \lambda o v ~ \pi o เ o u ́ \sigma n ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o s ~$ 




 tò ö̀ov $\pi$ oเoũ $\sigma \alpha ́ ~ \varepsilon ̇ \sigma \tau เ \nu . ~$
＂E $\sigma \tau \omega \gamma \dot{\alpha} \rho \tau \tilde{n} \mathrm{AB} \pi \rho o \sigma \alpha \rho \mu o ́ \zeta o v \sigma \alpha \dot{\eta} \mathrm{BE} \cdot \alpha \mathrm{AE}, \mathrm{EB}$ 的 $\alpha$
 $\tau \widetilde{\omega} \nu \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{AE}, \mathrm{EB} \tau \varepsilon \tau \rho \alpha \gamma \dot{\omega} \nu \omega \nu \mu \varepsilon ́ \sigma o \nu$ ，七ò $\delta^{\prime} \dot{\cup} \pi^{\prime} \alpha \cup \tau \widetilde{\omega} \nu$

 EB，xai $\sigma u ́ \mu \mu \varepsilon \tau \rho o ́ v ~ \varepsilon ̇ \sigma \tau l ~ t o ̀ ~ \sigma u \gamma x \varepsilon i ́ \mu \varepsilon v o v ~ \varepsilon ่ x ~ \tau \widetilde{\omega} \nu ~ \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu$
 $\mathrm{Z} \Delta$ тєтраү⿳㇒⿵冂一兀⿴囗⿱一一






$$
\rho \zeta^{\prime} .
$$




${ }^{3} \mathrm{E} \sigma \tau \omega \mu \varepsilon \tau \grave{\alpha} \mu \varepsilon ́ \sigma o \cup \mu \varepsilon ́ \sigma o \nu$ đò ö̀ov $\pi$ oเoṽ $\sigma \alpha \dot{\eta} \mathrm{AB}, x \alpha \grave{\imath} \tau \tilde{\eta}$
$F D$ are thus（straight－lines which are）incommensurable in square，making the sum of the squares on them ratio－ nal，and the（rectangle contained）by them medial．

Thus，$C D$ is a minor（straight－line）［Prop．10．76］． （Which is）the very thing it was required to show．

## Proposition 106

A（straight－line）commensurable（in length）with a （straight－line）which with a rational（area）makes a me－ dial whole is a（straight－line）which with a rational（area） makes a medial whole．


Let $A B$ be a（straight－line）which with a rational （area）makes a medial whole，and（let）$C D$（be）com－ mensurable（in length）with $A B$ ．I say that $C D$ is also a （straight－line）which with a rational（area）makes a me－ dial（whole）．

For let $B E$ be an attachment to $A B$ ．Thus，$A E$ and $E B$ are（straight－lines which are）incommensurable in square，making the sum of the squares on $A E$ and $E B$ medial，and the（rectangle contained）by them rational ［Prop．10．77］．And let the same construction have been made（as in the previous propositions）．So，similarly to the previous（propositions），we can show that $C F$ and $F D$ are in the same ratio as $A E$ and $E B$ ，and the sum of the squares on $A E$ and $E B$ is commensurable with the sum of the squares on $C F$ and $F D$ ，and the（rectangle contained）by $A E$ and $E B$ with the（rectangle contained） by $C F$ and $F D$ ．Hence，$C F$ and $F D$ are also（straight－ lines which are）incommensurable in square，making the sum of the squares on $C F$ and $F D$ medial，and the（rect－ angle contained）by them rational．
$C D$ is thus a（straight－line）which with a rational （area）makes a medial whole［Prop．10．77］．（Which is） the very thing it was required to show．

## Proposition 107

A（straight－line）commensurable（in length）with a （straight－line）which with a medial（area）makes a me－ dial whole is itself also a（straight－line）which with a me－ dial（area）makes a medial whole．


Let $A B$ be a（straight－line）which with a medial（area）
 $\mu \varepsilon ́ \sigma o \nu ~ \tau o ̀ ~ o ̈ \lambda o \nu ~ \pi o เ o u ̃ \sigma \alpha ́ ~ \varepsilon ̇ \sigma \tau \tau \nu . ~$


 $\mu \varepsilon ́ \sigma o \nu ~ \chi \alpha i ̀ ~ t o ̀ ~ ن ́ \pi ’ ~ \alpha u ̉ \tau \widetilde{\omega \nu} \mu \varepsilon ́ \sigma o \nu ~ \chi \alpha i ~ ह ै \tau ा ~ \alpha ̇ \sigma u ́ \mu \mu \varepsilon \tau р о \nu ~ \tau o ̀ ~$







 $\tau \rho \alpha \gamma \omega \dot{\nu} \omega \nu] \tau \widetilde{\varphi} \dot{\sim} \pi^{\prime} \alpha \cup \tau \widetilde{\omega} \nu$.
 ӧ $\pi \varepsilon \rho$ है $\delta \varepsilon \iota ~ \delta \varepsilon і ̈ \xi \alpha$.

$$
\rho \eta^{\prime}
$$




'A ơtı $\dot{\eta}$ tò $\lambda о \iota \pi o ̀ v ~ \delta u v \alpha \mu e ́ v \eta ~ \tau o ̀ ~ E \Gamma ~ \mu i ́ \alpha ~ \delta u ́ o ~ \alpha ́ \lambda o ́ \gamma \omega \nu ~ \gamma i ́ v e \tau \alpha l ~$









makes a medial whole, and let $C D$ be commensurable (in length) with $A B$. I say that $C D$ is also a (straight-line) which with a medial (area) makes a medial whole.

For let $B E$ be an attachment to $A B$. And let the same construction have been made (as in the previous propositions). Thus, $A E$ and $E B$ are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.78]. And, as was shown (previously), $A E$ and $E B$ are commensurable (in length) with $C F$ and $F D$ (respectively), and the sum of the squares on $A E$ and $E B$ with the sum of the squares on $C F$ and $F D$, and the (rectangle contained) by $A E$ and $E B$ with the (rectangle contained) by $C F$ and $F D$. Thus, $C F$ and $F D$ are also (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the [squares] on them incommensurable with the (rectangle contained) by them.

Thus, $C D$ is a (straight-line) which with a medial (area) makes a medial whole [Prop. 10.78]. (Which is) the very thing it was required to show.

## Proposition 108

A medial (area) being subtracted from a rational (area), one of two irrational (straight-lines) arise (as) the square-root of the remaining area-either an apotome, or a minor (straight-line).


For let the medial (area) $B D$ have been subtracted from the rational (area) $B C$. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), $E C$-either an apotome, or a minor (straight-line).

For let the rational (straight-line) $F G$ have been laid out, and let the right-angled parallelogram $G H$, equal to $B C$, have been applied to $F G$, and let $G K$, equal to $D B$, have been subtracted (from $G H$ ). Thus, the remainder $E C$ is equal to $L H$. Therefore, since $B C$ is a rational (area), and $B D$ a medial (area), and $B C$ (is) equal to


 бициદ́трои $\hat{\eta}$ oű.
 ӨZ бú $\mu \mu \varepsilon \tau \rho о \varsigma ~ \tau \tilde{n}$ ह̉xx








$p \vartheta^{\prime}$.



 ớtı $\dot{\eta}$ tò $\lambda о \iota \pi o ̀ v ~ \tau o ̀ ~ E \Gamma ~ \delta u v \alpha \mu e ́ v \eta ~ \mu i ́ \alpha ~ \delta u ́ o ~ \alpha ̀ \lambda o ́ \gamma \omega \nu ~ \gamma i v e \tau \alpha l ~$


${ }^{3}$ Еxx




 ג̇лò $\alpha \sigma \cup \mu \mu$ étpou.
$G H$, and $B D$ to $G K, G H$ is thus a rational (area), and $G K$ a medial (area). And they are applied to the rational (straight-line) $F G$. Thus, $F H$ (is) rational, and commensurable in length with $F G$ [Prop. 10.20], and $F K$ (is) also rational, and incommensurable in length with $F G$ [Prop. 10.22]. Thus, $F H$ is incommensurable in length with $F K$ [Prop. 10.13]. $F H$ and $F K$ are thus rational (straight-lines which are) commensurable in square only. Thus, $K H$ is an apotome [Prop. 10.73], and $K F$ an attachment to it. So, the square on $H F$ is greater than (the square on) $F K$ by the (square) on (some straightline which is) either commensurable, or not (commensurable), (in length with $H F$ ).

First, let the square (on it) be (greater) by the (square) on (some straight-line which is) commensurable (in length with $H F$ ). And the whole of $H F$ is commensurable in length with the (previously) laid down rational (straight-line) $F G$. Thus, $K H$ is a first apotome [Def. 10.1]. And the square-root of an (area) contained by a rational (straight-line) and a first apotome is an apotome [Prop. 10.91]. Thus, the square-root of $L H$-that is to say, (of) $E C$-is an apotome.

And if the square on $H F$ is greater than (the square on) $F K$ by the (square) on (some straight-line which is) incommensurable (in length) with ( $H F$ ), and (since) the whole of FH is commensurable in length with the (previously) laid down rational (straight-line) $F G, K H$ is a fourth apotome [Prop. 10.14]. And the square-root of an (area) contained by a rational (straight-line) and a fourth apotome is a minor (straight-line) [Prop. 10.94]. (Which is) the very thing it was required to show.

## Proposition 109

A rational (area) being subtracted from a medial (area), two other irrational (straight-lines) arise (as the square-root of the remaining area)-either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (area) $B D$ have been subtracted from the medial (area) $B C$. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), $E C$-either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (straight-line) $F G$ be laid down, and let similar areas (to the preceding proposition) have been applied (to it). So, accordingly, $F H$ is rational, and incommensurable in length with $F G$, and $K F$ (is) also rational, and commensurable in length with $F G$. Thus, $F H$ and $F K$ are rational (straight-lines which are) com-









 ӧ $\pi \varepsilon \rho$ है $\delta \varepsilon \iota ~ \delta \varepsilon i ̋ \xi \alpha เ . ~$

## pi'.


 $\mu \varepsilon \tau \alpha ̀ \mu \varepsilon ́ \sigma o u ~ \mu \varepsilon ́ \sigma o v ~ t o ̀ ~ o ̈ \lambda o v ~ \pi o เ o v ̃ \sigma \alpha . ~$
'А $\varphi$ пр



mensurable in square only [Prop. 10.13]. $K H$ is thus an apotome [Prop. 10.73], and $F K$ an attachment to it. So, the square on $H F$ is greater than (the square on) $F K$ either by the (square) on (some straight-line) commensurable (in length) with ( $H F$ ), or by the (square) on (some straight-line) incommensurable (in length with $H F$ ).


Therefore, if the square on $H F$ is greater than (the square on) $F K$ by the (square) on (some straight-line) commensurable (in length) with ( $H F$ ), and (since) the attachment $F K$ is commensurable in length with the (previously) laid down rational (straight-line) $F G, K H$ is a second apotome [Def. 10.12]. And $F G$ (is) rational. Hence, the square-root of $L H$-that is to say, (of) $E C$-is a first apotome of a medial (straight-line) [Prop. 10.92].

And if the square on $H F$ is greater than (the square on) $F K$ by the (square) on (some straight-line) incommensurable (in length with $H F$ ), and (since) the attachment $F K$ is commensurable in length with the (previously) laid down rational (straight-line) $F G, K H$ is a fifth apotome [Def. 10.15]. Hence, the square-root of $E C$ is that (straight-line) which with a rational (area) makes a medial whole [Prop. 10.95]. (Which is) the very thing it was required to show.

## Proposition 110

A medial (area), incommensurable with the whole, being subtracted from a medial (area), the two remaining irrational (straight-lines) arise (as) the (square-root of the area) -either a second apotome of a medial (straightline), or that (straight-line) which with a medial (area) makes a medial whole.

For, as in the previous figures, let the medial (area) $B D$, incommensurable with the whole, have been subtracted from the medial (area) $B C$. I say that the squareroot of $E C$ is one of two irrational (straight-lines) either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.















 દ̇எтા $\delta \varepsilon \cup \tau \varepsilon \rho \alpha ́$.








For since $B C$ and $B D$ are each medial (areas), and $B C$ (is) incommensurable with $B D$, accordingly, $F H$ and $F K$ will each be rational (straight-lines), and incommensurable in length with $F G$ [Prop. 10.22]. And since $B C$ is incommensurable with $B D$-that is to say, $G H$ with $G K-H F$ (is) also incommensurable (in length) with $F K$ [Props. 6.1, 10.11]. Thus, $F H$ and $F K$ are rational (straight-lines which are) commensurable in square only. $K H$ is thus as apotome [Prop. 10.73], [and $F K$ an attachment (to it). So, the square on $F H$ is greater than (the square on) $F K$ either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with $(F H)$.]

So, if the square on $F H$ is greater than (the square on) $F K$ by the (square) on (some straight-line) commensurable (in length) with ( $F H$ ), and (since) neither of $F H$ and $F K$ is commensurable in length with the (previously) laid down rational (straight-line) $F G, K H$ is a third apotome [Def. 10.3]. And $K L$ (is) rational. And the rectangle contained by a rational (straight-line) and a third apotome is irrational, and the square-root of it is that irrational (straight-line) called a second apotome of a medial (straight-line) [Prop. 10.93]. Hence, the squareroot of $L H$-that is to say, (of) $E C$-is a second apotome of a medial (straight-line).

And if the square on $F H$ is greater than (the square on) $F K$ by the (square) on (some straight-line) incommensurable [in length] with ( $F H$ ), and (since) neither of $H F$ and $F K$ is commensurable in length with $F G, K H$ is a sixth apotome [Def. 10.16]. And the square-root of the (rectangle contained) by a rational (straight-line) and a sixth apotome is that (straight-line) which with a medial (area) makes a medial whole [Prop. 10.96]. Thus, the square-root of $L H$-that is to say, (of) $E C$-is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to
show.
pi $\alpha^{\prime}$.


 $\tau \tilde{n}$ èx $\delta$ ல́o óvoú́t $\tau \omega$.























## Proposition 111

An apotome is not the same as a binomial.


Let $A B$ be an apotome. I say that $A B$ is not the same as a binomial.

For, if possible, let it be (the same). And let a rational (straight-line) $D C$ be laid down. And let the rectangle $C E$, equal to the (square) on $A B$, have been applied to $C D$, producing $D E$ as breadth. Therefore, since $A B$ is an apotome, $D E$ is a first apotome [Prop. 10.97]. Let $E F$ be an attachment to it. Thus, $D F$ and $F E$ are rational (straight-lines which are) commensurable in square only, and the square on $D F$ is greater than (the square on) $F E$ by the (square) on (some straight-line) commensurable (in length) with ( $D F$ ), and $D F$ is commensurable in length with the (previously) laid down rational (straightline) $D C$ [Def. 10.10]. Again, since $A B$ is a binomial, $D E$ is thus a first binomial [Prop. 10.60]. Let ( $D E$ ) have been divided into its (component) terms at $G$, and let $D G$ be the greater term. Thus, $D G$ and $G E$ are rational (straight-lines which are) commensurable in square only, and the square on $D G$ is greater than (the square on) $G E$ by the (square) on (some straight-line) commensurable (in length) with ( $D G$ ), and the greater (term) $D G$ is commensurable in length with the (previously) laid down rational (straight-line) $D C$ [Def. 10.5]. Thus, $D F$ is also commensurable in length with $D G$ [Prop. 10.12]. The remainder $G F$ is thus commensurable in length with $D F$ [Prop. 10.15]. [Therefore, since $D F$ is commensurable with $G F$, and $D F$ is rational, $G F$ is thus also rational. Therefore, since $D F$ is commensurable in length with $G F] D$,$F (is) incommensurable in length with E F$. Thus, $F G$ is also incommensurable in length with $E F$ [Prop. 10.13]. $G F$ and $F E$ [are] thus rational (straightlines which are) commensurable in square only. Thus,


#### Abstract

[Пópıб $\mu$.]  

Tò $\mu \varepsilon ̀ v ~ \gamma \alpha ̀ \rho ~ \alpha ̇ \pi o ̀ ~ \mu \varepsilon ́ \sigma \eta s ~ \pi \alpha p \alpha ̀ ~ \rho ं \eta t \eta ̀ \nu ~ \pi \alpha p \alpha \beta \alpha \lambda \lambda o ́ \mu \varepsilon v o v ~$          то $\mu \grave{\eta} \nu \pi \varepsilon ́ \mu \pi \tau \eta \nu$, тò $\delta \varepsilon ̀ ~ \alpha ́ \alpha o ̀ ~ \tau n ̃ s ~ \mu \varepsilon \tau \alpha ̀ ~ \mu \varepsilon ́ \sigma o u ~ \mu \varepsilon ́ \sigma o v ~ \tau o ̀ ~ o ̋ \lambda o v ~$            $\pi \dot{\alpha} \sigma \alpha \varsigma \dot{\alpha} \lambda o ́ \gamma o u s \overline{\mathfrak{l}}$,


$E G$ is an apotome [Prop. 10.73]. But, (it is) also rational. The very thing is impossible.

Thus, an apotome is not the same as a binomial. (Which is) the very thing it was required to show.

## [Corollary]

The apotome and the irrational (straight-lines) after it are neither the same as a medial (straight-line) nor (the same) as one another.

For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) incommensurable in length with the (straight-line) to which (the area) is applied [Prop. 10.22]. And the (square) on an apotome, applied to a rational (straight-line), produces as breadth a first apotome [Prop. 10.97]. And the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a second apotome [Prop. 10.98]. And the (square) on a second apotome of a medial (straight-line), applied to a rational (straightline), produces as breadth a third apotome [Prop. 10.99]. And (square) on a minor (straight-line), applied to a rational (straight-line), produces as breadth a fourth apotome [Prop. 10.100]. And (square) on that (straight-line) which with a rational (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a fifth apotome [Prop. 10.101]. And (square) on that (straight-line) which with a medial (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a sixth apotome [Prop. 10.102]. Therefore, since the aforementioned breadths differ from the first (breadth), and from one another-from the first, because it is rational, and from one another since they are not the same in order-clearly, the irrational (straightlines) themselves also differ from one another. And since it has been shown that an apotome is not the same as a binomial [Prop. 10.111], and (that) the (irrational straight-lines) after the apotome, being applied to a rational (straight-line), produce as breadth, each according to its own (order), apotomes, and (that) the (irrational straight-lines) after the binomial themselves also (produce as breadth), according (to their) order, binomials, the (irrational straight-lines) after the apotome are thus different, and the (irrational straight-lines) after the binomial (are also) different, so that there are, in order, 13 irrational (straight-lines) in all:

Méбทレ，

＇Еぇ $\delta$ ט́o $\mu \varepsilon ́ \sigma \omega \nu \pi \rho \omega ́ \tau \eta \nu$,
＇Еथ ठúo $\mu \varepsilon ́ \sigma \omega \nu ~ \delta \varepsilon \cup \tau \varepsilon ́ p \alpha \nu$,
Mعіًova，

$\Delta$ ט́o $\mu \varepsilon ́ \sigma \alpha ~ \delta u v \alpha \mu \varepsilon ́ v \eta \nu, ~$
＇Алотоци́v，
Méons à $\pi о \tau о \mu \grave{\nu} \nu \pi \rho \omega ́ \tau \eta \nu$ ，

＇E入人́ббov $\alpha$ ，


Meт $\alpha \mu \varepsilon ́ \sigma o u ~ \mu \varepsilon ́ \sigma o \nu ~ \tau o ̀ ~ o ̈ \lambda o \nu ~ \pi o เ o u ̃ \sigma \alpha \nu . ~$
pi $\beta^{\prime}$ ．
Tò á $\pi o ̀ ~ p ̊ \eta \tau n ̃ s ~ \pi \alpha p \alpha ̀ ~ \tau \eta ̀ \nu ~ ह ̉ x ~ \delta u ́ o ~ o ̉ v o \mu \alpha ́ \tau \omega \nu ~ \pi \alpha p \alpha-~$


 ėx ठúo ỏvo $\alpha \dot{\alpha} \tau \omega \nu$ ．













 ӨK $\pi \rho o ̀ s ~ o ̈ \lambda \eta \nu ~ \tau \grave{\eta} \nu \mathrm{KZ}$ ह̇ $\sigma \tau \nu, \dot{\omega} \varsigma \dot{\eta} \mathrm{ZK} \pi \rho o ̀ s ~ K E \cdot \dot{\omega} \varsigma ~ \gamma \dot{\alpha} \rho$ हैv






Medial，
Binomial，
First bimedial，
Second bimedial，
Major，
Square－root of a rational plus a medial（area），
Square－root of（the sum of）two medial（areas），
Apotome，
First apotome of a medial，
Second apotome of a medial，
Minor，
That which with a rational（area）produces a medial whole，

That which with a medial（area）produces a medial whole．

## Proposition $112^{\dagger}$

The（square）on a rational（straight－line），applied to a binomial（straight－line），produces as breadth an apo－ tome whose terms are commensurable（in length）with the terms of the binomial，and，furthermore，in the same ratio．Moreover，the created apotome will have the same order as the binomial．


Let $A$ be a rational（straight－line），and $B C$ a binomial （straight－line），of which let $D C$ be the greater term．And let the（rectangle contained）by $B C$ and $E F$ be equal to the（square）on $A$ ．I say that $E F$ is an apotome whose terms are commensurable（in length）with $C D$ and $D B$ ， and in the same ratio，and，moreover，that $E F$ will have the same order as $B C$ ．

For，again，let the（rectangle contained）by $B D$ and $G$ be equal to the（square）on $A$ ．Therefore，since the（rect－ angle contained）by $B C$ and $E F$ is equal to the（rectan－ gle contained）by $B D$ and $G$ ，thus as $C B$ is to $B D$ ，so $G$ （is）to $E F$［Prop．6．16］．And $C B$（is）greater than $B D$ ． Thus，$G$ is also greater than $E F$［Props．5．16，5．14］．Let $E H$ be equal to $G$ ．Thus，as $C B$ is to $B D$ ，so $H E$（is）to $E F$ ．Thus，via separation，as $C D$ is to $B D$ ，so $H F$（is） to $F E$［Prop．5．17］．Let it have been contrived that as $H F$（is）to $F E$ ，so $F K$（is）to $K E$ ．And，thus，the whole $H K$ is to the whole $K F$ ，as $F K$（is）to $K E$ ．For as one of the leading（proportional magnitudes is）to one of the

 KE $\alpha \nu \alpha ́ \lambda o \gamma o ́ v ~ \varepsilon i \sigma เ \nu . ~ \sigma u ́ \mu \mu \varepsilon \tau p o s ~ \alpha ้ p \alpha ~ \dot{\eta} \Theta K ~ \tau \tilde{n} K E \mu \eta ́ x \varepsilon เ . ~$









 «̈p $\alpha$ モ̇ $\sigma \tau i \nu \dot{\eta}$ EZ.
 $\dot{\varepsilon} \alpha \cup \tau \tilde{n} \hat{\eta} \tau \widetilde{\varphi} \alpha \dot{\alpha} \pi \grave{\alpha} \alpha{ }^{\alpha} \sigma \cup \mu \mu \varepsilon ́ \tau \rho o \cup$.














following, so all of the leading (magnitudes) are to all of the following [Prop. 5.12]. And as $F K$ (is) to $K E$, so $C D$ is to $D B$ [Prop. 5.11]. And, thus, as $H K$ (is) to $K F$, so $C D$ is to $D B$ [Prop. 5.11]. And the (square) on $C D$ (is) commensurable with the (square) on $D B$ [Prop. 10.36]. The (square) on $H K$ is thus also commensurable with the (square) on $K F$ [Props. 6.22, 10.11]. And as the (square) on $H K$ is to the (square) on $K F$, so $H K$ (is) to $K E$, since the three (straight-lines) $H K, K F$, and $K E$ are proportional [Def. 5.9]. $H K$ is thus commensurable in length with $K E$ [Prop. 10.11]. Hence, $H E$ is also commensurable in length with $E K$ [Prop. 10.15]. And since the (square) on $A$ is equal to the (rectangle contained) by $E H$ and $B D$, and the (square) on $A$ is rational, the (rectangle contained) by $E H$ and $B D$ is thus also rational. And it is applied to the rational (straight-line) $B D$. Thus, $E H$ is rational, and commensurable in length with $B D$ [Prop. 10.20]. And, hence, the (straight-line) commensurable (in length) with it, $E K$, is also rational [Def. 10.3], and commensurable in length with $B D$ [Prop. 10.12]. Therefore, since as $C D$ is to $D B$, so $F K$ (is) to $K E$, and $C D$ and $D B$ are (straight-lines which are) commensurable in square only, $F K$ and $K E$ are also commensurable in square only [Prop. 10.11]. And $K E$ is rational. Thus, $F K$ is also rational. $F K$ and $K E$ are thus rational (straight-lines which are) commensurable in square only. Thus, $E F$ is an apotome [Prop. 10.73].

And the square on $C D$ is greater than (the square on) $D B$ either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with ( $C D$ ).

Therefore, if the square on $C D$ is greater than (the square on) $D B$ by the (square) on (some straight-line) commensurable (in length) with [ $C D$ ] then the square on $F K$ will also be greater than (the square on) $K E$ by the (square) on (some straight-line) commensurable (in length) with ( $F K$ ) [Prop. 10.14]. And if $C D$ is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) $F K$ [Props. 10.11, 10.12]. And if $B D$ (is commensurable), (so) also (is) $K E$ [Prop. 10.12]. And if neither of $C D$ or $D B$ (is commensurable), neither also (are) either of $F K$ or $K E$.

And if the square on $C D$ is greater than (the square on) $D B$ by the (square) on (some straight-line) incommensurable (in length) with ( $C D$ ) then the square on $F K$ will also be greater than (the square on) $K E$ by the (square) on (some straight-line) incommensurable (in length) with ( $F K$ ) [Prop. 10.14]. And if $C D$ is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) $F K$ [Props. 10.11, 10.12]. And if $B D$ (is commensurable), (so) also (is) $K E$
[Prop. 10.12]. And if neither of $C D$ or $D B$ (is commensurable), neither also (are) either of $F K$ or $K E$. Hence, $F E$ is an apotome whose terms, $F K$ and $K E$, are commensurable (in length) with the terms, $C D$ and $D B$, of the binomial, and in the same ratio. And $(F E)$ has the same order as $B C$ [Defs. 10.5-10.10]. (Which is) the very thing it was required to show.
${ }^{\dagger}$ Heiberg considers this proposition, and the succeeding ones, to be relatively early interpolations into the original text.

## pir'.






$\underset{\vdash}{\mathrm{K}} \underset{\sim}{\mathrm{Z}} \quad \Theta$











 $\mathrm{B} \Delta, \mathrm{K} \Theta$, $\alpha v \alpha ́ \lambda o \gamma o v ~ \alpha \alpha p \alpha ~ \varepsilon ́ \sigma \tau i \nu ~ \dot{\omega} \varsigma \dot{\eta} \Gamma \mathrm{~B}$ трòs $\mathrm{B} \Delta$, oưt $\omega \varsigma \dot{\eta}$



 $\Gamma \Delta$, oút $\omega$ ц $\dot{\eta} \mathrm{K} \Theta$ трòs $\Theta \mathrm{E}$. үعүovét $\omega \dot{\omega} \varsigma \dot{\eta} \mathrm{K} \Theta$ трòs $\Theta \mathrm{E}$,

 ВГ, Г $\Delta$ бuvá $\mu \varepsilon ı \mu o ́ v o v ~[\varepsilon i \sigma i] ~ \sigma u ́ \mu \mu \varepsilon \tau \rho o l \cdot ~ \chi \alpha i ~ \alpha i ~ K Z, ~ Z \Theta ~ \alpha ̈ p \alpha ~$
 $\Theta \mathrm{E}, \dot{\eta} \mathrm{KZ}$ трòs $\mathrm{Z} \Theta, \dot{\alpha} \lambda \lambda^{\circ}$ ís $\dot{\eta} \mathrm{K} \Theta$ трòs $\Theta \mathrm{E}, \dot{\eta} \Theta \mathrm{Z}$ трòs







## Proposition 113

The (square) on a rational (straight-line), applied to an apotome, produces as breadth a binomial whose terms are commensurable with the terms of the apotome, and in the same ratio. Moreover, the created binomial has the same order as the apotome.


Let $A$ be a rational (straight-line), and $B D$ an apotome. And let the (rectangle contained) by $B D$ and $K H$ be equal to the (square) on $A$, such that the square on the rational (straight-line) $A$, applied to the apotome $B D$, produces $K H$ as breadth. I say that $K H$ is a binomial whose terms are commensurable with the terms of $B D$, and in the same ratio, and, moreover, that $K H$ has the same order as $B D$.

For let $D C$ be an attachment to $B D$. Thus, $B C$ and $C D$ are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let the (rectangle contained) by $B C$ and $G$ also be equal to the (square) on $A$. And the (square) on $A$ (is) rational. The (rectangle contained) by $B C$ and $G$ (is) thus also rational. And it has been applied to the rational (straight-line) $B C$. Thus, $G$ is rational, and commensurable in length with $B C$ [Prop. 10.20]. Therefore, since the (rectangle contained) by $B C$ and $G$ is equal to the (rectangle contained) by $B D$ and $K H$, thus, proportionally, as $C B$ is to $B D$, so $K H$ (is) to $G$ [Prop. 6.16]. And $B C$ (is) greater than $B D$. Thus, $K H$ (is) also greater than $G$ [Prop. 5.16, 5.14]. Let $K E$ be made equal to $G$. $K E$ is thus commensurable in length with $B C$. And since as $C B$ is to $B D$, so $H K$ (is) to $K E$, thus, via conversion, as $B C$ (is) to $C D$, so $K H$ (is) to $H E$ [Prop. 5.19 corr.]. Let it have been contrived that as $K H$ (is) to $H E$, so $H F$ (is) to $F E$. And thus the remainder $K F$ is to $F H$, as $K H$ (is) to $H E$-that is to say, [as] $B C$ (is) to $C D$ [Prop. 5.19]. And $B C$ and $C D$ [are] commensurable in square only.






 ठúo ỏvouát $\omega \nu$ ह̇бтì $\alpha \not p \alpha \dot{\eta} K \Theta$.



 $\tau \tilde{n}$ モ̇xx




 $\tau \widetilde{\omega} \nu \mathrm{B}, \Gamma \Delta$, oủठє $\frac{1}{\rho} \rho \alpha \tau \widetilde{\omega} \nu \mathrm{KZ}, \mathrm{Z} \Theta$.




$K F$ and $F H$ are thus also commensurable in square only [Prop. 10.11]. And since as $K H$ is to $H E$, (so) $K F$ (is) to $F H$, but as $K H$ (is) to $H E$, (so) $H F$ (is) to $F E$, thus, also as $K F$ (is) to $F H$, (so) $H F$ (is) to $F E$ [Prop. 5.11]. And hence as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9]. And thus as $K F$ (is) to $F E$, so the (square) on $K F$ (is) to the (square) on $F H$. And the (square) on $K F$ is commensurable with the (square) on $F H$. For $K F$ and $F H$ are commensurable in square. Thus, $K F$ is also commensurable in length with $F E$ [Prop. 10.11]. Hence, $K F$ [is] also commensurable in length with $K E$ [Prop. 10.15]. And $K E$ is rational, and commensurable in length with $B C$. Thus, $K F$ (is) also rational, and commensurable in length with $B C$ [Prop. 10.12]. And since as $B C$ is to $C D$, (so) $K F$ (is) to $F H$, alternately, as $B C$ (is) to $K F$, so $D C$ (is) to $F H$ [Prop. 5.16]. And $B C$ (is) commensurable (in length) with $K F$. Thus, $F H$ (is) also commensurable in length with $C D$ [Prop. 10.11]. And $B C$ and $C D$ are rational (straight-lines which are) commensurable in square only. $K F$ and $F H$ are thus also rational (straight-lines which are) commensurable in square only [Def. 10.3, Prop. 10.13]. Thus, $K H$ is a binomial [Prop. 10.36].

Therefore, if the square on $B C$ is greater than (the square on) $C D$ by the (square) on (some straight-line) commensurable (in length) with ( $B C$ ), then the square on $K F$ will also be greater than (the square on) $F H$ by the (square) on (some straight-line) commensurable (in length) with ( $K F$ ) [Prop. 10.14]. And if $B C$ is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) $K F$ [Prop. 10.12]. And if $C D$ is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) $F H$ [Prop. 10.12]. And if neither of $B C$ or $C D$ (are commensurable), neither also (are) either of $K F$ or $F H$ [Prop. 10.13].

And if the square on $B C$ is greater than (the square on) $C D$ by the (square) on (some straight-line) incommensurable (in length) with ( $B C$ ) then the square on $K F$ will also be greater than (the square on) $F H$ by the (square) on (some straight-line) incommensurable (in length) with ( $K F$ ) [Prop. 10.14]. And if $B C$ is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) $K F$ [Prop. 10.12]. And if $C D$ is commensurable, (so) also (is) $F H$ [Prop. 10.12]. And if neither of $B C$ or $C D$ (are commensurable), neither also (are) either of $K F$ or $F H$ [Prop. 10.13].
$K H$ is thus a binomial whose terms, $K F$ and $F H$, [are] commensurable (in length) with the terms, $B C$ and $C D$, of the apotome, and in the same ratio. Moreover,
pio'.














 $\lambda o ́ \gamma \varphi$.















 p̀ntท́ モ̇எтเข.
$K H$ will have the same order as $B C$ [Defs. 10.5-10.10]. (Which is) the very thing it was required to show.

## Proposition 114

If an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome then the square-root of the area is a rational (straight-line).


For let an area, the (rectangle contained) by $A B$ and $C D$, have been contained by the apotome $A B$, and the binomial $C D$, of which let the greater term be $C E$. And let the terms of the binomial, $C E$ and $E D$, be commensurable with the terms of the apotome, $A F$ and $F B$ (respectively), and in the same ratio. And let the square-root of the (rectangle contained) by $A B$ and $C D$ be $G$. I say that $G$ is a rational (straight-line).

For let the rational (straight-line) $H$ be laid down. And let (some rectangle), equal to the (square) on $H$, have been applied to $C D$, producing $K L$ as breadth. Thus, $K L$ is an apotome, of which let the terms, $K M$ and $M L$, be commensurable with the terms of the binomial, $C E$ and $E D$ (respectively), and in the same ratio [Prop. 10.112]. But, $C E$ and $E D$ are also commensurable with $A F$ and $F B$ (respectively), and in the same ratio. Thus, as $A F$ is to $F B$, so $K M$ (is) to $M L$. Thus, alternately, as $A F$ is to $K M$, so $B F$ (is) to $L M$ [Prop. 5.16]. Thus, the remainder $A B$ is also to the remainder $K L$ as $A F$ (is) to $K M$ [Prop. 5.19]. And $A F$ (is) commensurable with $K M$ [Prop. 10.12]. $A B$ is thus also commensurable with $K L$ [Prop. 10.11]. And as $A B$ is to $K L$, so the (rectangle contained) by $C D$ and $A B$ (is) to the (rectangle contained) by $C D$ and $K L$ [Prop. 6.1]. Thus, the (rectangle contained) by $C D$ and $A B$ is also commensurable with the (rectangle contained) by $C D$ and $K L$ [Prop. 10.11]. And the (rectangle contained) by $C D$ and $K L$ (is) equal to the (square) on $H$. Thus, the (rectangle contained) by $C D$ and $A B$ is commensurable with the (square) on $H$. And the (square) on $G$ is equal to the (rectangle contained) by $C D$ and $A B$. The (square) on $G$

## По́рьб $\mu$.

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is thus commensurable with the (square) on $H$. And the (square) on $H$ (is) rational. Thus, the (square) on $G$ is also rational. $G$ is thus rational. And it is the square-root of the (rectangle contained) by $C D$ and $A B$.

Thus, if an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome, then the square-root of the area is a rational (straight-line).

## Corollary

And it has also been made clear to us, through this, that it is possible for a rational area to be contained by irrational straight-lines. (Which is) the very thing it was required to show.

## Proposition 115

An infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and none of them is the same as any of the preceding (straight-lines).


Let $A$ be a medial (straight-line). I say that an infinite (series) of irrational (straight-lines) can be created from $A$, and that none of them is the same as any of the preceding (straight-lines).

Let the rational (straight-line) $B$ be laid down. And let the (square) on $C$ be equal to the (rectangle contained) by $B$ and $A$. Thus, $C$ is irrational [Def. 10.4]. For an (area contained) by an irrational and a rational (straight-line) is irrational [Prop. 10.20]. And ( $C$ is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces a medial (straight-line) as breadth. So, again, let the (square) on $D$ be equal to the (rectangle contained) by $B$ and $C$. Thus, the (square) on $D$ is irrational [Prop. 10.20]. $D$ is thus irrational [Def. 10.4]. And ( $D$ is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces $C$ as breadth. So, similarly, this arrangement being advanced to infinity, it is clear that an infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and that none of them is the same as any of the preceding (straight-lines). (Which is) the very thing it was required to show.

## ELEMENTS BOOK 11

## Elementary Stereometry

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 $\pi \varepsilon ́ p \alpha s ~ \tau \tilde{\eta} \varsigma ~ \varepsilon u ̉ \vartheta \varepsilon i ́ \alpha s ~ \varepsilon u ̉ \vartheta \varepsilon i ̃ \alpha ~ \varepsilon ̇ \pi ı \zeta \varepsilon \cup \chi \vartheta \tilde{n}, \dot{\eta} \pi \varepsilon \rho เ \varepsilon \chi \circ \mu \varepsilon ́ v \eta ~ \gamma \omega v i ́ \alpha$







$\eta^{\prime} . \Pi \alpha p \dot{\alpha} \lambda \lambda \eta \lambda \alpha \dot{\varepsilon} \pi \tau^{\prime} \pi \varepsilon \delta \dot{\alpha}$ ह̇ $\sigma \tau \iota \tau \dot{\alpha} \dot{\alpha} \sigma \cup ́ \mu \pi \tau \omega \tau \alpha$.
 $\pi \varepsilon \rho เ \varepsilon \chi o ́ \mu \varepsilon \nu \alpha$ औ $\sigma \omega \nu$ тò $\pi \lambda \tilde{\eta} \vartheta \circ \varsigma$.



 $\pi \rho o ̀ s ~ \pi \alpha ́ \sigma \alpha l s ~ \tau \alpha i ̈ s ~ \gamma p \alpha \mu \mu \alpha i ̆ s ~ x \lambda i ́ \sigma \iota s . ~ \alpha \lambda \lambda \omega \varsigma$. $\sigma \tau \varepsilon \rho \varepsilon \alpha ̀ ~ \gamma \omega v i ́ \alpha$

 $\sigma \tau \alpha \mu \varepsilon ́ v \omega \nu$.



 $\tau \dot{\alpha} \delta \dot{\varepsilon} \lambda$ лol $\pi \dot{\alpha} \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu \alpha$.

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 ท้̈ 七ò ทं $\mu \iota x \cup ́ x \lambda$ เov $\sigma \tau \rho \varepsilon ́ \varphi \varepsilon \tau \alpha$.
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## Definitions

1. A solid is a (figure) having length and breadth and depth.
2. The extremity of a solid (is) a surface.
3. A straight-line is at right-angles to a plane when it makes right-angles with all of the straight-lines joined to it which are also in the plane.
4. A plane is at right-angles to a(nother) plane when (all of) the straight-lines drawn in one of the planes, at right-angles to the common section of the planes, are at right-angles to the remaining plane.
5. The inclination of a straight-line to a plane is the angle contained by the drawn and standing (straightlines), when a perpendicular is lead to the plane from the end of the (standing) straight-line raised (out of the plane), and a straight-line is (then) joined from the point (so) generated to the end of the (standing) straight-line (lying) in the plane.
6. The inclination of a plane to a(nother) plane is the acute angle contained by the (straight-lines), (one) in each of the planes, drawn at right-angles to the common segment (of the planes), at the same point.
7. A plane is said to have been similarly inclined to a plane, as another to another, when the aforementioned angles of inclination are equal to one another.
8. Parallel planes are those which do not meet (one another).
9. Similar solid figures are those contained by equal numbers of similar planes (which are similarly arranged).
10. But equal and similar solid figures are those contained by similar planes equal in number and in magnitude (which are similarly arranged).
11. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point.
12. A pyramid is a solid figure, contained by planes, (which is) constructed from one plane to one point.
13. A prism is a solid figure, contained by planes, of which the two opposite (planes) are equal, similar, and parallel, and the remaining (planes are) parallelograms.
14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.
15. And the axis of the sphere is the fixed straight-line about which the semicircle is turned.






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 ้̋ $\sigma \omega \nu$ кגi i $\sigma о \pi \lambda \varepsilon u ́ p \omega \nu \pi \varepsilon p เ \varepsilon \chi o ́ \mu \varepsilon v o v . ~$





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16. And the center of the sphere is the same as that of the semicircle.
17. And the diameter of the sphere is any straightline which is drawn through the center and terminated in both directions by the surface of the sphere.
18. A cone is the figure enclosed when, one of the sides of a right-angled triangle about the right-angle remaining (fixed), the triangle is carried around, and again established at the same (position) from which it began to be moved. And if the fixed straight-line is equal to the remaining (straight-line) about the right-angle, (which is) carried around, then the cone will be right-angled, and if less, obtuse-angled, and if greater, acute-angled.
19. And the axis of the cone is the fixed straight-line about which the triangle is turned.
20. And the base (of the cone is) the circle described by the (remaining) straight-line (about the right-angle which is) carried around (the axis).
21. A cylinder is the figure enclosed when, one of the sides of a right-angled parallelogram about the rightangle remaining (fixed), the parallelogram is carried around, and again established at the same (position) from which it began to be moved.
22. And the axis of the cylinder is the stationary straight-line about which the parallelogram is turned.
23. And the bases (of the cylinder are) the circles described by the two opposite sides (which are) carried around.
24. Similar cones and cylinders are those for which the axes and the diameters of the bases are proportional.
25. A cube is a solid figure contained by six equal squares.
26. An octahedron is a solid figure contained by eight equal and equilateral triangles.
27. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.
28. A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

## Proposition $1^{\dagger}$

Some part of a straight-line cannot be in a reference plane, and some part in a more elevated (plane).

For, if possible, let some part, $A B$, of the straight-line $A B C$ be in a reference plane, and some part, $B C$, in a more elevated (plane).

In the reference plane, there will be some straight-line continuous with, and straight-on to, $A B .^{\ddagger}$ Let it be $B D$. Thus, $A B$ is a common segment of the two (different) straight-lines $A B C$ and $A B D$. The very thing is impossible, inasmuch as if we draw a circle with center $B$ and
$\pi \varepsilon \rho \iota \varphi$ рві́кц.



radius $A B$ then the diameters ( $A B D$ and $A B C$ ) will cut off unequal circumferences of the circle.


Thus, some part of a straight-line cannot be in a reference plane, and (some part) in a more elevated (plane). (Which is) the very thing it was required to show.

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## Proposition 2

If two straight-lines cut one another then they are in one plane, and every triangle (formed using segments of both lines) is in one plane.


For let the two straight-lines $A B$ and $C D$ have cut one another at point $E$. I say that $A B$ and $C D$ are in one plane, and that every triangle (formed using segments of both lines) is in one plane.

For let the random points $F$ and $G$ have been taken on $E C$ and $E B$ (respectively). And let $C B$ and $F G$ have been joined, and let $F H$ and $G K$ have been drawn across. I say, first of all, that triangle $E C B$ is in one (reference) plane. For if part of triangle $E C B$, either $F H C$










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or $G B K$, is in the reference [plane], and the remainder in a different (plane) then a part of one the straight-lines $E C$ and $E B$ will also be in the reference plane, and (a part) in a different (plane). And if the part $F C B G$ of triangle $E C B$ is in the reference plane, and the remainder in a different (plane) then parts of both of the straightlines $E C$ and $E B$ will also be in the reference plane, and (parts) in a different (plane). The very thing was shown to be absurb [Prop. 11.1]. Thus, triangle $E C B$ is in one plane. And in whichever (plane) triangle $E C B$ is (found), in that (plane) $E C$ and $E B$ (will) each also (be found). And in whichever (plane) $E C$ and $E B$ (are) each (found), in that (plane) $A B$ and $C D$ (will) also (be found) [Prop. 11.1]. Thus, the straight-lines $A B$ and $C D$ are in one plane, and every triangle (formed using segments of both lines) is in one plane. (Which is) the very thing it was required to show.

## Proposition 3

If two planes cut one another then their common section is a straight-line.


For let the two planes $A B$ and $B C$ cut one another, and let their common section be the line $D B$. I say that the line $D B$ is straight.

For, if not, let the straight-line $D E B$ have been joined from $D$ to $B$ in the plane $A B$, and the straight-line $D F B$ in the plane $B C$. So two straight-lines, $D E B$ and $D F B$, will have the same ends, and they will clearly enclose an area. The very thing (is) absurd. Thus, $D E B$ and $D F B$ are not straight-lines. So, similarly, we can show than no other straight-line can be joined from $D$ to $B$ except $D B$, the common section of the planes $A B$ and $B C$.

Thus, if two planes cut one another then their common section is a straight-line. (Which is) the very thing it was required to show.

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 $\dot{\varepsilon} \varphi \varepsilon \sigma \tau \alpha ́ \tau \omega \cdot \lambda \varepsilon ́ \gamma \omega$, o้ $\tau \iota \dot{\eta} \mathrm{EZ}$ каì $\tau \widetilde{\varphi} \delta i \dot{\alpha} \tau \widetilde{\omega} \nu \mathrm{AB}, \Gamma \Delta \varepsilon \varepsilon \pi \iota \pi \varepsilon ́ \delta \omega$ $\pi \rho o ̀ s ~ o ̉ \rho \vartheta \alpha ́ ́ s ~ \varepsilon ̇ \sigma \tau เ \nu . ~$








 тpí $\gamma \omega v \alpha ́$ ह̇бтı $\tau \grave{\alpha} \mathrm{AHE}, \mathrm{BE} \Theta$ тàs $\delta u ́ o ~ \gamma \omega v i ́ \alpha s ~ \delta u \sigma i ̀ ~ \gamma \omega v i ́ \alpha ı \varsigma ~$















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## Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).


For let some straight-line $E F$ have (been) set up at right-angles to two straight-lines, $A B$ and $C D$, cutting one another at point $E$, at $E$. I say that $E F$ is also at right-angles to the plane (passing) through $A B$ and $C D$.

For let $A E, E B, C E$ and $E D$ have been cut off from (the two straight-lines so as to be) equal to one another. And let $G E H$ have been drawn, at random, through $E$ (in the plane passing through $A B$ and $C D$ ). And let $A D$ and $C B$ have been joined. And, furthermore, let $F A$, $F G, F D, F C, F H$, and $F B$ have been joined from the random (point) $F$ (on $E F$ ).

For since the two (straight-lines) $A E$ and $E D$ are equal to the two (straight-lines) $C E$ and $E B$, and they enclose equal angles [Prop. 1.15], the base $A D$ is thus equal to the base $C B$, and triangle $A E D$ will be equal to triangle $C E B$ [Prop. 1.4]. Hence, the angle $D A E$ [is] equal to the angle $E B C$. And the angle $A E G$ (is) also equal to the angle $B E H$ [Prop. 1.15]. So $A G E$ and $B E H$ are two triangles having two angles equal to two angles, respectively, and one side equal to one side(namely), those by the equal angles, $A E$ and $E B$. Thus, they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, $G E$ (is) equal to $E H$, and $A G$ to $B H$. And since $A E$ is equal to $E B$, and $F E$ is common and at right-angles, the base $F A$ is thus equal to the base $F B$ [Prop. 1.4]. So, for the same (reasons), $F C$ is also equal to $F D$. And since $A D$ is equal to $C B$, and $F A$ is also equal to $F B$, the two (straight-lines) $F A$ and $A D$ are equal to the two (straight-lines) $F B$ and $B C$, respectively. And the base $F D$ was shown (to be) equal to the base $F C$. Thus, the angle $F A D$ is also equal to the angle $F B C$ [Prop. 1.8]. And, again, since $A G$ was shown (to be) equal to $B H$, but $F A$ (is) also equal to
$\pi \rho o ̀ \varsigma ~ \pi \alpha ́ \sigma \alpha \varsigma ~ \tau \grave{\alpha} \varsigma \dot{\alpha} \pi \tau о \mu \varepsilon ́ v \alpha \varsigma ~ \alpha u ̉ \tau \tilde{n} \varsigma ~ \varepsilon u ̛ v \varepsilon i ́ \alpha s ~ x \alpha i ̀ ~ o u ̈ \sigma \alpha \varsigma ~ \varepsilon ُ v ~ \tau \widetilde{̣}$










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$F B$, the two (straight-lines) $F A$ and $A G$ are equal to the two (straight-lines) $F B$ and $B H$ (respectively). And the angle $F A G$ was shown (to be) equal to the angle $F B H$. Thus, the base $F G$ is equal to the base $F H$ [Prop. 1.4]. And, again, since $G E$ was shown (to be) equal to $E H$, and $E F$ (is) common, the two (straight-lines) $G E$ and $E F$ are equal to the two (straight-lines) $H E$ and $E F$ (respectively). And the base $F G$ (is) equal to the base $F H$. Thus, the angle $G E F$ is equal to the angle $H E F$ [Prop. 1.8]. Each of the angles $G E F$ and $H E F$ (are) thus right-angles [Def. 1.10]. Thus, $F E$ is at right-angles to $G H$, which was drawn at random through $E$ (in the reference plane passing though $A B$ and $A C$ ). So, similarly, we can show that $F E$ will make right-angles with all straight-lines joined to it which are in the reference plane. And a straight-line is at right-angles to a plane when it makes right-angles with all straight-lines joined to it which are in the plane [Def. 11.3]. Thus, $F E$ is at right-angles to the reference plane. And the reference plane is that (passing) through the straight-lines $A B$ and $C D$. Thus, $F E$ is at right-angles to the plane (passing) through $A B$ and $C D$.

Thus, if a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both). (Which is) the very thing it was required to show.

## Proposition 5

If a straight-line is set up at right-angles to three straight-lines cutting one another, at the common point of section, then the three straight-lines are in one plane.


For let some straight-line $A B$ have been set up at right-angles to three straight-lines $B C, B D$, and $B E$, at the (common) point of section $B$. I say that $B C, B D$, and $B E$ are in one plane.

For (if) not, and if possible, let $B D$ and $B E$ be in the reference plane, and $B C$ in a more elevated (plane).

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And let the plane through $A B$ and $B C$ have been produced. So it will make a straight-line as a common section with the reference plane [Def. 11.3]. Let it make $B F$. Thus, the three straight-lines $A B, B C$, and $B F$ are in one plane-(namely), that drawn through $A B$ and $B C$. And since $A B$ is at right-angles to each of $B D$ and $B E, A B$ is thus also at right-angles to the plane (passing) through $B D$ and $B E$ [Prop. 11.4]. And the plane (passing) through $B D$ and $B E$ is the reference plane. Thus, $A B$ is at right-angles to the reference plane. Hence, $A B$ will also make right-angles with all straight-lines joined to it which are also in the reference plane [Def. 11.3]. And $B F$, which is in the reference plane, is joined to it. Thus, the angle $A B F$ is a right-angle. And $A B C$ was also assumed to be a right-angle. Thus, angle $A B F$ (is) equal to $A B C$. And they are in one plane. The very thing is impossible. Thus, $B C$ is not in a more elevated plane. Thus, the three straight-lines $B C, B D$, and $B E$ are in one plane.

Thus, if a straight-line is set up at right-angles to three straight-lines cutting one another, at the (common) point of section, then the three straight-lines are in one plane. (Which is) the very thing it was required to show.

## Proposition 6

If two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. ${ }^{\dagger}$


For let the two straight-lines $A B$ and $C D$ be at rightangles to a reference plane. I say that $A B$ is parallel to $C D$.

For let them meet the reference plane at points $B$ and $D$ (respectively). And let the straight-line $B D$ have been joined. And let $D E$ have been drawn at right-angles to $B D$ in the reference plane. And let $D E$ be made equal to $A B$. And let $B E, A E$, and $A D$ have been joined.

And since $A B$ is at right-angles to the reference plane, it will [thus] also make right-angles with all straight-lines joined to it which are in the reference plane [Def. 11.3].











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And $B D$ and $B E$, which are in the reference plane, are each joined to $A B$. Thus, each of the angles $A B D$ and $A B E$ are right-angles. So, for the same (reasons), each of the angles $C D B$ and $C D E$ are also right-angles. And since $A B$ is equal to $D E$, and $B D$ (is) common, the two (straight-lines) $A B$ and $B D$ are equal to the two (straight-lines) $E D$ and $D B$ (respectively). And they contain right-angles. Thus, the base $A D$ is equal to the base $B E$ [Prop. 1.4]. And since $A B$ is equal to $D E$, and $A D$ (is) also (equal) to $B E$, the two (straight-lines) $A B$ and $B E$ are thus equal to the two (straight-lines) $E D$ and $D A$ (respectively). And their base $A E$ (is) common. Thus, angle $A B E$ is equal to angle $E D A$ [Prop. 1.8]. And $A B E$ (is) a right-angle. Thus, $E D A$ (is) also a rightangle. $E D$ is thus at right-angles to $D A$. And it is also at right-angles to each of $B D$ and $D C$. Thus, $E D$ is standing at right-angles to the three straight-lines $B D, D A$, and $D C$ at the (common) point of section. Thus, the three straight-lines $B D, D A$, and $D C$ are in one plane [Prop. 11.5]. And in which(ever) plane $D B$ and $D A$ (are found), in that (plane) $A B$ (will) also (be found). For every triangle is in one plane [Prop. 11.2]. And each of the angles $A B D$ and $B D C$ is a right-angle. Thus, $A B$ is parallel to $C D$ [Prop. 1.28].

Thus, if two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. (Which is) the very thing it was required to show.

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## Proposition 7

If there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines).


Let $A B$ and $C D$ be two parallel straight-lines, and let the random points $E$ and $F$ have been taken on each of them (respectively). I say that the straight-line joining points $E$ and $F$ is in the same (reference) plane as the parallel (straight-lines).

For (if) not, and if possible, let it be in a more elevated











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 x $\grave{\text { è }} \pi \varepsilon \zeta \varepsilon \iota^{\prime} \chi \vartheta \omega \sigma \alpha \nu \alpha i \mathrm{BE}, \mathrm{AE}, \mathrm{A} \Delta$.








(plane), such as $E G F$. And let a plane have been drawn through $E G F$. So it will make a straight cutting in the reference plane [Prop. 11.3]. Let it make $E F$. Thus, two straight-lines (with the same end-points), $E G F$ and $E F$, will enclose an area. The very thing is impossible. Thus, the straight-line joining $E$ to $F$ is not in a more elevated plane. The straight-line joining $E$ to $F$ is thus in the plane through the parallel (straight-lines) $A B$ and $C D$.

Thus, if there are two parallel straight-lines, and random points are taken on each of them, then the straightline joining the two points is in the same plane as the parallel (straight-lines). (Which is) the very thing it was required to show.

## Proposition 8

If two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane.


Let $A B$ and $C D$ be two parallel straight-lines, and let one of them, $A B$, be at right-angles to a reference plane. I say that the remaining (one), $C D$, will also be at rightangles to the same plane.

For let $A B$ and $C D$ meet the reference plane at points $B$ and $D$ (respectively). And let $B D$ have been joined. $A B, C D$, and $B D$ are thus in one plane [Prop. 11.7]. Let $D E$ have been drawn at right-angles to $B D$ in the reference plane, and let $D E$ be made equal to $A B$, and let $B E, A E$, and $A D$ have been joined.

And since $A B$ is at right-angles to the reference plane, $A B$ is thus also at right-angles to all of the straight-lines joined to it which are in the reference plane [Def. 11.3]. Thus, the angles $A B D$ and $A B E$ [are] each right-angles. And since the straight-line $B D$ has met the parallel (straight-lines) $A B$ and $C D$, the (sum of the) angles $A B D$ and $C D B$ is thus equal to two right-angles


 $\mu \varepsilon ̀ v \mathrm{AB} \tau \tilde{n} \Delta \mathrm{E}, \dot{\eta} \delta \grave{̀} \mathrm{BE} \tau \tilde{n} \mathrm{~A} \Delta$, $\delta u ́ o ~ \delta \grave{\eta} \alpha i \mathrm{AB}, \mathrm{BE} \delta u \sigma \grave{i}$
 кowǹ $\dot{\eta}$ AE $\gamma \omega v i ́ \alpha$ äp $\alpha \dot{\eta}$ ú $\pi o ̀ ~ A B E ~ \gamma \omega v i ́ \alpha ~ \tau \tilde{n} ~ u ́ \pi o ̀ ~ E ~ A ~ A ~$













 $\pi \rho o ̀ s ~ o ̉ p \vartheta \alpha ́ s ~ \varepsilon ̇ \sigma \tau เ \nu . ~$

 $\pi \rho o ̀ \varsigma ~ o ̉ \rho \vartheta \grave{\alpha} \varsigma$ है $\sigma \tau \alpha l \cdot$ ö $\pi \varepsilon \rho$ عै $\delta \varepsilon \iota ~ \delta \varepsilon \tilde{\epsilon} \xi \alpha l$.

## $\vartheta^{\prime}$.




${ }^{\prime} E \sigma \tau \omega \gamma \dot{\alpha} p$ ह̇ $x \alpha \tau \varepsilon ́ p \alpha \tau \widetilde{\omega} \nu \mathrm{AB}, \Gamma \Delta \tau \tilde{n} \mathrm{EZ} \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o \varsigma$

[Prop. 1.29]. And $A B D$ (is) a right-angle. Thus, $C D B$ (is) also a right-angle. $C D$ is thus at right-angles to $B D$. And since $A B$ is equal to $D E$, and $B D$ (is) common, the two (straight-lines) $A B$ and $B D$ are equal to the two (straight-lines) $E D$ and $D B$ (respectively). And angle $A B D$ (is) equal to angle $E D B$. For each (is) a rightangle. Thus, the base $A D$ (is) equal to the base $B E$ [Prop. 1.4]. And since $A B$ is equal to $D E$, and $B E$ to $A D$, the two (sides) $A B, B E$ are equal to the two (sides) $E D, D A$, respectively. And their base $A E$ is common. Thus, angle $A B E$ is equal to angle $E D A$ [Prop. 1.8]. And $A B E$ (is) a right-angle. $E D A$ (is) thus also a rightangle. Thus, $E D$ is at right-angles to $A D$. And it is also at right-angles to $D B$. Thus, $E D$ is also at right-angles to the plane through $B D$ and $D A$ [Prop. 11.4]. And $E D$ will thus make right-angles with all of the straightlines joined to it which are also in the plane through $B D A$. And $D C$ is in the plane through $B D A$, inasmuch as $A B$ and $B D$ are in the plane through $B D A$ [Prop. 11.2], and in which(ever plane) $A B$ and $B D$ (are found), $D C$ is also (found). Thus, $E D$ is at right-angles to $D C$. Hence, $C D$ is also at right-angles to $D E$. And $C D$ is also at right-angles to $B D$. Thus, $C D$ is standing at right-angles to two straight-lines, $D E$ and $D B$, which meet one another, at the (point) of section, $D$. Hence, $C D$ is also at right-angles to the plane through $D E$ and $D B$ [Prop. 11.4]. And the plane through $D E$ and $D B$ is the reference (plane). $C D$ is thus at right-angles to the reference plane.

Thus, if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

## Proposition 9

(Straight-lines) parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another.


For let $A B$ and $C D$ each be parallel to $E F$, not being in the same plane as it. I say that $A B$ is parallel to $C D$.

ย̇のт兀レ $\dot{\eta} \mathrm{AB} \tau \tilde{\eta} \Gamma \Delta$ ．


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 $\gamma \omega v i ́ \alpha s ~ \pi \varepsilon p เ \varepsilon ́ \xi ু о \cup \sigma เ \nu$.

$\Delta u ́ o ~ \gamma \grave{\alpha} \rho$ عủvعĩal $\alpha i \operatorname{AB}, B \Gamma \dot{\alpha} \pi \tau o ́ \mu \varepsilon v \alpha l ~ \dot{\alpha} \lambda \lambda \dot{\eta} \lambda \omega \nu \pi \alpha \rho \dot{\alpha}$ ठúo عủvعías tàs $\Delta \mathrm{E}, \mathrm{EZ} \dot{\alpha} \pi \tau o \mu \varepsilon ́ v \alpha s ~ \alpha \dot{\alpha} \lambda \eta{ }_{\eta} \lambda \omega \nu$ है $\sigma \tau \omega \sigma \alpha \nu \mu \dot{\eta}$
 $\tau \tilde{n}$ újò $\Delta \mathrm{EZ}$ ．
 ж $\alpha i$ غ่ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ A \Delta, ~ Г Z, ~ В Е, ~ А Г, ~ \Delta Z . ~$











For let some point $G$ have been taken at random on $E F$ ．And from it let $G H$ have been drawn at right－angles to $E F$ in the plane through $E F$ and $A B$ ．And let $G K$ have been drawn，again at right－angles to $E F$ ，in the plane through $F E$ and $C D$ ．

And since $E F$ is at right－angles to each of $G H$ and $G K, E F$ is thus also at right－angles to the plane through $G H$ and $G K$［Prop．11．4］．And $E F$ is parallel to $A B$ ． Thus，$A B$ is also at right－angles to the plane through $H G K$［Prop．11．8］．So，for the same（reasons），$C D$ is also at right－angles to the plane through $H G K$ ．Thus， $A B$ and $C D$ are each at right－angles to the plane through $H G K$ ．And if two straight－lines are at right－angles to the same plane then the straight－lines are parallel ［Prop．11．6］．Thus，$A B$ is parallel to $C D$ ．（Which is） the very thing it was required to show．

## Proposition 10

If two straight－lines joined to one another are（respec－ tively）parallel to two straight－lines joined to one another， （but are）not in the same plane，then they will contain equal angles．


For let the two straight－lines joined to one another， $A B$ and $B C$ ，be（respectively）parallel to the two straight－lines joined to one another，$D E$ and $E F$ ，（but） not in the same plane．I say that angle $A B C$ is equal to （angle）$D E F$ ．

For let $B A, B C, E D$ ，and $E F$ have been cut off（so as to be，respectively）equal to one another．And let $A D$ ， $C F, B E, A C$ ，and $D F$ have been joined．

And since $B A$ is equal and parallel to $E D, A D$ is thus also equal and parallel to $B E$［Prop．1．33］．So，for the same reasons，$C F$ is also equal and parallel to $B E$ ．Thus， $A D$ and $C F$ are each equal and parallel to $B E$ ．And straight－lines parallel to the same straight－line，and which are not in the same plane as it，are also parallel to one an－ other［Prop．11．9］．Thus，$A D$ is parallel and equal to $C F$ ． And $A C$ and $D F$ join them．Thus，$A C$ is also equal and

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## $\alpha^{\prime}$













 $\ddot{\eta} \chi \vartheta \omega \dot{\eta} \mathrm{H} \Theta$.










parallel to $D F$ [Prop. 1.33]. And since the two (straightlines) $A B$ and $B C$ are equal to the two (straight-lines) $D E$ and $E F$ (respectvely), and the base $A C$ (is) equal to the base $D F$, the angle $A B C$ is thus equal to the (angle) $D E F$ [Prop. 1.8].

Thus, if two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles. (Which is) the very thing it was required to show.

## Proposition 11

To draw a perpendicular straight-line from a given raised point to a given plane.


Let $A$ be the given raised point, and the given plane the reference (plane). So, it is required to draw a perpendicular straight-line from point $A$ to the reference plane.

Let some random straight-line $B C$ have been drawn across in the reference plane, and let the (straight-line) $A D$ have been drawn from point $A$ perpendicular to $B C$ [Prop. 1.12]. If, therefore, $A D$ is also perpendicular to the reference plane then that which was prescribed will have occurred. And, if not, let $D E$ have been drawn in the reference plane from point $D$ at right-angles to $B C$ [Prop. 1.11], and let the (straight-line) $A F$ have been drawn from $A$ perpendicular to $D E$ [Prop. 1.12], and let $G H$ have been drawn through point $F$, parallel to $B C$ [Prop. 1.31].

And since $B C$ is at right-angles to each of $D A$ and $D E, B C$ is thus also at right-angles to the plane through $E D A$ [Prop. 11.4]. And $G H$ is parallel to it. And if two straight-lines are parallel, and one of them is at rightangles to some plane, then the remaining (straight-line) will also be at right-angles to the same plane [Prop. 11.8]. Thus, $G H$ is also at right-angles to the plane through







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$E D$ and $D A$. And $G H$ is thus at right-angles to all of the straight-lines joined to it which are also in the plane through $E D$ and $A D$ [Def. 11.3]. And $A F$, which is in the plane through $E D$ and $D A$, is joined to it. Thus, $G H$ is at right-angles to $F A$. Hence, $F A$ is also at right-angles to $H G$. And $A F$ is also at right-angles to $D E$. Thus, $A F$ is at right-angles to each of $G H$ and $D E$. And if a straightline is set up at right-angles to two straight-lines cutting one another, at the point of section, then it will also be at right-angles to the plane through them [Prop. 11.4]. Thus, $F A$ is at right-angles to the plane through $E D$ and $G H$. And the plane through $E D$ and $G H$ is the reference (plane). Thus, $A F$ is at right-angles to the reference plane.

Thus, the straight-line $A F$ has been drawn from the given raised point $A$ perpendicular to the reference plane. (Which is) the very thing it was required to do.

## Proposition 12

To set up a straight-line at right-angles to a given plane from a given point in it.


Let the given plane be the reference (plane), and $A$ a point in it. So, it is required to set up a straight-line at right-angles to the reference plane at point $A$.

Let some raised point $B$ have been assumed, and let the perpendicular (straight-line) $B C$ have been drawn from $B$ to the reference plane [Prop. 11.11]. And let $A D$ have been drawn from point $A$ parallel to $B C$ [Prop. 1.31].

Therefore, since $A D$ and $C B$ are two parallel straightlines, and one of them, $B C$, is at right-angles to the reference plane, the remaining (one) $A D$ is thus also at rightangles to the reference plane [Prop. 11.8].

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 غ̇ $\sigma \tau i v ~ \alpha ̀ \delta o ́ v \alpha \tau o v . ~$




## $10^{\prime}$.

 है $\sigma \tau \alpha \iota \tau \dot{\alpha}$ ह̀ $\pi i \not \tau \varepsilon \delta \alpha$.

Eủveĩa үáp $\tau \iota \varsigma \dot{\eta} \mathrm{AB}$ трòs घं $x \alpha ́ \tau \varepsilon \rho o \nu ~ \tau \widetilde{\omega} \nu \Gamma \Delta, \mathrm{EZ}$
 $\varepsilon ̇ \pi i \pi \varepsilon \delta \alpha$.

Thus, $A D$ has been set up at right-angles to the given plane, from the point in it, $A$. (Which is) the very thing it was required to do.

## Proposition 13

Two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side.


For, if possible, let the two straight-lines $A B$ and $A C$ have been set up at the same point $A$ at right-angles to the reference plane, on the same side. And let the plane through $B A$ and $A C$ have been drawn. So it will make a straight cutting (passing) through (point) $A$ in the reference plane [Prop. 11.3]. Let it have made $D A E$. Thus, $A B, A C$, and $D A E$ are straight-lines in one plane. And since $C A$ is at right-angles to the reference plane, it will thus also make right-angles with all of the straightlines joined to it which are also in the reference plane [Def. 11.3]. And $D A E$, which is in the reference plane, is joined to it. Thus, angle $C A E$ is a right-angle. So, for the same (reasons), $B A E$ is also a right-angle. Thus, $C A E$ (is) equal to $B A E$. And they are in one plane. The very thing is impossible.

Thus, two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side. (Which is) the very thing it was required to show.

## Proposition 14

Planes to which the same straight-line is at rightangles will be parallel planes.

For let some straight-line $A B$ be at right-angles to each of the planes $C D$ and $E F$. I say that the planes are parallel.


Ei $\gamma \grave{\alpha} \rho \mu \dot{\eta}, ~ \dot{\varepsilon} x \beta \alpha \lambda \lambda o ́ \mu \varepsilon \nu \alpha$ $\sigma \cup \mu \pi \varepsilon \sigma o u ̃ \nu \tau \alpha l . ~ \sigma \cup \mu \pi \iota \pi \tau \varepsilon ́ \tau-$
 $\mathrm{H} \Theta$, x $\alpha \grave{~ \varepsilon i ̉ \lambda n ́ \varphi \vartheta \omega ~ \varepsilon ̇ \pi i ̀ ~ \tau n ̃ s ~} \mathrm{H} \Theta$ тuðòv onueĩov tò K , x $\alpha \grave{~}$ ย̇ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu$ גi AK, BK.






 EZ غ̇лínєठ $\alpha$.



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$\Delta$ úo $\gamma \dot{\alpha} \rho$ عủvعĩal $\dot{\alpha} \pi \tau o ́ \mu \varepsilon \nu \alpha l ~ \dot{\alpha} \lambda \lambda \eta \dot{\eta} \lambda \omega \nu$ 人i $\mathrm{AB}, ~ В \Gamma ~ \pi \alpha \rho \dot{\alpha}$ ठúo عủvعías $\dot{\alpha} \pi \tau \sigma \mu \varepsilon ́ v \alpha s ~ \alpha \dot{\alpha} \lambda \eta \dot{\eta} \lambda \omega \nu \tau \grave{\alpha} \varsigma \Delta \mathrm{E}, \mathrm{EZ}$ है $\sigma \tau \omega \sigma \alpha \nu \mu \dot{\eta}$
 $\tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma}, \Delta \mathrm{E}, \mathrm{EZ}$ غ่ $\pi i ́ \pi \varepsilon \delta \alpha$ oủ $\sigma \cup \mu \pi \varepsilon \sigma \varepsilon i ̃ \tau \alpha l ~ \alpha \dot{\alpha} \lambda \eta \dot{\eta} \lambda o l s$.


 $\dot{\eta} H \Theta, \tau \tilde{n}$ סغ̀ EZ $\dot{\eta} H K$.


For, if not, being produced, they will meet. Let them have met. So they will make a straight-line as a common section [Prop. 11.3]. Let them have made $G H$. And let some random point $K$ have been taken on $G H$. And let $A K$ and $B K$ have been joined.

And since $A B$ is at right-angles to the plane $E F, A B$ is thus also at right-angles to $B K$, which is a straight-line in the produced plane $E F$ [Def. 11.3]. Thus, angle $A B K$ is a right-angle. So, for the same (reasons), $B A K$ is also a right-angle. So the (sum of the) two angles $A B K$ and $B A K$ in the triangle $A B K$ is equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, planes $C D$ and $E F$, being produced, will not meet. Planes $C D$ and $E F$ are thus parallel [Def. 11.8].

Thus, planes to which the same straight-line is at right-angles are parallel planes. (Which is) the very thing it was required to show.

## Proposition 15

If two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another).

For let the two straight-lines joined to one another, $A B$ and $B C$, be parallel to the two straight-lines joined to one another, $D E$ and $E F$ (respectively), not being in the same plane. I say that the planes through $A B, B C$ and $D E, E F$ will not meet one another (when) produced.

For let $B G$ have been drawn from point $B$ perpendicular to the plane through $D E$ and $E F$ [Prop. 11.11], and let it meet the plane at point $G$. And let $G H$ have been drawn through $G$ parallel to $E D$, and $G K$ (parallel) to $E F$ [Prop. 1.31].
























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 $\mathrm{EZ}, \mathrm{H} \Theta \cdot \lambda \varepsilon ́ \gamma \omega$, ơтı $\tau \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o ́ s ~ \varepsilon ̇ \sigma \tau \iota \nu ~ \dot{\eta} \mathrm{EZ}$ т $\tilde{n} \mathrm{H} \Theta$.


And since $B G$ is at right-angles to the plane through $D E$ and $E F$, it will thus also make right-angles with all of the straight-lines joined to it, which are also in the plane through $D E$ and $E F$ [Def. 11.3]. And each of $G H$ and $G K$, which are in the plane through $D E$ and $E F$, are joined to it. Thus, each of the angles $B G H$ and $B G K$ are right-angles. And since $B A$ is parallel to $G H$ [Prop. 11.9], the (sum of the) angles $G B A$ and $B G H$ is equal to two right-angles [Prop. 1.29]. And $B G H$ (is) a right-angle. $G B A$ (is) thus also a right-angle. Thus, $G B$ is at right-angles to $B A$. So, for the same (reasons), $G B$ is also at right-angles to $B C$. Therefore, since the straight-line $G B$ has been set up at right-angles to two straight-lines, $B A$ and $B C$, cutting one another, $G B$ is thus at right-angles to the plane through $B A$ and $B C$ [Prop. 11.4]. [So, for the same (reasons), $B G$ is also at right-angles to the plane through $G H$ and $G K$. And the plane through $G H$ and $G K$ is the (plane) through $D E$ and $E F$. And it was also shown that $G B$ is at rightangles to the plane through $A B$ and $B C$.] And planes to which the same straight-line is at right-angles are parallel planes [Prop. 11.14]. Thus, the plane through $A B$ and $B C$ is parallel to the (plane) through $D E$ and $E F$.

Thus, if two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another). (Which is) the very thing it was required to show.

## Proposition 16

If two parallel planes are cut by some plane then their common sections are parallel.

For let the two parallel planes $A B$ and $C D$ have been cut by the plane $E F G H$. And let $E F$ and $G H$ be their common sections. I say that $E F$ is parallel to $G H$.













 $\pi \alpha \rho \alpha \dot{\lambda} \lambda \eta \lambda$ дos $\ddot{\alpha} \rho \alpha$ है $\sigma \tau i v \dot{n} \mathrm{EZ} \tau \tilde{n} \mathrm{H} \Theta$.

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 عis toùs aútoùs $\lambda$ órous tun७̀ñovtal.
 $\tau \widetilde{\omega} \nu \mathrm{H} \Theta, \mathrm{K} \Lambda, \mathrm{MN} \tau \varepsilon \mu \nu \hat{\varepsilon} \sigma \vartheta \omega \sigma \alpha \nu \varkappa \alpha \tau \grave{\alpha} \tau \grave{\alpha} \mathrm{A}, \mathrm{E}, \mathrm{B}, \Gamma, \mathrm{Z}$,



 ai $E \Xi, \Xi \mathrm{Z}$.









For, if not, being produced, $E F$ and $G H$ will meet either in the direction of $F, H$, or of $E, G$. Let them be produced, as in the direction of $F, H$, and let them, first of all, have met at $K$. And since $E F K$ is in the plane $A B$, all of the points on $E F K$ are thus also in the plane $A B$ [Prop. 11.1]. And $K$ is one of the points on EFK. Thus, $K$ is in the plane $A B$. So, for the same (reasons), $K$ is also in the plane $C D$. Thus, the planes $A B$ and $C D$, being produced, will meet. But they do not meet, on account of being (initially) assumed (to be mutually) parallel. Thus, the straight-lines $E F$ and $G H$, being produced in the direction of $F, H$, will not meet. So, similarly, we can show that the straight-lines $E F$ and $G H$, being produced in the direction of $E, G$, will not meet either. And (straight-lines in one plane which), being produced, do not meet in either direction are parallel [Def. 1.23]. EF is thus parallel to $G H$.

Thus, if two parallel planes are cut by some plane then their common sections are parallel. (Which is) the very thing it was required to show.

## Proposition 17

If two straight-lines are cut by parallel planes then they will be cut in the same ratios.

For let the two straight-lines $A B$ and $C D$ be cut by the parallel planes $G H, K L$, and $M N$ at the points $A, E, B$, and $C, F, D$ (respectively). I say that as the straight-line $A E$ is to $E B$, so $C F$ (is) to $F D$.

For let $A C, B D$, and $A D$ have been joined, and let $A D$ meet the plane $K L$ at point $O$, and let $E O$ and $O F$ have been joined.

And since two parallel planes $K L$ and $M N$ are cut by the plane $E B D O$, their common sections $E O$ and $B D$ are parallel [Prop. 11.16]. So, for the same (reasons), since two parallel planes $G H$ and $K L$ are cut by the plane $A O F C$, their common sections $A C$ and $O F$ are parallel [Prop. 11.16]. And since the straight-line $E O$ has been drawn parallel to one of the sides $B D$ of trian-



 $\pi \rho o ̀ s ~ E B, ~ o u ́ \tau \omega s ~ \dot{\eta} \Gamma Z \pi \rho o ̀ s ~ Z \Delta$.




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gle $A B D$, thus, proportionally, as $A E$ is to $E B$, so $A O$ (is) to $O D$ [Prop. 6.2]. Again, since the straight-line $O F$ has been drawn parallel to one of the sides $A C$ of triangle $A D C$, proportionally, as $A O$ is to $O D$, so $C F$ (is) to $F D$ [Prop. 6.2]. And it was also shown that as $A O$ (is) to $O D$, so $A E$ (is) to $E B$. And thus as $A E$ (is) to $E B$, so $C F$ (is) to $F D$ [Prop. 5.11].


Thus, if two straight-lines are cut by parallel planes then they will be cut in the same ratios. (Which is) the very thing it was required to show.

## Proposition 18

If a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at rightangles to the same plane.

For let some straight-line $A B$ be at right-angles to a reference plane. I say that all of the planes (passing) through $A B$ are also at right-angles to the reference plane.

For let the plane $D E$ have been produced through $A B$. And let $C E$ be the common section of the plane $D E$ and the reference (plane). And let some random point $F$ have been taken on $C E$. And let $F G$ have been drawn from $F$, at right-angles to $C E$, in the plane $D E$ [Prop. 1.11].

And since $A B$ is at right-angles to the reference plane, $A B$ is thus also at right-angles to all of the straightlines joined to it which are also in the reference plane [Def. 11.3]. Hence, it is also at right-angles to $C E$. Thus, angle $A B F$ is a right-angle. And $G F B$ is also a rightangle. Thus, $A B$ is parallel to $F G$ [Prop. 1.28]. And $A B$ is at right-angles to the reference plane. Thus, $F G$ is also










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at right-angles to the reference plane [Prop. 11.8]. And a plane is at right-angles to a(nother) plane when the straight-lines drawn at right-angles to the common section of the planes, (and lying) in one of the planes, are at right-angles to the remaining plane [Def. 11.4]. And $F G$, (which was) drawn at right-angles to the common section of the planes, $C E$, in one of the planes, $D E$, was shown to be at right-angles to the reference plane. Thus, plane $D E$ is at right-angles to the reference (plane). So, similarly, it can be shown that all of the planes (passing) at random through $A B$ (are) at right-angles to the reference plane.


Thus, if a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

## Proposition 19

If two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane.


For let the two planes $A B$ and $B C$ be at right-angles to a reference plane, and let their common section be $B D$. I say that $B D$ is at right-angles to the reference











 $\tau \circ \mu \tilde{n} \varsigma \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \Gamma$ غ̇ $\pi \tau \pi \varepsilon ́ \delta \omega \omega$.




## $x^{\prime}$.


 $\beta \alpha v o ́ u \varepsilon v \alpha$.


 oัtı т $\omega$ ข













plane.
For (if) not, let $D E$ also have been drawn from point $D$, in the plane $A B$, at right-angles to the straight-line $A D$, and $D F$, in the plane $B C$, at right-angles to $C D$.

And since the plane $A B$ is at right-angles to the reference (plane), and $D E$ has been drawn at right-angles to their common section $A D$, in the plane $A B, D E$ is thus at right-angles to the reference plane [Def. 11.4]. So, similarly, we can show that $D F$ is also at right-angles to the reference plane. Thus, two (different) straight-lines are set up, at the same point $D$, at right-angles to the reference plane, on the same side. The very thing is impossible [Prop. 11.13]. Thus, no (other straight-line) except the common section $D B$ of the planes $A B$ and $B C$ can be set up at point $D$, at right-angles to the reference plane.

Thus, if two planes cutting one another are at rightangles to some plane then their common section will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

## Proposition 20

If a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way).


For let the solid angle $A$ have been contained by the three plane angles $B A C, C A D$, and $D A B$. I say that (the sum of) any two of the angles $B A C, C A D$, and $D A B$ is greater than the remaining (one), (the angles) being taken up in any (possible way).

For if the angles $B A C, C A D$, and $D A B$ are equal to one another then (it is) clear that (the sum of) any two is greater than the remaining (one). But, if not, let $B A C$ be greater (than $C A D$ or $D A B$ ). And let (angle) BAE, equal to the angle $D A B$, have been constructed in the plane through $B A C$, on the straight-line $A B$, at the point $A$ on it. And let $A E$ be made equal to $A D$. And $B E C$ being drawn across through point $E$, let it cut the straightlines $A B$ and $A C$ at points $B$ and $C$ (respectively). And let $D B$ and $D C$ have been joined.

And since $D A$ is equal to $A E$, and $A B$ (is) common,








 $\mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu o ́ \mu \varepsilon \nu \alpha l \cdot$ öт $\varepsilon \rho$ है $\delta \varepsilon \iota ~ \delta \varepsilon i ̋ \xi \alpha l$.
$\chi \alpha^{\prime}$.
"A $\pi \alpha \sigma \alpha$ $\sigma \tau \varepsilon p \varepsilon \grave{\alpha} \gamma \omega v i ́ \alpha$ ú $\pi o ̀ ~ \grave{~} \lambda \lambda \alpha \sigma \sigma o ́ v \omega \nu[\hat{\eta}] \tau \varepsilon \sigma \sigma \alpha ́ p \omega \nu$









 $\mu \varepsilon i \zeta o v e ́ s ~ \varepsilon i \sigma ル v . ~ \delta ı \alpha ̀ ~ \tau \alpha ̀ ~ \alpha u ̛ \tau \alpha ̀ ~ \delta \grave{\eta}$ x $\alpha i ̀ ~ \alpha i ~ \mu e ̀ v ~ u ́ \pi o ̀ ~ В Г А, ~ А Г \Delta ~$

 ВГА, АГ $\Delta, Г \Delta А, ~ А \Delta В ~ \tau р ı డ ั \nu ~ \tau \widetilde{\omega} \nu ~ ن ́ \pi o ̀ ~ Г В ~ В, ~ В Г А, ~ Г \Delta В ~$





the two (straight-lines $A D$ and $A B$ are) equal to the two (straight-lines $E A$ and $A B$, respectively). And angle $D A B$ (is) equal to angle $B A E$. Thus, the base $D B$ is equal to the base $B E$ [Prop. 1.4]. And since the (sum of the) two (straight-lines) $B D$ and $D C$ is greater than $B C$ [Prop. 1.20], of which $D B$ was shown (to be) equal to $B E$, the remainder $D C$ is thus greater than the remainder $E C$. And since $D A$ is equal to $A E$, but $A C$ (is) common, and the base $D C$ is greater than the base $E C$, the angle $D A C$ is thus greater than the angle $E A C$ [Prop. 1.25]. And $D A B$ was also shown (to be) equal to $B A E$. Thus, (the sum of) $D A B$ and $D A C$ is greater than $B A C$. So, similarly, we can also show that the remaining (angles), being taken in pairs, are greater than the remaining (one).

Thus, if a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way). (Which is) the very thing it was required to show.

## Proposition 21

Any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. ${ }^{\dagger}$


Let the solid angle $A$ be contained by the plane angles $B A C, C A D$, and $D A B$. I say that (the sum of) $B A C$, $C A D$, and $D A B$ is less than four right-angles.

For let the random points $B, C$, and $D$ have been taken on each of (the straight-lines) $A B, A C$, and $A D$ (respectively). And let $B C, C D$, and $D B$ have been joined. And since the solid angle at $B$ is contained by the three plane angles $C B A, A B D$, and $C B D$, (the sum of) any two is greater than the remaining (one) [Prop. 11.20]. Thus, (the sum of) $C B A$ and $A B D$ is greater than $C B D$. So, for the same (reasons), (the sum of) $B C A$ and $A C D$ is also greater than $B C D$, and (the sum of) $C D A$ and $A D B$ is greater than $C D B$. Thus, the (sum of the) six angles $C B A, A B D, B C A, A C D$, $C D A$, and $A D B$ is greater than the (sum of the) three (angles) $C B D, B C D$, and $C D B$. But, the (sum of the) three (angles) $C B D, B D C$, and $B C D$ is equal to two
$Г В А, ~ А Г В, ~ В А Г, ~ А Г \Delta, ~ Г \Delta А, ~ Г А ~ \Delta, ~ А ~ \triangle B, ~ \triangle В А, ~ В А ~ ص ~ \varepsilon ॄ ~ \xi ~$

 ВАГ, ГА $\Delta, \Delta \mathrm{AB}$ трєǐs [ $\gamma \omega v i ́ \alpha l] \pi \varepsilon \rho เ \varepsilon ́ \chi o \cup \sigma \alpha l ~ \tau ウ ̀ \nu ~ \sigma \tau \varepsilon р \varepsilon \alpha ̀ ~ \nu ~$



right-angles [Prop. 1.32]. Thus, the (sum of the) six angles $C B A, A B D, B C A, A C D, C D A$, and $A D B$ is greater than two right-angles. And since the (sum of the) three angles of each of the triangles $A B C, A C D$, and $A D B$ is equal to two right-angles, the (sum of the) nine angles $C B A, A C B, B A C, A C D, C D A, C A D, A D B, D B A$, and $B A D$ of the three triangles is equal to six right-angles, of which the (sum of the) six angles $A B C, B C A, A C D$, $C D A, A D B$, and $D B A$ is greater than two right-angles. Thus, the (sum of the) remaining three [angles] $B A C$, $C A D$, and $D A B$, containing the solid angle, is less than four right-angles.

Thus, any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalization to a solid angle contained by more than three plane angles is straightforward.

$$
x \beta^{\prime}
$$










 EZ, $\mathrm{H} \Theta, ~ \Theta K ~ \varepsilon u ̉ \vartheta \varepsilon i ̃ \alpha l, ~ \chi \alpha i ~ \varepsilon ̇ \pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ A \Gamma, ~ \Delta Z, ~ H K \cdot ~$














## Proposition 22

If there are three plane angles, of which (the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way), and if equal straight-lines contain them, then it is possible to construct a triangle from (the straight-lines created by) joining the (ends of the) equal straight-lines.


Let $A B C, D E F$, and $G H K$ be three plane angles, of which the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way)-(that is), $A B C$ and $D E F$ (greater) than $G H K$, $D E F$ and $G H K$ (greater) than $A B C$, and, further, $G H K$ and $A B C$ (greater) than $D E F$. And let $A B, B C, D E$, $E F, G H$, and $H K$ be equal straight-lines. And let $A C$, $D F$, and $G K$ have been joined. I say that that it is possible to construct a triangle out of (straight-lines) equal to $A C, D F$, and $G K$-that is to say, that (the sum of) any two of $A C, D F$, and $G K$ is greater than the remaining (one).

Now, if the angles $A B C, D E F$, and $G H K$ are equal to one another then (it is) clear that, (with) $A C, D F$, and $G K$ also becoming equal, it is possible to construct a triangle from (straight-lines) equal to $A C, D F$, and $G K$. And if not, let them be unequal, and let $K H L$, equal to angle $A B C$, have been constructed on the straight-line $H K$, at the point $H$ on it. And let $H L$ be made equal to



 $\tau \tilde{\eta} s \Delta Z \mu \varepsilon i \zeta \omega \nu$ ह̇бтív. $\alpha \lambda \lambda \alpha ̀ \alpha i ~ H K, ~ K \Lambda ~ \tau n ̃ s ~ H \Lambda ~ \mu \varepsilon i ́ \zeta o v e ́ s ~$







$$
x \gamma^{\prime}
$$

 $\mu \varepsilon i \zeta o v e ́ s ~ \varepsilon i \sigma \iota ~ \pi \alpha ́ v \tau \eta ~ \mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu o ́ \mu \varepsilon \nu \alpha l, ~ \sigma \tau \varepsilon \rho \varepsilon \alpha ̀ \nu ~ \gamma \omega v i ́ \alpha \nu$
 ovas हival.




 $\sigma \tau \varepsilon \rho \varepsilon \alpha ̀ \nu \gamma \omega v i ́ \alpha \nu$ $\sigma \cup \sigma \tau \eta ́ \sigma \alpha \sigma \vartheta \alpha \iota$.




 тò $\Lambda \mathrm{MN}$ трír $\omega$ vov xúx

one of $A B, B C, D E, E F, G H$, and $H K$. And let $K L$ and $G L$ have been joined. And since the two (straightlines) $A B$ and $B C$ are equal to the two (straight-lines) $K H$ and $H L$ (respectively), and the angle at $B$ (is) equal to $K H L$, the base $A C$ is thus equal to the base $K L$ [Prop. 1.4]. And since (the sum of) $A B C$ and $G H K$ is greater than $D E F$, and $A B C$ equal to $K H L, G H L$ is thus greater than $D E F$. And since the two (straightlines) $G H$ and $H L$ are equal to the two (straight-lines) $D E$ and $E F$ (respectively), and angle $G H L$ (is) greater than $D E F$, the base $G L$ is thus greater than the base $D F$ [Prop. 1.24]. But, (the sum of) $G K$ and $K L$ is greater than $G L$ [Prop. 1.20]. Thus, (the sum of) $G K$ and $K L$ is much greater than $D F$. And $K L$ (is) equal to $A C$. Thus, (the sum of) $A C$ and $G K$ is greater than the remaining (straight-line) $D F$. So, similarly, we can show that (the sum of) $A C$ and $D F$ is greater than $G K$, and, further, that (the sum of) $D F$ and $G K$ is greater than $A C$. Thus, it is possible to construct a triangle from (straight-lines) equal to $A C, D F$, and $G K$. (Which is) the very thing it was required to show.

## Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].


Let $A B C, D E F$, and $G H K$ be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to $A B C, D E F$, and $G H K$.

Let $A B, B C, D E, E F, G H$, and $H K$ be cut off (so as to be) equal (to one another). And let $A C, D F$, and $G K$ have been joined. It is, thus, possible to construct a triangle from (straight-lines) equal to $A C, D F$, and $G K$ [Prop. 11.22]. Let (such a triangle), $L M N$, have be constructed, such that $A C$ is equal to $L M, D F$ to $M N$, and, further, $G K$ to $N L$. And let the circle $L M N$ have been circumscribed about triangle $L M N$ [Prop. 4.5]. And let








 ن́đò $\mathrm{AB} \mathrm{\Gamma}, \Delta \mathrm{EZ}, \mathrm{H} \Theta \mathrm{K} \gamma \omega v i ́ \alpha l$ тpıoì $\tau \alpha i ̃ s ~ i ́ \pi o ̀ ~ \Lambda \Xi M, ~ M \Xi N, ~$









 $\Lambda \mathrm{M}$, oưt $\omega \varsigma \dot{\eta} \Xi \mathrm{O}$ трòs ОП• $\mathfrak{\varepsilon} v \alpha \lambda \lambda \grave{\alpha} \xi$ ìs $\dot{\eta} \Lambda \Xi \pi \rho o ̀ \varsigma ~ \Xi O$, ойт $\omega \varsigma \dot{\eta} \Lambda \mathrm{M} \pi \rho o ̀ \varsigma ~ О П . ~ \mu \varepsilon i \zeta \omega \nu ~ \delta \grave{\varepsilon} \dot{\eta} \Lambda \Xi \tau \tilde{\eta} \varsigma \Xi \mathrm{O} \cdot \mu \varepsilon i \zeta \omega \nu \alpha{ }^{\prime} \rho \alpha$






 $\mathrm{M} \Xi \mathrm{N}, \mathrm{N} \Xi \Lambda \mu \varepsilon i \zeta o v e ́ s ~ \varepsilon i \sigma เ \nu . \alpha \dot{\alpha} \lambda \lambda \alpha ̀ \alpha i ~ u ́ \pi o ̀ ~ A B \Gamma, ~ \Delta E Z, ~ H \Theta K ~$







its center have been found, and let it be (at) $O$. And let $L O, M O$, and $N O$ have been joined.


I say that $A B$ is greater than $L O$. For, if not, $A B$ is either equal to, or less than, $L O$. Let it, first of all, be equal. And since $A B$ is equal to $L O$, but $A B$ is equal to $B C$, and $O L$ to $O M$, so the two (straight-lines) $A B$ and $B C$ are equal to the two (straight-lines) $L O$ and $O M$, respectively. And the base $A C$ was assumed (to be) equal to the base $L M$. Thus, angle $A B C$ is equal to angle $L O M$ [Prop. 1.8]. So, for the same (reasons), $D E F$ is also equal to $M O N$, and, further, $G H K$ to $N O L$. Thus, the three angles $A B C, D E F$, and $G H K$ are equal to the three angles $L O M, M O N$, and $N O L$, respectively. But, the (sum of the) three angles $L O M, M O N$, and $N O L$ is equal to four right-angles. Thus, the (sum of the) three angles $A B C, D E F$, and $G H K$ is also equal to four rightangles. And it was also assumed (to be) less than four right-angles. The very thing (is) absurd. Thus, $A B$ is not equal to $L O$. So, I say that $A B$ is not less than $L O$ either. For, if possible, let it be (less). And let $O P$ be made equal to $A B$, and $O Q$ equal to $B C$, and let $P Q$ have been joined. And since $A B$ is equal to $B C, O P$ is also equal to $O Q$. Hence, the remainder $L P$ is also equal to (the remainder) $Q M . L M$ is thus parallel to $P Q$ [Prop. 6.2], and (triangle) $L M O$ (is) equiangular with (triangle) $P Q O$ [Prop. 1.29]. Thus, as $O L$ is to $L M$, so $O P$ (is) to $P Q$ [Prop. 6.4]. Alternately, as $L O$ (is) to $O P$, so $L M$ (is) to $P Q$ [Prop. 5.16]. And $L O$ (is) greater than $O P$. Thus, $L M$ (is) also greater than $P Q$ [Prop. 5.14]. But $L M$ was made equal to $A C$. Thus, $A C$ is also greater than $P Q$. Therefore, since the two (straight-lines) $A B$ and $B C$ are equal to the two (straight-lines) $P O$ and $O Q$ (respectively), and the base $A C$ is greater than the base $P Q$, the angle $A B C$ is thus greater than the angle $P O Q$ [Prop. 1.25]. So, similarly, we can show that $D E F$ is also greater than $M O N$, and $G H K$ than $N O L$. Thus, the (sum of the) three angles $A B C, D E F$, and $G H K$ is greater than the (sum of the) three angles $L O M, M O N$,
$\tau \tilde{\eta} \varsigma \Xi \mathrm{P}, x \alpha i$ ह̀ $\pi \varepsilon \zeta \varepsilon u ́ \chi \vartheta \omega \sigma \alpha \nu$ 人i $\mathrm{P} \Lambda, \mathrm{PM}, \mathrm{PN}$.











 $\dot{\varepsilon} x \alpha \tau \varepsilon ́ \rho \alpha \tau \widetilde{\omega} \nu \mathrm{PM}, \mathrm{PN} \cdot \dot{\varepsilon} x \alpha ́ \sigma \tau \eta$ 人̀ $\rho \alpha \tau \widetilde{\omega} \nu \mathrm{AB}, \mathrm{B} \mathrm{\Gamma}, \Delta \mathrm{E}, \mathrm{EZ}$,






 $\mathrm{H} \Theta \mathrm{K}, ~ \sigma \tau \varepsilon \rho \varepsilon \dot{\alpha} \gamma \omega v i ́ \alpha ~ \sigma u v \varepsilon ́ \sigma \tau \alpha \tau \alpha \downarrow \dot{\eta} \pi \rho o ̀ \varsigma \tau \widetilde{\varphi} \mathrm{P} \pi \varepsilon \rho เ \varepsilon \chi \circ \mu \varepsilon ́ v \eta$

and $N O L$. But, (the sum of) $A B C, D E F$, and $G H K$ was assumed (to be) less than four right-angles. Thus, (the sum of) $L O M, M O N$, and $N O L$ is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, $A B$ is not less than $L O$. And it was shown (to be) not equal either. Thus, $A B$ (is) greater than $L O$.

So let $O R$ have been set up at point $O$ at rightangles to the plane of circle $L M N$ [Prop. 11.12]. And let the (square) on $O R$ be equal to that (area) by which the square on $A B$ is greater than the (square) on $L O$ [Prop. 11.23 lem.]. And let $R L, R M$, and $R N$ have been joined.

And since $R O$ is at right-angles to the plane of circle $L M N, R O$ is thus also at right-angles to each of $L O$, $M O$, and $N O$. And since $L O$ is equal to $O M$, and $O R$ is common and at right-angles, the base $R L$ is thus equal to the base $R M$ [Prop. 1.4]. So, for the same (reasons), $R N$ is also equal to each of $R L$ and $R M$. Thus, the three (straight-lines) $R L, R M$, and $R N$ are equal to one another. And since the (square) on $O R$ was assumed to be equal to that (area) by which the (square) on $A B$ is greater than the (square) on $L O$, the (square) on $A B$ is thus equal to the (sum of the squares) on $L O$ and $O R$. And the (square) on $L R$ is equal to the (sum of the squares) on $L O$ and $O R$. For $L O R$ (is) a right-angle [Prop. 1.47]. Thus, the (square) on $A B$ is equal to the (square) on $R L$. Thus, $A B$ (is) equal to $R L$. But, each of $B C, D E, E F, G H$, and $H K$ is equal to $A B$, and each of $R M$ and $R N$ equal to $R L$. Thus, each of $A B, B C$, $D E, E F, G H$, and $H K$ is equal to each of $R L, R M$, and $R N$. And since the two (straight-lines) $L R$ and $R M$ are equal to the two (straight-lines) $A B$ and $B C$ (respectively), and the base $L M$ was assumed (to be) equal to the base $A C$, the angle $L R M$ is thus equal to the angle $A B C$ [Prop. 1.8]. So, for the same (reasons), $M R N$ is also equal to $D E F$, and $L R N$ to $G H K$.

Thus, the solid angle $R$, contained by the angles $L R M, M R N$, and $L R N$, has been constructed out of the three plane angles $L R M, M R N$, and $L R N$, which are equal to the three given (plane angles) $A B C, D E F$, and $G H K$ (respectively). (Which is) the very thing it was required to do.














$\chi \delta^{\prime}$.

 ย่ $\sigma \tau \iota$.











And we can demonstrate, thusly, in which manner to take the (square) on $O R$ equal to that (area) by which the (square) on $A B$ is greater than the (square) on $L O$. Let the straight-lines $A B$ and $L O$ be set out, and let $A B$ be greater, and let the semicircle $A B C$ have been drawn around it. And let $A C$, equal to the straight-line $L O$, which is not greater than the diameter $A B$, have been inserted into the semicircle $A B C$ [Prop. 4.1]. And let $C B$ have been joined. Therefore, since the angle $A C B$ is in the semicircle $A C B, A C B$ is thus a right-angle [Prop. 3.31]. Thus, the (square) on $A B$ is equal to the (sum of the) squares on $A C$ and $C B$ [Prop. 1.47]. Hence, the (square) on $A B$ is greater than the (square) on $A C$ by the (square) on $C B$. And $A C$ (is) equal to $L O$. Thus, the (square) on $A B$ is greater than the (square) on $L O$ by the (square) on $C B$. Therefore, if we take $O R$ equal to $B C$ then the (square) on $A B$ will be greater than the (square) on $L O$ by the (square) on $O R$. (Which is) the very thing it was prescribed to do.

## Proposition 24

If a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic.


For let the solid (figure) $C D H G$ have been contained by the parallel planes $A C, G F$, and $A H, D F$, and $B F$, $A E$. I say that its opposite planes are both equal and parallelogrammic.

For since the two parallel planes $B G$ and $C E$ are cut by the plane $A C$, their common sections are parallel [Prop. 11.16]. Thus, $A B$ is parallel to $D C$. Again, since the two parallel planes $B F$ and $A E$ are cut by the plane


 AE $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ о́ $ү \rho \alpha \mu \mu o ́ v ~ \varepsilon ̇ \sigma \tau \iota \nu . ~$
 $\mu \varepsilon ̀ \nu \mathrm{AB} \tau \tilde{n} \Delta \Gamma, \dot{\eta} \delta \dot{\varepsilon} \mathrm{~B} \Theta \tau \tilde{n} \Gamma \mathrm{Z}, \delta u ́ o ~ \delta \dot{\eta} \alpha i \mathrm{AB}, \mathrm{B} \Theta \dot{\alpha} \pi \tau o ́ \mu \varepsilon \nu \alpha \downarrow$







 « $\rho \alpha$ тò ВН $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu о \nu \tau \widetilde{\varphi}$ ГЕ $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$.
 тò $\delta$ ¿̀ $\mathrm{AE} \tau \widetilde{( }) \mathrm{BZ}$.




## $\chi \varepsilon^{\prime}$.


 $\pi \rho o ̀ \varsigma ~ \tau \grave{v} \nu \beta \alpha ́ \sigma เ \nu$, oútcus tò $\sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi \rho o ̀ \varsigma ~ \tau o ̀ ~ \sigma \tau \varepsilon \rho \varepsilon o ́ v . ~$



 ЕӨГZ $\beta \alpha ́ \sigma \iota \nu, ~ o u ̛ \tau \omega \varsigma ~ \tau o ̀ ~ A B Z \Upsilon ~ \sigma \tau \varepsilon p \varepsilon o ̀ v ~ \pi p o ̀ s ~ \tau o ̀ ~ Е Н Г \Delta ~$ $\sigma \tau \varepsilon \rho \varepsilon o ́ v$.


 $\tau \dot{\alpha} \Lambda \mathrm{O}, \mathrm{K} \Phi, \Theta \mathrm{X}, \mathrm{M} \Sigma \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu \alpha$ x $\alpha \grave{\imath} \tau \dot{\alpha} \Lambda \Pi, \mathrm{KP}$,
$A C$, their common sections are parallel [Prop. 11.16]. Thus, $B C$ is parallel to $A D$. And $A B$ was also shown (to be) parallel to $D C$. Thus, $A C$ is a parallelogram. So, similarly, we can also show that $D F, F G, G B, B F$, and $A E$ are each parallelograms.

Let $A H$ and $D F$ have been joined. And since $A B$ is parallel to $D C$, and $B H$ to $C F$, so the two (straight-lines) joining one another, $A B$ and $B H$, are parallel to the two straight-lines joining one another, $D C$ and $C F$ (respectively), not (being) in the same plane. Thus, they will contain equal angles [Prop. 11.10]. Thus, angle $A B H$ (is) equal to (angle) $D C F$. And since the two (straightlines) $A B$ and $B H$ are equal to the two (straight-lines) $D C$ and $C F$ (respectively) [Prop. 1.34], and angle $A B H$ is equal to angle $D C F$, the base $A H$ is thus equal to the base $D F$, and triangle $A B H$ is equal to triangle $D C F$ [Prop. 1.4]. And parallelogram $B G$ is double (triangle) $A B H$, and parallelogram $C E$ double (triangle) $D C F$ [Prop. 1.34]. Thus, parallelogram $B G$ (is) equal to parallelogram $C E$. So, similarly, we can show that $A C$ is also equal to $G F$, and $A E$ to $B F$.

Thus, if a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic. (Which is) the very thing it was required to show.

## Proposition 25

If a parallelipiped solid is cut by a plane which is parallel to the opposite planes (of the parallelipiped) then as the base (is) to the base, so the solid will be to the solid.


For let the parallelipiped solid $A B C D$ have been cut by the plane $F G$ which is parallel to the opposite planes $R A$ and $D H$. I say that as the base $A E F V$ (is) to the base $E H C F$, so the solid $A B F U$ (is) to the solid $E G C D$.

For let $A H$ have been produced in each direction. And let any number whatsoever (of lengths), $A K$ and $K L$, be made equal to $A E$, and any number whatsoever (of lengths), $H M$ and $M N$, equal to $E H$. And let the parallelograms $L P, K V, H W$, and $M S$ have been completed,
$\Delta \mathrm{M}, \mathrm{MT}$ бтєрعд́.



























and the solids $L Q, K R, D M$, and $M T$.
And since the straight-lines $L K, K A$, and $A E$ are equal to one another, the parallelograms $L P, K V$, and $A F$ are also equal to one another, and $K O, K B$, and $A G$ (are equal) to one another, and, further, $L X, K Q$, and $A R$ (are equal) to one another. For (they are) opposite [Prop. 11.24]. So, for the same (reasons), the parallelograms $E C, H W$, and $M S$ are also equal to one another, and $H G, H I$, and $I N$ are equal to one another, and, further, $D H, M Y$, and $N T$ (are equal to one another). Thus, three planes of (one of) the solids $L Q, K R$, and $A U$ are equal to the (corresponding) three planes (of the others). But, the three planes (in one of the soilds) are equal to the three opposite planes [Prop. 11.24]. Thus, the three solids $L Q, K R$, and $A U$ are equal to one another [Def. 11.10]. So, for the same (reasons), the three solids $E D, D M$, and $M T$ are also equal to one another. Thus, as many multiples as the base $L F$ is of the base $A F$, so many multiples is the solid $L U$ also of the the solid $A U$. So, for the same (reasons), as many multiples as the base $N F$ is of the base $F H$, so many multiples is the solid $N U$ also of the solid $H U$. And if the base $L F$ is equal to the base $N F$ then the solid $L U$ is also equal to the solid $N U .{ }^{\dagger}$ And if the base $L F$ exceeds the base $N F$ then the solid $L U$ also exceeds the solid $N U$. And if $(L F)$ is less than $(N F)$ then $(L U)$ is (also) less than $(N U)$. So, there are four magnitudes, the two bases $A F$ and $F H$, and the two solids $A U$ and $U H$, and equal multiples have been taken of the base $A F$ and the solid $A U$ - (namely), the base $L F$ and the solid $L U$-and of the base $H F$ and the solid $H U$-(namely), the base $N F$ and the solid $N U$. And it has been shown that if the base $L F$ exceeds the base $F N$ then the solid $L U$ also exceeds the [solid] $N U$, and if ( $L F$ is) equal (to $F N$ ) then ( $L U$ is) equal (to $N U$ ), and if ( $L F$ is) less than $(F N)$ then ( $L U$ is) less than ( $N U$ ). Thus, as the base $A F$ is to the base $F H$, so the solid $A U$ (is) to the solid $U H$ [Def. 5.5]. (Which is) the very thing it was required to show.
$\dagger$ Here, Euclid assumes that $L F \gtreqless N F$ implies $L U \gtreqless N U$. This is easily demonstrated.
$\chi \xi^{\prime}$.






 $\gamma \omega v i \alpha \nu \sigma \cup \sigma \tau \dot{\jmath} \sigma \alpha \sigma \vartheta \alpha l$.

## Proposition 26

To construct a solid angle equal to a given solid angle on a given straight-line, and at a given point on it.

Let $A B$ be the given straight-line, and $A$ the given point on it, and $D$ the given solid angle, contained by the plane angles $E D C, E D F$, and $F D C$. So, it is necessary to construct a solid angle equal to the solid angle $D$ on the straight-line $A B$, and at the point $A$ on it.


Eỉnń $\varphi \vartheta \omega$ ү̀̀̀






 $\dot{\eta} \Theta \mathrm{A} \cdot \lambda \varepsilon ́ \gamma \omega$, ötı $\dot{\eta} \pi \rho o ̀ s \tau \widetilde{\varphi} \mathrm{~A} \sigma \tau \varepsilon \rho \varepsilon \alpha ̀ \gamma \omega v i ́ \alpha \pi \varepsilon \rho เ \varepsilon \chi \circ \mu \varepsilon ́ v \eta$
 $\sigma \tau \varepsilon \rho \varepsilon \tilde{\alpha} ~ \gamma \omega \nu i ́ \alpha$ т $\tilde{n} \pi \varepsilon \rho เ \varepsilon \chi o \mu \varepsilon ́ \nu \eta$ ن́ $\pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{E} \Delta \Gamma, \mathrm{E} \Delta \mathrm{Z}, \mathrm{Z} \Delta \Gamma$ $\gamma \omega \nu / \widetilde{\omega} \nu$.






















For let some random point $F$ have been taken on $D F$, and let $F G$ have been drawn from $F$ perpendicular to the plane through $E D$ and $D C$ [Prop. 11.11], and let it meet the plane at $G$, and let $D G$ have been joined. And let $B A L$, equal to the angle $E D C$, and $B A K$, equal to $E D G$, have been constructed on the straight-line $A B$ at the point $A$ on it [Prop. 1.23]. And let $A K$ be made equal to $D G$. And let $K H$ have been set up at the point $K$ at right-angles to the plane through $B A L$ [Prop. 11.12]. And let $K H$ be made equal to $G F$. And let $H A$ have been joined. I say that the solid angle at $A$, contained by the (plane) angles $B A L, B A H$, and $H A L$, is equal to the solid angle at $D$, contained by the (plane) angles $E D C$, $E D F$, and $F D C$.

For let $A B$ and $D E$ have been cut off (so as to be) equal, and let $H B, K B, F E$, and $G E$ have been joined. And since $F G$ is at right-angles to the reference plane $(E D C)$, it will also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Thus, the angles $F G D$ and $F G E$ are right-angles. So, for the same (reasons), the angles $H K A$ and $H K B$ are also right-angles. And since the two (straight-lines) $K A$ and $A B$ are equal to the two (straight-lines) $G D$ and $D E$, respectively, and they contain equal angles, the base $K B$ is thus equal to the base $G E$ [Prop. 1.4]. And $K H$ is also equal to $G F$. And they contain right-angles (with the respective bases). Thus, $H B$ (is) also equal to $F E$ [Prop. 1.4]. Again, since the two (straight-lines) $A K$ and $K H$ are equal to the two (straight-lines) $D G$ and $G F$ (respectively), and they contain right-angles, the base $A H$ is thus equal to the base $F D$ [Prop. 1.4]. And $A B$ (is) also equal to $D E$. So, the two (straight-lines) $H A$ and $A B$ are equal to the two (straight-lines) $D F$ and $D E$ (respectively). And the base $H B$ (is) equal to the base $F E$. Thus, the angle $B A H$ is equal to the angle $E D F$ [Prop. 1.8]. So, for the same (reasons), $H A L$ is also equal to $F D C$. And $B A L$ is also equal to $E D C$.

Thus, (a solid angle) has been constructed, equal to the given solid angle at $D$, on the given straight-line $A B$,

$$
x \zeta^{\prime} .
$$


 $\lambda \varepsilon \pi i \pi \varepsilon \delta o \nu \alpha \dot{\alpha} \alpha \gamma \rho \alpha ́ \not \psi \alpha$.












 $\pi \varepsilon \pi \lambda \eta \rho \omega ́ \sigma \vartheta \omega$ тò $\Theta \mathrm{B} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ ó $\gamma \rho \alpha \mu \mu \circ \nu$ каì tò $\mathrm{A} \Lambda \sigma \tau \varepsilon \rho \varepsilon o ́ v$.










 غ่ $\sigma \tau \iota$.




$$
x \eta^{\prime}
$$

'Е $\grave{\alpha} \nu$ $\sigma \tau \varepsilon \rho \varepsilon o ̀ \nu ~ \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi i ́ \pi \varepsilon \delta o \nu ~ \varepsilon ̇ \pi เ \pi \varepsilon ́ \delta \omega ~ \tau \mu \eta \vartheta \tilde{n} \not \approx \alpha \tau \grave{\alpha}$
at the given point $A$ on it. (Which is) the very thing it was required to do.

## Proposition 27

To describe a parallelepiped solid similar, and similarly laid out, to a given parallelepiped solid on a given straight-line.

Let the given straight-line be $A B$, and the given parallelepiped solid $C D$. So, it is necessary to describe a parallelepiped solid similar, and similarly laid out, to the given parallelepiped solid $C D$ on the given straight-line $A B$.

For, let a (solid angle) contained by the (plane angles) $B A H, H A K$, and $K A B$ have been constructed, equal to solid angle at $C$, on the straight-line $A B$ at the point $A$ on it [Prop. 11.26], such that angle $B A H$ is equal to $E C F$, and $B A K$ to $E C G$, and $K A H$ to $G C F$. And let it have been contrived that as $E C$ (is) to $C G$, so $B A$ (is) to $A K$, and as $G C$ (is) to $C F$, so $K A$ (is) to $A H$ [Prop. 6.12]. And thus, via equality, as $E C$ is to $C F$, so $B A$ (is) to $A H$ [Prop. 5.22]. And let the parallelogram $H B$ have been completed, and the solid $A L$.


And since as $E C$ is to $C G$, so $B A$ (is) to $A K$, and the sides about the equal angles $E C G$ and $B A K$ are (thus) proportional, the parallelogram $G E$ is thus similar to the parallelogram $K B$. So, for the same (reasons), the parallelogram $K H$ is also similar to the parallelogram $G F$, and, further, $F E$ (is similar) to $H B$. Thus, three of the parallelograms of solid $C D$ are similar to three of the parallelograms of solid $A L$. But, the (former) three are equal and similar to the three opposite, and the (latter) three are equal and similar to the three opposite. Thus, the whole solid $C D$ is similar to the whole solid $A L$ [Def. 11.9].

Thus, $A L$, similar, and similarly laid out, to the given parallelepiped solid $C D$, has been described on the given straight-lines $A B$. (Which is) the very thing it was required to do.

## Proposition 28

If a parallelepiped solid is cut by a plane (passing)
 tò $\sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \dot{́ n o ̀ ~ \tau o u ̃ ~ غ ̇ \pi ル \tau \varepsilon ́ o ́ o u . ~}$

$\Sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \gamma \grave{\alpha} \rho \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi i \pi \varepsilon \delta \partial \nu$ tò AB ह̀ $\pi \pi \varepsilon \delta \delta \omega$ т $\tau$














through the diagonals of (a pair of) opposite planes then the solid will be cut in half by the plane.


For let the parallelepiped solid $A B$ have been cut by the plane $C D E F$ (passing) through the diagonals of the opposite planes $C F$ and $D E .^{\dagger}$ I say that the solid $A B$ will be cut in half by the plane $C D E F$.

For since triangle $C G F$ is equal to triangle $C F B$, and $A D E$ (is equal) to $D E H$ [Prop. 1.34], and parallelogram $C A$ is also equal to $E B$-for (they are) opposite [Prop. 11.24]-and $G E$ (equal) to $C H$, thus the prism contained by the two triangles $C G F$ and $A D E$, and the three parallelograms $G E, A C$, and $C E$, is also equal to the prism contained by the two triangles $C F B$ and $D E H$, and the three parallelograms $C H, B E$, and $C E$. For they are contained by planes (which are) equal in number and in magnitude [Def. 11.10]. ${ }^{\ddagger}$ Thus, the whole of solid $A B$ is cut in half by the plane $C D E F$. (Which is) the very thing it was required to show.
${ }^{\dagger}$ Here, it is assumed that the two diagonals lie in the same plane. The proof is easily supplied.
$\ddagger$ However, strictly speaking, the prisms are not similarly arranged, being mirror images of one another.

## $\chi \vartheta^{\prime}$.







## Proposition 29

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are on the same straight-lines, are equal to one another.


For let the parallelepiped solids $C M$ and $C N$ be on


 $\sigma \tau \varepsilon \rho \varepsilon \grave{v} \tau \widetilde{\varrho}$ ГN $\sigma \tau \varepsilon р \varepsilon \widetilde{\varphi}$.





















$$
\lambda^{\prime} .
$$






 $\mathrm{AZ}, \mathrm{AH}, \Lambda \mathrm{M}, \Lambda \mathrm{N}, Г \Delta, Г \mathrm{E}, \mathrm{B} \Theta, \mathrm{BK} \mu \grave{\eta}$ है $\sigma \tau \omega \sigma \alpha \nu$ ह̇ni $\tau \tilde{\omega} \nu$
the same base $A B$, and (have) the same height, and let the (ends of the straight-lines) standing up in them, $A G$, $A F, L M, L N, C D, C E, B H$, and $B K$, be on the same straight-lines, $F N$ and $D K$. I say that solid $C M$ is equal to solid $C N$.

For since $C H$ and $C K$ are each parallelograms, $C B$ is equal to each of $D H$ and $E K$ [Prop. 1.34]. Hence, $D H$ is also equal to $E K$. Let $E H$ have been subtracted from both. Thus, the remainder $D E$ is equal to the remainder $H K$. Hence, triangle $D C E$ is also equal to triangle $H B K$ [Props. 1.4, 1.8], and parallelogram $D G$ to parallelogram $H N$ [Prop. 1.36]. So, for the same (reasons), traingle $A F G$ is also equal to triangle $M L N$. And parallelogram $C F$ is also equal to parallelogram $B M$, and $C G$ to $B N$ [Prop. 11.24]. For they are opposite. Thus, the prism contained by the two triangles $A F G$ and $D C E$, and the three parallelograms $A D, D G$, and $C G$, is equal to the prism contained by the two triangles $M L N$ and $H B K$, and the three parallelograms $B M, H N$, and $B N$. Let the solid whose base (is) parallelogram $A B$, and (whose) opposite (face is) $G E H M$, have been added to both (prisms). Thus, the whole parallelepiped solid $C M$ is equal to the whole parallelepiped solid $C N$.

Thus, parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up (are) on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

## Proposition 30

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straightlines, are equal to one another.


Let the parallelepiped solids $C M$ and $C N$ be on the same base, $A B$, and (have) the same height, and let the (ends of the straight-lines) standing up in them, $A F, A G$,
 $\sigma \tau \varepsilon \rho \varepsilon \widetilde{( })$.




 $\beta \alpha ́ \sigma \iota s ~ \mu \varepsilon ̀ \nu ~ \tau o ̀ ~ А Г В \Lambda ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu, ~ \alpha ́ \pi \varepsilon \nu \alpha \nu \tau i ́ o v ~ \delta \grave{\varepsilon ̀ ~ t o ̀ ~}$









 $\tau \widetilde{\varphi} \Gamma \mathrm{N}$ бтعрєఢॅ.



$\lambda \alpha^{\prime}$.










 $\tau \varepsilon \mathrm{PX} \beta \alpha ́ \sigma \iota \varsigma ~ x \alpha i ~ \tau o ̀ ~ \Psi \Upsilon ~ \sigma \tau \varepsilon р \varepsilon o ́ v . ~$
$L M, L N, C D, C E, B H$, and $B K$, not be on the same straight-lines. I say that the solid $C M$ is equal to the solid $C N$.

For let $N K$ and $D H$ have been produced, and let them have joined one another at $R$. And, further, let $F M$ and $G E$ have been produced to $P$ and $Q$ (respectively). And let $A O, L P, C Q$, and $B R$ have been joined. So, solid $C M$, whose base (is) parallelogram $A C B L$, and opposite (face) $F D H M$, is equal to solid $C P$, whose base (is) parallelogram $A C B L$, and opposite (face) $O Q R P$. For they are on the same base, $A C B L$, and (have) the same height, and the (ends of the straight-lines) standing up in them, $A F, A O, L M, L P, C D, C Q, B H$, and $B R$, are on the same straight-lines, $F P$ and $D R$ [Prop. 11.29]. But, solid $C P$, whose base is parallelogram $A C B L$, and opposite (face) $O Q R P$, is equal to solid $C N$, whose base (is) parallelogram $A C B L$, and opposite (face) $G E K N$. For, again, they are on the same base, $A C B L$, and (have) the same height, and the (ends of the straight-lines) standing up in them, $A G, A O, C E, C Q, L N, L P, B K$, and $B R$, are on the same straight-lines, $G Q$ and $N R$ [Prop. 11.29]. Hence, solid $C M$ is also equal to solid $C N$.

Thus, parallelepiped solids (which are) on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

## Proposition 31

Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

Let the parallelepiped solids $A E$ and $C F$ be on the equal bases $A B$ and $C D$ (respectively), and (have) the same height. I say that solid $A E$ is equal to solid $C F$.

So, let the (straight-lines) standing up, $H K, B E, A G$, $L M, P Q, D F, C O$, and $R S$, first of all, be at right-angles to the bases $A B$ and $C D$. And let $R T$ have been produced in a straight-line with $C R$. And let (angle) $T R U$, equal to angle $A L B$, have been constructed on the straight-line $R T$, at the point $R$ on it [Prop. 1.23]. And let $R T$ be made equal to $A L$, and $R U$ to $L B$. And let the base $R W$, and the solid $X U$, have been completed.


 $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ о́ $\rho \alpha \mu \mu о \nu \tau \widetilde{\varphi} \Theta \Lambda \pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$. $\alpha \alpha i ̀ ~ \varepsilon ̇ \pi \varepsilon \grave{~}$

 $\lambda \eta \lambda о ́ \gamma \rho \alpha \mu \mu о \nu \tau \widetilde{\varphi}$ AM $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \omega$. $\delta \grave{\alpha} \tau \grave{\alpha} \alpha \cup \not \tau \dot{\alpha} \delta \dot{\eta}$







 $\dot{\varepsilon} x \beta \varepsilon \beta \lambda \eta \dot{\eta} \sigma \omega \dot{\eta} \mathrm{O} \Delta x \alpha \tau \grave{\alpha}$ tò $\alpha, x \alpha \grave{\imath} \sigma \cup \mu \pi \varepsilon \pi \lambda \eta \rho \omega \dot{\sigma} \vartheta \omega \tau \grave{\alpha} \Omega \Psi$,







 тò P؟XT $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ ó $ү \rho \alpha \mu \mu о \nu ~ \tau \widetilde{\varphi} \Omega T$ т $\alpha \rho \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \alpha \mu \varphi$.





And since the two (straight-lines) $T R$ and $R U$ are equal to the two (straight-lines) $A L$ and $L B$ (respectively), and they contain equal angles, parallelogram $R W$ is thus equal and similar to parallelogram $H L$ [Prop. 6.14]. And, again, since $A L$ is equal to $R T$, and $L M$ to $R S$, and they contain right-angles, parallelogram $R X$ is thus equal and similar to parallelogram $A M$ [Prop. 6.14]. So, for the same (reasons), $L E$ is also equal and similar to $S U$. Thus, three parallelograms of solid $A E$ are equal and similar to three parallelograms of solid $X U$. But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid $A E$ is equal to the whole parallelepiped solid $X U$ [Def. 11.10]. Let $D R$ and $W U$ have been drawn across, and let them have met one another at $Y$. And let $a T b$ have been drawn through $T$ parallel to $D Y$. And let $P D$ have been produced to $a$. And let the solids $Y X$ and $R I$ have been completed. So, solid $X Y$, whose base is parallelogram $R X$, and opposite (face) $Y c$, is equal to solid $X U$, whose base (is) parallelogram $R X$, and opposite (face) $U V$. For they are on the same base $R X$, and (have) the same height, and the (ends of the straightlines) standing up in them, $R Y, R U, T b, T W, S e, S d$, $X c$ and $X V$, are on the same straight-lines, $Y W$ and $e V$ [Prop. 11.29]. But, solid $X U$ is equal to $A E$. Thus,










 $\pi \rho o ̀ \varsigma ~ \tau o ̀ ~ P I . ~ غ ́ x \alpha ́ \tau \varepsilon \rho o \nu ~ \alpha ̈ \rho \alpha ~ \tau \widetilde{\omega} \nu ~ Г Z, ~ \Omega \Psi ~ \sigma \tau \varepsilon р \varepsilon \widetilde{\omega} \nu ~ \pi \rho o ̀ s ~ \tau o ̀ ~$




$\mathrm{M} \grave{\eta}$ हैఠт $\omega \sigma \alpha \nu$ ס̀̀ $\alpha i$ ह̇ $\varphi \varepsilon \sigma \tau \eta x \cup i ̃ \alpha \iota ~ \alpha i ~ A H, ~ \Theta K, ~ B E, ~ \Lambda M, ~$ $\Gamma \Xi, ~ О П, \Delta Z, ~ Р \Sigma \pi p o ̀ \varsigma ~ o ̉ p \vartheta \grave{\alpha} \varsigma ~ \tau \alpha i ̆ \varsigma ~ А В, ~ Г \Delta ~ \beta \alpha ́ \sigma \varepsilon \sigma เ v \cdot ~ \lambda \varepsilon ́ \gamma \omega ~$














solid $X Y$ is also equal to solid $A E$. And since parallelogram $R U W T$ is equal to parallelogram $Y T$. For they are on the same base $R T$, and between the same parallels $R T$ and $Y W$ [Prop. 1.35]. But, $R U W T$ is equal to $C D$, since (it is) also (equal) to $A B$. Parallelogram $Y T$ is thus also equal to $C D$. And $D T$ is another (parallelogram). Thus, as base $C D$ is to $D T$, so $Y T$ (is) to $D T$ [Prop. 5.7]. And since the parallelepiped solid $C I$ has been cut by the plane $R F$, which is parallel to the opposite planes (of $C I$ ), as base $C D$ is to base $D T$, so solid $C F$ (is) to solid $R I$ [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane $R X$, which is parallel to the opposite planes (of $Y I$ ), as base $Y T$ is to base $T D$, so solid $Y X$ (is) to solid $R I$ [Prop. 11.25]. But, as base $C D$ (is) to $D T$, so $Y T$ (is) to $D T$. And, thus, as solid $C F$ (is) to solid $R I$, so solid $Y X$ (is) to solid $R I$. Thus, solids $C F$ and $Y X$ each have the same ratio to $R I$ [Prop. 5.11]. Thus, solid $C F$ is equal to solid $Y X$ [Prop. 5.9]. But, $Y X$ was show (to be) equal to $A E$. Thus, $A E$ is also equal to $C F$.


And so let the (straight-lines) standing up, $A G, H K$, $B E, L M, C O, P Q, D F$, and $R S$, not be at right-angles to the bases $A B$ and $C D$. Again, I say that solid $A E$ (is) equal to solid $C F$. For let $K N, E T, G U, M V, Q W$, $F X, O Y$, and $S I$ have been drawn from points $K, E$, $G, M, Q, F, O$, and $S$ (respectively) perpendicular to the reference plane (i.e., the plane of the bases $A B$ and $C D)$, and let them have met the plane at points $N, T$, $U, V, W, X, Y$, and $I$ (respectively). And let $N T, N U$, $U V, T V, W X, W Y, Y I$, and $I X$ have been joined. So solid $K V$ is equal to solid $Q I$. For they are on the equal bases $K M$ and $Q S$, and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid $K V$ is equal to solid $A E$, and $Q I$ to $C F$. For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid $A E$ is also equal to solid $C F$.

Thus, parallelepiped solids which are on equal bases,

$$
\lambda \beta^{\prime}
$$

T $\alpha$ ú $\pi o ̀ ~ \tau o ̀ ~ \alpha u ̉ \tau o ̀ ~ u ̛ \psi o s ~ o ̋ v \tau \alpha ~ \sigma \tau \varepsilon p \varepsilon \alpha ̀ ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda \varepsilon \pi i ́ \pi \varepsilon \delta \alpha ~$


"Eбт $\omega$ ú $\mathrm{AB}, \Gamma \Delta \cdot \lambda \varepsilon ́ \gamma \omega$, ö $\tau \iota \frac{\downarrow}{\alpha} \mathrm{AB}, \Gamma \Delta \sigma \tau \varepsilon \rho \varepsilon \alpha \dot{\alpha} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi i ́ \pi \varepsilon \delta \alpha$

 тò $\Gamma \Delta$ бєєрعóv.

 $\sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda \varepsilon \pi i \pi \varepsilon \delta o \nu ~ \sigma \cup \mu \pi \varepsilon \pi \lambda \eta \rho \omega ́ \sigma \vartheta \omega$ tò HK. औैбov






 тท̀v $\Gamma$ Z $\beta \alpha \dot{\sigma} \sigma \nu$, oút $\tau \varsigma$ тò $\mathrm{AB} \sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \Gamma \Delta ~ \sigma \tau \varepsilon p \varepsilon o ́ v . ~$



$$
\lambda \gamma^{\prime}
$$

 $\pi \lambda \alpha \sigma$ óovt $\lambda o ́ \gamma \varphi$ عíбi т $\widetilde{\omega} \nu$ ó $\mu о \lambda o ́ \gamma \omega \nu \pi \lambda \varepsilon \cup p \widetilde{\omega} \nu$.


 трòs tウ̀レ $\Gamma$.




and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

## Proposition 32

Parallelepiped solids which (have) the same height are to one another as their bases.


Let $A B$ and $C D$ be parallelepiped solids (having) the same height. I say that the parallelepiped solids $A B$ and $C D$ are to one another as their bases. That is to say, as base $A E$ is to base $C F$, so solid $A B$ (is) to solid $C D$.

For let $F H$, equal to $A E$, have been applied to $F G$ (in the angle $F G H$ equal to angle $L C G$ ) [Prop. 1.45]. And let the parallelepiped solid $G K$, (having) the same height as $C D$, have been completed on the base $F H$. So solid $A B$ is equal to solid $G K$. For they are on the equal bases $A E$ and $F H$, and (have) the same height [Prop. 11.31]. And since the parallelepiped solid $C K$ has been cut by the plane $D G$, which is parallel to the opposite planes (of $C K$ ), thus as the base $C F$ is to the base $F H$, so the solid $C D$ (is) to the solid $D H$ [Prop. 11.25]. And base $F H$ (is) equal to base $A E$, and solid $G K$ to solid $A B$. And thus as base $A E$ is to base $C F$, so solid $A B$ (is) to solid $C D$.

Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

## Proposition 33

Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let $A B$ and $C D$ be similar parallelepiped solids, and let $A E$ correspond to $C F$. I say that solid $A B$ has to solid $C D$ the cubed ratio that $A E$ (has) to $C F$.

For let $E K, E L$, and $E M$ have been produced in a straight-line with $A E, G E$, and $H E$ (respectively). And let $E K$ be made equal to $C F$, and $E L$ equal to $F N$, and, further, $E M$ equal to $F R$. And let the parallelogram $K L$ have been completed, and the solid $K P$.




 тò $\mathrm{K} \Lambda \pi \alpha р \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu о \nu \tau \widetilde{\varphi}$ ГN $\tau \alpha p \alpha \lambda \lambda \eta \lambda о \gamma \rho \alpha ́ \mu \mu \varphi$. $\delta ı \grave{\alpha}$














 AH [ $\pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu]$ тpòs tò HK $\pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o v$,

 tò AH $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o v ~ \pi \rho o ̀ s ~ t o ̀ ~ H K, ~ o u ̈ \tau \omega s ~ t o ̀ ~ H K ~ \pi \rho o ̀ s ~$

 tò $\mathrm{HK} \pi \rho o ̀ s ~ t o ̀ ~ K \Lambda, ~ o u ̈ \tau \omega s ~ t o ̀ ~ \Xi \mathrm{E} \sigma \tau \rho \varepsilon o ̀ v ~ \pi \rho o ̀ s ~ t o ̀ ~ \Pi \Lambda ~$

 $\mathrm{E} \Xi$, oưt $\omega \varsigma$ tò $\mathrm{E} \Xi$ тpòs tò $\Pi \Lambda$ xaì tò $\Pi \Lambda$ tpòs tò KO .



 tò $\mathrm{E} \Xi$, oüt $\omega \omega$ тò $\mathrm{AH} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu$ тpòs tò HK ж $\alpha i \dot{\eta}$ AE عủveĩ $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{EK} \cdot \omega ̈ \sigma \tau \varepsilon$ xaì tò $\mathrm{AB} \sigma \tau \varepsilon \rho \varepsilon o ̀ \nu ~ \pi \rho o ̀ s ~ \tau o ̀ ~$





And since the two (straight-lines) $K E$ and $E L$ are equal to the two (straight-lines) $C F$ and $F N$, but angle $K E L$ is also equal to angle $C F N$, inasmuch as $A E G$ is also equal to $C F N$, on account of the similarity of the solids $A B$ and $C D$, parallelogram $K L$ is thus equal [and similar] to parallelogram $C N$. So, for the same (reasons), parallelogram $K M$ is also equal and similar to [parallelogram] $C R$, and, further, $E P$ to $D F$. Thus, three parallelograms of solid $K P$ are equal and similar to three parallelograms of solid $C D$. But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid $K P$ is equal and similar to the whole of solid $C D$ [Def. 11.10]. Let parallelogram $G K$ have been completed. And let the the solids $E O$ and $L Q$, with bases the parallelograms $G K$ and $K L$ (respectively), and with the same height as $A B$, have been completed. And since, on account of the similarity of solids $A B$ and $C D$, as $A E$ is to $C F$, so $E G$ (is) to $F N$, and $E H$ to $F R$ [Defs. 6.1, 11.9], and $C F$ (is) equal to $E K$, and $F N$ to $E L$, and $F R$ to $E M$, thus as $A E$ is to $E K$, so $G E$ (is) to $E L$, and $H E$ to $E M$. But, as $A E$ (is) to $E K$, so [parallelogram] $A G$ (is) to parallelogram $G K$, and as $G E$ (is) to $E L$, so $G K$ (is) to $K L$, and as $H E$ (is) to $E M$, so $Q E$ (is) to $K M$ [Prop. 6.1]. And thus as parallelogram $A G$ (is) to $G K$, so $G K$ (is) to $K L$, and $Q E$ (is) to $K M$. But, as $A G$ (is) to $G K$, so solid $A B$ (is) to solid $E O$, and as $G K$ (is) to $K L$, so solid $O E$ (is) to solid $Q L$, and as $Q E$ (is) to $K M$, so solid $Q L$ (is) to solid $K P$ [Prop. 11.32]. And, thus, as solid $A B$ is to $E O$, so $E O$ (is) to $Q L$, and $Q L$ to $K P$. And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid $A B$ has to $K P$ the cubed ratio which $A B$ (has) to $E O$. But, as $A B$ (is) to $E O$, so parallelogram $A G$ (is) to $G K$, and the straight-line $A E$ to $E K$ [Prop. 6.1]. Hence, solid $A B$ also has to $K P$ the cubed ratio that $A E$ (has) to $E K$. And solid $K P$ (is)
 тpòs t $\grave{\nu} \nu \dot{o} \mu o ́ \lambda o \gamma o v \pi \lambda \varepsilon u p \alpha ̀ \nu ~ \tau \grave{\eta} \nu \Gamma Z$.




## По́pıбца.


 тò $\alpha \pi o ̀ ~ \tau n ̃ ऽ ~ \pi p \omega ́ t \eta s ~ \sigma \tau \varepsilon p \varepsilon o ̀ v ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda \varepsilon \pi i ́ \pi \varepsilon \delta o \nu ~ \pi p o ̀ \varsigma ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~$

 $\pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ \delta \varepsilon u \tau e ́ p \alpha \nu . ~$

## $\lambda \delta^{\prime}$.




 ő $\iota \iota \widetilde{\omega} \nu \mathrm{AB}, \Gamma \Delta \sigma \tau \varepsilon \rho \varepsilon \widetilde{\omega} \nu \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi \iota \pi \varepsilon \delta \sigma \omega \nu \alpha \nu \tau \iota \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma \iota$

 $\sigma \tau \varepsilon р \varepsilon \circ$ ũ ü廿oऽ.
 $\Theta К, ~ Г М, ~ N \Xi, ~ О \Delta, ~ П Р ~ \pi р o ̀ s ~ o ̉ p \vartheta ̀ ̀ s ~ \tau \alpha i ̃ s ~ \beta \alpha ́ \sigma \varepsilon \sigma ル ~ \alpha u ̉ \tau \widetilde{\omega} v$.
 $\dot{\eta} \Gamma \mathrm{M}$ трòs tr̀ AH .





equal to solid $C D$, and straight-line $E K$ to $C F$. Thus, solid $A B$ also has to solid $C D$ the cubed ratio which its corresponding side $A E$ (has) to the corresponding side $C F$.


Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

## Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

## Proposition $34^{\dagger}$

The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let $A B$ and $C D$ be equal parallelepiped solids. I say that the bases of the parallelepiped solids $A B$ and $C D$ are reciprocally proportional to their heights, and (so) as base $E H$ is to base $N Q$, so the height of solid $C D$ (is) to the height of solid $A B$.

For, first of all, let the (straight-lines) standing up, $A G, E F, L B, H K, C M, N O, P D$, and $Q R$, be at rightangles to their bases. I say that as base $E H$ is to base $N Q$, so $C M$ (is) to $A G$.

Therefore, if base $E H$ is equal to base $N Q$, and solid $A B$ is also equal to solid $C D, C M$ will also be equal to $A G$. For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base
$\tau \widetilde{\omega} \mathrm{AB}, \Gamma \Delta \sigma \tau \varepsilon \rho \varepsilon \widetilde{\omega} \nu \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi \iota \pi \varepsilon \delta \partial \omega \nu \dot{\alpha} \nu \tau \iota \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma \iota \nu \alpha i$ $\beta \alpha ́ \sigma \varepsilon ı$ тоі̆s ü $\psi \varepsilon \sigma$ เน．








 бтєрєòv трòs tò $\Gamma \Phi$ бтєрєóv．$\alpha \lambda \lambda \lambda^{\circ} \omega \varsigma \mu \varepsilon ̀ \nu ~ \tau o ̀ ~ A B ~ \sigma \tau \varepsilon р \varepsilon o ̀ v ~$






 $\dot{\alpha} \nu \tau \iota \pi \varepsilon \pi o ́ v \vartheta \alpha \sigma ル ~ \alpha i ~ \beta \alpha ́ \sigma \varepsilon ı \varsigma ~ \tau о і ̃ \varsigma ~ \cup ̈ \psi \varepsilon \sigma เ \nu . ~$

П⿰́д入っレ $\delta \dot{\eta} \tau \widetilde{\omega} \nu \mathrm{AB}, \Gamma \Delta \sigma \tau \varepsilon \rho \varepsilon \widetilde{\omega} \nu \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi \iota \tau \varepsilon ́ \delta \omega \nu \alpha \dot{\alpha} \tau \tau-$










 $\tau \widetilde{\varphi} \Gamma \Delta$ бтعрє巛్．






$E H$（is）to $N Q$ ，so $C M$ will be to $A G$ ．And（so it is）clear that the bases of the parallelepiped solids $A B$ and $C D$ are reciprocally proportional to their heights．


So let base $E H$ not be equal to base $N Q$ ，but let $E H$ be greater．And solid $A B$ is also equal to solid $C D$ ．Thus， $C M$ is also greater than $A G$ ．Therefore，let $C T$ be made equal to $A G$ ．And let the parallelepiped solid $V C$ have been completed on the base $N Q$ ，with height $C T$ ．And since solid $A B$ is equal to solid $C D$ ，and $C V$（is）extrinsic （to them），and equal（magnitudes）have the same ratio to the same（magnitude）［Prop．5．7］，thus as solid $A B$ is to solid $C V$ ，so solid $C D$（is）to solid $C V$ ．But，as solid $A B$ （is）to solid $C V$ ，so base $E H$（is）to base $N Q$ ．For the solids $A B$ and $C V$（are）of equal height［Prop．11．32］． And as solid $C D$（is）to solid $C V$ ，so base $M Q$（is）to base $T Q$［Prop．11．25］，and $C M$ to $C T$［Prop．6．1］．And，thus， as base $E H$ is to base $N Q$ ，so $M C$（is）to $A G$ ．And $C T$ （is）equal to $A G$ ．And thus as base $E H$（is）to base $N Q$ ， so $M C$（is）to $A G$ ．Thus，the bases of the parallelepiped solids $A B$ and $C D$ are reciprocally proportional to their heights．

So，again，let the bases of the parallelepipid solids $A B$ and $C D$ be reciprocally proportional to their heights，and let base $E H$ be to base $N Q$ ，as the height of solid $C D$（is） to the height of solid $A B$ ．I say that solid $A B$ is equal to solid $C D$ ．［For］let the（straight－lines）standing up again be at right－angles to the bases．And if base $E H$ is equal to base $N Q$ ，and as base $E H$ is to base $N Q$ ，so the height of solid $C D$（is）to the height of solid $A B$ ，the height of solid $C D$ is thus also equal to the height of solid $A B$ ． And parallelepiped solids on equal bases，and also with the same height，are equal to one another［Prop．11．31］． Thus，solid $A B$ is equal to solid $C D$ ．

So，let base $E H$ not be equal to［base］$N Q$ ，but let $E H$ be greater．Thus，the height of solid $C D$ is also greater than the height of solid $A B$ ，that is to say $C M$ （greater）than $A G$ ．Let $C T$ again be made equal to $A G$ ， and let the solid $C V$ have been similarly completed． Since as base $E H$ is to base $N Q$ ，so $M C$（is）to $A G$ ，










 $\Xi \mathrm{N}, \Delta \mathrm{O}, \mathrm{M}, ~ Р П ~ \pi \rho o ̀ s ~ o ̉ \rho \vartheta \grave{\alpha} \varsigma ~ \tau \alpha u ̈ \varsigma ~ \beta \alpha ́ \sigma \varepsilon \sigma \sigma \nu ~ \alpha u ̉ \tau \widetilde{\omega} \nu, ~ x \alpha i ̀ ~$















 NП $\beta \alpha ́ \sigma เ \nu, ~ o u ̛ \tau \omega \varsigma ~ \tau o ̀ ~ \tau o u ̃ ~ \Delta \Psi ~ \sigma \tau \varepsilon p \varepsilon o u ̃ ~ u ̛ \psi o s ~ \pi p o ̀ \varsigma ~ \tau o ̀ ~ \tau o u ̃ ~ B T ~$ $\sigma \tau \varepsilon \rho \varepsilon \circ \tilde{u}$ ưqos. $\tau \grave{\alpha} \delta^{\prime} \alpha u ̛ \tau \alpha ̀ ~ u ̛ \psi \eta ~ \varepsilon ̇ \sigma \tau i ~ \tau \widetilde{\omega} \nu \Delta \Psi$, BT $\sigma \tau \varepsilon \rho \varepsilon \widetilde{\omega} \nu$

and $A G$ (is) equal to $C T$, thus as base $E H$ (is) to base $N Q$, so $C M$ (is) to $C T$. But, as [base] $E H$ (is) to base $N Q$, so solid $A B$ (is) to solid $C V$. For solids $A B$ and $C V$ are of equal heights [Prop. 11.32]. And as $C M$ (is) to $C T$, so (is) base $M Q$ to base $Q T$ [Prop. 6.1], and solid $C D$ to solid $C V$ [Prop. 11.25]. And thus as solid $A B$ (is) to solid $C V$, so solid $C D$ (is) to solid $C V$. Thus, $A B$ and $C D$ each have the same ratio to $C V$. Thus, solid $A B$ is equal to solid $C D$ [Prop. 5.9].


So, let the (straight-lines) standing up, $F E, B L, G A$, $K H, O N, D P, M C$, and $R Q$, not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through $E H$ and $N Q$ from points $F, G, B, K$, $O, M, R$, and $D$, and let them have joined the planes at (points) $S, T, U, V, W, X, Y$, and $a$ (respectively). And let the solids $F V$ and $O Y$ have been completed. In this case, also, I say that the solids $A B$ and $C D$ being equal, their bases are reciprocally proportional to their heights, and (so) as base $E H$ is to base $N Q$, so the height of solid $C D$ (is) to the height of solid $A B$.

Since solid $A B$ is equal to solid $C D$, but $A B$ is equal to $B T$. For they are on the same base $F K$, and (have) the same height [Props. 11.29, 11.30]. And solid $C D$ is equal is equal to $D X$. For, again, they are on the same base $R O$, and (have) the same height [Props. 11.29, 11.30]. Solid $B T$ is thus also equal to solid $D X$. Thus, as base $F K$ (is) to base $O R$, so the height of solid $D X$ (is) to the height of solid $B T$ (see first part of proposition). And base $F K$ (is) equal to base $E H$, and base $O R$ to $N Q$. Thus, as base $E H$ is to base $N Q$, so the height of solid $D X$ (is) to






 $\sigma \tau \varepsilon \rho \varepsilon o ̀ \nu \tau \widetilde{ب}$ Г $\Delta$ бєєрє $\widetilde{\varphi}$.















the height of solid $B T$. And solids $D X, B T$ are the same height as (solids) $D C, B A$ (respectively). Thus, as base $E H$ is to base $N Q$, so the height of solid $D C$ (is) to the height of solid $A B$. Thus, the bases of the parallelepiped solids $A B$ and $C D$ are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids $A B$ and $C D$ be reciprocally proportional to their heights, and (so) let base $E H$ be to base $N Q$, as the height of solid $C D$ (is) to the height of solid $A B$. I say that solid $A B$ is equal to solid $C D$.

For, with the same construction (as before), since as base $E H$ is to base $N Q$, so the height of solid $C D$ (is) to the height of solid $A B$, and base $E H$ (is) equal to base $F K$, and $N Q$ to $O R$, thus as base $F K$ is to base $O R$, so the height of solid $C D$ (is) to the height of solid $A B$. And solids $A B, C D$ are the same height as (solids) $B T$, $D X$ (respectively). Thus, as base $F K$ is to base $O R$, so the height of solid $D X$ (is) to the height of solid $B T$. Thus, the bases of the parallelepiped solids $B T$ and $D X$ are reciprocally proportional to their heights. Thus, solid $B T$ is equal to solid $D X$ (see first part of proposition). But, $B T$ is equal to $B A$. For [they are] on the same base $F K$, and (have) the same height [Props. 11.29, 11.30]. And solid $D X$ is equal to solid $D C$ [Props. 11.29, 11.30]. Thus, solid $A B$ is also equal to solid $C D$. (Which is) the very thing it was required to show.
$\dagger$ This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of
two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

## $\lambda \varepsilon^{\prime}$.

 риழ $\widetilde{\omega}$ 人




 $\pi \varepsilon \rho เ \varepsilon ́ \xi o \cup \sigma \iota \mu \varepsilon \tau \grave{\alpha} \tau \widetilde{\omega} \nu \mu \varepsilon \tau \varepsilon \omega ́ \rho \omega \nu$.
"Eбт $\omega \sigma \alpha \nu$ ठúo $\gamma \omega v i ́ \alpha l ~ \varepsilon u ̛ \vartheta u ́ \gamma p \alpha \mu \mu o l ~ " ̂ \sigma \alpha l ~ \alpha i ~ u ́ \pi o ̀ ~ В А Г, ~$




 $\dot{\alpha} \pi o ̀ \tau \tau \nu \mathrm{H}, \mathrm{M} \sigma \eta \mu \varepsilon i ́ \omega \nu$ ह̇ $\pi \grave{\imath} \tau \dot{\alpha} \delta \dot{\alpha} \tau \widetilde{\omega} \nu \mathrm{BA}, \mathrm{E} \Delta \mathrm{Z}$ है $\pi i ́ \pi \varepsilon \delta \alpha$




## Proposition 35

If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straightlines).

Let $B A C$ and $E D F$ be two equal rectilinear angles. And let the raised straight-lines $A G$ and $D M$ have been stood on points $A$ and $D$, containing equal angles respectively with the original straight-lines. (That is) $M D E$ (equal) to $G A B$, and $M D F$ (to) $G A C$. And let the random points $G$ and $M$ have been taken on $A G$ and $D M$ (respectively). And let the $G L$ and $M N$ have been drawn from points $G$ and $M$ perpendicular to the planes through

































$B A C$ and $E D F$ (respectively). And let them have joined the planes at points $L$ and $N$ (respectively). And let $L A$ and $N D$ have been joined. I say that angle $G A L$ is equal to angle $M D N$.


Let $A H$ be made equal to $D M$. And let $H K$ have been drawn through point $H$ parallel to $G L$. And $G L$ is perpendicular to the plane through $B A C$. Thus, $H K$ is also perpendicular to the plane through $B A C$ [Prop. 11.8]. And let $K C, N F, K B$, and $N E$ have been drawn from points $K$ and $N$ perpendicular to the straight-lines $A C$, $D F, A B$, and $D E$. And let $H C, C B, M F$, and $F E$ have been joined. Since the (square) on $H A$ is equal to the (sum of the squares) on $H K$ and $K A$ [Prop. 1.47], and the (sum of the squares) on $K C$ and $C A$ is equal to the (square) on $K A$ [Prop. 1.47], thus the (square) on $H A$ is equal to the (sum of the squares) on $H K, K C$, and $C A$. And the (square) on $H C$ is equal to the (sum of the squares) on $H K$ and $K C$ [Prop. 1.47]. Thus, the (square) on $H A$ is equal to the (sum of the squares) on $H C$ and $C A$. Thus, angle $H C A$ is a right-angle [Prop. 1.48]. So, for the same (reasons), angle DFM is also a right-angle. Thus, angle $A C H$ is equal to (angle) $D F M$. And $H A C$ is also equal to $M D F$. So, $M D F$ and $H A C$ are two triangles having two angles equal to two angles, respectively, and one side equal to one side(namely), that subtending one of the equal angles -(that is), $H A$ (equal) to $M D$. Thus, they will also have the remaining sides equal to the remaining sides, respectively [Prop. 1.26]. Thus, $A C$ is equal to $D F$. So, similarly, we can show that $A B$ is also equal to $D E$. Therefore, since $A C$ is equal to $D F$, and $A B$ to $D E$, so the two (straightlines) $C A$ and $A B$ are equal to the two (straight-lines) $F D$ and $D E$ (respectively). But, angle $C A B$ is also equal to angle $F D E$. Thus, base $B C$ is equal to base $E F$, and triangle $(A C B)$ to triangle ( $D F E$ ), and the remaining angles to the remaining angles (respectively) [Prop. 1.4].













 $\pi \rho о \tau \alpha ́ \sigma \varepsilon \omega \varsigma ~[о ̈ \pi \varepsilon \rho ~ \varepsilon ̋ \delta \varepsilon เ ~ \delta \varepsilon і ̈ \xi \alpha l] . ~$

## По́рьб $\alpha$.





 ठعï̌ $\alpha$.

## $\lambda \epsilon^{\prime}$.






Thus, angle $A C B$ (is) equal to $D F E$. And the right-angle $A C K$ is also equal to the right-angle $D F N$. Thus, the remainder $B C K$ is equal to the remainder $E F N$. So, for the same (reasons), $C B K$ is also equal to $F E N$. So, $B C K$ and $E F N$ are two triangles having two angles equal to two angles, respectively, and one side equal to one side-(namely), that by the equal angles-(that is), $B C$ (equal) to $E F$. Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus, $C K$ is equal to $F N$. And $A C$ (is) also equal to $D F$. So, the two (straight-lines) $A C$ and $C K$ are equal to the two (straight-lines) $D F$ and $F N$ (respectively). And they enclose right-angles. Thus, base $A K$ is equal to base $D N$ [Prop. 1.4]. And since $A H$ is equal to $D M$, the (square) on $A H$ is also equal to the (square) on $D M$. But, the the (sum of the squares) on $A K$ and $K H$ is equal to the (square) on $A H$. For angle $A K H$ (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on $D N$ and $N M$ (is) equal to the square on $D M$. For angle $D N M$ (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on $A K$ and $K H$ is equal to the (sum of the squares) on $D N$ and $N M$, of which the (square) on $A K$ is equal to the (square) on $D N$. Thus, the remaining (square) on $K H$ is equal to the (square) on $N M$. Thus, $H K$ (is) equal to $M N$. And since the two (straight-lines) $H A$ and $A K$ are equal to the two (straight-lines) $M D$ and $D N$, respectively, and base $H K$ was shown (to be) equal to base $M N$, angle $H A K$ is thus equal to angle $M D N$ [Prop. 1.8].

Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

## Corollary

So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

## Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).






 $\tau \widetilde{\omega} \nu \Delta \mathrm{E}, \mathrm{HE}, \mathrm{EZ}$, «גi $\sigma \cup \mu \pi \varepsilon \pi \lambda n \rho \omega \dot{\omega} \sigma \vartheta \omega$ tò $\mathrm{EK} \sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi \alpha-$





















 ӧл

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\lambda \zeta^{\prime} .
$$




Let $A, B$, and $C$ be three (continuously) proportional straight-lines, (such that) as $A$ (is) to $B$, so $B$ (is) to $C$. I say that the (parallelepiped) solid (formed) from $A, B$, and $C$ is equal to the equilateral solid on $B$ (which is) equiangular with the aforementioned (solid).

Let the solid angle at $E$, contained by $D E G, G E F$, and $F E D$, be set out. And let $D E, G E$, and $E F$ each be made equal to $B$. And let the parallelepiped solid $E K$ have been completed. And (let) $L M$ (be made) equal to $A$. And let the solid angle contained by $N L O$, $O L M$, and $M L N$ have been constructed on the straightline $L M$, and at the point $L$ on it, (so as to be) equal to the solid angle $E$ [Prop. 11.23]. And let $L O$ be made equal to $B$, and $L N$ equal to $C$. And since as $A$ (is) to $B$, so $B$ (is) to $C$, and $A$ (is) equal to $L M$, and $B$ to each of $L O$ and $E D$, and $C$ to $L N$, thus as $L M$ (is) to $E F$, so $D E$ (is) to $L N$. And (so) the sides around the equal angles $N L M$ and $D E F$ are reciprocally proportional. Thus, parallelogram $M N$ is equal to parallelogram $D F$ [Prop. 6.14]. And since the two plane rectilinear angles $D E F$ and $N L M$ are equal, and the raised straight-lines stood on them (at their apexes), $L O$ and $E G$, are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points $G$ and $O$ to the planes through $N L M$ and $D E F$ (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids $L H$ and $E K$ (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid $H L$ is equal to solid $E K$. And $L H$ is the solid (formed) from $A, B$, and $C$, and $E K$ the solid on $B$. Thus, the parallelepiped solid (formed) from $A, B$, and $C$ is equal to the equilateral solid on $B$ (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

## Proposition $37{ }^{\dagger}$

If four straight-lines are proportional then the similar,





 $\mathrm{H} \Theta, \dot{\omega} \varsigma \dot{\eta} \mathrm{AB} \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \Gamma \Delta$, oüt $\omega \varsigma \dot{\eta} \mathrm{EZ} \pi \rho o ̀ \varsigma \tau \dot{\eta} \nu \mathrm{H} \Theta$, $x \alpha \grave{l}$


 roòs tò NH.

 $\dot{\eta} \mathrm{AB} \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \Gamma \Delta$. $\delta ı \grave{\alpha} \tau \grave{\alpha} \alpha u ̉ \tau \grave{\alpha} \delta \grave{\eta}$ x $\alpha i ̀ ~ \tau o ̀ ~ M E ~ \pi p o ̀ \varsigma ~ \tau o ̀ ~ N H ~$
 ஸ́s $\dot{\eta} \mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \Delta$, oüt äp $\alpha$ tò $\mathrm{AK} \pi \rho o ̀ s ~ t o ̀ ~ \Lambda \Gamma, ~ o u ̛ t \omega \varsigma ~ t o ̀ ~ M E ~ \pi \rho o ̀ s ~ t o ̀ ~ N H . ~$
 oưt $\omega \varsigma$ tò ME $\sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi \rho o ̀ \varsigma ~ \tau o ̀ ~ N H \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~ o ̈ \tau \iota ~ \varepsilon ̇ \sigma t i \nu ~ ஸ ́ \omega \varsigma ~ \dot{\eta}$ AB عưvモĩ $\pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \Gamma \Delta$, oữ $\tau \omega \varsigma \dot{\eta} \mathrm{EZ} \pi \rho o ̀ s ~ \tau \grave{\nu} \nu \mathrm{H} \Theta$.
’Елєi $\gamma \grave{a} p \pi \alpha ́ \lambda \iota v ~ \tau o ̀ ~ K A ~ \pi p o ̀ s ~ t o ̀ ~ \Lambda \Gamma ~ \tau p ı \pi \lambda \alpha \sigma i ́ o v \alpha ~ \lambda o ́ \gamma o v ~$


 वैр $\dot{\eta} \mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Gamma \Delta$, oüt $\omega \varsigma \dot{\eta} \mathrm{EZ} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{H} \Theta$.


and similarly described, parallelepiped solids on them will also be proportional. And if the similar, and similarly described, parallelepiped solids on them are proportional then the straight-lines themselves will be proportional.


Let $A B, C D, E F$, and $G H$, be four proportional straight-lines, (such that) as $A B$ (is) to $C D$, so $E F$ (is) to $G H$. And let the similar, and similarly laid out, parallelepiped solids $K A, L C, M E$ and $N G$ have been described on $A B, C D, E F$, and $G H$ (respectively). I say that as $K A$ is to $L C$, so $M E$ (is) to $N G$.

For since the parallelepiped solid $K A$ is similar to $L C$, $K A$ thus has to $L C$ the cubed ratio that $A B$ (has) to $C D$ [Prop. 11.33]. So, for the same (reasons), $M E$ also has to $N G$ the cubed ratio that $E F$ (has) to $G H$ [Prop. 11.33]. And since as $A B$ is to $C D$, so $E F$ (is) to $G H$, thus, also, as $A K$ (is) to $L C$, so $M E$ (is) to $N G$.

And so let solid $A K$ be to solid $L C$, as solid $M E$ (is) to $N G$. I say that as straight-line $A B$ is to $C D$, so $E F$ (is) to $G H$.

For, again, since $K A$ has to $L C$ the cubed ratio that $A B$ (has) to $C D$ [Prop. 11.33], and $M E$ also has to $N G$ the cubed ratio that $E F$ (has) to $G H$ [Prop. 11.33], and as $K A$ is to $L C$, so $M E$ (is) to $N G$, thus, also, as $A B$ (is) to $C D$, so $E F$ (is) to $G H$.

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show.
${ }^{\dagger}$ This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and vice versa.

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\lambda \eta^{\prime}
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 $\dot{\alpha} \lambda \lambda \dot{\eta} \lambda \alpha \varsigma$.

## Proposition 38

If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.


Kúßou ү $\alpha$ р тoũ $A Z \tau \widetilde{\omega} \nu \alpha \dot{\alpha} \pi \varepsilon \nu \alpha \nu \tau i ́ o v ~ \varepsilon ̇ \pi \iota \tau \varepsilon ́ \delta \omega \nu ~ \tau \widetilde{\omega} \nu \Gamma Z$,



访 T $\Sigma, \dot{\eta}$ ס $̇ \Delta \mathrm{~T} \tau \tilde{n} \mathrm{TH}$.
















 $\tau \grave{\eta} \nu \Delta \Upsilon \tau \tilde{n} \mathrm{H} \Sigma \cdot \dot{\eta} \mu i \sigma \varepsilon \iota \alpha l$ үáp عíбl $\tau \widetilde{\omega} \nu \Delta \mathrm{E}, \mathrm{BH} \cdot x \alpha \grave{\imath} \tau \grave{\alpha} \varsigma$
 $\mu \varepsilon ̀ v \Delta \mathrm{~T} \tau \tilde{n} \mathrm{TH}, \dot{\eta} \delta \check{~} \Upsilon \mathrm{\Upsilon T} \tau \tilde{n} \mathrm{~T} \Sigma$.






For let the opposite planes $C F$ and $A H$ of the cube $A F$ have been cut in half at the points $K, L, M, N, O$, $Q, P$, and $R$. And let the planes $K N$ and $O R$ have been produced through the pieces. And let $U S$ be the common section of the planes, and $D G$ the diameter of cube $A F$. I say that $U T$ is equal to $T S$, and $D T$ to $T G$.

For let $D U, U E, B S$, and $S G$ have been joined. And since $D O$ is parallel to $P E$, the alternate angles $D O U$ and $U P E$ are equal to one another [Prop. 1.29]. And since $D O$ is equal to $P E$, and $O U$ to $U P$, and they contain equal angles, base $D U$ is thus equal to base $U E$, and triangle $D O U$ is equal to triangle $P U E$, and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle $O U D$ (is) equal to angle $P U E$. So, for this (reason), $D U E$ is a straight-line [Prop. 1.14]. So, for the same (reason), $B S G$ is also a straight-line, and $B S$ equal to $S G$. And since $C A$ is equal and parallel to $D B$, but $C A$ is also equal and parallel to $E G, D B$ is thus also equal and parallel to $E G$ [Prop. 11.9]. And the straightlines $D E$ and $B G$ join them. $D E$ is thus parallel to $B G$ [Prop. 1.33]. Thus, angle $E D T$ (is) equal to $B G T$. For (they are) alternate [Prop. 1.29]. And (angle) $D T U$ (is equal) to $G T S$ [Prop. 1.15]. So, $D T U$ and $G T S$ are two triangles having two angles equal to two angles, and one side equal to one side-(namely), that subtended by one of the equal angles-(that is), $D U$ (equal) to $G S$. For they are halves of $D E$ and $B G$ (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, $D T$ (is) equal to $T G$, and $U T$ to $T S$.

Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

## $\lambda \vartheta^{\prime}$.

























## Proposition 39

If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.


Let $A B C D E F$ and $G H K L M N$ be two equal height prisms, and let the former have the parallelogram $A F$, and the latter the triangle $G H K$, as a base. And let parallelogram $A F$ be twice triangle $G H K$. I say that prism $A B C D E F$ is equal to prism $G H K L M N$.

For let the solids $A O$ and $G P$ have been completed. Since parallelogram $A F$ is double triangle $G H K$, and parallelogram $H K$ is also double triangle $G H K$ [Prop. 1.34], parallelogram $A F$ is thus equal to parallelogram $H K$. And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid $A O$ is equal to solid $G P$. And prism $A B C D E F$ is half of solid $A O$, and prism GHKLMN half of solid GP [Prop. 11.28]. Prism $A B C D E F$ is thus equal to prism $G H K L M N$.

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.

## ELEMENTS BOOK 12

Proportional Stereometry

[^34] attributed to Eudoxus of Cnidus.
$\alpha^{\prime}$.
 $\dot{\omega} \varsigma \tau \dot{\alpha} \alpha \dot{\alpha} \sigma ̀ \tau \tau \nu \nu \delta \alpha \mu \varepsilon ́ \tau \rho \omega \nu \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \alpha$.



















 HN, oút $\omega \varsigma \dot{\eta}$ BA $\pi p o ̀ s ~ \tau \eta ̀ \nu ~ H Z . ~ \alpha ̀ \lambda \lambda \grave{\alpha} ~ \tau o u ̃ ~ \mu \varepsilon ̀ v ~ \tau n ̃ s ~ B M ~$


 $\pi \rho o ̀ s ~ \tau o ̀ ~ Z H \Theta K \Lambda ~ \pi o \lambda u ́ \gamma \omega v o v \cdot ~ \varkappa \alpha u ̀ ~ c i ́ s ~ \alpha " p \alpha ~ \tau o ̀ ~ \alpha ́ \alpha o ̀ ~ \tau n ̃ s ~ B M ~$ $\tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \circ \nu \pi \rho o ̀ \varsigma ~ \tau o ̀ ~ \alpha ́ \alpha o ̀ ~ \tau \tilde{\eta} \varsigma ~ H N ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o \nu, ~ o u ̛ \tau \omega \varsigma ~ \tau o ̀ ~$



$$
\beta^{\prime}
$$
 $\tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu \alpha$.


## Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).


Let $A B C$ and $F G H$ be circles, and let $A B C D E$ and $F G H K L$ be similar polygons (inscribed) in them (respectively), and let $B M$ and $G N$ be the diameters of the circles (respectively). I say that as the square on $B M$ is to the square on $G N$, so polygon $A B C D E$ (is) to polygon FGHKL.

For let $B E, A M, G L$, and $F N$ have been joined. And since polygon $A B C D E$ (is) similar to polygon $F G H K L$, angle $B A E$ is also equal to (angle) $G F L$, and as $B A$ is to $A E$, so $G F$ (is) to $F L$ [Def. 6.1]. So, $B A E$ and $G F L$ are two triangles having one angle equal to one angle, (namely), $B A E$ (equal) to $G F L$, and the sides around the equal angles proportional. Triangle $A B E$ is thus equiangular with triangle $F G L$ [Prop. 6.6]. Thus, angle $A E B$ is equal to (angle) $F L G$. But, $A E B$ is equal to $A M B$, and $F L G$ to $F N G$, for they stand on the same circumference [Prop. 3.27]. Thus, $A M B$ is also equal to $F N G$. And the right-angle $B A M$ is also equal to the right-angle $G F N$ [Prop. 3.31]. Thus, the remaining (angle) is also equal to the remaining (angle) [Prop. 1.32]. Thus, triangle $A B M$ is equiangular with triangle $F G N$. Thus, proportionally, as $B M$ is to $G N$, so $B A$ (is) to $G F$ [Prop. 6.4]. But, the (ratio) of the square on $B M$ to the square on $G N$ is the square of the ratio of $B M$ to $G N$, and the (ratio) of polygon $A B C D E$ to polygon $F G H K L$ is the square of the (ratio) of $B A$ to $G F$ [Prop. 6.20]. And, thus, as the square on $B M$ (is) to the square on $G N$, so polygon $A B C D E$ (is) to polygon $F G H K L$.

Thus, similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles). (Which is) the very thing it was required to show.

## Proposition 2

Circles are to one another as the squares on (their) diameters.

Let $A B C D$ and $E F G H$ be circles, and [let] $B D$ and

 $\pi \rho o ̀ \varsigma ~ \tau o ̀ ~ \alpha ́ \pi o ̀ ~ \tau n ̃ \varsigma ~ Z \Theta ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o v . ~$




 $\chi \omega \rho i ́ o \nu$ そे $\pi \rho o ̀ s ~ \mu \varepsilon і ̈ \zeta о \nu . ~ ह ै \sigma \tau \omega ~ \pi \rho o ́ \tau \varepsilon \rho o \nu ~ \pi \rho o ̀ s ~ ह ै \lambda ~ \alpha \sigma \sigma o \nu ~ t o ̀ ~$








 $\tau \grave{\alpha} \mathrm{K}, \Lambda, \mathrm{M}, \mathrm{N} \quad \sigma \eta \mu \varepsilon i ̃ \alpha, \chi \alpha i ̀ ~ \varepsilon ̇ \pi \varepsilon \zeta \varepsilon u ́ \chi \vartheta \omega \sigma \alpha \nu$ 人i EK，KZ，


 $\mathrm{K}, \Lambda, \mathrm{M}, \mathrm{N}$ бпиعí $\omega \nu$ ह́ $\varphi \alpha \pi \tau о \mu \varepsilon ́ v a s ~ \tau о \tilde{v} ~ \chi \cup ́ x \lambda о \cup ~ \alpha ̉ \gamma \alpha ́ \gamma \omega \mu \varepsilon \nu$
 $\pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu \alpha$ ，ย̇ห $\alpha \sigma \tau o \nu \tau \widetilde{\omega} \nu \mathrm{EKZ}, \mathrm{Z} \Lambda \mathrm{H}, \mathrm{HM} \Theta, ~ \Theta \mathrm{NE}$
 $\alpha \dot{\alpha} \lambda \dot{\alpha}$ тò $x \alpha \vartheta^{\prime}$ हं $\alpha \cup \tau o ̀ ~ \tau \mu \tilde{\eta} \mu \alpha$ है入 $\lambda \tau \tau o ́ v$ ह̇ $\sigma \tau \iota$ тоũ $\pi \alpha \rho \alpha \lambda \lambda \eta$－






$F H$［be］their diameters．I say that as circle $A B C D$ is to circle $E F G H$ ，so the square on $B D$（is）to the square on FH．


For if the circle $A B C D$ is not to the（circle）$E F G H$ ， as the square on $B D$（is）to the（square）on $F H$ ，then as the（square）on $B D$（is）to the（square）on $F H$ ，so circle $A B C D$ will be to some area either less than，or greater than，circle $E F G H$ ．Let it，first of all，be（in that ratio）to （some）lesser（area），$S$ ．And let the square $E F G H$ have been inscribed in circle $E F G H$［Prop．4．6］．So the in－ scribed square is greater than half of circle $E F G H$ ，inas－ much as if we draw tangents to the circle through the points $E, F, G$ ，and $H$ ，then square $E F G H$ is half of the square circumscribed about the circle［Prop．1．47］，and the circle is less than the circumscribed square．Hence， the inscribed square $E F G H$ is greater than half of cir－ cle $E F G H$ ．Let the circumferences $E F, F G, G H$ ，and $H E$ have been cut in half at points $K, L, M$ ，and $N$ （respectively），and let $E K, K F, F L, L G, G M, M H$ ， $H N$ ，and $N E$ have been joined．And，thus，each of the triangles $E K F, F L G, G M H$ ，and $H N E$ is greater than half of the segment of the circle about it，inasmuch as if we draw tangents to the circle through points $K$ ， $L, M$ ，and $N$ ，and complete the parallelograms on the straight－lines $E F, F G, G H$ ，and $H E$ ，then each of the triangles $E K F, F L G, G M H$ ，and $H N E$ will be half of the parallelogram about it，but the segment about it is less than the parallelogram．Hence，each of the tri－ angles $E K F, F L G, G M H$ ，and $H N E$ is greater than half of the segment of the circle about it．So，by cutting the circumferences remaining behind in half，and joining straight－lines，and doing this continually，we will（even－





 $\mathrm{HM}, \mathrm{M} \Theta, ~ \Theta \mathrm{~N}, \mathrm{NE} \tau \mu \dot{n} \mu \alpha \tau \alpha$ тõ̃ $\mathrm{EZH} \mathrm{\Theta}$ xúx入ou ह่̇ $\lambda \dot{\alpha} \tau \tau o v \alpha$
入oぃтòv äp $\alpha$ тò EKZイHM






 $\chi$（

 тò EKZムHMӨN ro入úr $\omega v o v$ ．$\mu \varepsilon i \zeta \omega \nu$ dè ò АВГ $\Delta$ xúx入os




 ஸ́s tò $\dot{\alpha} \pi o ̀ ~ Z \Theta ~ \pi \rho o ̀ s ~ t o ̀ ~ \alpha ́ \pi o ̀ ~ B \Delta, ~ o u ̛ \tau \omega \varsigma ~ o ́ ~ E Z H \Theta ~ x u ́ x \lambda o s ~$ трòs है入 $\lambda \sigma \sigma o ́ v ~ \tau \iota ~ \tau o u ̃ ~ А В Г \Delta ~ x u ́ x \lambda o u ~ \chi \omega p i ́ o v . ~$

 хи́жлои $\chi$ шрі́о。．


 $\Sigma \chi$ крíov $\pi \rho o ̀ s ~ t o ̀ v ~ А В Г \Delta ~ x u ́ x \lambda o v, ~ o u ̈ \tau \omega \varsigma ~ o ́ ~ E Z H \Theta ~ x u ́ x \lambda о \varsigma ~$




 ЕZH xúx
 $\mathrm{Z} \Theta$ ，оӥт $\omega \varsigma$ ó $\mathrm{AB} \mathrm{\Gamma} \Delta$ xúx入os тpòs тòv $\mathrm{EZH} \Theta$ xúx入ov．


tually）leave behind some segments of the circle whose （sum）will be less than the excess by which circle $E F G H$ exceeds the area $S$ ．For we showed in the first theo－ rem of the tenth book that if two unequal magnitudes are laid out，and if（a part）greater than a half is sub－ tracted from the greater，and（if from）the remainder（a part）greater than a half（is subtracted），and this hap－ pens continually，then some magnitude will（eventually） be left which will be less than the lesser laid out mag－ nitude［Prop．10．1］．Therefore，let the（segments）have been left，and let the（sum of the）segments of the circle $E F G H$ on $E K, K F, F L, L G, G M, M H, H N$ ，and $N E$ be less than the excess by which circle $E F G H$ exceeds area $S$ ．Thus，the remaining polygon $E K F L G M H N$ is greater than area $S$ ．And let the polygon $A O B P C Q D R$ ， similar to the polygon EKFLGMHN，have been in－ scribed in circle $A B C D$ ．Thus，as the square on $B D$ is to the square on $F H$ ，so polygon $A O B P C Q D R$（is）to polygon EKFLGMHN［Prop．12．1］．But，also，as the square on $B D$（is）to the square on $F H$ ，so circle $A B C D$ （is）to area $S$ ．And，thus，as circle $A B C D$（is）to area $S$ ， so polygon $A O B P G Q D R$（is）to polygon $E K F L G M H N$ ［Prop．5．11］．Thus，alternately，as circle $A B C D$（is）to the polygon（inscribed）within it，so area $S$（is）to poly－ gon EKFLGMHN［Prop．5．16］．And circle $A B C D$（is） greater than the polygon（inscribed）within it．Thus，area $S$ is also greater than polygon $E K F L G M H N$ ．But，（it is） also less．The very thing is impossible．Thus，the square on $B D$ is not to the（square）on $F H$ ，as circle $A B C D$（is） to some area less than circle $E F G H$ ．So，similarly，we can show that the（square）on $F H$（is）not to the（square）on $B D$ as circle $E F G H$（is）to some area less than circle $A B C D$ either．

So，I say that neither（is）the（square）on $B D$ to the（square）on $F H$ ，as circle $A B C D$（is）to some area greater than circle $E F G H$ ．

For，if possible，let it be（in that ratio）to（some） greater（area），$S$ ．Thus，inversely，as the square on $F H$ ［is］to the（square）on $D B$ ，so area $S$（is）to circle $A B C D$ ［Prop． 5.7 corr．］．But，as area $S$（is）to circle $A B C D$ ，so circle $E F G H$（is）to some area less than circle $A B C D$ （see lemma）．And，thus，as the（square）on $F H$（is）to the（square）on $B D$ ，so circle $E F G H$（is）to some area less than circle $A B C D$［Prop．5．11］．The very thing was shown（to be）impossible．Thus，as the square on $B D$ is to the（square）on $F H$ ，so circle $A B C D$（is）not to some area greater than circle $E F G H$ ．And it was shown that neither（is it in that ratio）to（some）lesser（area）．Thus， as the square on $B D$ is to the（square）on $F H$ ，so circle $A B C D$（is）to circle $E F G H$ ．

Thus，circles are to one another as the squares on
$\Lambda \tilde{\eta} \mu \mu \alpha$.


 $\chi$ (upíov.





 «pòs tò T $\chi$ copíov. $\mu$ हĭ





$$
\gamma^{\prime}
$$
















(their) diameters. (Which is) the very thing it was required to show.

## Lemma

So, I say that, area $S$ being greater than circle $E F G H$, as area $S$ is to circle $A B C D$, so circle $E F G H$ (is) to some area less than circle $A B C D$.

For let it have been contrived that as area $S$ (is) to circle $A B C D$, so circle $E F G H$ (is) to area $T$. I say that area $T$ is less than circle $A B C D$. For since as area $S$ is to circle $A B C D$, so circle $E F G H$ (is) to area $T$, alternately, as area $S$ is to circle $E F G H$, so circle $A B C D$ (is) to area $T$ [Prop. 5.16]. And area $S$ (is) greater than circle $E F G H$. Thus, circle $A B C D$ (is) also greater than area $T$ [Prop. 5.14]. Hence, as area $S$ is to circle $A B C D$, so circle $E F G H$ (is) to some area less than circle $A B C D$. (Which is) the very thing it was required to show.

## Proposition 3

Any pyramid having a triangular base is divided into two pyramids having triangular bases (which are) equal, similar to one another, and [similar] to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.


Let there be a pyramid whose base is triangle $A B C$, and (whose) apex (is) point $D$. I say that pyramid $A B C D$ is divided into two pyramids having triangular bases (which are) equal to one another, and similar to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.

For let $A B, B C, C A, A D, D B$, and $D C$ have been cut in half at points $E, F, G, H, K$, and $L$ (respectively). And let $H E, E G, G H, H K, K L, L H, K F$, and $F G$ have been joined. Since $A E$ is equal to $E B$, and $A H$ to $D H$,



















 тои̃ $\mathrm{A} \Delta \mathrm{B} \pi \alpha \rho \dot{\alpha} \mu i \alpha \nu \tau \widetilde{\omega} \nu \pi \lambda \varepsilon \cup \cup \widetilde{\omega} \nu \tau \dot{\eta} \nu \mathrm{AB} \tilde{\eta}^{\prime} \kappa \tau \alpha \iota \dot{\eta} \Theta K$,










 ó $\mu o i ́ \alpha ~ \varepsilon ̇ \sigma \tau i ̀ ~ \pi u p \alpha \mu i ́ \delta ı, ~ \eta ั ऽ ~ \beta \alpha ́ \sigma ı s ~ \mu \varepsilon ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ \Theta K \Lambda ~ \tau p i ́ \gamma \omega \nu o v, ~$








 $\lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu$ тои̃ трıүஸ́vou, "ै $\sigma \alpha$ ह̇ $\sigma i ̀ ~ \tau \grave{\alpha} \pi \rho i ́ \sigma \mu \alpha \tau \alpha$, "̂$\sigma o v$
 $\tau \widetilde{\omega} \nu \mathrm{BKZ}, \mathrm{E} \Theta \mathrm{H}, \tau \rho \stackrel{\omega}{\nu} \delta$ 文 $\tau \alpha p \alpha \lambda \lambda \eta \lambda o \gamma \rho \alpha ́ \mu \mu \omega \nu \tau \widetilde{\omega} \nu \mathrm{EBZH}$,

 $\tau \widetilde{\omega} \nu \mathrm{KZГ} \mathrm{\Lambda}, \Lambda Г Н \Theta, ~ \Theta К Z Н . ~ x \alpha i ̀ ~ \varphi \alpha v \varepsilon \rho o ́ v, ~ o ̋ \tau \iota ~ \varepsilon ́ x \alpha ́ \tau p o \nu ~ \tau \widetilde{\omega} \nu$ $\pi \rho เ \sigma \mu \alpha ́ \tau \omega \nu$, oũ $\tau \varepsilon \beta \alpha ́ \sigma \iota \varsigma ~ \tau o ̀ ~ E B Z H ~ \pi \alpha р \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu, \alpha \dot{\alpha} \pi \varepsilon-$


$E H$ is thus parallel to $D B$ [Prop. 6.2]. So, for the same (reasons), $H K$ is also parallel to $A B$. Thus, $H E B K$ is a parallelogram. Thus, $H K$ is equal to $E B$ [Prop. 1.34]. But, $E B$ is equal to $E A$. Thus, $A E$ is also equal to $H K$. And $A H$ is also equal to $H D$. So the two (straight-lines) $E A$ and $A H$ are equal to the two (straight-lines) $K H$ and $H D$, respectively. And angle $E A H$ (is) equal to angle $K H D$ [Prop. 1.29]. Thus, base $E H$ is equal to base $K D$ [Prop. 1.4]. Thus, triangle $A E H$ is equal and similar to triangle $H K D$ [Prop. 1.4]. So, for the same (reasons), triangle $A H G$ is also equal and similar to triangle $H L D$. And since $E H$ and $H G$ are two straight-lines joining one another (which are respectively) parallel to two straight-lines joining one another, $K D$ and $D L$, not being in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle $E H G$ is equal to angle $K D L$. And since the two straight-lines $E H$ and $H G$ are equal to the two straight-lines $K D$ and $D L$, respectively, and angle $E H G$ is equal to angle $K D L$, base $E G$ [is] thus equal to base $K L$ [Prop. 1.4]. Thus, triangle $E H G$ is equal and similar to triangle $K D L$. So, for the same (reasons), triangle $A E G$ is also equal and similar to triangle $H K L$. Thus, the pyramid whose base is triangle $A E G$, and apex the point $H$, is equal and similar to the pyramid whose base is triangle $H K L$, and apex the point $D$ [Def. 11.10]. And since $H K$ has been drawn parallel to one of the sides, $A B$, of triangle $A D B$, triangle $A D B$ is equiangular to triangle $D H K$ [Prop. 1.29], and they have proportional sides. Thus, triangle $A D B$ is similar to triangle $D H K$ [Def. 6.1]. So, for the same (reasons), triangle $D B C$ is also similar to triangle $D K L$, and $A D C$ to $D L H$. And since two straight-lines joining one another, $B A$ and $A C$, are parallel to two straight-lines joining one another, $K H$ and $H L$, not in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle $B A C$ is equal to (angle) $K H L$. And as $B A$ is to $A C$, so $K H$ (is) to $H L$. Thus, triangle $A B C$ is similar to triangle $H K L$ [Prop. 6.6]. And, thus, the pyramid whose base is triangle $A B C$, and apex the point $D$, is similar to the pyramid whose base is triangle $H K L$, and apex the point $D$ [Def. 11.9]. But, the pyramid whose base [is] triangle $H K L$, and apex the point $D$, was shown (to be) similar to the pyramid whose base is triangle $A E G$, and apex the point $H$. Thus, each of the pyramids $A E G H$ and $H K L D$ is similar to the whole pyramid $A B C D$.

And since $B F$ is equal to $F C$, parallelogram $E B F G$ is double triangle $G F C$ [Prop. 1.41]. And since, if two prisms (have) equal heights, and the former has a parallelogram as a base, and the latter a triangle, and the parallelogram (is) double the triangle, then the prisms are equal [Prop. 11.39], the prism contained by the two

 т̀̀s EZ, EK عưvعías, тò $\mu \varepsilon ̀ v ~ \pi p i ́ \sigma \mu \alpha, ~ o u ̃ ~ \beta \alpha ́ \sigma ı s ~ \tau o ̀ ~ E B Z H ~ \pi \alpha-~$














 $\pi \cup p \alpha \mu i ́ \delta \omega \nu, \check{\sim} \nu \beta \alpha ́ \sigma \varepsilon ı \varsigma \mu \varepsilon ̀ v \tau \alpha ̀ ~ A E H, ~ \Theta K \Lambda ~ \tau р i ́ \gamma \omega v \alpha, ~ x о р и \varphi \alpha \grave{~}$ ठ̀̀ $\tau \grave{\alpha} \Theta, \Delta$ бпиعĩ $\alpha$.






## $\delta^{\prime}$.





 $\pi \rho i ́ \sigma \mu \alpha \tau \alpha \pi \alpha ́ \nu \tau \alpha \pi \rho o ̀ \varsigma ~ \tau \alpha ̀ ~ \varepsilon ُ v ~ \tau n ̃ ~ \varepsilon ̇ \tau \alpha ́ \rho \alpha ~ \pi \cup p \alpha \mu i ́ o l ~ \pi \rho i ́ \sigma \mu \alpha \tau \alpha$ $\pi \alpha ́ \nu \tau \alpha$ iбоо $\lambda \eta \vartheta \tilde{\eta}$.




triangles $B K F$ and $E H G$, and the three parallelograms $E B F G, E B K H$, and $H K F G$, is thus equal to the prism contained by the two triangles $G F C$ and $H K L$, and the three parallelograms $K F C L, L C G H$, and $H K F G$. And (it is) clear that each of the prisms whose base (is) parallelogram $E B F G$, and opposite (side) straight-line $H K$, and whose base (is) triangle $G F C$, and opposite (plane) triangle $H K L$, is greater than each of the pyramids whose bases are triangles $A E G$ and $H K L$, and apexes the points $H$ and $D$ (respectively), inasmuch as, if we [also] join the straight-lines $E F$ and $E K$ then the prism whose base (is) parallelogram $E B F G$, and opposite (side) straight-line $H K$, is greater than the pyramid whose base (is) triangle $E B F$, and apex the point $K$. But the pyramid whose base (is) triangle $E B F$, and apex the point $K$, is equal to the pyramid whose base is triangle $A E G$, and apex point $H$. For they are contained by equal and similar planes. And, hence, the prism whose base (is) parallelogram $E B F G$, and opposite (side) straightline $H K$, is greater than the pyramid whose base (is) triangle $A E G$, and apex the point $H$. And the prism whose base is parallelogram $E B F G$, and opposite (side) straight-line $H K$, (is) equal to the prism whose base (is) triangle $G F C$, and opposite (plane) triangle $H K L$. And the pyramid whose base (is) triangle $A E G$, and apex the point $H$, is equal to the pyramid whose base (is) triangle $H K L$, and apex the point $D$. Thus, the (sum of the) aforementioned two prisms is greater than the (sum of the) aforementioned two pyramids, whose bases (are) triangles $A E G$ and $H K L$, and apexes the points $H$ and $D$ (respectively).

Thus, the whole pyramid, whose base (is) triangle $A B C$, and apex the point $D$, has been divided into two pyramids (which are) equal to one another [and similar to the whole], and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid. (Which is) the very thing it was required to show.

## Proposition 4

If there are two pyramids with the same height, having trianglular bases, and each of them is divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms then as the base of one pyramid (is) to the base of the other pyramid, so (the sum of) all the prisms in one pyramid will be to (the sum of all) the equal number of prisms in the other pyramid.

Let there be two pyramids with the same height, having the triangular bases $A B C$ and $D E F$, (with) apexes the points $G$ and $H$ (respectively). And let each of them have been divided into two pyramids equal to one an-

 $\pi \cup р \alpha \mu i ́ \sigma i \quad \pi \rho i \sigma \mu \alpha \tau \alpha$ iбо $\pi \lambda \eta \vartheta \tilde{\eta}$.















 tò OMN , $\pi \rho o ̀ \varsigma ~ \tau o ̀ ~ \pi \rho i ́ \sigma \mu \alpha, ~ o u ̛ ~ \beta \alpha ́ \sigma \iota \varsigma ~ \mu غ ̀ \nu ~ \tau o ̀ ~ P \Phi Z ~ \tau p i ́ \gamma \omega \nu o v, ~$


 $\mu \varepsilon ̀ v ~ \tau o ̀ ~ P \Phi Z ~ \tau p i ́ \gamma \omega \nu o v, ~ \alpha ́ \pi \varepsilon v \alpha \nu \tau i ́ o v ~ \delta e ̀ ~ \tau o ̀ ~ \Sigma T \Upsilon . ~ © ́ s ~ \delta ~ \delta e ̀ ~ \tau \alpha ̀ ~$
 $\mu \varepsilon ̀ v ~ t o ̀ ~ K B \Xi \Lambda ~ \pi \alpha р \alpha \lambda \lambda \eta \lambda o ́ \gamma p \alpha \mu \mu o \nu, ~ \alpha ́ \pi \varepsilon v \alpha \nu \tau i ́ o v ~ \delta غ ̀ ~ \dot{\eta} \mathrm{OM}$ عủvะĩ $\pi$,










other, and similar to the whole, and into two equal prisms [Prop. 12.3]. I say that as base $A B C$ is to base $D E F$, so (the sum of) all the prisms in pyramid $A B C G$ (is) to (the sum of) all the equal number of prisms in pyramid DEFH.


For since $B O$ is equal to $O C$, and $A L$ to $L C, L O$ is thus parallel to $A B$, and triangle $A B C$ similar to triangle $L O C$ [Prop. 12.3]. So, for the same (reasons), triangle $D E F$ is also similar to triangle $R V F$. And since $B C$ is double $C O$, and $E F$ (double) $F V$, thus as $B C$ (is) to $C O$, so $E F$ (is) to $F V$. And the similar, and similarly laid out, rectilinear (figures) $A B C$ and $L O C$ have been described on $B C$ and $C O$ (respectively), and the similar, and similarly laid out, [rectilinear] (figures) $D E F$ and $R V F$ on $E F$ and $F V$ (respectively). Thus, as triangle $A B C$ is to triangle $L O C$, so triangle $D E F$ (is) to triangle $R V F$ [Prop. 6.22]. Thus, alternately, as triangle $A B C$ is to [triangle] $D E F$, so [triangle] $L O C$ (is) to triangle $R V F$ [Prop. 5.16]. But, as triangle $L O C$ (is) to triangle $R V F$, so the prism whose base [is] triangle $L O C$, and opposite (plane) $P M N$, (is) to the prism whose base (is) triangle $R V F$, and opposite (plane) $S T U$ (see lemma). And, thus, as triangle $A B C$ (is) to triangle $D E F$, so the prism whose base (is) triangle $L O C$, and opposite (plane) $P M N$, (is) to the prism whose base (is) triangle $R V F$, and opposite (plane) $S T U$. And as the aforementioned prisms (are) to one another, so the prism whose base (is) parallelogram $K B O L$, and opposite (side) straight-line $P M$, (is) to the prism whose base (is) parallelogram $Q E V R$, and opposite (side) straightline $S T$ [Props. 11.39, 12.3]. Thus, also, (is) the (sum of the) two prisms-that whose base (is) parallelogram $K B O L$, and opposite (side) $P M$, and that whose base (is) $L O C$, and opposite (plane) $P M N$-to (the sum of) the (two) prisms-that whose base (is) $Q E V R$, and opposite (side) straight-line $S T$, and that whose base (is) triangle $R V F$, and opposite (plane) $S T U$ [Prop. 5.12]. And, thus, as base $A B C$ (is) to base $D E F$, so the (sum

OMN $\beta \alpha \dot{\alpha} \sigma \iota \varsigma ~ \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ \Sigma T \Upsilon ~ \beta \alpha ́ \sigma เ \nu, ~ o u ̛ \tau \omega \varsigma ~ \tau \grave{\alpha} \varepsilon ่ \nu \tau n ̃ ~ O M N H ~$ $\pi \cup p \alpha \mu i ́ \delta l$ ठúo $\pi p i ́ \sigma \mu \alpha \tau \alpha \pi \rho o ̀ s ~ \tau \alpha ̀ ~ \varepsilon ้ v ~ \tau n ̃ ~ \Sigma T \Upsilon \Theta ~ \pi u p \alpha \mu i ́ \delta ı ~ \delta u ́ o ~$



 $\pi \rho i ́ \sigma \mu \alpha \tau \alpha \pi \rho o ̀ s ~ \tau \alpha ̀ ~ \tau \varepsilon ́ \sigma \sigma \alpha p \alpha ~ \pi \rho i ́ \sigma \mu \alpha \tau \alpha$. ó $\mu o i ́ \omega s$ ठ气̀ $\chi \ddot{\alpha} \nu \tau \alpha ̆ \varsigma$






## $\Lambda \tilde{n} \mu \mu \alpha$.









 тoùs $\alpha u$ toùs $\lambda o ́ \gamma o u s ~ \tau \mu \eta \vartheta ウ ́ n o v \tau \alpha ı . ~ x \alpha i ̀ ~ \tau \varepsilon ́ \tau \mu \eta \tau \alpha ı ~ \dot{\eta}$ НГ ठí $\chi \alpha$













of the first) aforementioned two prisms (is) to the (sum of the second) aforementioned two prisms.

And, similarly, if pyramids $P M N G$ and $S T U H$ are divided into two prisms, and two pyramids, as base PMN (is) to base $S T U$, so (the sum of) the two prisms in pyramid $P M N G$ will be to (the sum of) the two prisms in pyramid $S T U H$. But, as base $P M N$ (is) to base $S T U$, so base $A B C$ (is) to base $D E F$. For the triangles $P M N$ and $S T U$ (are) equal to $L O C$ and $R V F$, respectively. And, thus, as base $A B C$ (is) to base $D E F$, so (the sum of) the four prisms (is) to (the sum of) the four prisms [Prop. 5.12]. So, similarly, even if we divide the pyramids left behind into two pyramids and into two prisms, as base $A B C$ (is) to base $D E F$, so (the sum of) all the prisms in pyramid $A B C G$ will be to (the sum of) all the equal number of prisms in pyramid $D E F H$. (Which is) the very thing it was required to show.

## Lemma

And one may show, as follows, that as triangle $L O C$ is to triangle $R V F$, so the prism whose base (is) triangle $L O C$, and opposite (plane) $P M N$, (is) to the prism whose base (is) [triangle] $R V F$, and opposite (plane) STU.

For, in the same figure, let perpendiculars have been conceived (drawn) from (points) $G$ and $H$ to the planes $A B C$ and $D E F$ (respectively). These clearly turn out to be equal, on account of the pyramids being assumed (to be) of equal height. And since two straight-lines, $G C$ and the perpendicular from $G$, are cut by the parallel planes $A B C$ and $P M N$ they will be cut in the same ratios [Prop. 11.17]. And $G C$ was cut in half by the plane $P M N$ at $N$. Thus, the perpendicular from $G$ to the plane $A B C$ will also be cut in half by the plane $P M N$. So, for the same (reasons), the perpendicular from $H$ to the plane $D E F$ will also be cut in half by the plane $S T U$. And the perpendiculars from $G$ and $H$ to the planes $A B C$ and $D E F$ (respectively) are equal. Thus, the perpendiculars from the triangles $P M N$ and $S T U$ to $A B C$ and $D E F$ (respectively, are) also equal. Thus, the prisms whose bases are triangles $L O C$ and $R V F$, and opposite (sides) $P M N$ and $S T U$ (respectively), [are] of equal height. And, hence, the parallelepiped solids described on the aforementioned prisms [are] of equal height and (are) to one another as their bases [Prop. 11.32]. Likewise, the halves (of the solids) [Prop. 11.28]. Thus, as base LOC is to base $R V F$, so the aforementioned prisms (are) to one another. (Which is) the very thing it was required to show.

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\varepsilon^{\prime} .
$$








 АВГН тираиis трòs тウ̀v $\Delta \mathrm{EZ}$ тирацíठ $\alpha$.












 $\tau \tilde{n} \Delta \mathrm{EZ} \Theta$ тирацíठı $\pi \rho i ́ \sigma \mu \alpha \tau \alpha \mu \varepsilon i \zeta о \nu \alpha ́ ~ \varepsilon ̇ \sigma \tau \iota ~ \tau о u ̃ ~ X ~ \sigma \tau \varepsilon р \varepsilon о и ̃ . ~$

 $\beta \alpha \dot{\alpha} \sigma \nu$, oút $\omega \varsigma \tau \dot{\alpha}$ हैv $\tau \tilde{n}$ АВГН $\pi \cup \rho \alpha \mu i ́ \delta \iota ~ \pi \rho i ́ \sigma \mu \alpha \tau \alpha \pi \rho o ̀ \varsigma ~ \tau \grave{\alpha}$ हैv









 $\beta \alpha ́ \sigma \iota \nu$, oưt $\omega \varsigma \dot{\eta}$ АВГН $\tau \cup \rho \alpha \mu i \varsigma ~ \pi \rho o ̀ \varsigma ~ c ̌ \lambda \alpha \sigma \sigma o ́ v ~ \tau l ~ \tau \tilde{\eta} \varsigma ~ \triangle E Z \Theta ~$




 $\Delta \mathrm{EZ}$ т тирацíסos $\sigma \tau \varepsilon р \varepsilon o ́ v$.

## Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.


Let there be pyramids of the same height whose bases (are) the triangles $A B C$ and $D E F$, and apexes the points $G$ and $H$ (respectively). I say that as base $A B C$ is to base $D E F$, so pyramid $A B C G$ (is) to pyramid $D E F H$.

For if base $A B C$ is not to base $D E F$, as pyramid $A B C G$ (is) to pyramid $D E F H$, then base $A B C$ will be to base $D E F$, as pyramid $A B C G$ (is) to some solid either less than, or greater than, pyramid $D E F H$. Let it, first of all, be (in this ratio) to (some) lesser (solid), $W$. And let pyramid $D E F H$ have been divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms. So, the (sum of the) two prisms is greater than half of the whole pyramid [Prop. 12.3]. And, again, let the pyramids generated by the division have been similarly divided, and let this be done continually until some pyramids are left from pyramid $D E F H$ which (when added together) are less than the excess by which pyramid $D E F H$ exceeds the solid $W$ [Prop. 10.1]. Let them have been left, and, for the sake of argument, let them be $D Q R S$ and $S T U H$. Thus, the (sum of the) remaining prisms within pyramid $D E F H$ is greater than solid $W$. Let pyramid $A B C G$ also have been divided similarly, and a similar number of times, as pyramid $D E F H$. Thus, as base $A B C$ is to base $D E F$, so the (sum of the) prisms within pyramid $A B C G$ (is) to the (sum of the) prisms within pyramid $D E F H$ [Prop. 12.4]. But, also, as base $A B C$ (is) to base $D E F$, so pyramid $A B C G$ (is) to solid $W$. And, thus, as pyramid $A B C G$ (is) to solid $W$, so the (sum of the) prisms within pyramid $A B C G$ (is) to the (sum of the) prisms within pyramid $D E F H$ [Prop. 5.11]. Thus, alternately, as pyramid $A B C G$ (is) to the (sum of the) prisms within it, so solid $W$ (is) to the (sum of the) prisms within pyramid $D E F H$ [Prop. 5.16]. And pyramid $A B C G$ (is) greater than the (sum of the) prisms within it. Thus, solid $W$ (is) also greater than the (sum of the) prisms within pyramid DEFH [Prop. 5.14]. But, (it is) also less. This very thing is impossible. Thus, as base $A B C$ is to base $D E F$, so pyramid $A B C G$ (is)


 $\pi p o ̀ s ~ \tau \grave{\eta} \nu ~ А В Г Н ~ \pi u p \alpha \mu i ́ \delta \alpha, ~ o u ́ \tau \omega s ~ \dot{\eta} \Delta \mathrm{EZ} \mathrm{\Theta}$ тupaцis $\pi \rho o ̀ s$










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\varsigma^{\prime} .
$$







 K $\Lambda \mathrm{N} \pi \cup р \alpha \mu i ́ \delta \alpha$.






not to some solid less than pyramid $D E F H$. So, similarly, we can show that base $D E F$ is not to base $A B C$, as pyramid $D E F H$ (is) to some solid less than pyramid $A B C G$ either.

So, I say that neither is base $A B C$ to base $D E F$, as pyramid $A B C G$ (is) to some solid greater than pyramid DEFH.

For, if possible, let it be (in this ratio) to some greater (solid), $W$. Thus, inversely, as base $D E F$ (is) to base $A B C$, so solid $W$ (is) to pyramid $A B C G$ [Prop. 5.7. corr.]. And as solid $W$ (is) to pyramid $A B C G$, so pyramid $D E F H$ (is) to some (solid) less than pyramid $A B C G$, as shown before [Prop. 12.2 lem.]. And, thus, as base $D E F$ (is) to base $A B C$, so pyramid $D E F H$ (is) to some (solid) less than pyramid $A B C G$ [Prop. 5.11]. The very thing was shown (to be) absurd. Thus, base $A B C$ is not to base $D E F$, as pyramid $A B C G$ (is) to some solid greater than pyramid $D E F H$. And, it was shown that neither (is it in this ratio) to a lesser (solid). Thus, as base $A B C$ is to base $D E F$, so pyramid $A B C G$ (is) to pyramid $D E F H$. (Which is) the very thing it was required to show.

## Proposition 6

Pyramids which are of the same height, and have polygonal bases, are to one another as their bases.


Let there be pyramids of the same height whose bases (are) the polygons $A B C D E$ and $F G H K L$, and apexes the points $M$ and $N$ (respectively). I say that as base $A B C D E$ is to base $F G H K L$, so pyramid $A B C D E M$ (is) to pyramid $F G H K L N$.

For let $A C, A D, F H$, and $F K$ have been joined. Therefore, since $A B C M$ and $A C D M$ are two pyramids having triangular bases and equal height, they are to one another as their bases [Prop. 12.5]. Thus, as base $A B C$ is to base $A C D$, so pyramid $A B C M$ (is) to pyramid $A C D M$. And, via composition, as base $A B C D$




 $\mathrm{A} \Delta \mathrm{E} \beta \alpha \dot{\alpha} \sigma \nu$, oütcos $\dot{\eta} \mathrm{AB} Г \Delta \mathrm{EM} \pi \cup p \alpha \mu i \varsigma ~ \pi p o ̀ \varsigma ~ t \grave{̀ \nu} \mathrm{~A} \Delta \mathrm{EM}$







 $\dot{\eta}$ АВГ $\Delta \mathrm{E} \beta \dot{\alpha} \sigma \iota \varsigma ~ \pi \rho o ̀ s ~ \tau \grave{\eta} ข ~ Z H \Theta ~ \beta \alpha ́ \sigma \sigma v, ~ o u ̈ \tau \omega \varsigma ~ \dot{\eta}$ АВГ $\Delta \mathrm{EM}$

 рацis трòs $\tau \grave{\eta}$ ZHӨK
 $\pi \cup р \alpha u i \varsigma ~ \pi \rho o ̀ s ~ \tau \grave{̀}$ ZHӨK

$$
\zeta^{\prime} .
$$






 $\beta \alpha \dot{\sigma \varepsilon ı}$.



(is) to base $A C D$, so pyramid $A B C D M$ (is) to pyramid $A C D M$ [Prop. 5.18]. But, as base $A C D$ (is) to base $A D E$, so pyramid $A C D M$ (is) also to pyramid $A D E M$ [Prop. 12.5]. Thus, via equality, as base $A B C D$ (is) to base $A D E$, so pyramid $A B C D M$ (is) to pyramid $A D E M$ [Prop. 5.22]. And, again, via composition, as base $A B C D E$ (is) to base $A D E$, so pyramid $A B C D E M$ (is) to pyramid $A D E M$ [Prop. 5.18]. So, similarly, it can also be shown that as base $F G H K L$ (is) to base $F G H$, so pyramid $F G H K L N$ (is) also to pyramid $F G H N$. And since $A D E M$ and $F G H N$ are two pyramids having triangular bases and equal height, thus as base $A D E$ (is) to base $F G H$, so pyramid $A D E M$ (is) to pyramid $F G H N$ [Prop. 12.5]. But, as base $A D E$ (is) to base $A B C D E$, so pyramid $A D E M$ (was) to pyramid $A B C D E M$. Thus, via equality, as base $A B C D E$ (is) to base $F G H$, so pyramid $A B C D E M$ (is) also to pyramid $F G H N$ [Prop. 5.22]. But, furthermore, as base $F G H$ (is) to base $F G H K L$, so pyramid $F G H N$ was also to pyramid $F G H K L N$. Thus, via equality, as base $A B C D E$ (is) to base $F G H K L$, so pyramid $A B C D E M$ (is) also to pyramid $F G H K L N$ [Prop. 5.22]. (Which is) the very thing it was required to show.

## Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to one another.


Let there be a prism whose base (is) triangle $A B C$, and opposite (plane) $D E F$. I say that prism $A B C D E F$ is divided into three pyramids having triangular bases (which are) equal to one another.

For let $B D, E C$, and $C D$ have been joined. Since $A B E D$ is a parallelogram, and $B D$ is its diagonal, triangle $A B D$ is thus equal to triangle $E B D$ [Prop. 1.34].















 $\beta \alpha ́ \sigma \iota s ~ \mu \varepsilon ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ A B \Delta ~ \tau p i ́ \gamma \omega \nu o v, ~ x o p u \varphi \eta ̀ ~ \delta e ̀ ~ \tau o ̀ ~ \Gamma ~ \sigma \eta \mu \varepsilon i ̃ o v . ~$













 тò $\Delta \mathrm{EZ}$.

## Пópıб ${ }^{\text {a }}$.





## $\eta^{\prime}$.








And, thus, the pyramid whose base (is) triangle $A B D$, and apex the point $C$, is equal to the pyramid whose base is triangle $D E B$, and apex the point $C$ [Prop. 12.5]. But, the pyramid whose base is triangle $D E B$, and apex the point $C$, is the same as the pyramid whose base is triangle $E B C$, and apex the point $D$. For they are contained by the same planes. And, thus, the pyramid whose base is $A B D$, and apex the point $C$, is equal to the pyramid whose base is $E B C$ and apex the point $D$. Again, since $F C B E$ is a parallelogram, and $C E$ is its diagonal, triangle $C E F$ is equal to triangle $C B E$ [Prop. 1.34]. And, thus, the pyramid whose base is triangle $B C E$, and apex the point $D$, is equal to the pyramid whose base is triangle $E C F$, and apex the point $D$ [Prop. 12.5]. And the pyramid whose base is triangle $B C E$, and apex the point $D$, was shown (to be) equal to the pyramid whose base is triangle $A B D$, and apex the point $C$. Thus, the pyramid whose base is triangle $C E F$, and apex the point $D$, is also equal to the pyramid whose base [is] triangle $A B D$, and apex the point $C$. Thus, the prism $A B C D E F$ has been divided into three pyramids having triangular bases (which are) equal to one another.

And since the pyramid whose base is triangle $A B D$, and apex the point $C$, is the same as the pyramid whose base is triangle $C A B$, and apex the point $D$. For they are contained by the same planes. And the pyramid whose base (is) triangle $A B D$, and apex the point $C$, was shown (to be) a third of the prism whose base is triangle $A B C$, and opposite (plane) $D E F$, thus the pyramid whose base is triangle $A B C$, and apex the point $D$, is also a third of the pyramid having the same base, triangle $A B C$, and opposite (plane) $D E F$.

## Corollary

And, from this, (it is) clear that any pyramid is the third part of the prism having the same base as it, and an equal height. (Which is) the very thing it was required to show.

## Proposition 8

Similar pyramids which also have triangular bases are in the cubed ratio of their corresponding sides.

Let there be similar, and similarly laid out, pyramids whose bases are triangles $A B C$ and $D E F$, and apexes the points $G$ and $H$ (respectively). I say that pyramid $A B C G$ has to pyramid $D E F H$ the cubed ratio of that $B C$ (has) to $E F$.

$\Sigma \cup \mu \pi \varepsilon \pi \lambda \eta \rho \omega \dot{\sigma} \vartheta \omega$ ү $\alpha \rho ~ \tau \grave{\alpha}$ ВНМА, ЕӨПО бтєрєд̀ $\pi \alpha \rho \alpha \lambda$ -

 $\gamma \omega v i ́ \alpha, \dot{\eta} \delta$ è ú $\pi o ̀ ~ H B \Gamma ~ \tau \tilde{n} ~ u ́ \pi o ̀ ~ \Theta E Z, ~ \dot{\eta} ~ \delta e ̀ ~ u ́ \pi o ̀ ~ A B H ~ \tau \tilde{n} ~ u ́ \pi o ̀ ~$






 $\dot{\alpha} \lambda \lambda \dot{\alpha} \tau \dot{\alpha} \mu \varepsilon ̀ \nu \tau \rho i ́ \alpha \tau \dot{\alpha} \mathrm{MB}, \mathrm{BK}, \mathrm{BN}$ тpเఠì тoĩs $\dot{\alpha} \pi \varepsilon \nu \alpha \nu \tau i ́ o \nu$ "ै $\sigma \alpha$
















## Пópıбца.









For let the parallelepiped solids $B G M L$ and $E H Q P$ have been completed. And since pyramid $A B C G$ is similar to pyramid $D E F H$, angle $A B C$ is thus equal to angle $D E F$, and $G B C$ to $H E F$, and $A B G$ to $D E H$. And as $A B$ is to $D E$, so $B C$ (is) to $E F$, and $B G$ to $E H$ [Def. 11.9]. And since as $A B$ is to $D E$, so $B C$ (is) to $E F$, and (so) the sides around equal angles are proportional, parallelogram $B M$ is thus similar to paralleleogram $E Q$. So, for the same (reasons), $B N$ is also similar to $E R$, and $B K$ to $E O$. Thus, the three (parallelograms) $M B, B K$, and $B N$ are similar to the three (parallelograms) $E Q$, $E O, E R$ (respectively). But, the three (parallelograms) $M B, B K$, and $B N$ are (both) equal and similar to the three opposite (parallelograms), and the three (parallelograms) $E Q, E O$, and $E R$ are (both) equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the solids $B G M L$ and $E H Q P$ are contained by equal numbers of similar (and similarly laid out) planes. Thus, solid $B G M L$ is similar to solid $E H Q P$ [Def. 11.9]. And similar parallelepiped solids are in the cubed ratio of corresponding sides [Prop. 11.33]. Thus, solid BGML has to solid $E H Q P$ the cubed ratio that the corresponding side $B C$ (has) to the corresponding side $E F$. And as solid $B G M L$ (is) to solid $E H Q P$, so pyramid $A B C G$ (is) to pyramid $D E F H$, inasmuch as the pyramid is the sixth part of the solid, on account of the prism, being half of the parallelepiped solid [Prop. 11.28], also being three times the pyramid [Prop. 12.7]. Thus, pyramid $A B C G$ also has to pyramid $D E F H$ the cubed ratio that $B C$ (has) to $E F$. (Which is) the very thing it was required to show.

## Corollary

So, from this, (it is) also clear that similar pyramids having polygonal bases (are) to one another as the cubed ratio of their corresponding sides. For, dividing them into the pyramids (contained) within them which have triangular bases, with the similar polygons of the bases also being divided into similar triangles (which are)
©







 है $\chi o \cup \sigma \alpha$ т $\rho o ̀ s ~ \tau \grave{\eta} \nu ~ o ́ \mu o i ́ \alpha \nu ~ \beta \alpha ́ \sigma เ \nu ~ ह ै \chi o v \sigma \alpha \nu ~ \tau p ı \pi \lambda \alpha \sigma i o v \alpha ~ \lambda o ́ \gamma o v ~$ है $\chi \varepsilon ا$ ク̈ $\pi \varepsilon \rho \dot{\eta} \pi \lambda \varepsilon \cup \cup \alpha \dot{\alpha} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \pi \lambda \varepsilon \cup \rho \alpha ́ \alpha$.

$$
\vartheta^{\prime} .
$$









 oútcos tò $\tau \tilde{\eta} s ~ \triangle \mathrm{EZ} \Theta$ тupauíóos ü廿os $\pi p o ̀ s ~ t o ̀ ~ \tau \tilde{\eta} s ~ А В Г Н ~$ тирацíסos üчos.






both equal in number, and corresponding, to the wholes [Prop. 6.20]. As one pyramid having a triangular base in the former (pyramid having a polygonal base is) to one pyramid having a triangular base in the latter (pyramid having a polygonal base), so (the sum of) all the pyramids having triangular bases in the former pyramid will also be to (the sum of) all the pyramids having triangular bases in the latter pyramid [Prop. 5.12]-that is to say, the (former) pyramid itself having a polygonal base to the (latter) pyramid having a polygonal base. And a pyramid having a triangular base is to a (pyramid) having a triangular base in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, a (pyramid) having a polygonal base also has to to a (pyramid) having a similar base the cubed ratio of a (corresponding) side to a (corresponding) side.

## Proposition 9

The bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids which have triangular bases whose bases are reciprocally proportional to their heights are equal.


For let there be (two) equal pyramids having the triangular bases $A B C$ and $D E F$, and apexes the points $G$ and $H$ (respectively). I say that the bases of the pyramids $A B C G$ and $D E F H$ are reciprocally proportional to their heights, and (so) that as base $A B C$ is to base $D E F$, so the height of pyramid $D E F H$ (is) to the height of pyramid $A B C G$.

For let the parallelepiped solids $B G M L$ and $E H Q P$ have been completed. And since pyramid $A B C G$ is equal to pyramid $D E F H$, and solid $B G M L$ is six times pyramid $A B C G$ (see previous proposition), and solid $E H Q P$ (is) six times pyramid $D E F H$, solid $B G M L$ is
 $\beta \alpha ́ \sigma \iota s ~ \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ E П ~ \beta \alpha ́ \sigma \iota \nu, ~ o u ̛ \tau \omega s ~ \tau o ̀ ~ \tau o u ̃ ~ E ~ Е П О ~ \sigma \tau \varepsilon р \varepsilon о u ̃ ~$







 $\beta \alpha ́ \sigma \tau \nu$, oüt $\omega \varsigma$ tò $\tau \tilde{\jmath} \varsigma ~ \triangle E Z \Theta ~ \pi \cup p \alpha \mu i ́ \delta o s ~ u ̈ \psi o s ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \tau \eta ̃ s ~$





 $\pi \cup p \alpha \mu i \varsigma ~ \tau \tilde{n} \Delta \mathrm{EZ} \Theta$ тирацíó.
 АВГ $\beta \alpha ́ \sigma เ s ~ \pi p o ̀ s ~ \tau \grave{\eta} \nu \triangle \mathrm{EZ} \beta \alpha ́ \sigma เ \nu$, oưт $\tau \omega$ тò tñs $\triangle \mathrm{EZ} \Theta \pi \cup-$

 $\lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu \pi \rho o ̀ s ~ t o ̀ ~ E \Pi ~ \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu, ~ x \alpha i ̀ ~ c o s ~ \alpha ้ p \alpha ~$ тò $\mathrm{BM} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o \nu \pi \rho o ̀ s ~ t o ̀ ~ E П ~ \pi \alpha p \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu o \nu$,





 ư廿оऽ лрòs tò тoũ BHM $\pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi ı \pi \varepsilon ́ \delta o u ~ u ̈ \psi o \varsigma . ~ \widetilde{\omega \nu}$

 $\sigma \tau \varepsilon \rho \varepsilon o ̀ \nu \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi i ́ \pi \varepsilon \delta o \nu \tau \widetilde{\varphi}$ ЕӨПО $\sigma \tau \varepsilon \rho \varepsilon \widetilde{\varphi} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon-$ $\pi \iota \pi \varepsilon ́ \delta \omega . ~ \chi \alpha i ́ ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o u ̃ ~ \mu \varepsilon ̀ v ~ В Н М \Lambda ~ \varepsilon ้ x \tau o v ~ \mu \varepsilon ́ p o s ~ \dot{\eta}$ АВГН

 pauíor.





## $i^{\prime}$.




thus equal to solid $E H Q P$. And the bases of equal parallelepiped solids are reciprocally proportional to their heights [Prop. 11.34]. Thus, as base $B M$ is to base $E Q$, so the height of solid $E H Q P$ (is) to the height of solid $B G M L$. But, as base $B M$ (is) to base $E Q$, so triangle $A B C$ (is) to triangle $D E F$ [Prop. 1.34]. And, thus, as triangle $A B C$ (is) to triangle $D E F$, so the height of solid $E H Q P$ (is) to the height of solid $B G M L$ [Prop. 5.11]. But, the height of solid $E H Q P$ is the same as the height of pyramid $D E F H$, and the height of solid $B G M L$ is the same as the height of pyramid $A B C G$. Thus, as base $A B C$ is to base $D E F$, so the height of pyramid $D E F H$ (is) to the height of pyramid $A B C G$. Thus, the bases of pyramids $A B C G$ and $D E F H$ are reciprocally proportional to their heights.

And so, let the bases of pyramids $A B C G$ and $D E F H$ be reciprocally proportional to their heights, and (thus) let base $A B C$ be to base $D E F$, as the height of pyramid $D E F H$ (is) to the height of pyramid $A B C G$. I say that pyramid $A B C G$ is equal to pyramid $D E F H$.

For, with the same construction, since as base $A B C$ is to base $D E F$, so the height of pyramid $D E F H$ (is) to the height of pyramid $A B C G$, but as base $A B C$ (is) to base $D E F$, so parallelogram $B M$ (is) to parallelogram $E Q$ [Prop. 1.34], thus as parallelogram $B M$ (is) to parallelogram $E Q$, so the height of pyramid $D E F H$ (is) also to the height of pyramid $A B C G$ [Prop. 5.11]. But, the height of pyramid $D E F H$ is the same as the height of parallelepiped $E H Q P$, and the height of pyramid $A B C G$ is the same as the height of parallelepiped $B G M L$. Thus, as base $B M$ is to base $E Q$, so the height of parallelepiped $E H Q P$ (is) to the height of parallelepiped $B G M L$. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal [Prop. 11.34]. Thus, the parallelepiped solid $B G M L$ is equal to the parallelepiped solid $E H Q P$. And pyramid $A B C G$ is a sixth part of $B G M L$, and pyramid $D E F H$ a sixth part of parallelepiped $E H Q P$. Thus, pyramid $A B C G$ is equal to pyramid DEFH.

Thus, the bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids having triangular bases whose bases are reciprocally proportional to their heights are equal. (Which is) the very thing it was required to show.

## Proposition 10

Every cone is the third part of the cylinder which has the same base as it, and an equal height.

For let there be a cone (with) the same base as a cylin-

 xढ́vou трıл $\lambda \alpha \sigma i ́ \omega \nu$ ह̇ $\sigma \tau i ́ \nu . ~$








 xä้ $\pi \varepsilon \rho i ~ \tau o ̀ v ~ А В Г \Delta ~ x и ́ x \lambda о \nu ~ \tau \varepsilon \tau \rho \alpha ́ \gamma \omega v o v ~ \pi \varepsilon р เ ү р \alpha ́ q \omega \mu \varepsilon \nu, ~$

 $\dot{\alpha} \nu เ \sigma \tau \alpha ́ \mu \varepsilon \nu \alpha$ $\sigma \tau \varepsilon \rho \varepsilon \dot{\alpha} \pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi i \pi \varepsilon \delta \alpha \pi \rho i \sigma \mu \alpha \tau \alpha$ íooü $\psi \tilde{\eta} \cdot \tau \grave{\alpha}$


 $\pi p i ́ \sigma \mu \alpha \tau o s ~ \alpha ̀ \pi o ̀ ~ \tau o u ̃ ~ \pi \varepsilon p i ̀ ~ \tau o ̀ v ~ А В Г \Delta ~ x u ́ x \lambda o v ~ \pi \varepsilon p ı \gamma p \alpha \varphi ́ ́ v \tau o s ~$



 тои̃ $\dot{\eta} \mu i \sigma \varepsilon \omega \varsigma ~ \tau о \tilde{u} ~ \chi \cup \lambda i ́ v \delta \rho o u . ~ \tau \varepsilon \tau \mu \dot{n} \sigma \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ A B, ~ В Г, ~$
 غ̇ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ A E, ~ E B, ~ B Z, ~ Z Г, ~ Г Н, ~ H ~ \Delta, ~ \Delta \Theta, ~ Ө А \cdot ~$









der, (namely) the circle $A B C D$, and an equal height. I say that the cone is the third part of the cylinder-that is to say, that the cylinder is three times the cone.


For if the cylinder is not three times the cone then the cylinder will be either more than three times, or less than three times, (the cone). Let it, first of all, be more than three times (the cone). And let the square $A B C D$ have been inscribed in circle $A B C D$ [Prop. 4.6]. So, square $A B C D$ is more than half of circle $A B C D$ [Prop. 12.2]. And let a prism of equal height to the cylinder have been set up on square $A B C D$. So, the prism set up is more than half of the cylinder, inasmuch as if we also circumscribe a square around circle $A B C D$ [Prop. 4.7] then the square inscribed in circle $A B C D$ is half of the circumscribed (square). And the solids set up on them are parallelepiped prisms of equal height. And parallelepiped solids having the same height are to one another as their bases [Prop. 11.32]. And, thus, the prism set up on square $A B C D$ is half of the prism set up on the square circumscribed about circle $A B C D$. And the cylinder is less than the prism set up on the square circumscribed about circle $A B C D$. Thus, the prism set up on square $A B C D$ of the same height as the cylinder is more than half of the cylinder. Let the circumferences $A B, B C$, $C D$, and $D A$ have been cut in half at points $E, F, G$, and $H$. And let $A E, E B, B F, F C, C G, G D, D H$, and $H A$ have been joined. And thus each of the triangles $A E B, B F C, C G D$, and $D H A$ is more than half of the segment of circle $A B C D$ about it, as was shown previously [Prop. 12.2]. Let prisms of equal height to the cylinder have been set up on each of the triangles $A E B$, $B F C, C G D$, and $D H A$. And each of the prisms set up is greater than the half part of the segment of the cylinder about it-inasmuch as if we draw (straight-lines) parallel to $A B, B C, C D$, and $D A$ through points $E, F, G$, and $H$

 $\dot{\eta} \mu i \sigma \eta$ ह̇எтì $\tau \alpha ̀ \pi p i ́ \sigma \mu \alpha \tau \alpha \tau \dot{\alpha}$ ह̇ $\pi i ̀ \tau \widetilde{\omega} \nu \mathrm{AEB}, \mathrm{BZГ}, \Gamma \mathrm{H} \Delta, \Delta \Theta \mathrm{A}$

 $\tau \alpha ̀ ~ \varepsilon ̇ \pi i ̀ \tau \widetilde{\omega} \nu \mathrm{AEB}, \mathrm{BZ} \mathrm{\Gamma}, ~ Г Н \Delta, ~ \Delta \Theta А ~ \tau \rho ı \gamma \omega ́ \nu \omega \nu ~ \pi \rho i ́ \sigma \mu \alpha \tau \alpha$






 $\mathrm{EB}, \mathrm{BZ}, \mathrm{Z}, \Gamma, \Gamma, \mathrm{H} \Delta, \Delta \Theta, \Theta \mathrm{A} \cdot \lambda o \iota \pi o ̀ v$ äp $\alpha$ tò $\pi \rho i ́ \sigma \mu \alpha$, oư $\beta \alpha ́ \sigma \iota s ~ \mu e ̀ v ~ \tau o ̀ ~ А E B Z \Gamma Н \Delta \Theta ~ \pi o \lambda u ́ \gamma \omega v o v, ~ u ̛ \psi o s ~ \delta e ̀ ~ t o ̀ ~ \alpha u ̉ t o ̀ ~$
 тò $\pi \rho i ́ \sigma \mu \alpha$, oư $\beta \alpha ́ \sigma ı \varsigma ~ \mu \varepsilon ̀ v ~ \varepsilon ̇ \sigma \tau i ̀ ~ \tau o ̀ ~ А Е В Z Г Н \Delta \Theta ~ \pi о \lambda u ́ \gamma \omega \nu o v, ~$
 paцíoos, ñs $\beta \dot{\alpha} \sigma \iota s ~ \mu \varepsilon ́ v ~ \varepsilon ́ \sigma \tau l ~ t o ̀ ~ А Е В Z Г Н \Delta \Theta ~ \pi o \lambda u ́ \gamma \omega v o v, ~$






犭ú入เvס̄pos тoũ x $u$ vou.










 $\sigma \tau \varepsilon \rho \varepsilon \alpha \dot{\alpha} \pi \alpha p \alpha \lambda \lambda \eta \lambda \varepsilon \pi i \pi \varepsilon \delta \alpha \alpha \dot{\alpha} \alpha \sigma \tau \eta \dot{\eta} \sigma \mu \varepsilon \nu$ iбоӥ $\psi \tilde{\eta} \tau \widetilde{\varphi} \chi \omega ́ \nu \varphi, \ddot{\alpha}$












(respectively), and complete the parallelograms on $A B$, $B C, C D$, and $D A$, and set up parallelepiped solids of equal height to the cylinder on them, then the prisms on triangles $A E B, B F C, C G D$, and $D H A$ are each half of the set up (parallelepipeds). And the segments of the cylinder are less than the set up parallelepiped solids. Hence, the prisms on triangles $A E B, B F C, C G D$, and $D H A$ are also greater than half of the segments of the cylinder about them. So (if) the remaining circumferences are cut in half, and straight-lines are joined, and prisms of equal height to the cylinder are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cylinder whose (sum) is less than the excess by which the cylinder exceeds three times the cone [Prop. 10.1]. Let them have been left, and let them be $A E, E B, B F, F C, C G, G D$, $D H$, and $H A$. Thus, the remaining prism whose base (is) polygon $A E B F C G D H$, and height the same as the cylinder, is greater than three times the cone. But, the prism whose base is polygon $A E B F C G D H$, and height the same as the cylinder, is three times the pyramid whose base is polygon $A E B F C G D H$, and apex the same as the cone [Prop. 12.7 corr.]. And thus the pyramid whose base [is] polygon $A E B F C G D H$, and apex the same as the cone, is greater than the cone having (as) base circle $A B C D$. But (it is) also less. For it is encompassed by it. The very thing (is) impossible. Thus, the cylinder is not more than three times the cone.

So, I say that neither (is) the cylinder less than three times the cone.

For, if possible, let the cylinder be less than three times the cone. Thus, inversely, the cone is greater than the third part of the cylinder. So, let the square $A B C D$ have been inscribed in circle $A B C D$ [Prop. 4.6]. Thus, square $A B C D$ is greater than half of circle $A B C D$. And let a pyramid having the same apex as the cone have been set up on square $A B C D$. Thus, the pyramid set up is greater than the half part of the cone, inasmuch as we showed previously that if we circumscribe a square about the circle [Prop. 4.7] then the square $A B C D$ will be half of the square circumscribed about the circle [Prop. 12.2]. And if we set up on the squares parallelepiped solids-which are also called prisms-of the same height as the cone, then the (prism) set up on square $A B C D$ will be half of the (prism) set up on the square circumscribed about the circle. For they are to one another as their bases [Prop. 11.32]. Hence, (the same) also (goes for) the thirds. Thus, the pyramid whose base is square $A B C D$ is half of the pyramid set up on the square circumscribed about the circle [Prop. 12.7 corr.]. And the pyramid set up on the square circumscribed about the circle is greater


























 xu入ívopou．



[^35]than the cone．For it encompasses it．Thus，the pyramid whose base is square $A B C D$ ，and apex the same as the cone，is greater than half of the cone．Let the circum－ ferences $A B, B C, C D$ ，and $D A$ have been cut in half at points $E, F, G$ ，and $H$（respectively）．And let $A E$ ， $E B, B F, F C, C G, G D, D H$ ，and $H A$ have been joined． And，thus，each of the triangles $A E B, B F C, C G D$ ，and $D H A$ is greater than the half part of the segment of cir－ cle $A B C D$ about it［Prop．12．2］．And let pyramids having the same apex as the cone have been set up on each of the triangles $A E B, B F C, C G D$ ，and $D H A$ ．And，thus，in the same way，each of the pyramids set up is more than the half part of the segment of the cone about it．So，（if）the remaining circumferences are cut in half，and straight－ lines are joined，and pyramids having the same apex as the cone are set up on each of the triangles，and this is done continually，then we will（eventually）leave some segments of the cone whose（sum）is less than the excess by which the cone exceeds the third part of the cylinder ［Prop．10．1］．Let them have been left，and let them be the（segments）on $A E, E B, B F, F C, C G, G D, D H$ ，and $H A$ ．Thus，the remaining pyramid whose base is poly－ gon $A E B F C G D H$ ，and apex the same as the cone，is greater than the third part of the cylinder．But，the pyra－ mid whose base is polygon $A E B F C G D H$ ，and apex the same as the cone，is the third part of the prism whose base is polygon $A E B F C G D H$ ，and height the same as the cylinder［Prop． 12.7 corr．］．Thus，the prism whose base is polygon $A E B F C G D H$ ，and height the same as the cylinder，is greater than the cylinder whose base is circle $A B C D$ ．But，（it is）also less．For it is encompassed by it．The very thing is impossible．Thus，the cylinder is not less than three times the cone．And it was shown that neither（is it）greater than three times（the cone）．Thus， the cylinder（is）three times the cone．Hence，the cone is the third part of the cylinder．

Thus，every cone is the third part of the cylinder which has the same base as it，and an equal height．（Which is） the very thing it was required to show．

## Proposition 11

Cones and cylinders having the same height are to one another as their bases．

Let there be cones and cylinders of the same height whose bases［are］the circles $A B C D$ and $E F G H$ ，axes $K L$ and $M N$ ，and diameters of the bases $A C$ and $E G$（re－ spectively）．I say that as circle $A B C D$ is to circle $E F G H$ ， so cone $A L$（is）to cone $E N$ ．



















 غ́x $\alpha \sigma \tau o u ~ \tau \widetilde{\omega} \nu ~ \Theta O E, ~ E П Z, ~ Z P H, ~ H \Sigma \Theta ~ \tau p r \gamma ஸ ́ v \omega \nu ~ \pi u p a \mu i s ~$







 $\beta \alpha ́ \sigma \iota s ~ \tau o ̀ ~ \Theta O E П Z P H \Sigma ~ \pi o \lambda u ́ \gamma \omega v o v, ~ U ̛ \psi o s ~ \delta e ̀ ~ \tau o ̀ ~ \alpha u ̉ t o ̀ ~ \tau \widetilde{̣}$









 трòs tòv EZH xúx入ov, oütcs ó $\mathrm{A} \Lambda$ x $\widetilde{\omega} v o s ~ \pi \rho o ̀ s ~ t o ̀ ~ \Xi ~$





For if not, then as circle $A B C D$ (is) to circle $E F G H$, so cone $A L$ will be to some solid either less than, or greater than, cone $E N$. Let it, first of all, be (in this ratio) to (some) lesser (solid), $O$. And let solid $X$ be equal to that (magnitude) by which solid $O$ is less than cone $E N$. Thus, cone $E N$ is equal to (the sum of) solids $O$ and $X$. Let the square $E F G H$ have been inscribed in circle $E F G H$ [Prop. 4.6]. Thus, the square is greater than half of the circle [Prop. 12.2]. Let a pyramid of the same height as the cone have been set up on square $E F G H$. Thus, the pyramid set up is greater than half of the cone, inasmuch as, if we circumscribe a square about the circle [Prop. 4.7], and set up on it a pyramid of the same height as the cone, then the inscribed pyramid is half of the circumscribed pyramid. For they are to one another as their bases [Prop. 12.6]. And the cone (is) less than the circumscribed pyramid. Let the circumferences $E F, F G, G H$, and $H E$ have been cut in half at points $P, Q, R$, and $S$. And let $H P, P E, E Q, Q F, F R, R G$, $G S$, and $S H$ have been joined. Thus, each of the triangles $H P E, E Q F, F R G$, and $G S H$ is greater than half of the segment of the circle about it [Prop. 12.2]. Let pyramids of the same height as the cone have been set up on each of the triangles $H P E, E Q F, F R G$, and $G S H$. And, thus, each of the pyramids set up is greater than half of the segment of the cone about it [Prop. 12.10]. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids of equal height to the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone (the sum of) which is less than solid $X$ [Prop. 10.1]. Let them have been left, and let them be the (segments) on $H P E, E Q F, F R G$, and $G S H$. Thus, the remaining pyramid whose base is polygon $H P E Q F R G S$, and height the same as the cone, is greater than solid $O$ [Prop. 6.18]. And let the polygon $D T A U B V C W$, similar, and similarly laid out, to polygon $H P E Q F R G S$, have been inscribed in circle $A B C D$. And on it let a pyramid of the same height as cone $A L$ have been set up. Therefore, since as the (square) on $A C$ is to the (square) on $E G$, so polygon $D T A U B V C W$ (is) to polygon $H P E Q F R G S$ [Prop. 12.1], and as the (square) on $A C$ (is) to the (square) on $E G$, so circle $A B C D$ (is)













 A $\Lambda$ x $\omega$ vou бтєрєóv.

 кढ́vou $\sigma \tau \varepsilon ร$ ธ́v.










 $\chi \widetilde{\omega} \nu \circ$.


 $\alpha \cup ๋ \tau \widetilde{\nu}$ íのoü४عіॅऽ.



[^36]to circle $E F G H$ [Prop. 12.2], thus as circle $A B C D$ (is) to circle $E F G H$, so polygon $D T A U B V C W$ also (is) to polygon $H P E Q F R G S$. And as circle $A B C D$ (is) to circle $E F G H$, so cone $A L$ (is) to solid $O$. And as polygon $D T A U B V C W$ (is) to polygon $H P E Q F R G S$, so the pyramid whose base is polygon $D T A U B V C W$, and apex the point $L$, (is) to the pyramid whose base is polygon $H P E Q F R G S$, and apex the point $N$ [Prop. 12.6]. And, thus, as cone $A L$ (is) to solid $O$, so the pyramid whose base is $D T A U B V C W$, and apex the point $L$, (is) to the pyramid whose base is polygon $H P E Q F R G S$, and apex the point $N$ [Prop. 5.11]. Thus, alternately, as cone $A L$ is to the pyramid within it, so solid $O$ (is) to the pyramid within cone $E N$ [Prop. 5.16]. But, cone $A L$ (is) greater than the pyramid within it. Thus, solid $O$ (is) also greater than the pyramid within cone $E N$ [Prop. 5.14]. But, (it is) also less. The very thing (is) absurd. Thus, circle $A B C D$ is not to circle $E F G H$, as cone $A L$ (is) to some solid less than cone $E N$. So, similarly, we can show that neither is circle $E F G H$ to circle $A B C D$, as cone $E N$ (is) to some solid less than cone $A L$.

So, I say that neither is circle $A B C D$ to circle $E F G H$, as cone $A L$ (is) to some solid greater than cone $E N$.

For, if possible, let it be (in this ratio) to (some) greater (solid), $O$. Thus, inversely, as circle $E F G H$ is to circle $A B C D$, so solid $O$ (is) to cone $A L$ [Prop. 5.7 corr.]. But, as solid $O$ (is) to cone $A L$, so cone $E N$ (is) to some solid less than cone $A L$ [Prop. 12.2 lem.]. And, thus, as circle $E F G H$ (is) to circle $A B C D$, so cone $E N$ (is) to some solid less than cone $A L$. The very thing was shown (to be) impossible. Thus, circle $A B C D$ is not to circle $E F G H$, as cone $A L$ (is) to some solid greater than cone $E N$. And, it was shown that neither (is it in this ratio) to (some) lesser (solid). Thus, as circle $A B C D$ is to circle $E F G H$, so cone $A L$ (is) to cone $E N$.

But, as the cone (is) to the cone, (so) the cylinder (is) to the cylinder. For each (is) three times each [Prop. 12.10]. Thus, circle $A B C D$ (is) also to circle $E F G H$, as (the ratio of the cylinders) on them (having) the same height.

Thus, cones and cylinders having the same height are to one another as their bases. (Which is) the very thing it was required to show.

## Proposition 12

Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

Let there be similar cones and cylinders of which the bases (are) the circles $A B C D$ and $E F G H$, the diameters of the bases (are) $B D$ and $F H$, and the axes of the cones


 $\lambda o ́ \gamma o v$ है $\chi \varepsilon \iota ~ \eta ̈ \pi \varepsilon \rho \dot{\eta} \mathrm{~B} \Delta$ трòs т $\grave{\nu} \mathrm{Z} \Theta$.










 $\mathrm{H} \Theta, ~ \Theta Е ~ \pi \varepsilon р เ \varphi \varepsilon ́ p \varepsilon ı \alpha l ~ \delta i ́ \chi \alpha ~ \chi \alpha \tau \alpha ̀ ~ \tau \grave{\alpha} \mathrm{O}, \Pi, \mathrm{P}, \Sigma \sigma \eta \mu \varepsilon \tau \sim \alpha, x \alpha \grave{l}$ غ̇ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu$ 人i EO, OZ, ZП, ПН, HP, P $\Theta, \Theta \Sigma, \Sigma$. x $\alpha \grave{\imath}$










 EZHON x $\widetilde{\omega} \nu \circ \varsigma ~ \tau о u ̃ ~ \Xi ~ \sigma \tau \varepsilon р \varepsilon о и ̃ . ~ \lambda \varepsilon \lambda \varepsilon i ́ \varphi \vartheta \omega, ~ x \alpha i ̀ ~ ह ै \sigma \tau \omega ~ \tau \grave{\alpha}$ ย̇лì $\tau \widetilde{\omega} \nu \mathrm{EO}, \mathrm{OZ}, \mathrm{Z} \Pi, \Pi Н, \mathrm{HP}, \mathrm{P} \Theta, \Theta \Sigma, \Sigma \mathrm{E} \cdot \lambda o \iota \tau \dot{\eta} \alpha{ }^{\alpha} \rho \alpha \dot{\eta}$ $\pi \cup p \alpha \mu i s$, ñs $\beta \alpha ́ \sigma \iota s ~ \mu \varepsilon ́ v ~ \varepsilon ̇ \sigma \tau ו ~ t o ̀ ~ E O Z П H P \Theta \Sigma ~ \pi o \lambda u ́ \gamma \omega v o v, ~$







 pauíठ $\alpha$, ทัs $\beta \alpha ́ \sigma \iota s ~ \mu \varepsilon ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ E O Z П H P \Theta \Sigma ~ \pi o \lambda u ́ \gamma \omega v o v, ~$
and cylinders (are) $K L$ and $M N$ (respectively). I say that the cone whose base [is] circle $A B C D$, and apex the point $L$, has to the cone whose base [is] circle $E F G H$, and apex the point $N$, the cubed ratio that $B D$ (has) to FH.


For if cone $A B C D L$ does not have to cone $E F G H N$ the cubed ratio that $B D$ (has) to $F H$ then cone $A B C D L$ will have the cubed ratio to some solid either less than, or greater than, cone $E F G H N$. Let it, first of all, have (such a ratio) to (some) lesser (solid), $O$. And let the square $E F G H$ have been inscribed in circle $E F G H$ [Prop. 4.6]. Thus, square $E F G H$ is greater than half of circle $E F G H$ [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on square $E F G H$. Thus, the pyramid set up is greater than the half part of the cone [Prop. 12.10]. So, let the circumferences $E F, F G$, $G H$, and $H E$ have been cut in half at points $P, Q, R$, and $S$ (respectively). And let $E P, P F, F Q, Q G, G R$, $R H, H S$, and $S E$ have been joined. And, thus, each of the triangles $E P F, F Q G, G R H$, and $H S E$ is greater than the half part of the segment of circle $E F G H$ about it [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on each of the triangles $E P F$, $F Q G, G R H$, and $H S E$. And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone $E F G H N$ exceeds solid $O$ [Prop. 10.1]. Let them have been left, and let them be the (segments) on $E P, P F$, $F Q, Q G, G R, R H, H S$, and $S E$. Thus, the remaining pyramid whose base is polygon $E P F Q G R H S$, and apex the point $N$, is greater than solid $O$. And let the polygon $A T B U C V D W$, similar, and similarly laid out, to polygon $E P F Q G R H S$, have been inscribed in circle $A B C D$ [Prop. 6.18]. And let a pyramid having the same apex as the cone have been set up on polygon $A T B U C V D W$.





 $\dot{\eta} \mathrm{BK} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{~K} \Lambda$, oütcus $\dot{\eta} \mathrm{ZM} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{MN}$. xaì $\pi \varepsilon \rho \grave{i}$









 $\mathrm{BK} \tau \tilde{n} \mathrm{KT}, \dot{\eta} \delta$ ò $\mathrm{ZM} \tau \tilde{n} \mathrm{OM}$, है $\sigma \tau \nu$ äp $\alpha$ ìs $\dot{\eta} \mathrm{TK} \pi \rho o ̀ s$


































And let $L B T$ be one of the triangles containing the pyramid whose base is polygon $A T B U C V D W$, and apex the point $L$. And let $N F P$ be one of the triangles containing the pyramid whose base is triangle $E P F Q G R H S$, and apex the point $N$. And let $K T$ and $M P$ have been joined. And since cone $A B C D L$ is similar to cone $E F G H N$, thus as $B D$ is to $F H$, so axis $K L$ (is) to axis $M N$ [Def. 11.24]. And as $B D$ (is) to $F H$, so $B K$ (is) to $F M$. And, thus, as $B K$ (is) to $F M$, so $K L$ (is) to $M N$. And, alternately, as $B K$ (is) to $K L$, so $F M$ (is) to $M N$ [Prop. 5.16]. And the sides around the equal angles $B K L$ and $F M N$ are proportional. Thus, triangle $B K L$ is similar to triangle $F M N$ [Prop. 6.6]. Again, since as $B K$ (is) to $K T$, so $F M$ (is) to $M P$, and (they are) about the equal angles $B K T$ and $F M P$, inasmuch as whatever part angle $B K T$ is of the four right-angles at the center $K$, angle $F M P$ is also the same part of the four right-angles at the center $M$. Therefore, since the sides about equal angles are proportional, triangle $B K T$ is thus similar to traingle $F M P$ [Prop. 6.6]. Again, since it was shown that as $B K$ (is) to $K L$, so $F M$ (is) to $M N$, and $B K$ (is) equal to $K T$, and $F M$ to $P M$, thus as $T K$ (is) to $K L$, so $P M$ (is) to $M N$. And the sides about the equal angles $T K L$ and $P M N$-for (they are both) right-angles-are proportional. Thus, triangle $L K T$ (is) similar to triangle $N M P$ [Prop. 6.6]. And since, on account of the similarity of triangles $L K B$ and $N M F$, as $L B$ (is) to $B K$, so $N F$ (is) to $F M$, and, on account of the similarity of triangles $B K T$ and $F M P$, as $K B$ (is) to $B T$, so $M F$ (is) to $F P$ [Def. 6.1], thus, via equality, as $L B$ (is) to $B T$, so $N F$ (is) to $F P$ [Prop. 5.22]. Again, since, on account of the similarity of triangles $L T K$ and $N P M$, as $L T$ (is) to $T K$, so $N P$ (is) to $P M$, and, on account of the similarity of triangles $T K B$ and $P M F$, as $K T$ (is) to $T B$, so $M P$ (is) to $P F$, thus, via equality, as $L T$ (is) to $T B$, so $N P$ (is) to $P F$ [Prop. 5.22]. And it was shown that as $T B$ (is) to $B L$, so $P F$ (is) to $F N$. Thus, via equality, as $T L$ (is) to $L B$, so $P N$ (is) to $N F$ [Prop. 5.22]. Thus, the sides of triangles $L T B$ and NPF are proportional. Thus, triangles $L T B$ and NPF are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle $B K T$, and apex the point $L$, is similar to the pyramid whose base is triangle $F M P$, and apex the point $N$. For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid $B K T L$ has to pyramid $F M P N$ the cubed ratio that $B K$ (has) to $F M$. So, similarly, joining straight-lines from (points) $A, W, D, V, C$, and $U$ to (center) $K$, and from (points) $E, S, H, R, G$, and $Q$ to (center) $M$, and set-


 $\mu \varepsilon ̀ v ~ \tau o ̀ ~ E O Z П H P \Theta \Sigma ~ \pi o \lambda u ́ \gamma \omega \nu o v, ~ x o p u p \grave{\eta}$ ठè tò N бпиعі̃ov．













 $\pi \cup p \alpha \mu i ́ \delta \alpha, ~ \eta ̃ \varsigma ~ \beta \alpha ́ \sigma \iota \varsigma ~ \mu \varepsilon ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ E O Z П Н Р \Theta \Sigma ~ \pi o \lambda u ́ \gamma \omega v o v, ~$









 тpòs тr̀v $\mathrm{B} \Delta$ ．

 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{Z} \Theta$ ．
 tò $\Xi \sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi \rho o ̀ s ~ \tau o ̀ v ~ А В Г \Delta \Lambda ~ \varkappa \widetilde{\omega} \nu o \nu ~ \tau \rho เ \pi \lambda \alpha \sigma i ́ o v \alpha ~ \lambda o ́ \gamma o v ~$


 $\pi p o ̀ s ~ ह ै \lambda \alpha \tau \tau o ́ v ~ \tau \iota ~ \tau о u ̃ ~ А В Г \Delta \Lambda ~ x ढ ́ v o u ~ \sigma \tau \varepsilon p \varepsilon o ̀ v ~ \tau p ı \pi \lambda \alpha \sigma i ́ o v \alpha ~$




 グ $\pi \varepsilon \rho \dot{\eta} \mathrm{B} \Delta \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \mathrm{Z} \Theta$ ．



 тpòs tウ̀ $\nu \mathrm{Z} \Theta$ ．

ting up pyramids having the same apexes as the cones on each of the triangles（so formed），we can also show that each of the pyramids（on base $A B C D$ taken）in or－ der will have to each of the pyramids（on base $E F G H$ taken）in order the cubed ratio that the corresponding side $B K$（has）to the corresponding side $F M$－that is to say，that $B D$（has）to $F H$ ．And（for two sets of propor－ tional magnitudes）as one of the leading（magnitudes is） to one of the following，so（the sum of）all of the leading （magnitudes is）to（the sum of）all of the following（mag－ nitudes）［Prop．5．12］．And，thus，as pyramid BKTL（is） to pyramid $F M P N$ ，so the whole pyramid whose base is polygon $A T B U C V D W$ ，and apex the point $L$ ，（is）to the whole pyramid whose base is polygon $E P F Q G R H S$ ， and apex the point $N$ ．And，hence，the pyramid whose base is polygon $A T B U C V D W$ ，and apex the point $L$ ， has to the pyramid whose base is polygon $E P F Q G R H S$ ， and apex the point $N$ ，the cubed ratio that $B D$（has） to $F H$ ．And it was also assumed that the cone whose base is circle $A B C D$ ，and apex the point $L$ ，has to solid $O$ the cubed ratio that $B D$（has）to $F H$ ．Thus，as the cone whose base is circle $A B C D$ ，and apex the point $L$ ， is to solid $O$ ，so the pyramid whose base（is）［polygon］ $A T B U C V D W$ ，and apex the point $L$ ，（is）to the pyramid whose base is polygon $E P F Q G R H S$ ，and apex the point $N$ ．Thus，alternately，as the cone whose base（is）circle $A B C D$ ，and apex the point $L$ ，（is）to the pyramid within it whose base（is）the polygon $A T B U C V D W$ ，and apex the point $L$ ，so the［solid］$O$（is）to the pyramid whose base is polygon $E P F Q G R H S$ ，and apex the point $N$ ［Prop．5．16］．And the aforementioned cone（is）greater than the pyramid within it．For it encompasses it．Thus， solid $O$（is）also greater than the pyramid whose base is polygon $E P F Q G R H S$ ，and apex the point $N$ ．But，（it is）also less．The very thing is impossible．Thus，the cone whose base（is）circle $A B C D$ ，and apex the［point］$L$ ， does not have to some solid less than the cone whose base（is）circle $E F G H$ ，and apex the point $N$ ，the cubed ratio that $B D$（has）to $E H$ ．So，similarly，we can show that neither does cone $E F G H N$ have to some solid less than cone $A B C D L$ the cubed ratio that $F H$（has）to $B D$ ．

So，I say that neither does cone $A B C D L$ have to some solid greater than cone $E F G H N$ the cubed ratio that $B D$ （has）to $F H$ ．

For，if possible，let it have（such a ratio）to a greater （solid），$O$ ．Thus，inversely，solid $O$ has to cone $A B C D L$ the cubed ratio that $F H$（has）to $B D$［Prop． 5.7 corr．］． And as solid $O$（is）to cone $A B C D L$ ，so cone $E F G H N$ （is）to some solid less than cone $A B C D L$［12．2 lem．］． Thus，cone $E F G H N$ also has to some solid less than cone $A B C D L$ the cubed ratio that $F H$（has）to $B D$ ．The very



## $' \gamma^{\prime}$.













 oư $\beta \alpha ́ \sigma \varepsilon ı \varsigma ~ o i ~ О П, ~ \Phi X ~ x u ́ x \lambda o l . ~ x \alpha i ~ \varepsilon ̇ x \beta \varepsilon \beta \lambda \eta ́ \sigma \vartheta \omega ~ \delta i \alpha ̀ ~ \tau \widetilde{\omega} \nu$
 $\beta \alpha ́ \sigma \varepsilon \sigma \iota ~ \tau о \tilde{u}$ OX xu入ívסjpou xaì $\pi о เ \varepsilon i ́ t \omega \sigma \alpha \nu ~ \tau o u ̀ s ~ P \Sigma, ~ T \Upsilon ~$












thing was shown (to be) impossible. Thus, cone $A B C D L$ does not have to some solid greater than cone EFGHN the cubed ratio than $B D$ (has) to $F H$. And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone $A B C D L$ has to cone $E F G H N$ the cubed ratio that $B D$ (has) to $F G$.

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that $B D$ (has) to $F H$.

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.

## Proposition 13

If a cylinder is cut by a plane which is parallel to the opposite planes (of the cylinder) then as the cylinder (is) to the cylinder, so the axis will be to the axis.


For let the cylinder $A D$ have been cut by the plane $G H$ which is parallel to the opposite planes (of the cylinder), $A B$ and $C D$. And let the plane $G H$ have met the axis at point $K$. I say that as cylinder $B G$ is to cylinder $G D$, so axis $E K$ (is) to axis $K F$.

For let axis $E F$ have been produced in each direction to points $L$ and $M$. And let any number whatsoever (of lengths), $E N$ and $N L$, equal to axis $E K$, be set out (on the axis $E L$ ), and any number whatsoever (of lengths), $F O$ and $O M$, equal to (axis) $F K$, (on the axis $K M$ ). And let the cylinder $P W$, whose bases (are) the circles $P Q$ and $V W$, have been conceived on axis $L M$. And let planes parallel to $A B, C D$, and the bases of cylinder $P W$, have been produced through points $N$ and $O$, and let them have made the circles $R S$ and $T U$ around the centers $N$ and $O$ (respectively). And since axes $L N, N E$, and $E K$ are equal to one another, the cylinders $Q R, R B$, and $B G$ are to one another as their bases [Prop. 12.11]. But the bases are equal. Thus, the cylinders $Q R, R B$, and $B G$ (are) also equal to one another. Therefore, since the axes $L N, N E$, and $E K$ are equal to one another, and the cylinders $Q R, R B$, and $B G$ are also equal to one another, and the number (of the former) is equal to the number (of the latter), thus as many multiples as axis $K L$

 $\gamma \varepsilon \vartheta \widetilde{\omega} \nu$ oै$\tau \tau \omega \nu, \alpha \dot{\alpha} \xi o ́ v \omega \nu \mu \varepsilon ̀ \nu \tau \widetilde{\omega} \nu \mathrm{EK}, \mathrm{KZ}, x \cup \lambda i \downarrow \delta \rho \omega \nu$ ס̀̀ $\tau \widetilde{\omega} \nu$









$$
\delta^{\prime}
$$






 $\alpha \xi$ ova.












is of axis $E K$, so many multiples is cylinder $Q G$ also of cylinder $G B$. And so, for the same (reasons), as many multiples as axis $M K$ is of axis $K F$, so many multiples is cylinder $W G$ also of cylinder $G D$. And if axis $K L$ is equal to axis $K M$ then cylinder $Q G$ will also be equal to cylinder $G W$, and if the axis (is) greater than the axis then the cylinder (will also be) greater than the cylinder, and if (the axis is) less then (the cylinder will also be) less. So, there are four magnitudes-the axes $E K$ and $K F$, and the cylinders $B G$ and $G D$-and equal multiples have been taken of axis $E K$ and cylinder $B G$-(namely), axis $L K$ and cylinder $Q G$-and of axis $K F$ and cylinder $G D$-(namely), axis $K M$ and cylinder $G W$. And it has been shown that if axis $K L$ exceeds axis $K M$ then cylin$\operatorname{der} Q G$ also exceeds cylinder $G W$, and if (the axes are) equal then (the cylinders are) equal, and if ( $K L$ is) less then ( $Q G$ is) less. Thus, as axis $E K$ is to axis $K F$, so cylinder $B G$ (is) to cylinder $G D$ [Def. 5.5]. (Which is) the very thing it was required to show.

## Proposition 14

Cones and cylinders which are on equal bases are to one another as their heights.


For let $E B$ and $F D$ be cylinders on equal bases, (namely) the circles $A B$ and $C D$ (respectively). I say that as cylinder $E B$ is to cylinder $F D$, so axis $G H$ (is) to axis $K L$.

For let the axis $K L$ have been produced to point $N$. And let $L N$ be made equal to axis $G H$. And let the cylinder $C M$ have been conceived about axis $L N$. Therefore, since cylinders $E B$ and $C M$ have the same height they are to one another as their bases [Prop. 12.11]. And the bases are equal to one another. Thus, cylinders $E B$ and $C M$ are also equal to one another. And since cylinder $F M$ has been cut by the plane $C D$, which is parallel to its opposite planes, thus as cylinder $C M$ is to cylinder $F D$, so axis $L N$ (is) to axis $K L$ [Prop. 12.13]. And cylin$\operatorname{der} C M$ is equal to cylinder $E B$, and axis $L N$ to axis $G H$. Thus, as cylinder $E B$ is to cylinder $F D$, so axis $G H$ (is)





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to axis $K L$. And as cylinder $E B$ (is) to cylinder $F D$, so cone $A B G$ (is) to cone $C D K$ [Prop. 12.10]. Thus, also, as axis $G H$ (is) to axis $K L$, so cone $A B G$ (is) to cone $C D K$, and cylinder $E B$ to cylinder $F D$. (Which is) the very thing it was required to show.

## Proposition 15

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.


Let there be equal cones and cylinders whose bases are the circles $A B C D$ and $E F G H$, and the diameters of (the bases) $A C$ and $E G$, and (whose) axes (are) $K L$ and $M N$, which are also the heights of the cones and cylinders (respectively). And let the cylinders $A O$ and $E P$ have been completed. I say that the bases of cylinders $A O$ and $E P$ are reciprocally proportional to their heights, and (so) as base $A B C D$ is to base $E F G H$, so height $M N$ (is) to height $K L$.

For height $L K$ is either equal to height $M N$, or not. Let it, first of all, be equal. And cylinder $A O$ is also equal to cylinder $E P$. And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base $A B C D$ (is) also equal to base $E F G H$. And, hence, reciprocally, as base $A B C D$ (is) to base $E F G H$, so height $M N$ (is) to height $K L$. And so, let height $L K$ not be equal to $M N$, but let $M N$ be greater. And let $Q N$, equal to $K L$, have been cut off from height $M N$. And let the cylinder $E P$ have been cut, through point $Q$, by the plane TUS (which is) parallel to the planes of the circles $E F G H$ and $R P$. And let cylinder $E S$ have been conceived, with base the circle $E F G H$, and height $N Q$. And since cylinder $A O$ is equal to cylinder $E P$, thus, as cylinder $A O$ (is) to cylinder $E S$, so cylinder $E P$ (is) to cylinder $E S$ [Prop. 5.7]. But, as cylinder $A O$ (is) to cylinder $E S$, so base $A B C D$ (is) to base $E F G H$. For cylinders $A O$ and $E S$ (have) the same height [Prop. 12.11]. And as cylinder $E P$ (is) to (cylinder) $E S$, so height $M N$ (is) to height $Q N$. For cylinder $E P$ has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base $A B C D$ is to base $E F G H$, so height $M N$ (is) to height $Q N$ [Prop. 5.11]. And height $Q N$

 EZH $\beta \alpha \dot{\alpha} \sigma \iota$, oưt $\omega \varsigma$ tò MN ư $\psi o s ~ \pi \rho o ̀ s ~ \tau o ̀ ~ K \Lambda ~ u ̛ \psi o \varsigma \cdot ~ \lambda \varepsilon ́ \gamma \omega, ~$







 ư $\psi o \varsigma ~ \pi \rho o ̀ s ~ \tau o ̀ ~ \Pi N ~[u ̈ \psi o s], ~ o u ̈ \tau \omega \varsigma ~ o ́ ~ E O ~ x u ́ \lambda ı \nu \delta \rho o s ~ \pi \rho o ̀ s ~ \tau o ̀ v ~$





 غ̇Y $\gamma \rho \alpha ́ \psi \alpha ı ~ \mu n ̀ ~ \psi a v ̃ o v ~ t o u ̃ ~ غ ̀ \lambda \alpha ́ \sigma \sigma o v o s ~ x u ́ x \lambda o u . ~$












(is) equal to height $K L$. Thus, as base $A B C D$ is to base $E F G H$, so height $M N$ (is) to height $K L$. Thus, the bases of cylinders $A O$ and $E P$ are reciprocally proportional to their heights.

And, so, let the bases of cylinders $A O$ and $E P$ be reciprocally proportional to their heights, and (thus) let base $A B C D$ be to base $E F G H$, as height $M N$ (is) to height $K L$. I say that cylinder $A O$ is equal to cylinder $E P$.

For, with the same construction, since as base $A B C D$ is to base $E F G H$, so height $M N$ (is) to height $K L$, and height $K L$ (is) equal to height $Q N$, thus, as base $A B C D$ (is) to base $E F G H$, so height $M N$ will be to height $Q N$. But, as base $A B C D$ (is) to base $E F G H$, so cylinder $A O$ (is) to cylinder $E S$. For they are the same height [Prop. 12.11]. And as height $M N$ (is) to [height] $Q N$, so cylinder $E P$ (is) to cylinder $E S$ [Prop. 12.13]. Thus, as cylinder $A O$ is to cylinder $E S$, so cylinder $E P$ (is) to (cylinder) $E S$ [Prop. 5.11]. Thus, cylinder $A O$ (is) equal to cylinder $E P$ [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.

## Proposition 16

There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.


Let $A B C D$ and $E F G H$ be the given two circles, about the same center, $K$. So, it is necessary to inscribe an equilateral and even-sided polygon in the greater circle $A B C D$, not touching circle $E F G H$.

Let the straight-line $B K D$ have been drawn through the center $K$. And let $G A$ have been drawn, at rightangles to the straight-line $B D$, through point $G$, and let it have been drawn through to $C$. Thus, $A C$ touches circle $E F G H$ [Prop. 3.16 corr.]. So, (by) cutting circumference $B A D$ in half, and the half of it in half, and doing this continually, we will (eventually) leave a circumference less







 $\pi o เ \tilde{\eta} \sigma \alpha l$.
than $A D$ [Prop. 10.1]. Let it have been left, and let it be $L D$. And let $L M$ have been drawn, from $L$, perpendicular to $B D$, and let it have been drawn through to $N$. And let $L D$ and $D N$ have been joined. Thus, $L D$ is equal to $D N$ [Props. 3.3, 1.4]. And since $L N$ is parallel to $A C$ [Prop. 1.28], and $A C$ touches circle $E F G H, L N$ thus does not touch circle $E F G H$. Thus, even more so, $L D$ and $D N$ do not touch circle $E F G H$. And if we continuously insert (straight-lines) equal to straight-line $L D$ into circle $A B C D$ [Prop. 4.1] then an equilateral and evensided polygon, not touching the lesser circle $E F G H$, will have been inscribed in circle $A B C D .{ }^{\dagger}$ (Which is) the very thing it was required to do.
${ }^{\dagger}$ Note that the chord of the polygon, $L N$, does not touch the inner circle either.

$$
\zeta^{\prime}
$$

$\Delta$ úo $\sigma \varphi \alpha \iota \rho \widetilde{\omega} \nu ~ \pi \varepsilon p i ~ t o ̀ ~ \alpha u ̉ t o ̀ ~ \chi \varepsilon ́ v \tau p o v ~ o u ̉ \sigma \omega ̃ \nu ~ \varepsilon i \varsigma ~ \tau \eta ̀ \nu ~$














## Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.


Let two spheres have been conceived about the same center, $A$. So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres have been cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we conceive (of for) the semi-circle, the plane produced through it will make a






























犭úx入ou ह̇ $\pi i ́ \pi \varepsilon \delta \delta \nu$. $\pi เ \pi \tau \varepsilon ́ \tau \omega \sigma \alpha \nu, x \alpha \grave{~ ह ै \sigma \tau \omega \sigma \alpha \nu ~ \alpha i ~} \mathrm{O} \Phi, \Sigma \mathrm{X}$,




 $\pi \rho o ̀ s ~ \tau \grave{\nu} \nu \Phi \mathrm{~A}, ~ o u ̛ \tau \omega \varsigma \dot{\eta} \mathrm{KX} \pi \rho o ̀ s ~ \tau \eta \grave{\nu} \mathrm{XA} \cdot \pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o s$ «̈p $\alpha$













circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let $B C D E$ be the circle in the greater sphere, and $F G H$ the circle in the lesser sphere. And let two diameters of them have been drawn at right-angles to one another, (namely), $B D$ and $C E$. And there being two circles about the same center-(namely), $B C D E$ and $F G H$-let an equilateral and even-sided polygon have been inscribed in the greater circle, $B C D E$, not touching the lesser circle, $F G H$ [Prop. 12.16], of which let the sides in the quadrant $B E$ be $B K, K L, L M$, and $M E$. And, $K A$ being joined, let it have been drawn across to $N$. And let $A O$ have been set up at point $A$, at right-angles to the plane of circle $B C D E$. And let it meet the surface of the (greater) sphere at $O$. And let planes have been produced through $A O$ and each of $B D$ and $K N$. So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let $B O D$ and $K O N$ be semi-circles on the diameters $B D$ and $K N$ (respectively). And since $O A$ is at right-angles to the plane of circle $B C D E$, all of the planes through $O A$ are thus also at right-angles to the plane of circle $B C D E$ [Prop. 11.18]. And, hence, the semi-circles $B O D$ and $K O N$ are also at right-angles to the plane of circle $B C D E$. And since semi-circles $B E D$, $B O D$, and $K O N$ are equal-for (they are) on the equal diameters $B D$ and $K N$ [Def. 3.1]-the quadrants $B E$, $B O$, and $K O$ are also equal to one another. Thus, as many sides of the polygon as are in quadrant $B E$, so many are also in quadrants $B O$ and $K O$ equal to the straight-lines $B K, K L, L M$, and $M E$. Let them have been inscribed, and let them be $B P, P Q, Q R, R O, K S$, $S T, T U$, and $U O$. And let $S P, T Q$, and $U R$ have been joined. And let perpendiculars have been drawn from $P$ and $S$ to the plane of circle $B C D E$ [Prop. 11.11]. So, they will fall on the common sections of the planes $B D$ and $K N$ (with $B C D E$ ), inasmuch as the planes of $B O D$ and $K O N$ are also at right-angles to the plane of circle $B C D E$ [Def. 11.4]. Let them have fallen, and let them be $P V$ and $S W$. And let $W V$ have been joined. And since $B P$ and $K S$ are equal (circumferences) having been cut off in the equal semi-circles $B O D$ and $K O N$ [Def. 3.28], and $P V$ and $S W$ are perpendiculars having been drawn (from them), $P V$ is [thus] equal to $S W$, and $B V$ to $K W$ [Props. 3.27, 1.26]. And the whole of $B A$ is also equal to the whole of $K A$. And, thus, as $B V$ is to $V A$, so $K W$ (is) to $W A$. $W V$ is thus parallel to $K B$ [Prop. 6.2]. And


 $\beta \alpha ́ \sigma \varepsilon ı \varsigma ~ \mu \varepsilon ̀ \nu ~ \tau \alpha ̀ ~ K B O \Sigma, ~ \Sigma О П Т, ~ Т П Р \Upsilon ~ \tau \varepsilon \tau p \alpha ́ \pi \lambda \varepsilon \cup p \alpha ~ x \alpha i ̀ ~ \tau o ̀ ~$








 ZH丹 хи́х入оร．
＂Н $\chi \vartheta \omega$ 人̇лò тoũ A $\sigma \eta \mu \varepsilon i ́ o u ~ \varepsilon ̇ \pi i ~ \tau o ̀ ~ \tau o u ̃ ~ K B O \Sigma ~ \tau \varepsilon-~$









 $\dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{~A} \Psi, \Psi K$ ．xolvòv $\alpha \varphi n \rho \eta ́ \sigma \vartheta \omega$ tò $\alpha \pi o ̀ ~ \tau \tilde{\eta} \varsigma ~ A \Psi \cdot ~ \lambda o l \pi o ̀ v ~$
















 $\pi \alpha p \alpha \lambda \lambda \eta \lambda$ оүра́ $\mu \mu$ и каì tò ú ùò $\Delta \mathrm{B}, \mathrm{B} \Omega$ व̈p $\alpha$ тоũ ú $\pi$ ò $\Delta \Omega$ ，







since $P V$ and $S W$ are each at right－angles to the plane of circle $B C D E, P V$ is thus parallel to $S W$［Prop．11．6］． And it was also shown（to be）equal to it．And，thus， $W V$ and $S P$ are equal and parallel［Prop．1．33］．And since $W V$ is parallel to $S P$ ，but $W V$ is parallel to $K B$ ， $S P$ is thus also parallel to $K B$［Prop．11．1］．And $B P$ and $K S$ join them．Thus，the quadrilateral $K B P S$ is in one plane，inasmuch as if there are two parallel straight－ lines，and a random point is taken on each of them，then the straight－line joining the points is in the same plane as the parallel（straight－lines）［Prop．11．7］．So，for the same（reasons），each of the quadrilaterals $S P Q T$ and $T Q R U$ is also in one plane．And triangle $U R O$ is also in one plane［Prop．11．2］．So，if we conceive straight－ lines joining points $P, S, Q, T, R$ ，and $U$ to $A$ then some solid polyhedral figure will have been constructed between the circumferences $B O$ and $K O$ ，being com－ posed of pyramids whose bases（are）the quadrilaterals $K B P S, S P Q T, T Q R U$ ，and the triangle $U R O$ ，and apex the point $A$ ．And if we also make the same construction on each of the sides $K L, L M$ ，and $M E$ ，just as on $B K$ ， and，further，（repeat the construction）in the remaining three quadrants，then some polyhedral figure which has been inscribed in the sphere will have been constructed， being contained by pyramids whose bases（are）the afore－ mentioned quadrilaterals，and triangle $U R O$ ，and the （quadrilaterals and triangles）similarly arranged to them， and apex the point $A$ ．

So，I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle $F G H$ is（situated）．

Let the perpendicular（straight－line）$A X$ have been drawn from point $A$ to the plane $K B P S$ ，and let it meet the plane at point $X$［Prop．11．11］．And let $X B$ and $X K$ have been joined．And since $A X$ is at right－angles to the plane of quadrilateral $K B P S$ ，it is thus also at right－ angles to all of the straight－lines joined to it which are also in the plane of the quadrilateral［Def．11．3］．Thus， $A X$ is at right－angles to each of $B X$ and $X K$ ．And since $A B$ is equal to $A K$ ，the（square）on $A B$ is also equal to the（square）on $A K$ ．And the（sum of the squares）on $A X$ and $X B$ is equal to the（square）on $A B$ ．For the angle at $X$（is）a right－angle［Prop．1．47］．And the（sum of the squares）on $A X$ and $X K$ is equal to the（square）on $A K$［Prop．1．47］．Thus，the（sum of the squares）on $A X$ and $X B$ is equal to the（sum of the squares）on $A X$ and $X K$ ．Let the（square）on $A X$ have been subtracted from both．Thus，the remaining（square）on $B X$ is equal to the remaining（square）on $X K$ ．Thus，$B X$（is）equal to $X K$ ． So，similarly，we can show that the straight－lines joined from $X$ to $P$ and $S$ are equal to each of $B X$ and $X K$ ．
 $\tau \tilde{\eta} \varsigma \mathrm{KA}$ ' $\sigma \alpha \tau \dot{\alpha} \dot{\alpha} \pi \grave{o} \tau \widetilde{\omega} \nu \mathrm{~K} \Omega, \Omega \mathrm{~A} \cdot \tau \dot{\alpha} \alpha \not \alpha \alpha \alpha \dot{\alpha} \pi o ̀ \tau \widetilde{\omega} \nu \mathrm{~B} \Psi, \Psi \mathrm{~A}$

 тои̃ $\alpha \pi o ̀ ~ \tau \tilde{\eta} s \Psi A . \mu \varepsilon i \zeta \omega \nu$ 解 $\alpha \dot{\eta} A \Psi \tau \tilde{\eta} s A \Omega \cdot \pi o \lambda \lambda \widetilde{\varphi} \alpha{ }^{\alpha} p \alpha \dot{\eta}$




$\Delta u ́ o ~ \alpha ้ p \alpha ~ \sigma \varphi \alpha ı \rho \widetilde{\omega} \nu ~ \pi \varepsilon p i ̀ ~ t o ̀ ~ \alpha u ̉ \tau o ̀ ~ \chi e ́ v \tau \rho o v ~ o u ̉ \sigma \widetilde{\omega} \nu ~ \varepsilon i ऽ ~ \tau \grave{\nu}$ $\mu \varepsilon i \zeta o v \alpha ~ \sigma \varphi \alpha i ̃ \rho \alpha \nu ~ \sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \pi о \lambda u ́ \varepsilon \delta \rho o v ~ \varepsilon ̇ \gamma \gamma \varepsilon ́ \gamma p \alpha \pi \tau \alpha l ~ \mu \grave{\eta} \psi \alpha u ̃ o v$
 $\pi o เ ท ̃ \sigma \alpha l$.

Thus, a circle drawn (in the plane of the quadrilateral) with center $X$, and radius one of $X B$ or $X K$, will also pass through $P$ and $S$, and the quadrilateral $K B P S$ will be inside the circle.

And since $K B$ is greater than $W V$, and $W V$ (is) equal to $S P, K B$ (is) thus greater than $S P$. And $K B$ (is) equal to each of $K S$ and $B P$. Thus, $K S$ and $B P$ are each greater than $S P$. And since quadrilateral $K B P S$ is in a circle, and $K B, B P$, and $K S$ are equal (to one another), and $P S$ (is) less (than them), and $B X$ is the radius of the circle, the (square) on $K B$ is thus greater than double the (square) on $B X .{ }^{\dagger}$ Let the perpendicular $K Y$ have been drawn from $K$ to $B V . \ddagger$ And since $B D$ is less than double $D Y$, and as $B D$ is to $D Y$, so the (rectangle contained) by $D B$ and $B Y$ (is) to the (rectangle contained) by $D Y$ and $Y B$-a square being described on $B Y$, and a (rectangular) parallelogram (with short side equal to $B Y$ ) completed on $Y D$-the (rectangle contained) by $D B$ and $B Y$ is thus also less than double the (rectangle contained) by $D Y$ and $Y B$. And, $K D$ being joined, the (rectangle contained) by $D B$ and $B Y$ is equal to the (square) on $B K$, and the (rectangle contained) by $D Y$ and $Y B$ equal to the (square) on $K Y$ [Props. 3.31, 6.8 corr.]. Thus, the (square) on $K B$ is less than double the (square) on $K Y$. But, the (square) on $K B$ is greater than double the (square) on $B X$. Thus, the (square) on $K Y$ (is) greater than the (square) on $B X$. And since $B A$ is equal to $K A$, the (square) on $B A$ is equal to the (square) on $A K$. And the (sum of the squares) on $B X$ and $X A$ is equal to the (square) on $B A$, and the (sum of the squares) on $K Y$ and $Y A$ (is) equal to the (square) on $K A$ [Prop. 1.47]. Thus, the (sum of the squares) on $B X$ and $X A$ is equal to the (sum of the squares) on $K Y$ and $Y A$, of which the (square) on $K Y$ (is) greater than the (square) on $B X$. Thus, the remaining (square) on $Y A$ is less than the (square) on $X A$. Thus, $A X$ (is) greater than $A Y$. Thus, $A X$ is much greater than $A G .{ }^{\S}$ And $A X$ is (a perpendicular) on one of the bases of the polyhedron, and $A G$ (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do.

[^37]
## Пópıбนа.


























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 ¿סí $\omega \nu$ ठı $\alpha \mu$ ѓтр $\omega \nu$.


## Corollary

And, also, if a similar polyhedral solid to that in sphere $B C D E$ is inscribed in another sphere then the polyhedral solid in sphere $B C D E$ has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere $B C D E$ has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral $K B P S$, and apex the point $A$, will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius $A B$ of the sphere about center $A$ to the radius of the other sphere. And, similarly, each pyramid in the sphere about center $A$ will have to each similarly situated pyramid in the other sphere the cubed ratio that $A B$ (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center $A$ will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) $A B$ (has) to the radius of the other sphere. That is to say, that diameter $B D$ (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.

## Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.


Nevoń $\sigma \vartheta \omega \sigma \alpha \nu ~ \sigma \varphi \alpha i ̃ p \alpha l ~ \alpha i ~ A B \Gamma, ~ \Delta E Z, ~ \delta \iota \alpha ́ \mu \varepsilon \tau \rho o l ~ \delta e ̀ ~$



Ei $\gamma \dot{\alpha} \rho \mu \dot{\eta} \dot{\eta} \mathrm{AB}$ б $\varphi \alpha i ̃ \rho \alpha \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Delta \mathrm{EZ} \sigma \varphi \alpha i ̃ \rho \alpha \nu \tau \rho l-$
 АВГ $\sigma \varphi \alpha i ̃ \rho \alpha ~ \pi \rho o ̀ s ~ ह ̀ \lambda \alpha \alpha ́ \sigma \sigma o v \alpha ́ ~ \tau \tau v \alpha ~ \tau \tilde{\eta} s ~ \Delta \mathrm{EZ} \mathrm{\sigma} \mathrm{\varphi} \mathrm{\alpha ip} \mathrm{\alpha s} \mathrm{\tau pı-}$ $\pi \lambda \alpha \sigma$ óov $\lambda$ 入óүov $\ddot{\eta} \pi \rho o ̀ s ~ \mu \varepsilon i \zeta o v \alpha ~ \eta ̈ \pi \varepsilon \rho ~ \dot{\eta}$ ВГ $\pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ E Z . ~$
 $\Delta \mathrm{EZ} \tau \tilde{\eta} \mathrm{H} \Theta \mathrm{K} \pi \varepsilon \rho i ̀ ~ t o ̀ ~ \alpha u ̉ \tau o ̀ ~ \chi \varepsilon ́ v \tau p o v, ~ \chi \alpha i ̀ ~ \varepsilon ̀ \gamma \gamma \varepsilon \gamma \rho \alpha ́ \varphi \vartheta \omega ~ \varepsilon i \varsigma ~$


















 $\lambda o ́ \gamma o v$ है $\chi \varepsilon \iota$ グ $\pi \varepsilon \rho \dot{\eta} \mathrm{EZ} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu ~ В Г . ~$

 EZ．


 $\dot{\omega} \varsigma ~ \delta \grave{̀} \dot{\eta} \Lambda \mathrm{MN} \sigma \varphi \alpha \tilde{\rho} \rho \alpha \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{AB} \mathrm{\Gamma} \sigma \varphi \alpha i ̃ \rho \alpha \nu$ ，oữ $\omega \varsigma \dot{\eta} \Delta \mathrm{EZ}$


 трı $\pi \lambda \alpha \sigma$ iova $\lambda o ́ \gamma o v$ है $\chi \varepsilon \iota ~ ク ̈ \pi \varepsilon \rho ~ \grave{\eta} \mathrm{EZ} \pi \rho o ̀ s ~ \tau \eta ̀ \nu ~ B \Gamma . ~ o ̈ \pi \varepsilon \rho ~$






Let the spheres $A B C$ and $D E F$ have been conceived， and（let）their diameters（be）$B C$ and $E F$（respectively）． I say that sphere $A B C$ has to sphere $D E F$ the cubed ratio that $B C$（has）to $E F$ ．

For if sphere $A B C$ does not have to sphere $D E F$ the cubed ratio that $B C$（has）to $E F$ then sphere $A B C$ will have to some（sphere）either less than，or greater than， sphere $D E F$ the cubed ratio that $B C$（has）to $E F$ ．Let it，first of all，have（such a ratio）to a lesser（sphere）， $G H K$ ．And let $D E F$ have been conceived about the same center as $G H K$ ．And let a polyhedral solid have been inscribed in the greater sphere $D E F$ ，not touching the lesser sphere $G H K$ on its surface［Prop．12．17］．And let a polyhedral solid，similar to the polyhedral solid in sphere $D E F$ ，have also been inscribed in sphere $A B C$ ． Thus，the polyhedral solid in sphere $A B C$ has to the polyhedral solid in sphere $D E F$ the cubed ratio that $B C$ （has）to $E F$［Prop． 12.17 corr．］．And sphere $A B C$ also has to sphere $G H K$ the cubed ratio that $B C$（has）to $E F$ ． Thus，as sphere $A B C$ is to sphere $G H K$ ，so the polyhe－ dral solid in sphere $A B C$（is）to the polyhedral solid is sphere $D E F$ ．［Thus］，alternately，as sphere $A B C$（is）to the polygon within it，so sphere $G H K$（is）to the polyhe－ dral solid within sphere $D E F$［Prop．5．16］．And sphere $A B C$（is）greater than the polyhedron within it．Thus， sphere $G H K$（is）also greater than the polyhedron within sphere $D E F$［Prop．5．14］．But，（it is）also less．For it is encompassed by it．Thus，sphere $A B C$ does not have to （a sphere）less than sphere $D E F$ the cubed ratio that di－ ameter $B C$（has）to $E F$ ．So，similarly，we can show that sphere $D E F$ does not have to（a sphere）less than sphere $A B C$ the cubed ratio that $E F$（has）to $B C$ either．

So，I say that sphere $A B C$ does not have to some （sphere）greater than sphere $D E F$ the cubed ratio that $B C$（has）to $E F$ either．

For，if possible，let it have（the cubed ratio）to a greater（sphere），$L M N$ ．Thus，inversely，sphere $L M N$ （has）to sphere $A B C$ the cubed ratio that diameter $E F$（has）to diameter $B C$［Prop． 5.7 corr．］．And as sphere $L M N$（is）to sphere $A B C$ ，so sphere $D E F$ （is）to some（sphere）less than sphere $A B C$ ，inasmuch as $L M N$ is greater than $D E F$ ，as was shown before ［Prop． 12.2 lem．］．And，thus，sphere $D E F$ has to some （sphere）less than sphere $A B C$ the cubed ratio that $E F$ （has）to $B C$ ．The very thing was shown（to be）impossi－ ble．Thus，sphere $A B C$ does not have to some（sphere） greater than sphere $D E F$ the cubed ratio that $B C$（has） to $E F$ ．And it was shown that neither（does it have such a ratio）to a lesser（sphere）．Thus，sphere $A B C$ has to sphere $D E F$ the cubed ratio that $B C$（has）to $E F$ ． （Which is）the very thing it was required to show．

# ELEMENTS BOOK 13 

## The Platonic Solids

[^38]
## $\alpha^{\prime}$.

 tò $\mu \varepsilon і ̈ \zeta o \nu ~ \tau \mu \tilde{n} \mu \alpha ~ \pi \rho о \sigma \lambda \alpha \beta o ̀ v ~ \tau \eta ̀ \nu ~ n ́ \mu i ́ \sigma \varepsilon เ \alpha \nu ~ \tau n ̃ ऽ ~ o ̈ \lambda \eta S ~ \pi \varepsilon v-~$





 тò $\dot{\alpha} \pi o ̀ ~ \tau \tilde{n} \varsigma \Gamma \Delta$ тOŨ $\alpha \pi o ̀ ~ \tau n ̃ 今 ~ \Delta A . ~$

















 $\dot{\alpha} \pi o ̀ ~ \tau \eta ̃ s ~ \Delta \mathrm{~A}$.




## Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.


For let the straight-line $A B$ have been cut in extreme and mean ratio at point $C$, and let $A C$ be the greater piece. And let the straight-line $A D$ have been produced in a straight-line with $C A$. And let $A D$ be made (equal to) half of $A B$. I say that the (square) on $C D$ is five times the (square) on $D A$.

For let the squares $A E$ and $D F$ have been described on $A B$ and $D C$ (respectively). And let the figure in $D F$ have been drawn. And let $F C$ have been drawn across to $G$. And since $A B$ has been cut in extreme and mean ratio at $C$, the (rectangle contained) by $A B C$ is thus equal to the (square) on $A C$ [Def. 6.3, Prop. 6.17]. And $C E$ is the (rectangle contained) by $A B C$, and $F H$ the (square) on $A C$. Thus, $C E$ (is) equal to $F H$. And since $B A$ is double $A D$, and $B A$ (is) equal to $K A$, and $A D$ to $A H$, $K A$ (is) thus also double $A H$. And as $K A$ (is) to $A H$, so $C K$ (is) to $C H$ [Prop. 6.1]. Thus, $C K$ (is) double $C H$. And $L H$ plus $H C$ is also double $C H$ [Prop. 1.43]. Thus, $K C$ (is) equal to $L H$ plus $H C$. And $C E$ was also shown (to be) equal to $H F$. Thus, the whole square $A E$ is equal to the gnomon $M N O$. And since $B A$ is double $A D$, the (square) on $B A$ is four times the (square) on $A D$-that is to say, $A E$ (is four times) $D H$. And $A E$ (is) equal to gnomon $M N O$. And, thus, gnomon $M N O$ is also four times $A P$. Thus, the whole of $D F$ is five times $A P$. And $D F$ is the (square) on $D C$, and $A P$ the (square) on $D A$. Thus, the (square) on $C D$ is five times the (square) on $D A$.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of

## $\beta^{\prime}$ ．

＇Еฝ̀⿱




 $\tau \alpha \pi \lambda \alpha ́ \sigma \iota \circ$ ठuvá $\sigma \vartheta \omega$ ，$\tau \tilde{\eta} \varsigma ~ \delta \grave{\varepsilon} ~ А \Gamma ~ \delta \iota \pi \lambda \tilde{\eta}$ है $\sigma \tau \omega \dot{\eta} \Gamma \Delta$ ．$\lambda \varepsilon ́ \gamma \omega$ ，
 $\tau \mu \tilde{\eta} \mu \alpha ́ \dot{\varepsilon}$ धัбть $\dot{\eta}$ ГВ．


 тои̃ $\alpha \pi o ̀ ~ \tau \tilde{n} \varsigma ~ A \Gamma, \pi \varepsilon \nu \tau \alpha \pi \lambda \alpha ́ \sigma เ o ́ v ~ \varepsilon ̇ \sigma \tau \iota ~ \tau o ̀ ~ A Z ~ \tau о \tilde{u} A \Theta$ ．тє－














 $\tau \mu \tilde{\eta} \mu \alpha ́ \alpha$ モ̇бтル $\dot{\eta}$ ГВ．

 $\mu \varepsilon ́ \sigma o \nu ~ \lambda o ́ \gamma o v ~ \tau \varepsilon \mu \nu о \mu \varepsilon ́ v \eta s ~ \tau o ̀ ~ \mu \varepsilon і ̈ \zeta o \nu ~ \tau \mu n ̃ \mu \alpha ~ \tau o ̀ ~ \lambda o เ \pi o ̀ v ~ \mu \varepsilon ́ p o s ~$
the whole，is five times the square on the half．（Which is） the very thing it was required to show．

## Proposition 2

If the square on a straight－line is five times the （square）on a piece of it，and double the aforementioned piece is cut in extreme and mean ratio，then the greater piece is the remaining part of the original straight－line．


For let the square on the straight－line $A B$ be five times the（square）on the piece of it，$A C$ ．And let $C D$ be double $A C$ ．I say that if $C D$ is cut in extreme and mean ratio then the greater piece is $C B$ ．

For let the squares $A F$ and $C G$ have been described on each of $A B$ and $C D$（respectively）．And let the figure in $A F$ have been drawn．And let $B E$ have been drawn across．And since the（square）on $B A$ is five times the （square）on $A C, A F$ is five times $A H$ ．Thus，gnomon $M N O$（is）four times $A H$ ．And since $D C$ is double $C A$ ， the（square）on $D C$ is thus four times the（square）on $C A$－that is to say，$C G$（is four times）$A H$ ．And the gnomon $M N O$ was also shown（to be）four times $A H$ ． Thus，gnomon $M N O$（is）equal to $C G$ ．And since $D C$ is double $C A$ ，and $D C$（is）equal to $C K$ ，and $A C$ to $C H$ ， ［ $K C$（is）thus also double $C H$ ］，（and）$K B$（is）also dou－ ble $B H$［Prop．6．1］．And $L H$ plus $H B$ is also double $H B$ ［Prop．1．43］．Thus，$K B$（is）equal to $L H$ plus $H B$ ．And the whole gnomon $M N O$ was also shown（to be）equal to the whole of $C G$ ．Thus，the remainder $H F$ is also equal to（the remainder）$B G$ ．And $B G$ is the（rectangle contained）by $C D B$ ．For $C D$（is）equal to $D G$ ．And $H F$ （is）the square on $C B$ ．Thus，the（rectangle contained） by $C D B$ is equal to the（square）on $C B$ ．Thus，as $D C$ is to $C B$ ，so $C B$（is）to $B D$［Prop．6．17］．And $D C$（is） greater than $C B$（see lemma）．Thus，$C B$（is）also greater than $B D$［Prop．5．14］．Thus，if the straight－line $C D$ is cut

$\Lambda \tilde{\eta} \mu \mu \alpha$.
 ठєıxtéov.




 oưx $\alpha p \alpha \dot{\eta}$ ГВ $\delta เ \pi \lambda \alpha \sigma i ́ \alpha ~ \varepsilon ̇ \sigma \tau i ~ \tau \tilde{\eta} s ~ А Г . ~ o ́ \mu o i ́ \omega s ~ \delta \grave{\eta} \delta \varepsilon i \xi o \mu \varepsilon v$,
 үàp [ $\mu \varepsilon$ І̃ॅov] тò öтолоv.
 ठєi̧̋al.


 $\mu \varepsilon і \zeta о \nu о \varsigma ~ т \mu и ́ \mu \alpha \tau о \varsigma ~ \tau \varepsilon т р \alpha \gamma ஸ ́ v o u . ~$






in extreme and mean ratio then the greater piece is $C B$.
Thus, if the square on a straight-line is five times the (square) on a piece of itself, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line. (Which is) the very thing it was required to show.

## Lemma

And it can be shown that double $A C$ (i.e., $D C$ ) is greater than $B C$, as follows.

For if (double $A C$ is) not (greater than $B C$ ), if possible, let $B C$ be double $C A$. Thus, the (square) on $B C$ (is) four times the (square) on $C A$. Thus, the (sum of) the (squares) on $B C$ and $C A$ (is) five times the (square) on $C A$. And the (square) on $B A$ was assumed (to be) five times the (square) on $C A$. Thus, the (square) on $B A$ is equal to the (sum of) the (squares) on $B C$ and $C A$. The very thing (is) impossible [Prop. 2.4]. Thus, $C B$ is not double $A C$. So, similarly, we can show that a (straightline) less than $C B$ is not double $A C$ either. For (in this case) the absurdity is much [greater].

Thus, double $A C$ is greater than $C B$. (Which is) the very thing it was required to show.

## Proposition 3

If a straight-line is cut in extreme and mean ratio then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.


For let some straight-line $A B$ have been cut in extreme and mean ratio at point $C$. And let $A C$ be the greater piece. And let $A C$ have been cut in half at $D$. I say that the (square) on $B D$ is five times the (square) on $D C$.



















## $\delta^{\prime}$.



 тетраү⿳㇒⿵冂一兀寸。и．










For let the square $A E$ have been described on $A B$ ． And let the figure have been drawn double．Since $A C$ is double $D C$ ，the（square）on $A C$（is）thus four times the （square）on $D C$－that is to say，$R S$（is four times）$F G$ ． And since the（rectangle contained）by $A B C$ is equal to the（square）on $A C$［Def．6．3，Prop．6．17］，and $C E$ is the （rectangle contained）by $A B C, C E$ is thus equal to $R S$ ． And $R S$（is）four times $F G$ ．Thus，$C E$（is）also four times $F G$ ．Again，since $A D$ is equal to $D C, H K$ is also equal to $K F$ ．Hence，square $G F$ is also equal to square $H L$ ．Thus， $G K$（is）equal to $K L$－that is to say，$M N$ to $N E$ ．Hence， $M F$ is also equal to $F E$ ．But，$M F$ is equal to $C G$ ．Thus， $C G$ is also equal to $F E$ ．Let $C N$ have been added to both．Thus，gnomon $O P Q$ is equal to $C E$ ．But，$C E$ was shown（to be）equal to four times $G F$ ．Thus，gnomon $O P Q$ is also four times square $F G$ ．Thus，gnomon $O P Q$ plus square $F G$ is five times $F G$ ．But，gnomon $O P Q$ plus square $F G$ is（square）$D N$ ．And $D N$ is the（square）on $D B$ ，and $G F$ the（square）on $D C$ ．Thus，the（square）on $D B$ is five times the（square）on $D C$ ．（Which is）the very thing it was required to show．

## Proposition 4

If a straight－line is cut in extreme and mean ratio then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece．


Let $A B$ be a straight－line，and let it have been cut in extreme and mean ratio at $C$ ，and let $A C$ be the greater piece．I say that the（sum of the squares）on $A B$ and $B C$ is three times the（square）on $C A$ ．

For let the square $A D E B$ have been described on $A B$ ， and let the（remainder of the）figure have been drawn． Therefore，since $A B$ has been cut in extreme and mean ratio at $C$ ，and $A C$ is the greater piece，the（rectangle





 $\gamma \nu \omega ́ \mu \omega \nu$ каі [七ò ГК тєтра́ү $\omega v o \nu \delta \iota \pi \lambda \alpha ́ \sigma เ \alpha ́ \alpha ~ \varepsilon ̇ \sigma \tau \iota ~ \tau о \tilde{u} \Theta H \cdot$







## $\varepsilon^{\prime}$.



 $\alpha \dot{\alpha} \chi \tilde{n} \varsigma ~ \varepsilon \cup ̉ \cup \varepsilon \varepsilon i \alpha . ~$














contained) by $A B C$ is thus equal to the (square) on $A C$ [Def. 6.3, Prop. 6.17]. And $A K$ is the (rectangle contained) by $A B C$, and $H G$ the (square) on $A C$. Thus, $A K$ is equal to $H G$. And since $A F$ is equal to $F E$ [Prop. 1.43], let $C K$ have been added to both. Thus, the whole of $A K$ is equal to the whole of $C E$. Thus, $A K$ plus $C E$ is double $A K$. But, $A K$ plus $C E$ is the gnomon $L M N$ plus the square $C K$. Thus, gnomon $L M N$ plus square $C K$ is double $A K$. But, indeed, $A K$ was also shown (to be) equal to $H G$. Thus, gnomon $L M N$ plus [square $C K$ is double $H G$. Hence, gnomon $L M N$ plus] the squares $C K$ and $H G$ is three times the square $H G$. And gnomon $L M N$ plus the squares $C K$ and $H G$ is the whole of $A E$ plus $C K$-which are the squares on $A B$ and $B C$ (respectively) - and $G H$ (is) the square on $A C$. Thus, the (sum of the) squares on $A B$ and $B C$ is three times the square on $A C$. (Which is) the very thing it was required to show.

## Proposition 5

If a straight-line is cut in extreme and mean ratio, and a (straight-line) equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.


For let the straight-line $A B$ have been cut in extreme and mean ratio at point $C$. And let $A C$ be the greater piece. And let $A D$ be [made] equal to $A C$. I say that the straight-line $D B$ has been cut in extreme and mean ratio at $A$, and that the original straight-line $A B$ is the greater piece.

For let the square $A E$ have been described on $A B$, and let the (remainder of the) figure have been drawn. And since $A B$ has been cut in extreme and mean ratio at $C$, the (rectangle contained) by $A B C$ is thus equal to the (square) on $A C$ [Def. 6.3, Prop. 6.17]. And $C E$ is the (rectangle contained) by $A B C$, and $C H$ the (square) on $A C$. But, $H E$ is equal to $C E$ [Prop. 1.43], and $D H$ equal

 $\pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \mathrm{BA}$, oưt $\omega \varsigma \dot{\eta} \mathrm{BA} \pi \rho o ̀ \varsigma ~ \tau \grave{\eta} \nu \mathrm{~A} \Delta$. $\mu \varepsilon$ í̧ $\omega \nu$ ठè $\dot{\eta} \Delta \mathrm{B}$ $\tau \tilde{\eta} s \mathrm{BA} \cdot \mu \varepsilon i \zeta \omega \nu$ "̈ $\rho \alpha$ к $\dot{\eta} \dot{\eta} \mathrm{BA} \tau \tilde{\eta} s \mathrm{~A} \Delta$.



## $\varsigma^{\prime}$.


 тои'.












 тò $\alpha \pi o ̀ ~ \Delta A ~ \lambda o ́ \gamma o v ~ o u ̉ x ~ ह ै \chi \varepsilon เ, ~ o ̂ v ~ \tau \varepsilon \tau р \alpha ́ \gamma \omega v o s ~ \alpha ̉ p ı \vartheta \mu o ̀ s ~ \pi \rho o ̀ s ~$







 б̀̀ каi $\dot{\eta}$ ГА $\dot{\alpha} \pi о \tau о \mu \dot{\eta}$.



to $H C$. Thus, $D H$ is also equal to $H E$. [Let $H B$ have been added to both.] Thus, the whole of $D K$ is equal to the whole of $A E$. And $D K$ is the (rectangle contained) by $B D$ and $D A$. For $A D$ (is) equal to $D L$. And $A E$ (is) the (square) on $A B$. Thus, the (rectangle contained) by $B D A$ is equal to the (square) on $A B$. Thus, as $D B$ (is) to $B A$, so $B A$ (is) to $A D$ [Prop. 6.17]. And $D B$ (is) greater than $B A$. Thus, $B A$ (is) also greater than $A D$ [Prop. 5.14].

Thus, $D B$ has been cut in extreme and mean ratio at $A$, and the greater piece is $A B$. (Which is) the very thing it was required to show.

## Proposition 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straightline) called an apotome.


Let $A B$ be a rational straight-line cut in extreme and mean ratio at $C$, and let $A C$ be the greater piece. I say that $A C$ and $C B$ is each that irrational (straight-line) called an apotome.

For let $B A$ have been produced, and let $A D$ be made (equal) to half of $B A$. Therefore, since the straightline $A B$ has been cut in extreme and mean ratio at $C$, and $A D$, which is half of $A B$, has been added to the greater piece $A C$, the (square) on $C D$ is thus five times the (square) on $D A$ [Prop. 13.1]. Thus, the (square) on $C D$ has to the (square) on $D A$ the ratio which a number (has) to a number. The (square) on $C D$ (is) thus commensurable with the (square) on $D A$ [Prop. 10.6]. And the (square) on $D A$ (is) rational. For $D A$ [is] rational, being half of $A B$, which is rational. Thus, the (square) on $C D$ (is) also rational [Def. 10.4]. Thus, $C D$ is also rational. And since the (square) on $C D$ does not have to the (square) on $D A$ the ratio which a square number (has) to a square number, $C D$ (is) thus incommensurable in length with $D A$ [Prop. 10.9]. Thus, $C D$ and $D A$ are rational (straight-lines which are) commensurable in square only. Thus, $A C$ is an apotome [Prop. 10.73]. Again, since $A B$ has been cut in extreme and mean ratio, and $A C$ is the greater piece, the (rectangle contained) by $A B$ and $B C$ is thus equal to the (square) on $A C$ [Def. 6.3, Prop. 6.17]. Thus, the (square) on the apotome $A C$, applied to the rational (straight-line) $A B$, makes $B C$ as width. And the (square) on an apotome, applied to a rational (straight-line), makes a first apotome as width [Prop. 10.97]. Thus, $C B$ is a first apotome. And $C A$ was also shown (to be) an apotome.

## $\zeta^{\prime}$.


 है $\sigma \tau \alpha u$ tò $\pi \varepsilon \nu \tau \dot{\alpha} \gamma(\omega \nu 0 \nu$.



 $\pi \varepsilon \nu \tau \alpha ́ \gamma \omega v o v$.
















 äp $\alpha$ ह́бтì tò АВГ $\Delta \mathrm{E} \pi \varepsilon \nu \tau \alpha ́ \gamma \varphi \nu$ 人v.









Thus, if a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

## Proposition 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.


For let three angles of the equilateral pentagon $A B C D E$-first of all, the consecutive (angles) at $A, B$, and $C$--be equal to one another. I say that pentagon $A B C D E$ is equiangular.

For let $A C, B E$, and $F D$ have been joined. And since the two (straight-lines) $C B$ and $B A$ are equal to the two (straight-lines) $B A$ and $A E$, respectively, and angle $C B A$ is equal to angle $B A E$, base $A C$ is thus equal to base $B E$, and triangle $A B C$ equal to triangle $A B E$, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4], (that is), $B C A$ (equal) to $B E A$, and $A B E$ to $C A B$. And hence side $A F$ is also equal to side $B F$ [Prop. 1.6]. And the whole of $A C$ was also shown (to be) equal to the whole of $B E$. Thus, the remainder $F C$ is also equal to the remainder $F E$. And $C D$ is also equal to $D E$. So, the two (straight-lines) $F C$ and $C D$ are equal to the two (straight-lines) $F E$ and $E D$ (respectively). And $F D$ is their common base. Thus, angle $F C D$ is equal to angle $F E D$ [Prop. 1.8]. And $B C A$ was also shown (to be) equal to $A E B$. And thus the whole of $B C D$ (is) equal to the whole of $A E D$. But, (angle) $B C D$ was assumed (to be) equal to the angles at $A$ and $B$. Thus, (angle) $A E D$ is also equal to the angles at $A$ and $B$. So, similarly, we can show that angle $C D E$ is also equal to the angles at $A, B, C$. Thus, pentagon $A B C D E$ is equiangular.

And so let consecutive angles not be equal, but let the (angles) at points $A, C$, and $D$ be equal. I say that pentagon $A B C D E$ is also equiangular in this case.

For let $B D$ have been joined. And since the two









## $\eta^{\prime}$ ．

＇E这 $\pi \varepsilon \nu \tau \alpha \gamma \omega ́ v o u$ í $\sigma o \pi \lambda \varepsilon u ́ p o u ~ x \alpha i ~ i \sigma o \gamma \omega v i ́ o u ~ \tau \grave{\alpha} \varsigma ~ x \alpha \tau \grave{\alpha}$













 АВГ трıү⿳㇒⿵冂⿻兀






（straight－lines）$B A$ and $A E$ are equal to the（straight－ lines）$B C$ and $C D$ ，and they contain equal angles，base $B E$ is thus equal to base $B D$ ，and triangle $A B E$ is equal to triangle $B C D$ ，and the remaining angles will be equal to the remaining angles which the equal sides subtend ［Prop．1．4］．Thus，angle $A E B$ is equal to（angle）$C D B$ ． And angle $B E D$ is also equal to（angle）$B D E$ ，since side $B E$ is also equal to side $B D$［Prop．1．5］．Thus，the whole angle $A E D$ is also equal to the whole（angle）$C D E$ ．But， （angle）$C D E$ was assumed（to be）equal to the angles at $A$ and $C$ ．Thus，angle $A E D$ is also equal to the（angles） at $A$ and $C$ ．So，for the same（reasons），（angle）$A B C$ is also equal to the angles at $A, C$ ，and $D$ ．Thus，pentagon $A B C D E$ is equiangular．（Which is）the very thing it was required to show．

## Proposition 8

If straight－lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio，and their greater pieces are equal to the sides of the pentagon．


For let the two straight－lines，$A C$ and $B E$ ，cutting one another at point $H$ ，have subtended two consecutive an－ gles，at $A$ and $B$（respectively），of the equilateral and equiangular pentagon $A B C D E$ ．I say that each of them has been cut in extreme and mean ratio at point $H$ ，and that their greater pieces are equal to the sides of the pen－ tagon．

For let the circle $A B C D E$ have been circumscribed about pentagon $A B C D E$［Prop．4．14］．And since the two straight－lines $E A$ and $A B$ are equal to the two（straight－ lines）$A B$ and $B C$（respectively），and they contain equal angles，the base $B E$ is thus equal to the base $A C$ ，and tri－ angle $A B E$ is equal to triangle $A B C$ ，and the remaining angles will be equal to the remaining angles，respectively， which the equal sides subtend［Prop．1．4］．Thus，angle $B A C$ is equal to（angle）$A B E$ ．Thus，（angle）$A H E$（is） double（angle）$B A H$［Prop．1．32］．And $E A C$ is also dou－







 $\mathrm{E} \Theta \cdot \dot{\omega} \varsigma \nless \alpha \rho \alpha \dot{\eta} \mathrm{BE} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{E} \Theta$, oút $\omega \varsigma \dot{\eta} \mathrm{E} \Theta \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \Theta \mathrm{~B}$.







## $\vartheta^{\prime}$.






"Eбтढ xúx入os ó $\mathrm{AB} \mathrm{\Gamma}$, xai $\tau \widetilde{\omega} v$ हís tòv $\mathrm{AB} \mathrm{\Gamma}$ xúx入ov


 $\mu \varepsilon і ̈ \zeta o \nu ~ \alpha \cup ๋ \tau \tilde{n} \varsigma ~ \tau \mu \tilde{n} \mu \alpha ́ ~ \varepsilon ̇ \sigma \tau \tau \nu ~ \dot{\eta} Г \Delta$.


ble $B A C$, inasmuch as circumference $E D C$ is also double circumference $C B$ [Props. 3.28, 6.33]. Thus, angle $H A E$ (is) equal to (angle) $A H E$. Hence, straight-line $H E$ is also equal to (straight-line) $E A$-that is to say, to (straight-line) $A B$ [Prop. 1.6]. And since straight-line $B A$ is equal to $A E$, angle $A B E$ is also equal to $A E B$ [Prop. 1.5]. But, $A B E$ was shown (to be) equal to $B A H$. Thus, $B E A$ is also equal to $B A H$. And (angle) $A B E$ is common to the two triangles $A B E$ and $A B H$. Thus, the remaining angle $B A E$ is equal to the remaining (angle) $A H B$ [Prop. 1.32]. Thus, triangle $A B E$ is equiangular to triangle $A B H$. Thus, proportionally, as $E B$ is to $B A$, so $A B$ (is) to $B H$ [Prop. 6.4]. And $B A$ (is) equal to $E H$. Thus, as $B E$ (is) to $E H$, so $E H$ (is) to $H B$. And $B E$ (is) greater than $E H . E H$ (is) thus also greater than $H B$ [Prop. 5.14]. Thus, $B E$ has been cut in extreme and mean ratio at $H$, and the greater piece $H E$ is equal to the side of the pentagon. So, similarly, we can show that $A C$ has also been cut in extreme and mean ratio at $H$, and that its greater piece $C H$ is equal to the side of the pentagon. (Which is) the very thing it was required to show.

## Proposition 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together then the whole straight-line has been cut in extreme and mean ratio (at the junction point), and its greater piece is the side of the hexagon. ${ }^{\dagger}$


Let $A B C$ be a circle. And of the figures inscribed in circle $A B C$, let $B C$ be the side of a decagon, and $C D$ (the side) of a hexagon. And let them be (laid down) straighton (to one another). I say that the whole straight-line $B D$ has been cut in extreme and mean ratio (at $C$ ), and that $C D$ is its greater piece.

For let the center of the circle, point $E$, have been
 $\pi \lambda \alpha \sigma i ́ \omega \nu$ 人" $\alpha \dot{\eta}$ АГВ $\pi \varepsilon \rho เ \varphi \varepsilon ́ p \varepsilon ı \alpha ~ \tau \tilde{\eta} \varsigma ~ В Г ~ \pi \varepsilon р ь \varphi \varepsilon р \varepsilon i ́ \alpha \varsigma . ~ \tau \varepsilon-~$







 $\dot{\eta}$ únò ЕГВ $\gamma \omega v i ́ \alpha$ tñs únò $\mathrm{E} \Delta \Gamma$. $\dot{\alpha} \lambda \lambda \grave{\alpha}$ tñs ínò ЕГВ ठь-












found [Prop. 3.1], and let $E B, E C$, and $E D$ have been joined, and let $B E$ have been drawn across to $A$. Since $B C$ is a side on an equilateral decagon, circumference $A C B$ (is) thus five times circumference $B C$. Thus, circumference $A C$ (is) four times $C B$. And as circumference $A C$ (is) to $C B$, so angle $A E C$ (is) to $C E B$ [Prop. 6.33]. Thus, (angle) $A E C$ (is) four times $C E B$. And since angle $E B C$ (is) equal to $E C B$ [Prop. 1.5], angle $A E C$ is thus double $E C B$ [Prop. 1.32]. And since straight-line $E C$ is equal to $C D$-for each of them is equal to the side of the hexagon [inscribed] in circle $A B C$ [Prop. 4.15 corr.]angle $C E D$ is also equal to angle $C D E$ [Prop. 1.5]. Thus, angle $E C B$ (is) double $E D C$ [Prop. 1.32]. But, $A E C$ was shown (to be) double $E C B$. Thus, $A E C$ (is) four times $E D C$. And $A E C$ was also shown (to be) four times $B E C$. Thus, $E D C$ (is) equal to $B E C$. And angle $E B D$ (is) common to the two triangles $B E C$ and $B E D$. Thus, the remaining (angle) $B E D$ is equal to the (remaining angle) $E C B$ [Prop. 1.32]. Thus, triangle $E B D$ is equiangular to triangle $E B C$. Thus, proportionally, as $D B$ is to $B E$, so $E B$ (is) to $B C$ [Prop. 6.4]. And $E B$ (is) equal to $C D$. Thus, as $B D$ is to $D C$, so $D C$ (is) to $C B$. And $B D$ (is) greater than $D C$. Thus, $D C$ (is) also greater than $C B$ [Prop. 5.14]. Thus, the straight-line $B D$ has been cut in extreme and mean ratio [at $C$ ], and $D C$ is its greater piece. (Which is), the very thing it was required to show.
${ }^{\dagger}$ If the circle is of unit radius then the side of the hexagon is 1 , whereas the side of the decagon is $(1 / 2)(\sqrt{5}-1)$.

$$
\therefore \text {. }
$$






## Proposition 10

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle. ${ }^{\dagger}$














































 $\tau \tilde{\eta} \varsigma \mathrm{BZ} \cdot \tau o ̀ ~ \ddot{\alpha} \rho \alpha$ ú̃ò $\tau \widetilde{\omega} \nu \mathrm{ABN} \mu \varepsilon \tau \dot{\alpha}$ тoũ ú $\pi o ̀ ~ B A N$, ö $\pi \varepsilon \rho$





Let $A B C D E$ be a circle. And let the equilateral pentagon $A B C D E$ have been inscribed in circle $A B C D E$. I say that the square on the side of pentagon $A B C D E$ is the (sum of the squares) on the sides of the hexagon and of the decagon inscribed in circle $A B C D E$.

For let the center of the circle, point $F$, have been found [Prop. 3.1]. And, $A F$ being joined, let it have been drawn across to point $G$. And let $F B$ have been joined. And let $F H$ have been drawn from $F$ perpendicular to $A B$. And let it have been drawn across to $K$. And let $A K$ and $K B$ have been joined. And, again, let $F L$ have been drawn from $F$ perpendicular to $A K$. And let it have been drawn across to $M$. And let $K N$ have been joined.

Since circumference $A B C G$ is equal to circumference $A E D G$, of which $A B C$ is equal to $A E D$, the remaining circumference $C G$ is thus equal to the remaining (circumference) $G D$. And $C D$ (is the side) of the pentagon. $C G$ (is) thus (the side) of the decagon. And since $F A$ is equal to $F B$, and $F H$ is perpendicular (to $A B$ ), angle $A F K$ (is) thus also equal to $K F B$ [Props. 1.5, 1.26]. Hence, circumference $A K$ is also equal to $K B$ [Prop. 3.26]. Thus, circumference $A B$ (is) double circumference $B K$. Thus, straight-line $A K$ is the side of the decagon. So, for the same (reasons, circumference) $A K$ is also double $K M$. And since circumference $A B$ is double circumference $B K$, and circumference $C D$ (is) equal to circumference $A B$, circumference $C D$ (is) thus also double circumference $B K$. And circumference $C D$ is also double $C G$. Thus, circumference $C G$ (is) equal to circumference $B K$. But, $B K$ is double $K M$, since $K A$ (is) also (double $K M$ ). Thus, (circumference) $C G$ is also double $K M$. But, indeed, circumference $C B$ is also double circumference $B K$. For circumference $C B$ (is) equal to $B A$. Thus, the whole circumference $G B$ is also double $B M$. Hence, angle $G F B$ [is] also double angle $B F M$ [Prop. 6.33]. And $G F B$ (is) also double $F A B$. For $F A B$ (is) equal to $A B F$. Thus, $B F N$ is also equal to $F A B$. And angle $A B F$ (is) common to the two triangles $A B F$ and $B F N$. Thus, the remaining (angle) $A F B$ is equal to the remaining (angle) $B N F$ [Prop. 1.32]. Thus, triangle $A B F$ is equiangular to triangle $B F N$. Thus, proportionally, as straight-line $A B$ (is) to $B F$, so $F B$ (is) to $B N$ [Prop. 6.4]. Thus, the (rectangle contained) by $A B N$ is equal to the (square) on $B F$ [Prop. 6.17]. Again, since $A L$ is equal to $L K$, and $L N$ is common and at right-angles (to $K A$ ), base $K N$ is thus equal to base $A N$ [Prop. 1.4]. And, thus, angle $L K N$ is equal to angle $L A N$. But, $L A N$ is equal to $K B N$ [Props. 3.29, 1.5]. Thus, $L K N$ is also equal to $K B N$. And the (angle) at $A$ (is) common to the two triangles $A K B$ and $A K N$. Thus, the remaining (angle) $A K B$ is


equal to the remaining (angle) $K N A$ [Prop. 1.32]. Thus, triangle $K B A$ is equiangular to triangle $K N A$. Thus, proportionally, as straight-line $B A$ is to $A K$, so $K A$ (is) to $A N$ [Prop. 6.4]. Thus, the (rectangle contained) by $B A N$ is equal to the (square) on $A K$ [Prop. 6.17]. And the (rectangle contained) by $A B N$ was also shown (to be) equal to the (square) on $B F$. Thus, the (rectangle contained) by $A B N$ plus the (rectangle contained) by $B A N$, which is the (square) on $B A$ [Prop. 2.2], is equal to the (square) on $B F$ plus the (square) on $A K$. And $B A$ is the side of the pentagon, and $B F$ (the side) of the hexagon [Prop. 4.15 corr.], and $A K$ (the side) of the decagon.

Thus, the square on the side of the pentagon (inscribed in a circle) is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.
${ }^{\dagger}$ If the circle is of unit radius then the side of the pentagon is $(1 / 2) \sqrt{10-2 \sqrt{5}}$.
$i \alpha^{\prime}$.






 $\chi \alpha \lambda o u \mu \varepsilon ́ v \eta$ ह่̇ $\alpha \dot{\alpha} \sigma \sigma \omega \nu$.

Eỉńpvo ràp tò xévtpov toũ xúx入ou tò Z onuعiov,








## Proposition 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter then the side of the pentagon is that irrational (straight-line) called minor.


For let the equilateral pentagon $A B C D E$ have been inscribed in the circle $A B C D E$ which has a rational diameter. I say that the side of pentagon [ $A B C D E$ ] is that irrational (straight-line) called minor.

For let the center of the circle, point $F$, have been found [Prop. 3.1]. And let $A F$ and $F B$ have been joined. And let them have been drawn across to points $G$ and $H$ (respectively). And let $A C$ have been joined. And let $F K$ made (equal) to the fourth part of $A F$. And $A F$ (is) rational. $F K$ (is) thus also rational. And $B F$ is also rational. Thus, the whole of $B K$ is rational. And since circumference $A C G$ is equal to circumference $A D G$, of which







 $\delta \iota \pi \lambda \tilde{\eta} \pi \rho o ̀ s ~ \tau ท ̀ \nu ~ Г А, ~ o u ̈ \tau \omega \varsigma ~ \dot{\eta} \tau \tilde{\eta} \varsigma \mathrm{MZ} \delta \iota \pi \lambda \tilde{\eta} \pi \rho o ̀ s ~ \tau \dot{\eta} \nu \mathrm{ZA}$.










































$A B C$ is equal to $A E D$, the remainder $C G$ is thus equal to the remainder $G D$. And if we join $A D$ then the angles at $L$ are inferred (to be) right-angles, and $C D$ (is inferred to be) double $C L$ [Prop. 1.4]. So, for the same (reasons), the (angles) at $M$ are also right-angles, and $A C$ (is) double $C M$. Therefore, since angle $A L C$ (is) equal to $A M F$, and (angle) $L A C$ (is) common to the two triangles $A C L$ and $A M F$, the remaining (angle) $A C L$ is thus equal to the remaining (angle) $M F A$ [Prop. 1.32]. Thus, triangle $A C L$ is equiangular to triangle $A M F$. Thus, proportionally, as $L C$ (is) to $C A$, so $M F$ (is) to $F A$ [Prop. 6.4]. And (we can take) the doubles of the leading (magnitudes). Thus, as double $L C$ (is) to $C A$, so double $M F$ (is) to $F A$. And as double $M F$ (is) to $F A$, so $M F$ (is) to half of $F A$. And, thus, as double $L C$ (is) to $C A$, so $M F$ (is) to half of $F A$. And (we can take) the halves of the following (magnitudes). Thus, as double $L C$ (is) to half of $C A$, so $M F$ (is) to the fourth of $F A$. And $D C$ is double $L C$, and $C M$ half of $C A$, and $F K$ the fourth part of $F A$. Thus, as $D C$ is to $C M$, so $M F$ (is) to $F K$. Via composition, as the sum of $D C M$ (i.e., $D C$ and $C M$ ) (is) to $C M$, so $M K$ (is) to $K F$ [Prop. 5.18]. And, thus, as the (square) on the sum of $D C M$ (is) to the (square) on $C M$, so the (square) on $M K$ (is) to the (square) on $K F$. And since the greater piece of a (straight-line) subtending two sides of a pentagon, such as $A C$, (which is) cut in extreme and mean ratio is equal to the side of the pentagon [Prop. 13.8]that is to say, to $D C$--and the square on the greater piece added to half of the whole is five times the (square) on half of the whole [Prop. 13.1], and $C M$ (is) half of the whole, $A C$, thus the (square) on $D C M$, (taken) as one, is five times the (square) on $C M$. And the (square) on $D C M$, (taken) as one, (is) to the (square) on $C M$, so the (square) on $M K$ was shown (to be) to the (square) on $K F$. Thus, the (square) on $M K$ (is) five times the (square) on $K F$. And the square on $K F$ (is) rational. For the diameter (is) rational. Thus, the (square) on $M K$ (is) also rational. Thus, $M K$ is rational [in square only]. And since $B F$ is four times $F K, B K$ is thus five times $K F$. Thus, the (square) on $B K$ (is) twenty-five times the (square) on $K F$. And the (square) on $M K$ (is) five times the square on $K F$. Thus, the (square) on $B K$ (is) five times the (square) on $K M$. Thus, the (square) on $B K$ does not have to the (square) on $K M$ the ratio which a square number (has) to a square number. Thus, $B K$ is incommensurable in length with $K M$ [Prop. 10.9]. And each of them is a rational (straightline). Thus, $B K$ and $K M$ are rational (straight-lines which are) commensurable in square only. And if from a rational (straight-line) a rational (straight-line) is subtracted, which is commensurable in square only with the










 BM.


whole, then the remainder is that irrational (straight-line called) an apotome [Prop. 10.73]. Thus, $M B$ is an apotome, and $M K$ its attachment. So, I say that (it is) also a fourth (apotome). So, let the (square) on $N$ be (made) equal to that (magnitude) by which the (square) on $B K$ is greater than the (square) on $K M$. Thus, the square on $B K$ is greater than the (square) on $K M$ by the (square) on $N$. And since $K F$ is commensurable (in length) with $F B$ then, via composition, $K B$ is also commensurable (in length) with $F B$ [Prop. 10.15]. But, $B F$ is commensurable (in length) with $B H$. Thus, $B K$ is also commensurable (in length) with $B H$ [Prop. 10.12]. And since the (square) on $B K$ is five times the (square) on $K M$, the (square) on $B K$ thus has to the (square) on $K M$ the ratio which 5 (has) to one. Thus, via conversion, the (square) on $B K$ has to the (square) on $N$ the ratio which 5 (has) to 4 [Prop. 5.19 corr.], which is not (that) of a square (number) to a square (number). $B K$ is thus incommensurable (in length) with $N$ [Prop. 10.9]. Thus, the square on $B K$ is greater than the (square) on $K M$ by the (square) on (some straight-line which is) incommensurable (in length) with ( $B K$ ). Therefore, since the square on the whole, $B K$, is greater than the (square) on the attachment, $K M$, by the (square) on (some straightline which is) incommensurable (in length) with ( $B K$ ), and the whole, $B K$, is commensurable (in length) with the (previously) laid down rational (straight-line) $B H$, $M B$ is thus a fourth apotome [Def. 10.14]. And the rectangle contained by a rational (straight-line) and a fourth apotome is irrational, and its square-root is that irrational (straight-line) called minor [Prop. 10.94]. And the square on $A B$ is the rectangle contained by $H B M$, on account of joining $A H$, (so that) triangle $A B H$ becomes equiangular with triangle $A B M$ [Prop. 6.8], and (proportionally) as $H B$ is to $B A$, so $A B$ (is) to $B M$.

Thus, the side $A B$ of the pentagon is that irrational (straight-line) called minor. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the circle has unit radius then the side of the pentagon is $(1 / 2) \sqrt{10-2 \sqrt{5}}$. However, this length can be written in the "minor" form (see Prop. 10.94) $(\rho / \sqrt{2}) \sqrt{1+k / \sqrt{1+k^{2}}}-(\rho / \sqrt{2}) \sqrt{1-k / \sqrt{1+k^{2}}}$, with $\rho=\sqrt{5 / 2}$ and $k=2$.

$$
\beta^{\prime}
$$

’Eàv عís xúx трเүต́ஸvou $\pi \lambda \varepsilon u p \alpha ̀ ~ \delta u v \alpha ́ \mu \varepsilon \iota ~ \tau \rho เ \pi \lambda \alpha \sigma i \omega \nu ~ \varepsilon ̇ \sigma \tau i ~ \tau \tilde{\eta} ร$ ह̉x тоũ хย́vtpou тoũ xúx入ou.


 АВГ хи́хлои.

## Proposition 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the (square) on the radius of the circle.

Let there be a circle $A B C$, and let the equilateral triangle $A B C$ have been inscribed in it [Prop. 4.2]. I say that the square on one side of triangle $A B C$ is three times the (square) on the radius of circle $A B C$.

















$$
i \gamma^{\prime} \text {. }
$$


 $\tau \tilde{\eta} \varsigma ~ \pi \lambda \varepsilon \cup \rho \alpha \tilde{c} \tau \tilde{\eta} \varsigma ~ \pi \cup \rho \alpha u i ̂ o o s . ~$


For let the center, $D$, of circle $A B C$ have been found [Prop. 3.1]. And $A D$ (being) joined, let it have been drawn across to $E$. And let $B E$ have been joined.

And since triangle $A B C$ is equilateral, circumference $B E C$ is thus the third part of the circumference of circle $A B C$. Thus, circumference $B E$ is the sixth part of the circumference of the circle. Thus, straight-line $B E$ is (the side) of a hexagon. Thus, it is equal to the radius $D E$ [Prop. 4.15 corr.]. And since $A E$ is double $D E$, the (square) on $A E$ is four times the (square) on $E D$-that is to say, of the (square) on $B E$. And the (square) on $A E$ (is) equal to the (sum of the squares) on $A B$ and $B E$ [Props. 3.31, 1.47]. Thus, the (sum of the squares) on $A B$ and $B E$ is four times the (square) on $B E$. Thus, via separation, the (square) on $A B$ is three times the (square) on $B E$. And $B E$ (is) equal to $D E$. Thus, the (square) on $A B$ is three times the (square) on $D E$.

Thus, the square on the side of the triangle is three times the (square) on the radius [of the circle]. (Which is) the very thing it was required to show.

## Proposition 13

To construct a (regular) pyramid (i.e., a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.














 $\gamma \omega v i ́ \alpha \varsigma . \alpha \ddot{\alpha} \tau \tau \varepsilon \tau \alpha l$ ठ̇̀ $\alpha u ̋ \tau \tilde{\eta} s \dot{\varepsilon} x \alpha ́ \sigma \tau \eta \tau \widetilde{\omega} \nu \Theta E, \Theta Z, \Theta H \cdot \dot{\eta} \Theta K$












 KEZ, KZH, KEH. $\quad \pi \cup \rho \alpha u i s ~ a ̈ p \alpha ~ \sigma u v \varepsilon ́ \sigma \tau \alpha \tau \alpha l ~ غ ̇ x ~ \tau \varepsilon \sigma \sigma \alpha ́ p \omega v ~$



Let the diameter $A B$ of the given sphere be laid out, and let it have been cut at point $C$ such that $A C$ is double $C B$ [Prop. 6.10]. And let the semi-circle $A D B$ have been drawn on $A B$. And let $C D$ have been drawn from point $C$ at right-angles to $A B$. And let $D A$ have been joined. And let the circle $E F G$ be laid down having a radius equal to $D C$, and let the equilateral triangle $E F G$ have been inscribed in circle $E F G$ [Prop. 4.2]. And let the center of the circle, point $H$, have been found [Prop. 3.1]. And let $E H, H F$, and $H G$ have been joined. And let $H K$ have been set up, at point $H$, at right-angles to the plane of circle $E F G$ [Prop. 11.12]. And let $H K$, equal to the straight-line $A C$, have been cut off from $H K$. And let $K E, K F$, and $K G$ have been joined. And since $K H$ is at right-angles to the plane of circle $E F G$, it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle $E F G$ [Def. 11.3]. And $H E, H F$, and $H G$ each join it. Thus, $H K$ is at right-angles to each of $H E, H F$, and $H G$. And since $A C$ is equal to $H K$, and $C D$ to $H E$, and they contain right-angles, the base $D A$ is thus equal to the base $K E$ [Prop. 1.4]. So, for the same (reasons), $K F$ and $K G$ is each equal to $D A$. Thus, the three (straight-lines) $K E$, $K F$, and $K G$ are equal to one another. And since $A C$ is double $C B, A B$ (is) thus triple $B C$. And as $A B$ (is) to $B C$, so the (square) on $A D$ (is) to the (square) on $D C$, as will be shown later [see lemma]. Thus, the (square) on $A D$ (is) three times the (square) on $D C$. And the (square) on $F E$ is also three times the (square) on $E H$ [Prop. 13.12], and $D C$ is equal to $E H$. Thus, $D A$ (is)




 x














 б乇̀ $\Gamma \mathrm{B} \dot{\eta} \Theta \Lambda$.














also equal to $E F$. But, $D A$ was shown (to be) equal to each of $K E, K F$, and $K G$. Thus, $E F, F G$, and $G E$ are equal to $K E, K F$, and $K G$, respectively. Thus, the four triangles $E F G, K E F, K F G$, and $K E G$ are equilateral. Thus, a pyramid, whose base is triangle $E F G$, and apex the point $K$, has been constructed from four equilateral triangles.

So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For let the straight-line $H L$ have been produced in a straight-line with $K H$, and let $H L$ be made equal to $C B$. And since as $A C$ (is) to $C D$, so $C D$ (is) to $C B$ [Prop. 6.8 corr.], and $A C$ (is) equal to $K H$, and $C D$ to $H E$, and $C B$ to $H L$, thus as $K H$ is to $H E$, so $E H$ (is) to $H L$. Thus, the (rectangle contained) by $K H$ and $H L$ is equal to the (square) on $E H$ [Prop. 6.17]. And each of the angles $K H E$ and $E H L$ is a right-angle. Thus, the semi-circle drawn on $K L$ will also pass through $E$ [inasmuch as if we join $E L$ then the angle $L E K$ becomes a right-angle, on account of triangle $E L K$ becoming equiangular to each of the triangles $E L H$ and $E H K$ [Props. 6.8, 3.31] ]. So, if $K L$ remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points $F$ and $G$, (because) if $F L$ and $L G$ are joined, the angles at $F$ and $G$ will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, $K L$, of the sphere is equal to the diameter, $A B$, of the given sphereinasmuch as $K H$ was made equal to $A C$, and $H L$ to $C B$.

So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since $A C$ is double $C B, A B$ is thus triple $B C$. Thus, via conversion, $B A$ is one and a half times $A C$. And as $B A$ (is) to $A C$, so the (square) on $B A$ (is) to the (square) on $A D$ [inasmuch as if $D B$ is joined then as $B A$ is to $A D$, so $D A$ (is) to $A C$, on account of the similarity of triangles $D A B$ and $D A C$. And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on $B A$ (is) also one and a half times the (square) on $A D$. And $B A$ is the diameter of the given sphere, and $A D$ (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the radius of the sphere is unity then the side of the pyramid (i.e., tetrahedron) is $\sqrt{8 / 3}$.

$\Lambda \tilde{n} \mu \mu \alpha$.



 тò $\mathrm{E} \Gamma$, xаі $\sigma \cup \mu \pi \varepsilon \pi \lambda \eta \rho \omega ́ \sigma \vartheta \omega ~ t o ̀ ~ Z B ~ \pi \alpha р \alpha \lambda \lambda \eta \lambda o ́ \gamma \rho \alpha \mu \mu о \nu . ~$





 $\mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\nu} \nu \mathrm{~B} \mathrm{\Gamma}$, oüt $\omega \varsigma$ tò ú $\pi$ ò $\tau \widetilde{\omega} \nu \mathrm{BA}, \mathrm{A} \Gamma \pi \rho o ̀ ~ \tau o ̀ ~ u ́ \pi o ̀ ~$







## ' $\delta^{\prime}$.


 $\delta เ \pi \lambda \alpha \sigma i \alpha$ ह̇ $\sigma \tau i \not \tau \tilde{n} \varsigma \pi \lambda \varepsilon \cup p \alpha \tilde{\varsigma}$ тоט̃ ỏx $\tau \alpha \varepsilon ́ \delta \rho \circ \cup$.







Lemma
It must be shown that as $A B$ is to $B C$, so the (square) on $A D$ (is) to the (square) on $D C$.

For, let the figure of the semi-circle have been set out, and let $D B$ have been joined. And let the square $E C$ have been described on $A C$. And let the parallelogram $F B$ have been completed. Therefore, since, on account of triangle $D A B$ being equiangular to triangle $D A C$ [Props. 6.8, 6.4], (proportionally) as $B A$ is to $A D$, so $D A$ (is) to $A C$, the (rectangle contained) by $B A$ and $A C$ is thus equal to the (square) on $A D$ [Prop. 6.17]. And since as $A B$ is to $B C$, so $E B$ (is) to $B F$ [Prop. 6.1]. And $E B$ is the (rectangle contained) by $B A$ and $A C$-for $E A$ (is) equal to $A C$. And $B F$ the (rectangle contained) by $A C$ and $C B$. Thus, as $A B$ (is) to $B C$, so the (rectangle contained) by $B A$ and $A C$ (is) to the (rectangle contained) by $A C$ and $C B$. And the (rectangle contained) by $B A$ and $A C$ is equal to the (square) on $A D$, and the (rectangle contained) by $A C B$ (is) equal to the (square) on $D C$. For the perpendicular $D C$ is the mean proportional to the pieces of the base, $A C$ and $C B$, on account of $A D B$ being a right-angle [Prop. 6.8 corr.]. Thus, as $A B$ (is) to $B C$, so the (square) on $A D$ (is) to the (square) on $D C$. (Which is) the very thing it was required to show.

## Proposition 14

To construct an octahedron, and to enclose (it) in a (given) sphere, like in the preceding (proposition), and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

Let the diameter $A B$ of the given sphere be laid out, and let it have been cut in half at $C$. And let the semicircle $A D B$ have been drawn on $A B$. And let $C D$ be drawn from $C$ at right-angles to $A B$. And let $D B$ have




 $\Lambda \mathrm{E}, \Lambda \mathrm{Z}, \Lambda \mathrm{H}, \Lambda \Theta, \mathrm{ME}, \mathrm{MZ}, \mathrm{MH}, \mathrm{M} \Theta$.














 ỏx $\tau \alpha \varepsilon ́ \delta ̄ \rho o \cup \pi \lambda \varepsilon \cup p a ̃ s$.







been joined. And let the square $E F G H$, having each of its sides equal to $D B$, be laid out. And let $H F$ and $E G$ have been joined. And let the straight-line $K L$ have been set up, at point $K$, at right-angles to the plane of square $E F G H$ [Prop. 11.12]. And let it have been drawn across on the other side of the plane, like $K M$. And let $K L$ and $K M$, equal to one of $E K, F K, G K$, and $H K$, have been cut off from $K L$ and $K M$, respectively. And let $L E, L F$, $L G, L H, M E, M F, M G$, and $M H$ have been joined.


And since $K E$ is equal to $K H$, and angle $E K H$ is a right-angle, the (square) on the $H E$ is thus double the (square) on $E K$ [Prop. 1.47]. Again, since $L K$ is equal to $K E$, and angle $L K E$ is a right-angle, the (square) on $E L$ is thus double the (square) on $E K$ [Prop. 1.47]. And the (square) on $H E$ was also shown (to be) double the (square) on $E K$. Thus, the (square) on $L E$ is equal to the (square) on $E H$. Thus, $L E$ is equal to $E H$. So, for the same (reasons), $L H$ is also equal to $H E$. Triangle $L E H$ is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square $E F G H$, and apexes the points $L$ and $M$, are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) $L K, K M$, and $K E$ are equal to one another, the semi-circle drawn on $L M$ will thus also pass through $E$. And, for the same (reasons), if $L M$ remains (fixed), and the semi-circle is car-















${ }^{\dagger}$ If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

$$
\iota \varepsilon^{\prime}
$$

Kúßov $\sigma \cup \sigma \tau \eta ́ \sigma \alpha \sigma \vartheta \alpha l ~ x \alpha i ̀ ~ \sigma \varphi \alpha i p \alpha ~ \pi \varepsilon p ı \lambda \alpha \beta \varepsilon \tau ̃, ~ n ̃ ~ x \alpha i ̀ ~ \tau \eta ̀ \nu ~$




 $\tau \tilde{n} \mathrm{AB} \pi \rho o ̀ \varsigma ~ o ̉ \rho \vartheta \dot{\alpha} \varsigma ~ \tilde{\eta} \chi \vartheta \omega \dot{\eta} \Gamma \Delta$, xail ह̇л $\pi \zeta \varepsilon \cup ́ \chi \vartheta \omega \dot{\eta} \Delta \mathrm{~B}$, x $\alpha \grave{l}$



 EZ, $\mathrm{ZH}, \mathrm{H} \Theta, ~ \Theta \mathrm{E}$ 亿ै $\sigma \eta$ ह่ $x \alpha ́ \sigma \tau \eta ~ \tau \widetilde{\omega} \nu \mathrm{EK}, \mathrm{Z} \Lambda, \mathrm{HM}, \Theta \mathrm{N}, x \alpha \grave{\imath}$



 $\tau \tilde{n} \varsigma \pi \lambda \varepsilon \cup p \alpha ̃ \varsigma ~ \tau о \cup ̃ ~ \chi u ́ \beta o u . ~$
ried around, and again established at the same (position) from which it began to be moved, then it will also pass through points $F, G$, and $H$, and the octahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since $L K$ is equal to $K M$, and $K E$ (is) common, and they contain right-angles, the base $L E$ is thus equal to the base $E M$ [Prop. 1.4]. And since angle $L E M$ is a right-angle-for (it is) in a semi-circle [Prop. 3.31]-the (square) on $L M$ is thus double the (square) on $L E$ [Prop. 1.47]. Again, since $A C$ is equal to $C B, A B$ is double $B C$. And as $A B$ (is) to $B C$, so the (square) on $A B$ (is) to the (square) on $B D$ [Prop. 6.8, Def. 5.9]. Thus, the (square) on $A B$ is double the (square) on $B D$. And the (square) on $L M$ was also shown (to be) double the (square) on $L E$. And the (square) on $D B$ is equal to the (square) on $L E$. For $E H$ was made equal to $D B$. Thus, the (square) on $A B$ (is) also equal to the (square) on $L M$. Thus, $A B$ (is) equal to $L M$. And $A B$ is the diameter of the given sphere. Thus, $L M$ is equal to the diameter of the given sphere.

Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron. ${ }^{\dagger}$ (Which is) the very thing it was required to show.

## Proposition 15

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

Let the diameter $A B$ of the given sphere be laid out, and let it have been cut at $C$ such that $A C$ is double $C B$. And let the semi-circle $A D B$ have been drawn on $A B$. And let $C D$ have been drawn from $C$ at rightangles to $A B$. And let $D B$ have been joined. And let the square $E F G H$, having (its) side equal to $D B$, be laid out. And let $E K, F L, G M$, and $H N$ have been drawn from (points) $E, F, G$, and $H$, (respectively), at right-angles to the plane of square $E F G H$. And let $E K, F L, G M$, and $H N$, equal to one of $E F, F G, G H$, and $H E$, have been cut off from $E K, F L, G M$, and $H N$, respectively. And let $K L, L M, M N$, and $N K$ have been joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.







 тท̀̀ ZK. xaì ठì $\alpha$ тои̃тo $\pi \alpha ́ \lambda \iota \nu ~ \tau o ̀ ~ \varepsilon ̇ \pi i ~ \tau n ̃ s ~ H K ~ \gamma p \alpha \varphi o ́ \mu \varepsilon v o v ~$









 ठ乇̀ $\dot{\eta} \mathrm{AB} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{~B} \mathrm{\Gamma}$, oư $\tau \omega \varsigma$ tò $\alpha \pi o ̀ ~ \tau \tilde{\eta} s \mathrm{AB} \pi \rho o ̀ s ~ \tau o ̀ ~ \alpha ́ \alpha o ̀ ~$










For let $K G$ and $E G$ have been joined. And since angle $K E G$ is a right-angle-on account of $K E$ also being at right-angles to the plane $E G$, and manifestly also to the straight-line $E G$ [Def. 11.3]-the semi-circle drawn on $K G$ will thus also pass through point $E$. Again, since $G F$ is at right-angles to each of $F L$ and $F E, G F$ is thus also at right-angles to the plane $F K$ [Prop. 11.4]. Hence, if we also join $F K$ then $G F$ will also be at right-angles to $F K$. And, again, on account of this, the semi-circle drawn on $G K$ will also pass through point $F$. Similarly, it will also pass through the remaining (angular) points of the cube. So, if $K G$ remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then the cube will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since $G F$ is equal to $F E$, and the angle at $F$ is a right-angle, the (square) on $E G$ is thus double the (square) on $E F$ [Prop. 1.47]. And $E F$ (is) equal to $E K$. Thus, the (square) on $E G$ is double the (square) on $E K$. Hence, the (sum of the squares) on $G E$ and $E K$-that is to say, the (square) on $G K$ [Prop. 1.47]-is three times the (square) on $E K$. And since $A B$ is three times $B C$, and as $A B$ (is) to $B C$, so the (square) on $A B$ (is) to the (square) on $B D$ [Prop. 6.8, Def. 5.9], the (square) on $A B$ (is) thus three times the (square) on $B D$. And the (square) on $G K$ was also shown (to be) three times the (square) on $K E$. And $K E$ was made equal to $D B$. Thus, $K G$ (is) also equal to $A B$. And $A B$ is the radius of the given sphere. Thus, $K G$ is also equal to the diameter of the given sphere.

Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on
${ }^{\dagger}$ If the radius of the sphere is unity then the side of the cube is $\sqrt{4 / 3}$ ．

$$
i \varepsilon^{\prime} .
$$





${ }^{3}$ Еxx




 عis tòv EZHӨK xúx入ov $\pi \varepsilon v \tau \alpha ́ \gamma \omega \nu o v ~ i \sigma o ́ \pi \lambda \varepsilon \cup p o ́ v ~ \tau \varepsilon ~ x \alpha i ̀ ~$






 ЕZHӨK жúx入ov，x $\alpha \grave{~ \varepsilon ̇ \pi \varepsilon \zeta \varepsilon u ́ \chi \vartheta \omega \sigma \alpha \nu ~ \alpha i ~ П Р, ~ P \Sigma, ~} \Sigma \mathrm{~T}, \mathrm{~T}$ ， $\Upsilon \Pi, ~ П \Lambda, ~ \Lambda Р, ~ Р М, ~ М \Sigma, ~ \Sigma N, ~ N T, ~ Т \Xi, ~ \Xi \Upsilon, ~ \Upsilon О, ~ О П . ~$





 xaì $\dot{\eta} \Pi \Upsilon$ тоũ عis tòv EZHӨK xúx入ov ह̇rүpapouévou．








the side of the cube．${ }^{\dagger}$（Which is）the very thing it was required to show．

## Proposition 16

To construct an icosahedron，and to enclose（it）in a sphere，like the aforementioned figures，and to show that the side of the icosahedron is that irrational（straight－ line）called minor．


Let the diameter $A B$ of the given sphere be laid out， and let it have been cut at $C$ such that $A C$ is four times $C B$［Prop．6．10］．And let the semi－circle $A D B$ have been drawn on $A B$ ．And let the straight－line $C D$ have been drawn from $C$ at right－angles to $A B$ ．And let $D B$ have been joined．And let the circle $E F G H K$ be set down， and let its radius be equal to $D B$ ．And let the equilat－ eral and equiangular pentagon $E F G H K$ have been in－ scribed in circle $E F G H K$［Prop．4．11］．And let the cir－ cumferences $E F, F G, G H, H K$ ，and $K E$ have been cut in half at points $L, M, N, O$ ，and $P$（respectively）．And let $L M, M N, N O, O P, P L$ ，and $E P$ have been joined． Thus，pentagon $L M N O P$ is also equilateral，and $E P$（is） the side of the decagon（inscribed in the circle）．And let the straight－lines $E Q, F R, G S, H T$ ，and $K U$ ，which are equal to the radius of circle $E F G H K$ ，have been set up at right－angles to the plane of the circle，at points $E, F$ ， $G, H$ ，and $K$（respectively）．And let $Q R, R S, S T, T U$ ， $U Q, Q L, L R, R M, M S, S N, N T, T O, O U, U P$ ，and $P Q$ have been joined．

And since $E Q$ and $K U$ are each at right－angles to the same plane，$E Q$ is thus parallel to $K U$［Prop．11．6］．And it is also equal to it．And straight－lines joining equal and parallel（straight－lines）on the same side are（themselves） equal and parallel［Prop．1．33］．Thus，$Q U$ is equal and parallel to $E K$ ．And $E K$（is the side）of an equilateral pentagon（inscribed in circle $E F G H K$ ）．Thus，$Q U$（is） also the side of an equilateral pentagon inscribed in circle $E F G H K$ ．So，for the same（reasons），$Q R, R S, S T$ ，and $T U$ are also the sides of an equilateral pentagon inscribed in circle $E F G H K$ ．Pentagon $Q R S T U$（is）thus equilat－










 $\dot{\alpha} \varphi n \rho \eta \dot{\eta} \sigma \vartheta \omega \dot{\varepsilon} \xi \alpha \gamma \omega ́ v o u \mu \varepsilon ̀ \nu \dot{\eta} \Phi X, \delta \varepsilon x \alpha \gamma \omega ́ v o u ~ \delta \varepsilon ̀ ~ \varepsilon ̇ x \alpha \tau \varepsilon ́ p \alpha ~ \tau \widetilde{\omega} \nu$
 $\Psi \mathrm{M}$.







eral. And side $Q E$ is (the side) of a hexagon (inscribed in circle $E F G H K$ ), and $E P$ (the side) of a decagon, and (angle) $Q E P$ is a right-angle, thus $Q P$ is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), $P U$ is also the side of a pentagon. And $Q U$ is also (the side) of a pentagon. Thus, triangle $Q P U$ is equilateral. So, for the same (reasons), (triangles) $Q L R$, $R M S, S N T$, and $T O U$ are each also equilateral. And since $Q L$ and $Q P$ were each shown (to be the sides) of a pentagon, and $L P$ is also (the side) of a pentagon, triangle $Q L P$ is thus equilateral. So, for the same (reasons), triangles $L R M, M S N, N T O$, and $O U P$ are each also equilateral.


Let the center, point $V$, of circle $E F G H K$ have been found [Prop. 3.1]. And let $V Z$ have been set up, at (point) $V$, at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like $V X$. And let $V W$ have been cut off (from $X Z$ so as to be equal to the side) of a hexagon, and each of $V X$ and $W Z$ (so as to be equal to the side) of a decagon. And let $Q Z, Q W, U Z, E V, L V, L X$, and $X M$ have been joined.

And since $V W$ and $Q E$ are each at right-angles to the plane of the circle, $V W$ is thus parallel to $Q E$ [Prop. 11.6]. And they are also equal. $E V$ and $Q W$ are thus equal and parallel (to one another) [Prop. 1.33].















 iбот入єúp $\omega \nu \pi \varepsilon \rho เ \varepsilon \chi о ́ \mu \varepsilon v o v$.

 $\dot{\varepsilon} \lambda \alpha \dot{\alpha} \sigma \sigma \omega \nu$.




















 $\pi \rho о \sigma \lambda \alpha \beta о \tilde{\sigma} \sigma \alpha$ т $̀ \nu \dot{\eta} \mu i \sigma \varepsilon i \alpha \nu$ тои̃ $\mu \varepsilon$ íhovos т $\mu \dot{\eta} \mu \alpha \tau о \varsigma ~ \tau \grave{\eta} \nu \mathrm{X} \alpha$










And $E V$ (is the side) of a hexagon. Thus, $Q W$ (is) also (the side) of a hexagon. And since $Q W$ is (the side) of a hexagon, and $W Z$ (the side) of a decagon, and angle $Q W Z$ is a right-angle [Def. 11.3, Prop. 1.29], $Q Z$ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), $U Z$ is also (the side) of a pentagon-inasmuch as, if we join $V K$ and $W U$ then they will be equal and opposite. And $V K$, being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, $W U$ (is) also the side of a hexagon. And $W Z$ (is the side) of a decagon, and (angle) $U W Z$ (is) a right-angle. Thus, $U Z$ (is the side) of a pentagon [Prop. 13.10]. And $Q U$ is also (the side) of a pentagon. Triangle $Q U Z$ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines $Q R$, $R S, S T$, and $T U$, and apexes the point $Z$, are also equilateral. Again, since $V L$ (is the side) of a hexagon, and $V X$ (the side) of a decagon, and angle $L V X$ is a rightangle, $L X$ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join $M V$, which is (the side) of a hexagon, $M X$ is also inferred (to be the side) of a pentagon. And $L M$ is also (the side) of a pentagon. Thus, triangle $L M X$ is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) $M N, N O, O P$, and $P L$, and apexes the point $X$, are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since $V W$ is (the side) of a hexagon, and $W Z$ (the side) of a decagon, $V Z$ has thus been cut in extreme and mean ratio at $W$, and $V W$ is its greater piece [Prop. 13.9]. Thus, as $Z V$ is to $V W$, so $V W$ (is) to $W Z$. And $V W$ (is) equal to $V E$, and $W Z$ to $V X$. Thus, as $Z V$ is to $V E$, so $E V$ (is) to $V X$. And angles $Z V E$ and $E V X$ are right-angles. Thus, if we join straight-line $E Z$ then angle $X E Z$ will be a right-angle, on account of the similarity of triangles $X E Z$ and $V E Z$. [Prop. 6.8]. So, for the same (reasons), since as $Z V$ is to $V W$, so $V W$ (is) to $W Z$, and $Z V$ (is) equal to $X W$, and $V W$ to $W Q$, thus as $X W$ is to $W Q$, so $Q W$ (is) to $W Z$. And, again, on account of this, if we join $Q X$ then the angle at $Q$ will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on $X Z$ will also pass through $Q$ [Prop. 3.31]. And if $X Z$ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) $Q$, and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been en-














 غ่ $\lambda \alpha \dot{\alpha} \tau \tau \omega \nu$.

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closed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let $V W$ have been cut in half at $a$. And since the straight-line $V Z$ has been cut in extreme and mean ratio at $W$, and $Z W$ is its lesser piece, then the square on $Z W$ added to half of the greater piece, $W a$, is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on $Z a$ is five times the (square) on $a W$. And $Z X$ is double $Z a$, and $V W$ double $a W$. Thus, the (square) on $Z X$ is five times the (square) on $W V$. And since $A C$ is four times $C B, A B$ is thus five times $B C$. And as $A B$ (is) to $B C$, so the (square) on $A B$ (is) to the (square) on $B D$ [Prop. 6.8, Def. 5.9]. Thus, the (square) on $A B$ is five times the (square) on $B D$. And the (square) on $Z X$ was also shown (to be) five times the (square) on $V W$. And $D B$ is equal to $V W$. For each of them is equal to the radius of circle $E F G H K$. Thus, $A B$ (is) also equal to $X Z$. And $A B$ is the diameter of the given sphere. Thus, $X Z$ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $E F G H K$, the radius of circle $E F G H K$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $E F G H K$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

## Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle. ${ }^{\dagger}$
$\dagger$ If the radius of the sphere is unity then the radius of the circle is $2 / \sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2 / \sqrt{5}, 1-1 / \sqrt{5}$, and $(1 / \sqrt{5}) \sqrt{10-2 \sqrt{5}}$, respectively.
گ'.




## Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straightline) called an apotome.


 غ்xáбтท т $\widetilde{\omega} \nu \mathrm{AB}, \mathrm{B}, \Gamma, \Gamma, \Delta \mathrm{A}, \mathrm{EZ}, \mathrm{EB}, \mathrm{Z} \Gamma \pi \lambda \varepsilon \cup p \widetilde{\omega} \nu \delta i ́ \chi \alpha$ x $\alpha \tau \grave{\alpha} \tau \grave{\alpha} \mathrm{H}, \Theta, \mathrm{K}, \Lambda, \mathrm{M}, \mathrm{N}, \Xi$, , $\alpha i$ ह่̇ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu$ 人i HK , $\Theta \Lambda, \mathrm{M} \Theta, \mathrm{N} \Xi$, x $\alpha \grave{\imath} \tau \varepsilon \eta \eta \dot{\sigma} \vartheta \omega$ غ்x $\alpha \sigma \tau \eta ~ \tau \widetilde{\omega} \nu \mathrm{NO}, \mathrm{O} \Xi, \Theta \Pi$
 $\alpha \cup ๋ \tau \widetilde{\omega} \mu \varepsilon i \zeta o v \alpha \tau \mu \eta ́ \mu \alpha \tau \alpha \tau \grave{\alpha} \mathrm{PO}, \mathrm{O} \Sigma, \mathrm{T}$,,$x \alpha \grave{\alpha} \alpha \nu \varepsilon \sigma \tau \alpha ́ \tau \omega \sigma \alpha \nu$


 $\Upsilon B, B X, Х Г, Г \Phi, \Phi \Upsilon$.




 $\mu \varepsilon ̀ \nu ~ O N ~ \tau \tilde{n} \mathrm{NB}, \dot{\eta} \delta \check{~} \mathrm{OP} \tau \tilde{\eta} \mathrm{P} \mathrm{\Upsilon} \cdot \tau \alpha \alpha_{\alpha} p \alpha \alpha \dot{\alpha} \pi o ̀ ~ \tau \widetilde{\omega} \nu \mathrm{BN}, \mathrm{NP}$







 őтı x $\alpha$ غ́x $\alpha \sigma \tau \eta ~ \tau \widetilde{\omega} \nu ~ B X, ~ Х Г, ~ Г \Phi ~ \varepsilon ̇ x \alpha \tau \varepsilon ́ p \alpha ~ \tau \widetilde{\omega} \nu ~ В \Upsilon, ~ \Upsilon \Phi ~$


 $\mu \varepsilon ́ p \eta \dot{\eta} O \Psi, \chi \alpha i$ ह̇ $\pi \varepsilon \zeta \varepsilon \dot{\chi} \chi \vartheta \omega \sigma \alpha \nu$ 人i $\Psi \Theta, \Theta X \cdot \lambda \varepsilon ́ \gamma \omega$, ö $\tau \iota \dot{\eta}$





Let two planes of the aforementioned cube [Prop. 13.15], $A B C D$ and $C B E F$, (which are) at right-angles to one another, be laid out. And let the sides $A B, B C$, $C D, D A, E F, E B$, and $F C$ have each been cut in half at points $G, H, K, L, M, N$, and $O$ (respectively). And let $G K, H L, M H$, and $N O$ have been joined. And let $N P$, $P O$, and $H Q$ have each been cut in extreme and mean ratio at points $R, S$, and $T$ (respectively). And let their greater pieces be $R P, P S$, and $T Q$ (respectively). And let $R U, S V$, and $T W$ have been set up on the exterior side of the cube, at points $R, S$, and $T$ (respectively), at right-angles to the planes of the cube. And let them be made equal to $R P, P S$, and $T Q$. And let $U B, B W, W C$, $C V$, and $V U$ have been joined.

I say that the pentagon $U B W C V$ is equilateral, and in one plane, and, further, equiangular. For let $R B, S B$, and $V B$ have been joined. And since the straight-line $N P$ has been cut in extreme and mean ratio at $R$, and $R P$ is the greater piece, the (sum of the squares) on $P N$ and $N R$ is thus three times the (square) on $R P$ [Prop. 13.4]. And $P N$ (is) equal to $N B$, and $P R$ to $R U$. Thus, the (sum of the squares) on $B N$ and $N R$ is three times the (square) on $R U$. And the (square) on $B R$ is equal to the (sum of the squares) on $B N$ and $N R$ [Prop. 1.47]. Thus, the (square) on $B R$ is three times the (square) on $R U$. Hence, the (sum of the squares) on $B R$ and $R U$ is four times the (square) on $R U$. And the (square) on $B U$ is equal to the (sum of the squares) on $B R$ and $R U$ [Prop. 1.47]. Thus, the (square) on $B U$ is four times the (square) on $U R$. Thus, $B U$ is double $R U$. And $V U$ is also double $U R$, inasmuch as $S R$ is also double $P R$-that is to say, $R U$. Thus, $B U$ (is) equal to $U V$. So, similarly, it can be shown that each of $B W, W C, C V$ is equal to each















 $\pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{NO}$, oưt $\omega \omega$ ц $\dot{\eta} \mathrm{NO} \pi \rho o ̀ s ~ \tau \grave{\eta} \nu \mathrm{O} \Sigma], \dot{\eta} \mathrm{N} \Sigma$ äp $\alpha$ äxpov x人ì $\mu \varepsilon ́ \sigma o \nu ~ \lambda o ́ \gamma o v ~ \tau \varepsilon ́ \tau \mu \eta \tau \alpha l ~ \chi \alpha \tau \alpha ̀ ~ \tau o ̀ ~ O, ~ \chi \alpha i ̀ ~ \tau o ̀ ~ \mu \varepsilon і ̈ \zeta o \nu ~ \tau \mu n ̃ \mu \alpha ́ ~$


 $\tau \tilde{\eta} \varsigma \mathrm{NB} \cdot \omega ँ \sigma \tau \varepsilon \tau \dot{\alpha} \alpha \dot{\alpha} \pi \grave{\tau} \tau \widetilde{\nu} \Phi \Sigma, \Sigma \mathrm{~N}, \mathrm{NB} \tau \varepsilon \tau \rho \alpha \pi \lambda \alpha \alpha^{\prime} \sigma \dot{\alpha}$ ह̇ $\sigma \tau \iota$



















 $\dot{\alpha} \pi о \tau о \mu \dot{\eta}$.








of $B U$ and $U V$. Thus, pentagon $B U V C W$ is equilateral. So, I say that it is also in one plane. For let $P X$ have been drawn from $P$, parallel to each of $R U$ and $S V$, on the exterior side of the cube. And let $X H$ and $H W$ have been joined. I say that $X H W$ is a straight-line. For since $H Q$ has been cut in extreme and mean ratio at $T$, and $Q T$ is its greater piece, thus as $H Q$ is to $Q T$, so $Q T$ (is) to $T H$. And $H Q$ (is) equal to $H P$, and $Q T$ to each of $T W$ and $P X$. Thus, as $H P$ is to $P X$, so $W T$ (is) to $T H$. And $H P$ is parallel to $T W$. For of each of them is at right-angles to the plane $B D$ [Prop. 11.6]. And $T H$ (is parallel) to $P X$. For each of them is at right-angles to the plane $B F$ [Prop. 11.6]. And if two triangles, like $X P H$ and $H T W$, having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, $X H$ is straight-on to $H W$. And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon $U B W C V$ is in one plane.

So, I say that it is also equiangular.
For since the straight-line $N P$ has been cut in extreme and mean ratio at $R$, and $P R$ is the greater piece [thus as the sum of $N P$ and $P R$ is to $P N$, so $N P$ (is) to $P R$ ], and $P R$ (is) equal to $P S$ [thus as $S N$ is to $N P$, so $N P$ (is) to $P S], N S$ has thus also been cut in extreme and mean ratio at $P$, and $N P$ is the greater piece [Prop. 13.5]. Thus, the (sum of the squares) on $N S$ and $S P$ is three times the (square) on $N P$ [Prop. 13.4]. And $N P$ (is) equal to $N B$, and $P S$ to $S V$. Thus, the (sum of the) squares on $N S$ and $S V$ is three times the (square) on $N B$. Hence, the (sum of the squares) on $V S, S N$, and $N B$ is four times the (square) on $N B$. And the (square) on $S B$ is equal to the (sum of the squares) on $S N$ and $N B$ [Prop. 1.47]. Thus, the (sum of the squares) on $B S$ and $S V$-that is to say, the (square) on $B V$ [for angle $V S B$ (is) a right-angle]-is four times the (square) on $N B$ [Def. 11.3, Prop. 1.47]. Thus, $V B$ is double $B N$. And $B C$ (is) also double $B N$. Thus, $B V$ is equal to $B C$. And since the two (straight-lines) $B U$ and $U V$ are equal to the two (straight-lines) $B W$ and $W C$ (respectively), and the base $B V$ (is) equal to the base $B C$, angle $B U V$ is thus equal to angle $B W C$ [Prop. 1.8]. So, similarly, we can show that angle $U V C$ is equal to angle $B W C$. Thus, the three angles $B W C, B U V$, and $U V C$ are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon $B U V C W$ is equiangular. And it was also shown (to be) equilateral. Thus, pentagon $B U V C W$ is equilateral and equiangular, and it is on one of the sides, $B C$, of the cube. Thus, if we make the





 ठદ̀ xaì $\dot{\eta}$ દ̉x то





 бبaipas тñs $\pi \varepsilon p ı \lambda \alpha \mu \alpha \nu o u ́ \sigma \eta s$ tòv xúßov. xaí ध̇бтl tò $\Omega$



 «̈ $\rho \alpha$ тò $\delta \omega \delta \varepsilon x \alpha \varepsilon ́ \delta \rho \circ \nu$ тñ $\delta о \vartheta \varepsilon i ́ \sigma \eta ~ \sigma \varphi \alpha i p \alpha . ~$
 $\dot{\eta}$ калоине́vŋ $\alpha \pi о \tau о \mu \dot{\eta}$.






 P $\Sigma$ трòs $\sigma \cup \nu \alpha \mu \varphi o ́ \tau \varepsilon \rho o \nu ~ \tau \grave{\eta} \nu \mathrm{NP}, \Sigma \Xi . \mu \varepsilon i \zeta \omega \nu$ ठغ̀ $\dot{\eta} \mathrm{N} \Xi \tau \tilde{\eta} \varsigma$







白тотоий.
 $\dot{\alpha} \pi о \tau о \mu \dot{\eta}$.
same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will have been constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let $X P$ have been produced, and let (the produced straight-line) be $X Z$. Thus, $P Z$ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at $Z$. Thus, $Z$ is the center of the sphere enclosing the cube, and $Z P$ (is) half the side of the cube. So, let $U Z$ have been joined. And since the straight-line $N S$ has been cut in extreme and mean ratio at $P$, and its greater piece is $N P$, the (sum of the squares) on $N S$ and $S P$ is thus three times the (square) on $N P$ [Prop. 13.4]. And $N S$ (is) equal to $X Z$, inasmuch as $N P$ is also equal to $P Z$, and $X P$ to $P S$. But, indeed, $P S$ (is) also (equal) to $X U$, since (it is) also (equal) to $R P$. Thus, the (sum of the squares) on $Z X$ and $X U$ is three times the (square) on $N P$. And the (square) on $U Z$ is equal to the (sum of the squares) on $Z X$ and $X U$ [Prop. 1.47]. Thus, the (square) on $U Z$ is three times the (square) on $N P$. And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And $N P$ is half of the side of the cube. Thus, $U Z$ is equal to the radius of the sphere enclosing the cube. And $Z$ is the center of the sphere enclosing the cube. Thus, point $U$ is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since $R P$ is the greater piece of $N P$, which has been cut in extreme and mean ratio, and $P S$ is the greater piece of $P O$, which has been cut in extreme and mean ratio, $R S$ is thus the greater piece of the whole of $N O$, which has been cut in extreme and mean ratio. [Thus, since as $N P$ is to $P R$, (so) $P R$ (is) to $R N$, and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding

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order) [Prop. 5.15]. Thus, as $N O$ (is) to $R S$, so $R S$ (is) to the sum of $N R$ and $S O$. And $N O$ (is) greater than $R S$. Thus, $R S$ (is) also greater than the sum of $N R$ and $S O$ [Prop. 5.14]. Thus, $N O$ has been cut in extreme and mean ratio, and $R S$ is its greater piece.] And $R S$ (is) equal to $U V$. Thus, $U V$ is the greater piece of $N O$, which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, $N O$, which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

Thus, $U V$, which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

## Corollary

So, (it is) clear, from this, that the side of the dodecahedron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio. ${ }^{\dagger}$ (Which is) the very thing it was required to show.
${ }^{\dagger}$ If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4 / 3}$, and the side of the dodecahedron is $(1 / 3)(\sqrt{15}-\sqrt{3})$.

$$
i \eta^{\prime}
$$

 $\alpha l$ тpòs $\dot{\alpha} \lambda \lambda \dot{\eta} \lambda \alpha \varsigma$.




 $\pi \rho o ̀ s ~ o ̉ p \vartheta \grave{\alpha} \varsigma ~ \eta \eta \chi \vartheta \omega \sigma \alpha \nu$ 人i $\Gamma \mathrm{E}, \Delta \mathrm{Z}$, $\alpha \alpha i$ ध่ $\pi \varepsilon \zeta \varepsilon \cup ́ \chi \vartheta \omega \sigma \alpha \nu \alpha i \mathrm{AZ}$,




## Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another. ${ }^{\dagger}$


Let the diameter, $A B$, of the given sphere be laid out. And let it have been cut at $C$, such that $A C$ is equal to $C B$, and at $D$, such that $A D$ is double $D B$. And let the semi-circle $A E B$ have been drawn on $A B$. And let $C E$ and $D F$ have been drawn from $C$ and $D$ (respectively), at right-angles to $A B$. And let $A F, F B$, and $E B$ have been joined. And since $A D$ is double $D B, A B$ is thus triple $B D$. Thus, via conversion, $B A$ is one and a half







 $\dot{\alpha} \pi o ̀ ~ \tau \tilde{\eta} \varsigma \mathrm{AB}$ тоũ $\alpha \pi o ̀ ~ \tau \tilde{\eta} \varsigma \mathrm{BZ}$. है $\sigma \tau \iota ~ \delta \grave{\varepsilon}$ xaì $\dot{\eta} \tau \tilde{\eta} \varsigma ~ \sigma \varphi \alpha i p \alpha \varsigma$

 દ̇б立 $\pi \lambda \varepsilon u p \alpha ́$.





 غ̇бஎì $\pi \lambda \varepsilon u p \alpha ́$.
































times $A D$. And as $B A$ (is) to $A D$, so the (square) on $B A$ (is) to the (square) on $A F$ [Def. 5.9]. For triangle $A F B$ is equiangular to triangle $A F D$ [Prop. 6.8]. Thus, the (square) on $B A$ is one and a half times the (square) on $A F$. And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And $A B$ is the diameter of the sphere. Thus, $A F$ is equal to the side of the pyramid.

Again, since $A D$ is double $D B, A B$ is thus triple $B D$. And as $A B$ (is) to $B D$, so the (square) on $A B$ (is) to the (square) on $B F$ [Prop. 6.8, Def. 5.9]. Thus, the (square) on $A B$ is three times the (square) on $B F$. And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And $A B$ is the diameter of the sphere. Thus, $B F$ is the side of the cube.

And since $A C$ is equal to $C B, A B$ is thus double $B C$. And as $A B$ (is) to $B C$, so the (square) on $A B$ (is) to the (square) on $B E$ [Prop. 6.8, Def. 5.9]. Thus, the (square) on $A B$ is double the (square) on $B E$. And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And $A B$ is the diameter of the given sphere. Thus, $B E$ is the side of the octagon.

So let $A G$ have been drawn from point $A$ at rightangles to the straight-line $A B$. And let $A G$ be made equal to $A B$. And let $G C$ have been joined. And let $H K$ have been drawn from $H$, perpendicular to $A B$. And since $G A$ is double $A C$. For $G A$ (is) equal to $A B$. And as $G A$ (is) to $A C$, so $H K$ (is) to $K C$ [Prop. 6.4]. $H K$ (is) thus also double $K C$. Thus, the (square) on $H K$ is four times the (square) on $K C$. Thus, the (sum of the squares) on $H K$ and $K C$, which is the (square) on $H C$ [Prop. 1.47], is five times the (square) on $K C$. And $H C$ (is) equal to $C B$. Thus, the (square) on $B C$ (is) five times the (square) on $C K$. And since $A B$ is double $C B$, of which $A D$ is double $D B$, the remainder $B D$ is thus double the remainder $D C$. $B C$ (is) thus triple $C D$. The (square) on $B C$ (is) thus nine times the (square) on $C D$. And the (square) on $B C$ (is) five times the (square) on $C K$. Thus, the (square) on $C K$ (is) greater than the (square) on $C D . C K$ is thus greater than $C D$. Let $C L$ be made equal to $C K$. And let $L M$ have been drawn from $L$ at right-angles to $A B$. And let $M B$ have been joined. And since the (square) on $B C$ is five times the (square) on $C K$, and $A B$ is double $B C$, and $K L$ double $C K$, the (square) on $A B$ is thus five times the (square) on $K L$. And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And $A B$ is the diameter of the sphere. Thus, $K L$ is the radius of the circle from



 モ̇бтì $\dot{\eta} \mathrm{MB}$.
 $\mu \varepsilon ́ \sigma o v ~ \lambda o ́ \gamma o v ~ x \alpha \tau \alpha ̀ ~ \tau o ̀ ~ N, ~ x \alpha i ̀ ~ \varepsilon ै \sigma \tau \omega ~ \mu \varepsilon і ̈ \zeta o v ~ \tau \mu \tilde{n ̃ \mu ~ \tau o ̀ ~ N B \cdot ~ \dot{~}}$ NB $\alpha \rho \alpha ~ \delta \omega \delta \varepsilon x \alpha \varepsilon ́ \delta \rho o u ~ \varepsilon ̇ \sigma \tau i ~ \pi \lambda \varepsilon u p \alpha ́ . ~$





 $\pi \cup \rho \alpha \mu i ́ \delta o s ~ \pi \lambda \varepsilon \cup p \alpha ̀ ~ \tau \eta ̃ \varsigma ~ \mu \varepsilon ̀ v ~ \tau о u ̃ ~ o ̉ x \tau \alpha \varepsilon ́ \delta \rho o u ~ \pi \lambda \varepsilon \cup p a ̃ s ~ \delta u v \alpha ́ \mu \varepsilon \iota ~$






 $\dot{\varepsilon} \lambda \alpha \dot{\alpha} \tau \tau \omega \nu, \dot{\eta}$ ठ文 $\alpha \pi o \tau o \mu \dot{\eta}$.
 тои̃ $\delta \omega \delta \varepsilon \chi \alpha \varepsilon ́ \delta \rho o u ~ \tau \eta ̃ \varsigma ~ N B, ~ \delta \varepsilon i \xi о \mu \varepsilon \nu ~ o u ̛ \tau \omega \varsigma . ~$














 тò $\mu \varepsilon і ̈ \zeta о \nu ~ \tau \mu \tilde{\eta} \mu \alpha ́ ~ \varepsilon ̇ \sigma \tau \tau \nu ~ \dot{\eta}$ NB $\mu \varepsilon i \zeta \omega \nu$ äp $\dot{\eta}$ K $\Lambda$ т $\tilde{\eta}$ NB.


 б $\omega \delta \varepsilon \chi \alpha \varepsilon ́ \delta \rho \circ \cup$ öँ
which the icosahedron has been described. Thus, $K L$ is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and $A B$ is the diameter of the sphere, and $K L$ the side of the hexagon, and $A K$ (is) equal to $L B$, thus $A K$ and $L B$ are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since $L B$ is (the side) of the decagon. And $M L$ (is the side) of the hexagon-for (it is) equal to $K L$, since (it is) also (equal) to $H K$, for they are equally far from the center. And $H K$ and $K L$ are each double $K C . M B$ is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, $M B$ is (the side) of the icosahedron.

And since $F B$ is the side of the cube, let it have been cut in extreme and mean ratio at $N$, and let $N B$ be the greater piece. Thus, $N B$ is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, $A F$, of the pyramid, and twice the square on (the side), $B E$, of the octagon, and three times the square on (the side), $F B$, of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures-I mean, of the pyramid, and of the octahedron, and of the cubeare in rational ratios to one another. And (the sides of) the remaining two (figures)-I mean, of the icosahedron, and of the dodecahedron-are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straightlines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].
(And), we can show that the side, $M B$, of the icosahedron is greater that the (side), $N B$, or the dodecahedron, as follows.

For, since triangle $F D B$ is equiangular to triangle $F A B$ [Prop. 6.8], proportionally, as $D B$ is to $B F$, so $B F$ (is) to $B A$ [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third,
so the (square) on the first (is) to the (square) on the second [Def. 5.9 , Prop. 6.20 corr.]. Thus, as $D B$ is to $B A$, so the (square) on $D B$ (is) to the (square) on $B F$. Thus, inversely, as $A B$ (is) to $B D$, so the (square) on $F B$ (is) to the (square) on $B D$. And $A B$ (is) triple $B D$. Thus, the (square) on $F B$ (is) three times the (square) on $B D$. And the (square) on $A D$ is also four times the (square) on $D B$. For $A D$ (is) double $D B$. Thus, the (square) on $A D$ (is) greater than the (square) on $F B$. Thus, $A D$ (is) greater than $F B$. Thus, $A L$ is much greater than $F B$. And $K L$ is the greater piece of $A L$, which is cut in extreme and mean ratio-inasmuch as $L K$ is (the side) of the hexagon, and $K A$ (the side) of the decagon [Prop. 13.9]. And $N B$ is the greater piece of $F B$, which is cut in extreme and mean ratio. Thus, $K L$ (is) greater than $N B$. And $K L$ (is) equal to $L M$. Thus, $L M$ (is) greater than $N B$ [and $M B$ is greater than $L M$ ]. Thus, $M B$, which is (the side) of the icosahedron, is much greater than $N B$, which is (the side) of the dodecahedron. (Which is) the very thing it was required to show.
$\dagger$ If the radius of the given sphere is unity then the sides of the pyramid (i.e., tetrahedron), octahedron, cube, icosahedron, and dodecahedron, respectively, satisfy the following inequality: $\sqrt{8 / 3}>\sqrt{2}>\sqrt{4 / 3}>(1 / \sqrt{5}) \sqrt{10-2 \sqrt{5}}>(1 / 3)(\sqrt{15}-\sqrt{3})$.

 жаi í $\sigma o \gamma \omega v i ́ \omega \nu$ "̂ $\sigma \omega \nu \dot{\alpha} \lambda \lambda \eta \dot{\eta} \lambda o l s$.



 x $\alpha \grave{~ i} \sigma o \gamma \omega v i ́ \omega \nu ~ \pi \rho o ̀ s ~ \varepsilon ̇ v i ~ \sigma \eta \mu \varepsilon i ́ ̣ ~ \sigma u v เ \sigma \tau \alpha \mu e ́ v \omega \nu ~ o u ̉ x ~ ह ै \sigma \tau \alpha ル ~$








 oưons $\gamma \dot{\alpha} \rho$ тñs тoũ $\pi \varepsilon \nu \tau \alpha \gamma \omega ́ v o u$ í $\sigma o \pi \lambda \varepsilon u ́ p o u ~ \gamma \omega v i ́ \alpha s ~ o ̉ p \vartheta n ̃ s ~$


 $\pi o v$.
 $\sigma \tau \varepsilon \rho \varepsilon o ̀ v ~ \sigma \cup \sigma \tau \alpha \vartheta \eta ́ \sigma \varepsilon \tau \alpha l ~ i ́ \pi o ̀ ~ i \sigma o \pi \lambda \varepsilon u ́ p \omega \nu ~ \tau \varepsilon ~ \chi \alpha i ̀ ~ i \sigma o \gamma \omega v i ́ \omega \nu$


So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to twothirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four





 $\mathrm{ZA}, \mathrm{ZB}, \mathrm{Z}, \mathrm{Z} \Delta, \mathrm{ZE}$. סíx $\alpha$ äp $\alpha$ té $\mu \nu 0 \cup \sigma$ tàs $\pi p o ̀ s ~ \tau o i ̃ s ~ A, ~$ $\mathrm{B}, \Gamma, \Delta$, E тои̃ $\pi \varepsilon \nu \tau \alpha \gamma \dot{\omega} \nu \circ \cup \gamma \omega v i ́ \alpha \varsigma . ~ \chi \alpha i ~ \varepsilon ̇ \pi \varepsilon i ~ \alpha i ~ \pi \rho o ̀ \varsigma ~ \tau \widetilde{̣} ~ Z ~$






(equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of rightangle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.


It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a rightangle, as follows.

For let $A B C D E$ be an equilateral and equiangular pentagon, and let the circle $A B C D E$ have been circumscribed about it [Prop. 4.14]. And let its center, $F$, have been found [Prop. 3.1]. And let $F A, F B, F C, F D$, and $F E$ have been joined. Thus, they cut the angles of the pentagon in half at (points) $A, B, C, D$, and $E$ [Prop. 1.4]. And since the five angles at $F$ are equal (in sum) to four right-angles, and are also equal (to one another), (any) one of them, like $A F B$, is thus one less a fifth of a right-angle. Thus, the (sum of the) remaining (angles in triangle $A B F$ ), $F A B$ and $A B F$, is one plus a fifth of a right-angle [Prop. 1.32]. And $F A B$ (is) equal to $F B C$. Thus, the whole angle, $A B C$, of the pentagon is also one and one-fifth of a right-angle. (Which is) the very thing it was required to show.

## GREEK-ENGLISH LEXICON

ABBREVIATIONS：act－active；adj－adjective；adv－adverb；conj －conjunction；fut－future；gen－genitive；imperat－imperative； impf－imperfect；ind－indeclinable；indic－indicative；intr－in－ transitive；mid－middle；neut－neuter；no－noun；par－particle； part－participle；pass－passive；perf－perfect；pre－preposition； pres－present；pro－pronoun；sg－singular；tr－transitive；$v b$－ verb．


$\alpha \varepsilon^{\prime}: ~ a d \nu$ ，always，for ever．


$\alpha$ 人itn $\mu \alpha$－$\alpha$ тоऽ，тó ：no，postulate．
 with．

$\dot{\alpha} \lambda \lambda \dot{\alpha}$ ：conj，but，otherwise．
«̀ $\lambda$ oros－ov ：adj，irrational．
$\alpha \mu \alpha: a d v$ ，at once，at the same time，together．
$\alpha \dot{\alpha} \mu \lambda \cup \gamma \dot{\omega} v \iota o s-o v: a d j$ ，obtuse－angled；tò $\alpha \mu \beta \lambda \cup \gamma \omega ́ v เ o v, n o$ ，ob－ tuse angle．
$\alpha \mu \beta \lambda u ́ s-\varepsilon i ̃ \alpha-ט ́ ~: ~ a d j, ~ o b t u s e . ~$
дд $\mu о ́ \tau \varepsilon р о \varsigma ~-~-~-o v ~: ~ p r o, ~ b o t h . ~$
$\alpha \nu \alpha \gamma \rho \alpha ́ \varphi \omega: \nu b$ ，describe（a figure）；see $\gamma \rho \alpha ́ \varphi \omega$ ．
$\dot{\alpha} \alpha \lambda \gamma^{\prime} i \alpha, \dot{\eta}$ ：no，proportion，（geometric）progression．
$\dot{\alpha} \nu \dot{\alpha} \lambda o \gamma o s-o \nu: a d j$ ，proportional．
$\dot{\alpha} \nu \alpha \dot{\alpha} \pi \alpha \lambda \iota \nu: a d v$ ，inverse（ly）．
$\alpha \nu \alpha \pi \lambda \eta \rho o ́ \omega: ~ v b$ ，fill up．
$\alpha \dot{\alpha} \alpha \sigma \tau \rho \varepsilon ́ \varphi \omega: v b$ ，turn upside down，convert（ratio）；see $\sigma \tau \rho \varepsilon ́ \varphi \omega$ ．
$\dot{\alpha} \nu \alpha \sigma \tau \rho \circ \varphi \dot{\eta}, \dot{\eta}: n o$ ，turning upside down，conversion（of ratio）．
$\alpha \dot{\alpha} v \vartheta \cup \varphi \alpha \iota \rho \varepsilon ́ \omega: \nu b$ ，take away in turn；see aip $\omega$ ．$\omega$ ．

＂̀vtoos－ov ：adj，unequal，uneven．
$\dot{\alpha} \nu \tau \iota \pi \dot{\alpha} \sigma \chi \omega: v b$ ，be reciprocally proportional；see $\pi \dot{\alpha} \sigma \chi \omega$ ．
${ }_{\alpha} \xi \omega \nu$－ovos，$\delta: \nu b$ ，axis．
$\alpha \pi \alpha \xi$ ：$a d v$ ，once．
$\ddot{\alpha} \pi \alpha \varsigma, \ddot{\alpha}^{\alpha} \pi \alpha \sigma \alpha, \ddot{\alpha}^{\alpha} \pi \alpha \nu$ ：adj，quite all，the whole．
$\alpha \not \pi \varepsilon ı \rho o s-o \nu$ ：adj，infinite．
$\dot{\alpha} \pi \varepsilon \nu \alpha \nu \tau i o v: i n d$, opposite．
$\dot{\alpha} \pi \varepsilon \chi \omega$ ：$v b$ ，be far from，be away from；see $\varepsilon$ ह $\chi \omega$ ．
$\dot{\alpha} \pi \lambda \alpha \tau \dot{n} \leqslant-\varepsilon ́ \varsigma: a d j$ ，without breadth．
$\dot{\alpha} \pi o ́ \delta \varepsilon \varepsilon \iota \xi!\varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，proof．

$\dot{\alpha} \pi о \lambda \alpha \mu \beta \alpha \nu \omega$ ：$v b$ ，take from，subtract from，cut off from；see $\lambda \alpha \mu \beta \alpha ́ v \omega$ ．
$\dot{\alpha} \pi о \tau \varepsilon ́ \mu \nu \omega: \nu b$ ，cut off，subtend．
$\dot{\alpha} \pi o ́ \tau \mu \eta \mu \alpha-\alpha \tau o \varsigma$, tó ：no，piece cut off，segment．
$\dot{\alpha} \pi о \tau о \mu \dot{\gamma}, \dot{\eta}: v b$ ，piece cut off，apotome．
$\stackrel{\alpha}{\alpha} \pi \tau \omega, \dot{\alpha} \psi \omega, \tilde{n}^{\psi} \psi \alpha,-$ ，ñ $\mu \mu \alpha,-: v b$ ，touch，join，meet．
$\alpha \pi \omega \dot{\alpha} \tau \rho \circ \varsigma-\alpha-o v: a d j$ ，further off．
${ }_{\alpha} \rho \alpha$ ：par，thus，as it seems（inferential）．
ג̇рıधиós，$\delta: n o$, number．
ג̀ptıóxıs ：$a d v$ ，an even number of times．
артıóт $\lambda \varepsilon \cup \rho \circ \varsigma-$－ov：adj，having a even number of sides．
 gin．
वे $\sigma \cup ́ \mu \mu \varepsilon \tau р о \varsigma ~-o v: ~ a d j, ~ i n c o m m e n s u r a b l e . ~$
$\dot{\alpha} \sigma \dot{\jmath} \mu \pi \tau \omega \tau \circ \varsigma-o \nu: a d j$, not touching，not meeting．
«̈ртıos $-\alpha-$ ov ：adj，even，perfect．

人ं兀ótos－ov ：adj，absurd，paradoxical．
$\alpha \cup ̉ \tau o ́ v \varepsilon \nu: a d v$ ，immediately，obviously．
$\dot{\alpha} \varphi \alpha i p \varepsilon \omega: v b$ ，take from，subtract from，cut off from；see $\alpha i p \varepsilon ́ \omega$ ．
$\dot{\alpha} \varphi \dot{\eta}, \dot{\eta}: n o$ ，point of contact．
$\beta \dot{\alpha} \vartheta \bigcirc о \varsigma-\varepsilon о \varsigma$ ，тó ：no，depth，height．
$\beta \alpha i v \omega,-\beta \dot{\eta} \sigma o \mu \alpha l,-\varepsilon \dot{\beta} \eta \nu, \beta \dot{\varepsilon} \beta \eta \chi \alpha,-,-: v b$ ，walk；perf，stand （of angle）．
$\beta \dot{\alpha} \lambda \lambda \omega, \beta \alpha \lambda \widetilde{\omega}$, 关 $\beta \alpha \lambda o \nu, \beta \dot{\varepsilon} \beta \lambda \eta \chi \alpha, \beta \dot{\varepsilon} \beta \lambda \eta \mu \alpha L, \dot{\varepsilon} \beta \lambda \dot{\eta} \vartheta \eta \nu: v b$ ，throw．
$\beta \dot{\alpha} \sigma!\varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，base（of a triangle）．
$\gamma$ áp ：conj，for（explanatory）．
 happen，become．
$\gamma \nu \omega ́ \mu \omega \nu$－ovos，$\dot{\eta}$ ：no，gnomon．
үраций，$\dot{\eta}$ ：no，line．
$\gamma \rho \alpha ́ \varphi \omega, \gamma \rho \alpha ́ \psi \omega$ ，é $\gamma \rho \alpha[\psi / \varphi] \alpha$ ，$\gamma$ ह́ $\gamma \rho \alpha \varphi \alpha$ ，$\gamma$ ह́ $\gamma \rho \alpha \mu \mu \alpha \mathrm{L}$ ，غ̇ $\rho \alpha \psi \alpha ́ \mu \eta \nu: v b$ ， draw（a figure）．
$\gamma \omega v i ́ \alpha, \dot{\eta}$ ：no，angle．
$\delta \varepsilon \tilde{\iota}: \nu b$ ，be necessary；$\delta \varepsilon \tilde{\iota}$, it is necessary；$\varepsilon \delta \varepsilon \iota$ ，it was necassary； ठє́ov，being necessary．
 demonstrate．
бєเxtéov ：ind，one must show．

$\delta \varepsilon x \alpha \gamma \omega ́ v o s-o v: a d j$ ，ten－sided；tò $\delta \varepsilon \chi \alpha \gamma \omega ́ v o v, n o$ ，decagon．
 accept．
$\delta \dot{\eta}$ ：conj，so（explanatory）．
$\delta \eta \lambda \alpha \delta \dot{\eta}$ ：ind，quite clear，manifest．
$\delta \tilde{\eta} \lambda о \varsigma-\eta$－ov ：adj，clear．
$\delta \eta \lambda o v o ́ t \iota: a d v$ ，manifestly．
$\delta \iota \alpha ́ \gamma \omega: v b$ ，carry over，draw through，draw across；see ${ }_{\alpha}^{\alpha} \gamma \omega$ ．
$\delta \iota \alpha \gamma \omega ́ v$ vos－ov ：adj，diagonal．
$\delta \iota \alpha \lambda \varepsilon i ́ \pi \omega: v b$ ，leave an interval between．
 agonal．
$\delta \iota \alpha \dot{p} \varepsilon \sigma \iota \varsigma-\varepsilon \omega \varsigma, \dot{\eta}:$ no，division，separation．
$\delta \iota \alpha \iota \rho \varepsilon ́ \omega$ ：$v b$ ，divide（in two）；$\delta \iota \alpha \rho \varepsilon \vartheta \varepsilon ́ v \tau o \varsigma-\eta-o v$ ，adj，separated （ratio）；see $\alpha i p \varepsilon ́ \omega$.

$\delta \iota \propto \varepsilon ́ \rho \omega: ~ v b$ ，differ；see $\varphi \varepsilon ́ \rho \omega$ ．

бцроіроц－ov ：adj，two－thirds．
$\delta \iota \pi \lambda \alpha \sigma \iota \alpha ́ \zeta \omega: v b$ ，double．
$\delta \iota \pi \lambda \alpha ́ \sigma \iota o \varsigma-\alpha-o v: a d j$ ，double，twofold．
$\delta \iota \pi \lambda \alpha \sigma^{\prime} \omega \nu-o \nu$ ：adj，double，twofold．
$\delta \iota \pi \lambda o u ̃ \varsigma-\tilde{\eta}$－oũv ：adj，double．
ठis ：adv，twice．
$\delta_{i}^{\prime} \chi \alpha: a d v$ ，in two，in half．
$\delta$ боороцí $\alpha, \dot{\eta}$ ：no，point of bisection．
ठUác－$\alpha$ óos，$\dot{\eta}: n o$ ，the number two，dyad．
$\delta \dot{v} v \mu \alpha \iota: v b$ ，be able，be capable，generate，square，be when squared；סuvauévn，$\dot{\eta}, n o$ ，square－root（of area）－i．e．，strai－ ght－line whose square is equal to a given area．
ठঠ́vapus－$\varepsilon \omega \varsigma, \dot{\eta}:$ no，power（usually 2 nd power when used in mathematical sence，hence），square．
$\delta u v \alpha \tau o ́ s-n ́ n$－óv ：adj，possible．
б $\omega \delta \varepsilon \chi \alpha ́ \varepsilon \delta \partial \rho \circ \varsigma-o \nu: a d j$ ，twelve－sided．
غ́autoũ－$\tilde{\eta} s$－oũ ：adj，of him／her／it／self，his／her／its／own．
غ̀ $\gamma \gamma^{i} \omega \nu$－ov ：adj，nearer，nearest．
غ̇ $\gamma \gamma \rho \alpha ́ \varphi \omega: v b$ ，inscribe；see $\gamma \rho \alpha ́ \varphi \omega$ ．
عโัos－₹оऽ，七ó ：no，figure，form，shape．


say，speak；per pass part，вıpクuévos－$\eta$－ov，adj，said，afore－ mentioned．
ยไเะ ．．．$\varepsilon ้ \tau \varepsilon$ ：ind，either ．．．or．
ย้ $\chi \alpha \sigma \tau \circ \varsigma-\eta$－ov ：pro，each，every one．
غ́x $\alpha \tau$ ह́pos－$\alpha-o \nu$ ：pro，each（of two）．
$\dot{\varepsilon} \chi \beta \dot{\alpha} \lambda \lambda \omega: \nu b$ ，produce（a line）；see $\beta \dot{\alpha} \lambda \lambda \omega$ ．
غ̇ $\chi \vartheta \varepsilon ์ \omega$ ：$v b$ ，set out．
ย้ $\varkappa \varkappa \varepsilon ц \mu \alpha!: \nu b$ ，be set out，be taken；see $\varkappa \varepsilon \tilde{\mu} \mu \alpha$ ．
غ̇xti७nu：：vb，set out；see tivnuı．
غ̇x тós ：pre＋gen，outside，external．
$\dot{\varepsilon} \lambda \dot{\alpha} \sigma \sigma \omega \nu / \dot{\varepsilon} \lambda \dot{\alpha} \tau \tau \omega \nu-o \nu: a d j$ ，less，lesser．
غ̀ $\lambda \alpha$ 人 $!\sigma \tau о \varsigma-\eta$－ov ：adj，least．
$\dot{\varepsilon} \lambda \lambda \varepsilon i \pi \omega: \nu b$ ，be less than，fall short of．
$\dot{\varepsilon} \mu \pi i \pi \tau \omega: \nu b$ ，meet（of lines），fall on；see $\pi i \pi \tau \omega$ ．
ยै $\mu \pi \rho \circ \sigma \vartheta \varepsilon v: a d v$ ，in front．
$\dot{\varepsilon} \vee \alpha \lambda \lambda \alpha ́ \xi: a d v$ ，alternate（ly）．

ع่v $\delta \varepsilon ́ \chi o \mu \alpha \iota: v b$ ，admit，allow．
ย้vยหย ：ind，on account of，for the sake of．
$\dot{\varepsilon} \nu \nu \alpha \pi \lambda \alpha ́ \sigma เ o \varsigma-\alpha-o v: a d j$ ，nine－fold，nine－times．
हैvvot $\alpha, \dot{\eta}: n o$ ，notion．
$\varepsilon \nu \pi \varepsilon \rho เ \varepsilon ́ \chi \omega: \nu b$ ，encompass；see $\varepsilon ้ \chi \omega$ ．
$\dot{\varepsilon} v \pi i \pi \tau \omega$ ：see $\varepsilon$ ह̉ $\mu i \pi \tau \tau \omega$ ．
๕̇vtós ：pre + gen，inside，interior，within，internal．

$\varepsilon \xi \alpha \pi \lambda \alpha \dot{\alpha} เ \circ \varsigma-\alpha-o \nu$ ：adj，sixfold．
$\varepsilon \xi \tilde{n} \varsigma$ ：$a d v$ ，in order，successively，consecutively．
$\varepsilon \xi \omega \vartheta \varepsilon v: a d v$ ，outside，extrinsic．
$\dot{\varepsilon} \pi \alpha \dot{\alpha} \omega \omega$ ：$a d v$ ，above．
غ̇л $\alpha \varphi \dot{\eta}, \dot{\eta}: n o$ ，point of contact．
غ̇лદí ：conj，since（causal）．
$\dot{\varepsilon} \pi \varepsilon \iota \delta \dot{r} \pi \varepsilon \rho$ ：ind，inasmuch as，seeing that．
 $v b$ ，join（by a line）．

غ̇ $\pi เ \nu \circ$ é $\omega: v b$ ，think of，contrive．

غ่ $\pi เ \sigma \chi$ ह́лтоน $\alpha!: \nu b$ ，investigate．
$\varepsilon ่ \pi i \sigma \chi \varepsilon \psi レ \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，inspection，investigation．
غ̇ $\tau \iota \tau \alpha ́ \sigma \sigma \omega: v b$ ，put upon，enjoin；七ò $\dot{\varepsilon} \pi \iota \tau \alpha \chi \vartheta \varepsilon ́ v, ~ n o$ ，the（thing） prescribed；see $\tau \alpha ́ \sigma \sigma \omega$ ．
èrítpıтos－ov ：adj，one and a third times．
غ̇лเ甲́́veı $\alpha, \dot{\eta}: n o$ ，surface．
हैтоцаь：vb，follow．

है $\sigma \chi \alpha \tau \circ \varsigma-\eta$－ov ：adj，outermost，uttermost，last．

हัтєроऽ－$\alpha$－ov ：adj，other（of two）．
ह̌兀ı ：par，yet，still，besides．
 linear figure．
 عu่ทยĩ ，in a straight－line，straight－on．

$\dot{\varepsilon} \varphi \alpha ́ \pi \tau \omega: v b$ ，bind to；mid，touch；$\dot{\eta} \dot{\varepsilon} \varphi \alpha \pi \tau o \mu \varepsilon ́ v \eta$ ，no，tangent； see $\alpha \pi \tau \omega$ ．
 ：vb，coincide；pass，be applied．
$\dot{\varepsilon} \varphi \varepsilon \xi \tilde{\eta} \varsigma: a d v$ ，in order，adjacent．



そ้ $\bar{\eta} \eta$ ：ind，already，now．
$\dot{\eta} \chi \omega, \stackrel{\eta}{\eta} \xi \omega$, —，一，一，—：$v b$ ，have come，be present．
ท̀ $\mu \iota \varkappa \dot{\jmath} x \lambda \iota \circ \nu$ ，七ó ：no，semi－circle．
$\dot{\eta} \mu \iota o ́ \lambda \iota o s-\alpha-o v: a d j$ ，containing one and a half，one and a half times．

$\ddot{\eta} \pi \varepsilon \rho=\eta \quad \eta+\pi \varepsilon \rho:$ conj，than，than indeed．
 $\vartheta \varepsilon ́ \sigma \iota \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，placing，setting，position．
$\vartheta \varepsilon \omega \rho \eta \mu \alpha-\alpha$ тоऽ，七ó ：no，theorem．
¿欠̂ıos－$\alpha$－ov ：adj，one＇s own．
 the same multiples，equal multiples．
iбoү⿳㇒⿻⿰㇒乛小七七ıos－ov ：adj，equiangular．
iбónлєupos－ov ：adj，equilateral．



 thing）．
 up（oneself）；Note：perfect I have stood up can be taken to mean present I am standing．
iбoü廿и́s－દ́s ：adj，of equal height．
$\chi \alpha \vartheta \dot{\alpha} \pi \varepsilon \rho:$ ind，according as，just as．
犭⿱㇒́刂兀єтоऽ－ov：adj，perpendicular．
$\chi \alpha \vartheta$ ó $\bar{\lambda}$ ou ：$a d v$ ，on the whole，in general．
$\chi \alpha \lambda \varepsilon ́ \omega: v b$ ，call．

$x_{\alpha} \stackrel{\alpha}{\nu}=x \alpha i \stackrel{\alpha}{\alpha} \nu$ ：ind，even if，and if．
х $\alpha \tau \alpha \gamma \rho \alpha \varphi \dot{\eta}, \dot{\eta}: n o$ ，diagram，figure．
жатаүра́ф $\omega: \nu b$ ，describe／draw，inscribe（a figure）；see rpó $\varphi \omega$ ． $\chi \alpha \tau \alpha \chi \circ \lambda \circ \cup \vartheta \varepsilon ์ \omega: \nu b$ ，follow after．
$\chi \alpha \tau \alpha \lambda \varepsilon i ́ \pi \omega: v b$ ，leave behind；see $\lambda \varepsilon i ́ \pi \omega$ ；$\tau \grave{\alpha} \chi \alpha \tau \alpha \lambda \varepsilon เ \pi o ́ \mu \varepsilon \nu \alpha$, no， remainder．
$\chi \alpha \tau \alpha \lambda \lambda \lambda \eta \lambda \circ \varsigma-o \nu$ ：adj，in succession，in corresponding order．
$\chi \alpha \tau \alpha \mu \varepsilon \tau \rho \varepsilon ́ \omega: v b$ ，measure（exactly）．
$\chi \alpha \tau \alpha \nu \tau \alpha ́ \omega: v b$ ，come to，arrive at．
$\chi \alpha \tau \alpha \sigma \varkappa \varepsilon \cup \alpha ́ \zeta \omega: v b$ ，furnish，construct．
 made；see tînuu．
иદ́vtpov，тó ：no，center．
$\chi \lambda \dot{\alpha} \omega: \nu b$ ，break off，inflect．
 cline．
$\chi \lambda i \sigma t \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，inclination，bending．
кої̀оऽ－$\eta$－ov ：adj，hollow，concave．
xорич $\dot{n}, \dot{\eta}:$ no，top，summit，apex；x $\alpha \tau \grave{\alpha}$ xopu甲 $\dot{\eta} \nu$, vertically opposite（of angles）．

хи́ßоя，ó ：no，cube．
хи́хлоऽ，ó ：no，circle．
хи́льレסроя，ó：no，cylinder．
xuptós－n̆－óv：adj，convex．
$\chi \widetilde{\omega} \nu \circ \varsigma$, ó ：no，cone．
 take．
$\lambda \varepsilon ́ \gamma \omega: v b$ ，say；pres pass part，$\lambda \varepsilon \gamma o ́ \mu \varepsilon v o s-\eta$－ov，adj，so－called； see हैıمん．
 leave behind．
$\lambda \eta \mu \mu \dot{\alpha} \tau \iota o \nu$ ，七ó ：no，diminutive of $\lambda \tilde{n} \mu \mu \alpha$ ．
$\lambda \tilde{n} \mu \mu \alpha-\alpha \tau о \varsigma$ ，тó ：no，lemma．
$\lambda \tilde{n} \psi \iota \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，taking，catching．
$\lambda o ́ \gamma o s, \delta: n o$ ，ratio，proportion，argument．
$\lambda o u \pi o ́ s-\dot{\eta}$－óv ：adj，remaining．

$\mu \varepsilon ́ \gamma \varepsilon \vartheta ั \circ \varsigma-\varepsilon \circ \varsigma$, tó ：no，magnitude，size．

$\mu \varepsilon ́ v \omega, \mu \varepsilon \nu \widetilde{\omega}$ ，$\varepsilon^{\prime} \mu \varepsilon เ \nu \alpha, \mu \varepsilon \mu \varepsilon ́ v \eta \gamma \alpha,-,-: v b$ ，stay，remain．
$\mu$ ќpos－ous，tó ：no，part，direction，side．
$\mu \varepsilon ́ \sigma o s-\eta$－ov ：adj，middle，mean，medial；ह̇x ठóo $\mu \varepsilon ́ \sigma \omega \nu$ ，bime－ dial．
$\mu \varepsilon \tau \alpha \lambda \alpha \mu \beta \alpha \nu \omega: \nu b$ ，take up．
$\mu \varepsilon \tau \alpha \xi \dot{\jmath}$ ：$a d v$ ，between．
$\mu \varepsilon \tau \varepsilon ́ \omega \rho o \varsigma-o v: a d j$ ，raised off the ground．
$\mu \varepsilon \tau \rho \varepsilon ́ \omega: v b$ ，measure．
$\mu$ ц́троv，тó ：no，measure．
$\mu \eta \delta \varepsilon i ́, \mu \eta \delta \varepsilon \mu i \alpha, \mu \eta \delta \bar{\varepsilon} v: a d j$ ，not even one，（neut．）nothing．
$\mu \eta \delta \dot{\varepsilon ́ \pi о т \varepsilon ~: ~ a d v, ~ n e v e r . ~}$
$\mu \eta \delta$ ह́tєроs－$\alpha-o \nu: p r o$, neither（of two）．
$\mu \tilde{\eta} x о \varsigma-\varepsilon о \varsigma$, tó ：no，length．
$\mu \dot{\eta} \nu$ ：par，truely，indeed．
$\mu o v \alpha ́ \varsigma-\alpha ́ \delta o s, \dot{\eta}: n o$ ，unit，unity．
uovaxós－म́n－óv：adj，unique．
$\mu o v \alpha \chi \widetilde{\omega} \varsigma$ ：$a d v$ ，uniquely．
uóvos－$\eta$－ov ：adj，alone．
 conceive．
oios－$\alpha$－ov ：pre，such as，of what sort．
ỏ $\chi \tau \alpha ́ \varepsilon \delta \delta \rho o s ~-o v: a d j, ~ e i g h t-s i d e d . ~$
ö $\lambda$ os－$\eta$－ov ：adj，whole．
ouorevńs－$\varepsilon \varsigma$ ：adj，of the same kind．
ö $\mu$ oıos－$\alpha$－ov：adj，similar．
$\dot{\delta} \mu$ оот $\lambda \eta \vartheta \forall \dot{\eta} \varsigma-\varepsilon ́ \varsigma: a d j$ ，similar in number．

óuotótns－$\eta$ tos，$\dot{\eta}$ ：no similarity．
б $\mu$ oí $\omega \varsigma$ ：$a d v$ ，similarly．
ó ód oros－ov ：adj，corresponding，homologous．$_{\text {on }}$

$\dot{\delta} \mu \dot{\omega} v \cup \mu o s-o v: a d j$, having the same name．

 gle．
ỏ̧ús－モĩ－ú ：adj，acute．
$\dot{\delta} \pi о \iota o \sigma o \tilde{\nu} \nu=\delta \pi \sigma \tilde{o} \circ \varsigma-\alpha-o \nu+o u ̃ \nu: a d j$ ，of whatever kind，any kind whatsoever．
$\dot{\delta}$ ócos $-\eta$－ov ：pro，as many，as many as．
$\delta \pi о \sigma о \sigma \delta \eta \pi о \tau о \tilde{u} \nu=\delta \pi \delta ́ \sigma o s-\eta-o \nu+\delta \dot{\eta}+\pi o \tau \varepsilon ́ \varepsilon+o u ̛ ้ \nu: a d j$ ，of whatever number，any number whatsoever．
$\dot{\delta} \pi о \sigma о \sigma o u ̃ \nu=\delta \pi \sigma ́ \sigma o \varsigma-\eta-o \nu+o \tilde{v} \nu: a d j$ ，of whatever number， any number whatsoever．
о́о́тєроऽ $-\alpha-$ ov ：pro，either（of two），which（of two）．
òpษorต́vıov，七ó ：no，rectangle，right－angle．
ỏค७ós－$\dot{\eta}$－óv ：adj，straight，right－angled，perpendicular；$\pi \rho o ̀ \varsigma$ ópけ̀̀ $\varsigma \omega v i \alpha \varsigma$ ，at right－angles．
őpos，$\delta: n o$ ，boundary，definition，term（of a ratio）．
 whatsoever．
óóxus ：ind，as many times as，as often as．
$\delta \sigma \alpha \pi \lambda \alpha \dot{\sigma} \sigma \circ \varsigma-o \nu:$ pro，as many times as．
őoos－$\eta$－ov ：pro，as many as．
ǒ $\sigma \pi \varepsilon \rho, \stackrel{\eta}{\eta} \pi \varepsilon \rho$ ，ö $\pi \varepsilon \rho$ ：pro，the very man who，the very thing which．
ö $\sigma \tau \iota \varsigma$, ク̈tus，ö tı ：pro，anyone who，anything which．
оัт $\alpha \nu$ ：$a d v$ ，when，whenever．
ótเoũv ：ind，whatsoever．



oủvย์v ：ind，nothing．
oưv ：adv，therefore，in fact．
oư $\tau \omega \varsigma: a d v$ ，thusly，in this case．
$\pi \alpha \lambda \iota \nu$ ：$a d \nu$ ，back，again．
$\pi \alpha \dot{\alpha} \tau \omega \varsigma: a d \nu$ ，in all ways．
$\pi \alpha p \dot{\alpha}:$ prep + acc，parallel to．
$\pi \alpha p \alpha \beta \alpha \dot{\lambda} \lambda \omega$ ：$\nu b$ ，apply（a figure）；see $\beta \dot{\alpha} \lambda \lambda \omega$ ．
$\pi \alpha p \alpha \beta o \lambda \dot{n}, \dot{\eta}$ ：no，application．
$\pi \alpha \rho \alpha ́ \varkappa \varepsilon!\mu \alpha \iota: \nu b$ ，lie beside，apply（a figure）；see $\chi \varepsilon \tilde{\mu} \alpha \iota$ ．
$\pi \alpha \rho \alpha \lambda \lambda \alpha ́ \sigma \sigma \omega, \pi \alpha \rho \alpha \lambda \lambda \dot{\alpha} \xi \omega,-, \pi \alpha \rho \dot{\eta} \lambda \lambda \alpha \alpha \alpha,-,-: v b$, miss，fall awry．
$\pi \alpha \rho \alpha \lambda \lambda \eta \lambda \varepsilon \pi i \pi \varepsilon \delta o \varsigma,-o \nu$ ：adj，with parallel surfaces；tò $\pi \alpha \rho \alpha \lambda$－ $\lambda \eta \lambda \varepsilon \pi i \pi \varepsilon \delta o \nu, n o$ ，parallelepiped．
$\pi \alpha \rho \alpha \lambda \lambda \eta \lambda$ ó $о р \alpha \mu \mu о \varsigma-o \nu: a d j$ ，bounded by parallel lines；tò $\pi \alpha-$ pa入入ท入órpaццоь，no，parallelogram．
$\pi \alpha \rho \alpha \dot{\lambda} \lambda \eta \lambda \circ \varsigma-o \nu$ ：adj，parallel；tò $\pi \alpha \rho \alpha ́ \lambda \lambda \eta \lambda o \nu, n o$ ，parallel， parallel－line．
$\pi \alpha \rho \alpha \pi \lambda n n^{\rho} \omega \mu \alpha-\alpha \tau o \varsigma$ ，七ó ：no，complement（of a parallelogram）．
$\pi \alpha p \alpha \tau \varepsilon ́ \lambda \cup \varepsilon \tau o \varsigma-o \nu: a d j$ ，penultimate．
$\pi \alpha p \varepsilon ́ \chi:$ prep + gen，except．
$\pi \alpha p \varepsilon \mu \pi i \pi \tau \omega: \nu b$ ，insert；see $\pi i \pi \tau \omega$ ．

$\pi \varepsilon \nu \tau \alpha ́ \gamma \omega \nu \circ \varsigma-o \nu: a d j$ ，pentagonal；тò $\pi \varepsilon \nu \tau \alpha ́ \gamma \omega \nu o v, n o$ ，pen－ tagon．
$\pi \varepsilon \nu \tau \alpha \pi \lambda \alpha ́ \sigma t o s-\alpha-o \nu: a d j$ ，five－fold，five－times．
$\pi \varepsilon \nu \tau \varepsilon \chi \alpha\llcorner\delta \varepsilon \chi \alpha ́ \gamma \omega \nu \circ \nu$ ，七ó ：no，fifteen－sided figure．
$\pi \varepsilon \pi \varepsilon \rho \alpha \sigma \mu \varepsilon ́ v o \varsigma-\eta$－оv ：adj，finite，limited；see $\pi \varepsilon \rho \alpha i v \omega$ ．
$\pi \varepsilon \rho \alpha i v \omega, \pi \varepsilon \rho \alpha \nu \widetilde{\omega}, \varepsilon ̇ \pi \varepsilon ́ \rho \alpha \nu \alpha,-, \pi \varepsilon \pi \varepsilon \rho \alpha \dot{\sigma} \sigma \mu \alpha$, ह̇ $\pi \varepsilon \rho \alpha \nu \alpha \dot{\nu} \vartheta \eta \nu: v b$ ，bring
to end，finish，complete；pass，be finite．
$\pi \varepsilon ́ p \alpha \varsigma-\alpha \tau о \varsigma$ ，tó ：no，end，extremity．
$\pi \varepsilon \rho \alpha \tau o ́ \omega,-,-,-,-,-: v b$ ，bring to an end．
$\pi \varepsilon \rho เ \gamma \rho \alpha ́ \varphi \omega: \nu b$ ，circumscribe；see $\gamma \rho \alpha ́ \varphi \omega$.
$\pi \varepsilon \rho เ \varepsilon ́ \chi \omega: v b$ ，encompass，surround，contain，comprise；see દ̌ $\chi \omega$ ．
$\pi \varepsilon \rho เ \lambda \alpha \mu \beta \alpha \dot{\nu} \omega: v b$ ，enclose；see $\lambda \alpha \mu \beta \alpha ́ \nu \omega$ ．
$\pi \varepsilon p \iota \sigma \sigma \alpha \dot{\alpha} \bullet \varsigma$ ：$a d v$ ，an odd number of times．
$\pi \varepsilon \rho เ \sigma \sigma o ́ s-n$－óv ：adj，odd．
$\pi \varepsilon р \iota \varphi$ е́рєı $\alpha, \dot{\eta}$ ：no，circumference．
$\pi \varepsilon р เ \varphi \varepsilon ́ \rho \omega: \nu b$ ，carry round；see $\varphi \varepsilon ́ \rho \omega$ ．
$\pi \eta \lambda \iota x$ ót $\eta$ s－$\eta$ тоs，$\dot{\eta}: n o$ ，magnitude，size．
$\pi i \pi \tau \omega, \pi \varepsilon \sigma \circ \tilde{\mu} \mu \alpha \mathrm{~L}$ ，光 $\pi \varepsilon \sigma o \nu, \pi \varepsilon ́ \pi \tau \omega \chi \alpha,-,-: v b$ ，fall．
$\pi \lambda \alpha ́ \tau o \varsigma-\varepsilon о \varsigma, ~ \tau o ́ ~: ~ n o, ~ b r e a d t h, ~ w i d t h . ~$
$\pi \lambda \varepsilon i \omega \nu$－ov ：adj，more，several．
$\pi \lambda \varepsilon u p \alpha ́, \dot{\eta}: n o$ ，side．
$\pi \lambda \tilde{\eta} \vartheta \circ \varsigma-\varepsilon \circ \varsigma$ ，tó ：no，great number，multitude，number．
$\pi \lambda \dot{\eta} \nu: a d \nu \& p r e p+g e n$, more than．
тotós－$\dot{\alpha}-o ́ v: a d j$ ，of a certain nature，kind，quality，type．
$\pi o \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \zeta \omega$ ：$v b$ ，multiply．
$\pi о \lambda \lambda \alpha \pi \lambda \alpha \sigma \iota \alpha \sigma \mu \sigma ́ \varsigma, \delta \quad: n o$ ，multiplication．
$\pi о \lambda \lambda \alpha \pi \lambda \alpha \dot{\sigma} \iota \circ \nu$ ，七ó ：no，multiple．

$\pi \circ \lambda \dot{\gamma} \gamma \omega \nu \circ \varsigma-o \nu$ ：adj，polygonal；tó $\pi o \lambda u ́ \gamma \omega \nu o v, n o$ ，polygon．
по入и́ $\lambda \lambda \varepsilon \cup \rho \circ \varsigma-o \nu: a d j$, multilateral．
$\pi o ́ p ı \sigma \mu \alpha-\alpha \tau o \varsigma$, тó ：no，corollary．
$\pi о \tau \varepsilon ́ \quad$ ：ind，at some time．
$\pi \rho \tilde{\imath} \sigma \mu \alpha-\alpha \tau о \varsigma$ ，七ó ：no，prism．
$\pi \rho o \beta \alpha i v \omega: v b$ ，step forward，advance．


троєрє́ $\omega$ ：vb，say beforehand；perf pass part，проє！рпиє́vos－$\eta$ －ov，adj，aforementioned；see ع＂p
$\pi \rho o \sigma \alpha \nu \alpha \pi \lambda \eta \rho o ́ \omega: ~ v b$ ，fill up，complete．
тробаvaүpáqן ：$v b$ ，complete（tracing of）；see rpá $\varphi \omega$ ．
$\pi \rho о \sigma \alpha \rho \mu о ́ \zeta \omega: \nu b$ ，fit to，attach to．
$\pi р о \sigma \varepsilon \chi \beta \dot{\alpha} \lambda \lambda \omega: \nu b$ ，produce（a line）；see $\varepsilon$ ย $\chi \beta \dot{\alpha} \lambda \lambda \omega$ ．
$\pi \rho о \sigma \varepsilon \cup р i ́ \sigma \chi \omega: v b$ ，find besides，find；see عípí $\sigma \omega$ ．
$\pi \rho о \sigma \lambda \alpha \mu \beta \alpha ́ \mu \omega: \nu b$ ，add．
$\pi \rho о ́ \varkappa \varepsilon ч \mu ц!: \nu b$ ，set before，prescribe；see $\chi \varepsilon \tilde{\mu} \mu เ$.
$\pi \rho o ́ \sigma \varkappa \varepsilon ч \alpha<: ~ v b$ ，be laid on，have been added to；see $\varkappa \varepsilon \tilde{\mu} \mu \alpha$.
$\pi \rho о \sigma \pi i \pi \tau \omega: v b$ ，fall on，fall toward，meet；see $\pi i \pi \tau \omega$ ．
$\pi \rho о \tau \alpha \sigma \iota \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$, proposition．
$\pi \rho \circ \sigma \tau \alpha ́ \sigma \sigma \omega$ ：$v b$ ，prescribe，enjoin；тò тpo $\tau \tau \alpha \vartheta \varepsilon ์ v$, no，the thing prescribed；see $\tau \alpha \dot{\sigma} \sigma \omega$ ．
$\pi \rho о \sigma \tau i \vartheta \eta \mu: \nu b$ ，add；see тi७nu．
тро́тєроs－$\alpha$－ov ：adj，first（comparative），before，former．

$\pi \rho \circ \chi \omega \rho \varepsilon ́ \omega$ ：$\nu b$ ，go／come forward，advance．
$\pi \rho \widetilde{\omega} \tau o \varsigma-\alpha-o v: a d j$, first，prime．
тираиіс－íoos，$\dot{\eta}: n o$, pyramid．
portós－$\dot{\eta}$－óv ：adj，expressible，rational．

póußos，ó no，rhombus．
бпиєі̃ov，тó ：no，point．
$\sigma \chi \alpha \lambda \eta \nu o ́ s-\dot{\eta}$－óv ：adj，scalene．
бтєрєós－$-\alpha$－óv ：adj，solid；тò $\sigma \tau \varepsilon \rho \varepsilon o ́ v, ~ n o, ~ s o l i d, ~ s o l i d ~ b o d y . ~$
бтоьхєiov，七ó ：no，element．

$\sigma \dot{\gamma} \gamma \varkappa \varepsilon!\mu \alpha: v b$ ，lie together，be the sum of，be composed； $\sigma \cup \gamma \varkappa \varepsilon ́ \mu \varepsilon \varepsilon$ vos $-\eta$－ov，adj，composed（ratio），compounded； see $\varkappa \varepsilon і ̈ \mu \alpha . ~$
бúrxpíve ：vb，compare；see xpív
$\sigma \cup \mu \beta \alpha i v \omega: v b$ ，come to pass，happen，follow；see $\beta \alpha i v \omega$ ．
$\sigma \cup \mu \beta \dot{\alpha} \lambda \lambda \omega: \nu b$ ，throw together，meet；see $\beta \dot{\alpha} \lambda \lambda \omega$ ．
$\sigma u ́ \mu \mu \varepsilon \tau \rho о \varsigma-o v: a d j$ ，commensurable．
$\sigma \cup ́ \mu \tau \alpha \varsigma-\alpha \nu \tau о \varsigma, \delta \quad:$ no，sum，whole．
$\sigma \cup \mu \pi i \pi \tau \omega: \nu b$ ，meet together（of lines）；see $\pi i \pi \tau \omega$ ．
$\sigma u \mu \pi \lambda$ noó $\omega: v b$ ，complete（a figure），fill in．
$\sigma u v \alpha ́ \gamma \omega: \nu b$ ，conclude，infer；see $\alpha \not \gamma \omega$ ．
 sum（of two things）．
$\sigma \cup v \alpha \pi о \delta \varepsilon i x \nu \cup \mu!: n o$, demonstrate together；see $\delta \varepsilon i x \nu \cup \mu \iota$.
$\sigma u v \alpha \varphi \dot{\eta}, \dot{\eta}: n o$ ，point of junction．
$\sigma u ́ v \delta v o, ~ o i, ~ \alpha i, ~ \tau \alpha ́ ~: ~ n o, ~ t w o ~ t o g e t h e r, ~ i n ~ p a i r s . ~$
$\sigma u v \varepsilon \chi \grave{n} \varsigma-\varepsilon ́ \varsigma:$ adj，continuous；$\chi \alpha \tau \grave{\alpha}$ tò $\sigma u v \varepsilon \chi \varepsilon ́ \varsigma$, continuously．
$\sigma \dot{v} \vartheta \varepsilon \sigma \iota \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，putting together，composition．
$\sigma$ Úvधモтos－ov ：adj，composite．
$\sigma v[\nu]$ í $\tau \eta u: ~: ~ v b$ ，construct（a figure），set up together；perf im－ perat pass 3rd sg，$\sigma u v \varepsilon \sigma \tau \alpha \dot{\alpha} \omega$ ；see ไ $\sigma \tau \eta u$ ．
ouvtînui ：vb，put together，add together，compound（ratio）； see тivnur．
$\sigma \chi \varepsilon ́ \sigma ル \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，state，condition．
$\sigma \chi \tilde{n} \mu \alpha-\alpha \tau o \varsigma$, тó ：no，figure．
$\sigma \varphi \alpha$ ïp $\alpha-\alpha \varsigma, \dot{\eta}: n o$ ，sphere．
$\tau \dot{\alpha} \xi\llcorner\varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，arrangement，order．
 ble，disturbe；тєтараүиє́vos－$\eta$－ov，adj，disturbed，per－ turbed．
 draw up．

$\tau \varepsilon ́ \mu \nu \omega$ ，$\tau \varepsilon \mu \nu \widetilde{\omega}$ ，غ̇ $\tau \varepsilon \mu \circ \nu,-\tau \varepsilon ́ \tau \mu \eta \nsim \alpha$ ，$\tau \varepsilon ́ \tau \mu \eta \mu \alpha \iota$ ，غ̀ $\tau \mu \mathfrak{\eta} \vartheta \eta \nu: \nu b$ ，cut； pres／fut indic act $3 r d$ sg，七દ́usı．
тєтартпиорьov，tó ：no，quadrant．
т $\varepsilon \tau \rho \alpha ́ \gamma \omega \nu \circ \varsigma-o \nu: a d j$ ，square；七ò $\tau \varepsilon \tau \rho \alpha ́ \gamma \omega \nu o \nu$, no，square．
$\tau \varepsilon \tau \rho \alpha ́ x \iota s: a d v$ ，four times．
$\tau \varepsilon \tau \rho \alpha \pi \lambda \alpha ́ \sigma t o \varsigma-\alpha-o \nu: a d j$ ，quadruple．
тєтра́ $\pi \lambda \varepsilon \cup \rho \circ \varsigma-o \nu: a d j$ ，quadrilateral．
$\tau \varepsilon \tau \rho \alpha \pi \lambda o ́ o s-\eta$－ov ：adj，fourfold．

$\tau \mu \tilde{\eta} \mu \alpha-\alpha \tau o \varsigma$ ，七ó ：no，part cut off，piece，segment．
toivuv ：par，accordingly．
тoooũtos－$\alpha$ útn－oũтo ：pro，such as this．
тоивús－દ́ $\omega \varsigma, \delta$ ：no，sector（of circle）．
тou $\dot{\eta}, \dot{\eta}: n o$ ，cutting，stump，piece．
то́тоя，$\delta:$ no，place，space．
тоб $\alpha \cup \tau \alpha$ xıs ：$a d v$ ，so many times．
тоб $\alpha \cup \tau \alpha \pi \lambda \alpha ́ \sigma \iota o s-\alpha-o \nu:$ pro，so many times．
тoбoũtos－$\alpha$ ט́tท－oũto ：pro，so many．
тouté $\sigma \tau \iota=$ тoั̃ $\tau^{\circ}$ है $\sigma \tau \iota:$ par，that is to say．


трเл $\lambda \alpha ́ \sigma เ \circ \varsigma-\alpha-o \nu: a d j$ ，triple，threefold．
трín $\lambda \varepsilon \cup \rho \circ \varsigma-o v: a d j$ ，trilateral．

тро́тоऽ，$\delta:$ ：no，way．
 $v b$ ，hit，happen to be at（a place）．
$\dot{\delta} \pi \dot{\alpha} \rho \chi \omega: v b$ ，begin，be，exist；see $\alpha \not \rho \chi \chi \omega$ ．
$\dot{\dot{\iota}} \pi \varepsilon \xi \alpha i \rho \varepsilon \sigma \iota \varsigma-\varepsilon \omega \varsigma, \dot{\eta}: n o$ ，removal．
$\dot{\dot{\delta}} \pi \varepsilon \rho \beta \alpha \dot{\lambda} \lambda \omega$ ：$v b$ ，overshoot，exceed；see $\beta \dot{\alpha} \lambda \lambda \omega$ ．
$\dot{\ddots} \pi \varepsilon \rho \circ \chi \dot{\eta}, \dot{\eta}: n o$ ，excess，difference．
ธ́ $\pi \varepsilon \rho \varepsilon ́ \chi \omega: \nu b$ ，exceed；see है $\chi \omega$ ．

ن́тóxєццаı ：vb，underlie，be assumed（as hypothesis）；see $\chi \varepsilon \tilde{\mu} \mu \alpha$.
$\dot{\delta} \pi о \lambda \varepsilon i \pi \omega: v b$ ，leave remaining．
 ：vb，subtend．
ư廿оऽ－६оৎ，七ó ：no，height．
بavepós－ג－óv：adj，visible，manifest．
$\varphi \eta \mu i, \varphi n ̃ \sigma \omega$ ，है $\varphi \eta \nu,-,-,-: v b$ ，say；है $\varphi \alpha \mu \varepsilon \nu$ ，we said．

$\chi \omega$ рьо⿱，тó ：no，place，spot，area，figure．
$\chi \omega$ pis ：pre + gen，apart from．
$\psi \alpha{ }^{\prime} \omega: v b$ ，touch．
$\dot{\omega}$ ：par，as，like，for instance．
$\dot{\omega}$ हัँ $\tau \cup \chi \varepsilon \nu$ ：par，at random．
$\dot{\omega} \sigma \alpha \dot{\tau} \tau \omega \varsigma$ ：adv，in the same manner，just so．
$\check{\omega} \sigma \tau \varepsilon:$ conj，so that（causal），hence．


[^0]:    ${ }^{\dagger}$ The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative ${ }^{\prime} H \iota \tau \dot{\eta} \sigma \vartheta \vartheta \omega$ could be translated as "let it be postulated", in the sense "let it stand as postulated", but not "let the postulate be now brought forward". The literal translation "let it have been postulated" sounds awkward in English, but more accurately captures the meaning of the Greek.
    ₹ This postulate effectively specifies that we are dealing with the geometry of flat, rather than curved, space.

[^1]:    ${ }^{\dagger}$ As an obvious extension of C.N.s $2 \& 3$-if equal things are added or subtracted from the two sides of an inequality then the inequality remains

[^2]:    † The Greek text has " $F B, B C$ ", which is obviously a mistake.
    $\ddagger$ This is an additional common notion.

[^3]:    ${ }^{\dagger}$ Here, use is made of the additional common notion that the squares of equal things are themselves equal. Later on, the inverse notion is used.

[^4]:    $\dagger$ Presumably, in an angular sense. $\ddagger$ This is not proved, except by reference to the figure.

[^5]:    $\dagger$ Presumably, by finding the center of $A B C$ [Prop. 3.1], drawing a straight-line between the center and point $B$, and then drawing $E F$ through

[^6]:    ${ }^{\dagger}$ Presumably, by finding the center of the circle [Prop. 3.1], drawing a line through it, and then drawing a second line through it, at right-angles to the first [Prop. 1.11].

[^7]:    $\dagger$ See the footnote to the previous proposition.
    $\ddagger$ See the footnote to Prop. 3.34.

[^8]:    ${ }^{\dagger}$ The theory of proportion set out in this book is generally attributed to Eudoxus of Cnidus. The novel feature of this theory is its ability to deal with irrational magnitudes, which had hitherto been a major stumbling block for Greek mathematicians. Throughout the footnotes in this book, $\alpha, \beta, \gamma$, etc., denote general (possibly irrational) magnitudes, whereas $m, n, l$, etc., denote positive integers.

[^9]:    ${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$ then $m \alpha: n \beta:: m \gamma: n \delta$, for all $m$ and $n$.

[^10]:    ${ }^{\dagger}$ In modern notation, this proposition reads that if $\alpha: \beta:: \gamma: \delta$, and $\alpha$ is the greatest and $\delta$ the least, then $\alpha+\delta>\beta+\gamma$.

[^11]:    ${ }^{\dagger}$ The fact that the two straight-lines meet follows because the sum of $A C E$ and $C A E$ is less than two right-angles, as can easily be demonstrated.

[^12]:    $\dagger$ This proposition is a geometric solution of the quadratic equation $x^{2}+\alpha x-\beta=0$. Here, $x$ is the ratio of a side of the excess to the corresponding side of figure $D, \alpha$ is the ratio of the length of $A B$ to the length of that side of figure $D$ which corresponds to the side of the excess running along $A B$, and $\beta$ is the ratio of the areas of figures $C$ and $D$. Only the positive root of the equation is found.

[^13]:    ${ }^{\dagger}$ This is a straight-forward generalization of Prop. 3.27

[^14]:    ${ }^{\dagger}$ The propositions contained in Books 7-9 are generally attributed to the school of Pythagoras.

[^15]:    ${ }^{\dagger}$ In modern notation, this proposition states that if $a=(1 / n) b$ and $c=(1 / n) d$ then if $a=(k / l) c$ then $b=(k / l) d$, where all symbols denote numbers.

[^16]:    ${ }^{\dagger}$ This proposition is a special case of Prop. 7.9.

[^17]:    ${ }^{\dagger}$ The propositions contained in Books 7-9 are generally attributed to the school of Pythagoras.

[^18]:    ${ }^{\dagger}$ In other words，between two given square numbers there exists a number in continued proportion．
    $\ddagger$ Literally，＂double＂．

[^19]:    ${ }^{\dagger}$ The propositions contained in Books 7-9 are generally attributed to the school of Pythagoras.

[^20]:    $\dagger$ This proposition demonstrates that perfect numbers take the form $2^{n-1}\left(2^{n}-1\right)$ provided that $2^{n}-1$ is a prime number. The ancient Greeks knew of four perfect numbers: $6,28,496$, and 8128 , which correspond to $n=2,3,5$, and 7 , respectively.

[^21]:    ${ }^{\dagger}$ The theory of incommensurable magntidues set out in this book is generally attributed to Theaetetus of Athens. In the footnotes throughout this book, $k, k^{\prime}$, etc. stand for distinct ratios of positive integers.

[^22]:    ${ }^{\dagger}$ In other words, two magnitudes $\alpha$ and $\beta$ are commensurable if $\alpha: \beta:: 1: k$, and incommensurable otherwise.
    $\ddagger$ Literally, "in power".
    § In other words, two straight-lines of length $\alpha$ and $\beta$ are commensurable in square if $\alpha: \beta:: 1: k^{1 / 2}$, and incommensurable in square otherwise. Likewise, the straight-lines are commensurable in length if $\alpha: \beta:: 1: k$, and incommensurable in length otherwise.
    『To be more exact, straight-lines can either be commensurable in square only, incommensurable in length only, or commenusrable/incommensurable in both length and square, with an assigned straight-line.

    * Let the length of the assigned straight-line be unity. Then rational straight-lines have lengths expressible as $k$ or $k^{1 / 2}$, depending on whether the lengths are commensurable in length, or in square only, respectively, with unity. All other straight-lines are irrational.
    \$ The square-root of an area is the length of the side of an equal area square.
    $\|$ The area of the square on the assigned straight-line is unity. Rational areas are expressible as $k$. All other areas are irrational. Thus, squares whose sides are of rational length have rational areas, and vice versa.

[^23]:    $\dagger$ Literally, "from two names".
    $\ddagger$ Thus, a binomial straight-line has a length expressible as $1+k^{1 / 2}$ [or, more generally, $\rho\left(1+k^{1 / 2}\right)$, where $\rho$ is rational-the same proviso applies to the definitions in the following propositions]. The binomial and the corresponding apotome, whose length is expressible as $1-k^{1 / 2}$

[^24]:    † Literally, "second from two medials".
    $\ddagger$ Since, by hypothesis, the squares on $A B$ and $B C$ are commensurable-see Props. 10.15, 10.23.
    ${ }^{\S}$ Thus, a second bimedial straight-line has a length expressible as $k^{1 / 4}+k^{1 / 2} / k^{1 / 4}$. The second bimedial and the corresponding second apotome of a medial, whose length is expressible as $k^{1 / 4}-k^{\prime 1 / 2} / k^{1 / 4}$ (see Prop. 10.75), are the positive roots of the quartic $x^{4}-2\left[\left(k+k^{\prime}\right) / \sqrt{k}\right] x^{2}+$

[^25]:    ${ }^{\dagger}$ If the rational straight-line has unit length then the length of a second binomial straight-line is $k / \sqrt{1-k^{\prime 2}}+k$. This, and the second apotome,

[^26]:    $\dagger$ If the rational straight-line has unit length then this proposition states that the square-root of a first binomial straight-line is a binomial straightline: i.e., a first binomial straight-line has a length $k+k \sqrt{1-k^{\prime 2}}$ whose square-root can be written $\rho\left(1+\sqrt{k^{\prime \prime}}\right)$, where $\rho=\sqrt{k\left(1+k^{\prime}\right) / 2}$ and $k^{\prime \prime}=\left(1-k^{\prime}\right) /\left(1+k^{\prime}\right)$. This is the length of a binomial straight-line (see Prop. 10.36), since $\rho$ is rational.

[^27]:    ${ }^{\dagger}$ If the rational straight-line has unit length then this proposition states that the square-root of a fourth binomial straight-line is a major straightline: i.e., a fourth binomial straight-line has a length $k\left(1+1 / \sqrt{1+k^{\prime}}\right)$ whose square-root can be written $\rho \sqrt{\left[1+k^{\prime \prime} /\left(1+k^{\prime \prime 2}\right)^{1 / 2}\right] / 2}+$ $\rho \sqrt{\left[1-k^{\prime \prime} /\left(1+k^{\prime \prime 2}\right)^{1 / 2}\right] / 2}$, where $\rho=\sqrt{k}$ and $k^{\prime \prime 2}=k^{\prime}$. This is the length of a major straight-line (see Prop. 10.39), since $\rho$ is rational.
    $\nu \eta^{\prime}$.
    
    
    
    

    ## Proposition 58

    If an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of a rational plus a medial (area). ${ }^{\dagger}$

[^28]:    ${ }^{\dagger}$ If the rational straight-line has unit length then this proposition states that the square-root of a fifth binomial straight-line is the square root of a rational plus a medial area: i.e., a fifth binomial straight-line has a length $k\left(\sqrt{1+k^{\prime}}+1\right)$ whose square-root can be written $\rho \sqrt{\left[\left(1+k^{\prime \prime 2}\right)^{1 / 2}+k^{\prime \prime}\right] /\left[2\left(1+k^{\prime \prime 2}\right)\right]}+\rho \sqrt{\left[\left(1+k^{\prime \prime 2}\right)^{1 / 2}-k^{\prime \prime}\right] /\left[2\left(1+k^{\prime \prime 2}\right)\right]}$, where $\rho=\sqrt{k\left(1+k^{\prime \prime 2}\right)}$ and $k^{\prime \prime 2}=k^{\prime}$. This is the length of

[^29]:    ${ }^{\dagger}$ If the rational straight-line has unit length then this proposition states that the square-root of a sixth binomial straight-line is the square root of the sum of two medial areas: i.e., a sixth binomial straight-line has a length $\sqrt{k}+\sqrt{k^{\prime}}$ whose square-root can be written
    $k^{1 / 4}\left(\sqrt{\left[1+k^{\prime \prime} /\left(1+k^{\prime 2}\right)^{1 / 2}\right] / 2}+\sqrt{\left[1-k^{\prime \prime} /\left(1+k^{\prime \prime 2}\right)^{1 / 2}\right] / 2}\right)$, where $k^{\prime \prime 2}=\left(k-k^{\prime}\right) / k^{\prime}$. This is the length of the square-root of the sum of

[^30]:    ${ }^{\dagger}$ See footnote to Prop. 10.37.

[^31]:    $\dagger$ See footnote to Prop. 10.53.

[^32]:    ${ }^{\dagger}$ The proofs of the first three propositions in this book are not at all rigorous. Hence, these three propositions should properly be regarded as additional axioms.
    ${ }^{\ddagger}$ This assumption essentially presupposes the validity of the proposition under discussion.

[^33]:    ${ }^{\dagger}$ In other words, the two straight-lines lie in the same plane, and never meet when produced in either direction.

[^34]:    ${ }^{\dagger}$ The novel feature of this book is the use of the so-called method of exhaustion (see Prop. 10.1), a precursor to integration which is generally

[^35]:    $i \alpha^{\prime}$.
     $\dot{\alpha} \lambda \lambda \dot{n} \lambda$ ous عíoiv $\dot{\omega} \varsigma$ 人i $\beta \dot{\alpha} \sigma \varepsilon ı s . ~$
    
     $\mathrm{K} \Lambda, \mathrm{MN}, \delta \iota \alpha ́ \mu \varepsilon \tau \rho o \iota ~ \delta \grave{\varepsilon} \tau \widetilde{\omega} \nu \beta \alpha ́ \sigma \varepsilon \omega \nu$ 人i $\mathrm{A} \mathrm{\Gamma}, \mathrm{EH} \cdot \lambda \varepsilon ́ \gamma \omega$ ，ö $\tau$
     ó $\mathrm{A} \Lambda$ x $\widetilde{\omega} v o s ~ \pi \rho o ̀ s ~ t o ̀ v ~ E N ~ x \widetilde{\omega} v o v . ~$

[^36]:    $\beta^{\prime}$.
    
    
     oi AB , $\Delta$, $\mathrm{EZH} \mathrm{\Theta} \mathrm{xúx} \mathrm{\lambda ol}, \mathrm{\delta ı} \mathrm{\alpha ́} \mathrm{\mu} \mathrm{\varepsilon} \mathrm{\tau pol} \mathrm{\delta è} \mathrm{\tau} \mathrm{\widetilde{ } \mathrm{\omega} \nu ~ \beta \alpha ́ \sigma \varepsilon \omega \nu ~ \alpha i ~} \mathrm{~B} \Delta$,
    

[^37]:    ${ }^{\dagger}$ Since $K B, B P$, and $K S$ are greater than the sides of an inscribed square, which are each of length $\sqrt{2} B X$.
    $\ddagger$ Note that points $Y$ and $V$ are actually identical.
    ${ }^{\S}$ This conclusion depends on the fact that the chord of the polygon in proposition 12.16 does not touch the inner circle.

[^38]:    ${ }^{\dagger}$ The five regular solids-the cube, tetrahedron (i.e., pyramid), octahedron, icosahedron, and dodecahedron-were problably discovered by the school of Pythagoras. They are generally termed "Platonic" solids because they feature prominently in Plato's famous dialogue Timaeus. Many of the theorems contained in this book-particularly those which pertain to the last two solids-are ascribed to Theaetetus of Athens.

