Infinity in Classical Indian Mathematics

By Avinash Sathaye

Professor of Mathematics
University of Kentucky

1. Introduction.
Even a child learns to count 1, 2, 3 and so on. The idea which leads to imagining the end of this process, or a lack thereof, leads to the idea of infinity and takes a greater sophistication. In the same fashion, the idea to extend the count backwards and imagine a 0 or even a -1, -2, and so on needs a different imaginative process. Both these ideas are, of course old and by now, quite familiar to everybody. One has, thus gotten used to these numbers called the ``Intregers''.

Mathematicians in ancient India are credited with being the pioneers in both these inventions. The whole world routinely uses the decimal number system with the ten digits 0,1,2,3,4,5,6,7,8,9 serving to build arbitrarily large numbers with the power of their place value. This is the so-called Hindu Arabic system, developed in India and propagated through the Arabic Mathematical sources across Europe.

Many of the standard techniques of algebraic calculations with integers written as decimal numbers are routinely taught in elementary schools and one no longer thinks about their power mainly due to their familiarity. These, however, were clearly developed in India; in view of the fact that without the place value system of number representation, they cannot exist!

The idea of infinity is a different story. If you open Mathematical Books of today, you will find the idea of infinity mentioned in somewhat higher level courses. In Calculus related courses, you will find the idea of the ``real'' infinity, the variables taking on larger and larger positive values. The resulting analysis of related variables leading to the notion of limits is at the heart of Calculus and thus, Modern Analysis. Similarly, the idea of a variable getting infinitesimally close to a finite number, like 0, is also at the heart of Calculus. You won't, however, find either the infinite or the infinitesimal in an elementary book on algebra, let alone arithmetic! The only thing you may find in an algebra book is a very stern warning about not ever dividing by zero!

On the other hand, in the algebra books of old times in India, we find both the infinite and the infinitesimal treated routinely. They also include some rather intriguing exercises. These appear to be shear nonsense, if one approaches them armed with modern conventions without recognizing the novelty of approach used in these ancient books. Often, these exercises are discarded, calling them unfortunate blemishes on the otherwise brilliant achievements of the authors. My aim in this short essay is to analyze these problems in detail and propose that they may be based on a potentially powerful modern algebraic idea. I am not proposing that the old books had developed the full algebraic machinery. It certainly never materialized among the known mathematical texts or their subsequent commentaries. But that may be only due to insufficient follow-up activity and understanding. There is a very interesting example of this phenomenon in Indian Astronomy.

Āryabhaṭa, the great mathematician and astronomer of the fifth had proposed a heliocentric model of the solar system as well as the notion that the earth was just a globe hanging in space. His ideas, however, were thoroughly rejected by the next mathematical genius Brahmagupta within a hundred years. The objections by Brahmagupta were the usual ``common sense'' arguments for a flat earth, some of which persist even today. But due to the reputation of Brahmagupta, the Āryabhaṭa theories never became widespread in India until they were imported from the European sources centuries later.

The ideas about infinity that I am proposing to discuss were certainly stated by Brahmagupta (sixth century) and the listed exercises below are to be found in the works of Bhāskarāchārya (II, of course) in the twelfth century. It is quite likely that Bhāskarāchārya's ideas were not grasped by his commentators. At the same time, his work came to be the central text for algebra as well as astronomy during the rest of history of India. As a result, his original ideas were probably not pursued any further.

There is yet another facet of the idea of the infinite, namely the counting infinity. In Modern Mathematics, this leads to the concepts of ordinal and cardinal numbers. In ancient Indian Mathematics, we find Jain texts discussing various such concepts of infinities. These texts are mainly religious or philosophical, but often carry a healthy amount of serious mathematics. They seem to introduce formal concepts of finite or enumerable, innumerable (very large but still finite) and infinite. They even classify multidimensional concepts for infinity. It is possible that they might have come close to the ideas of modern cardinal (or at least ordinal) numbers. However, I have not yet succeeded in finding explicit pointers to advanced ideas similar to the algebraic ideas discussed here. So a similar evaluation of Jain theories of infinities will have to await further evidence.

2. Basic Definitions.
To keep the discussion brief, I will give all the citations from Bhāskarāchārya's Bījagaõita (his book on Algebra) with some cross reference from his Līlāvatī (his book on Arithmetic). There is a small difference in the numbering of the verses in different editions, but the reader should be able to locate them near the indicated citations.

First, I collect the various defining properties of multiplication and division by zero.

वधादौ वियत् खस्य खं खेन घाते खहारो भवेत् खेन भक्तश्च राशिः॥ बीज २.१८

vadhādau viyat khasya khaṁ khena ghāte khahāro bhavet khena bhaktaśca rāśiḥ || bīj 2.18

A zero results when multiplied by zero, a **``khahara''** (zero-divided) results when a number (rāshi) is divided by zero.

In Līlāvatī, he gives more instruction about multiplying by zero.

योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः।

yoge khaṁ kṣepasamaṁ vargādau khaṁ khabhājito rāśiḥ |

खहरः स्यात् खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ॥ लीला.४६

khaharaḥ syāt khaguṇaḥ khaṁ khaguṇaścintyaśca śeṣavidhau || līlā.46

Zero plus (minus) zero is zero and powers of zero or zero. A number *divided by zero is* **``khahara''** (zero-divided). A number *multiplied by zero* is zero (but this) **khaguṇa** must be paid attention to in the rest of the calculation. In other words, Bhāskarāchārya recommends that one should wait to finish all operations before evaluating the khaguõa.

शून्ये गुणके जाते खं हारश्चेत् पुनस्तदा राशिः।

śūnye guṇake jāte khaṁ hāraścet punastadā rāśiḥ |

अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः॥ लीला. ४७

avikṛta eva jñeyastathaiva khenonitaśca yutaḥ || līlā. 47

If a zero becomes a multiplier and a number turns into zero, it should (really) be considered as unchanged if it is again divided by zero! Similarly, if a zero is subtracted off and added in (a number is considered unchanged.)

Effectively, Bhāskarāchārya is proposing two special terms to be called **khaguṇa** and **khahara** which require special algebraic manipulations.

For **khahara**, he explicitly adds a colorful description:

अस्मिन् विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु।
asmin vikāraḥ khahare na rāśāvapi praviṣṭeṣvapi niḥsṛteṣu |

बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत्॥ बीज. २।२०
bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat || bīja. 2|20

There is no change in this **khahara** by adding or subtracting (quantities), just like infinite immutable (Brahma or Viṣõu) which does not have any effect by the living beings entering or leaving it at the time of dissolution or creation of the world respectively.

For convenience, let us make our own formal definitions using modern terminology. It is clear that if we want to multiply or divide by zero then we need a place holder symbol for it since we want to be able to recover the original number by a reverse process later.

Definition.
Let $\Re$ stand for the usual set of real numbers (real numbers are common in Modern Mathematics, even though in old times, rational or algebraic numbers would be commonly used.)

Let ε stand for the multiplier zero. If x is any number (in $\Re$, i.e.), then by $x\varepsilon$ we shall denote the corresponding **khaguṇa** . The set of all **khaguṇa** is thus can be denoted as $\Re\varepsilon$.

Similarly, we write ∞ to denote the **khahara** $\frac{1}{0}$. Thus again every khahara can be represented as $x\infty$ as x varies over $\Re$. Thus, the set of khaharas can be written as $\Re\infty$.

We now have three kinds of numbers: ordinary, khaguṇa and khara.

The additional facts about the khahara can be presented thus.

$$x\infty + y = x\infty$$

for any real number y. We may call this the **winning rule**, it says that when a **khahara** is added to an ordinary number, then only the **khahara** survives!

The main issue to decide is what is the product of two of the new numbers.

We propose the following natural rules:

$$\varepsilon \cdot \varepsilon = \varepsilon \text{ , } \infty \cdot \infty = \infty \text{ and } \infty \cdot \varepsilon = \varepsilon \cdot \infty = 1.$$

Later on, we will explain why these rules are necessary. However, it is useful to note that such entities are known in Modern Algebra and are called **idempotents** or quantities whose squares equal themselves. Usually, 0 and 1 are the only two idempotents. Typically, if you have more of these idempotents, then the algebraic system gets too far away from the classical notions of number systems.

Now we shall take up the discussion of three ``exercises'' from Bhāskarāchārya's works to illustrate how the above rules help us get the correct answers. We shall also see how a strictly traditional interpretation of the algebraic rules will make the exercises as either nonsense, or, at best, mysterious!

3. The exercises.

Here are the three exercises. We give the original formulation as well as a translation using modern terminology. Each exercise has one special equation to solve which is of interest to us.

Problem 1.

खं पञ्चयुग्भवति किं वद खस्य वर्गं मूलं घनं घनपदं खगुणाश्च पञ्च।

khaṁ pañcayugbhavati kiṁ vada khasya vargaṁ mūlaṁ ghanaṁ ghanapadaṁ khaguṇāśca pañca |

खेनोद्धृता दश च कः खगुणो निजार्धयुक्तस्त्रिभिश्च गुणितः खहृतस्त्रिषष्टिः॥ लीला. ४८

khenoddhṛtā daśa ca kaḥ khaguṇo nijārdhayuktastribhiśca guṇitaḥ khahṛtastriṣaṣṭhiḥ ||līlā. 48

(1.1) What is 0 plus 5, (what are its) square and square roots, cube and cube root, (what is) 5 times 0 ?

(1.2) What is 10 divided by 0?

(1.3) What is the number (x), which, when **multiplied by 0** (x ε) **combined with its half** ($x\,\varepsilon + \frac{x\,\varepsilon}{2}$ or $\frac{3x\,\varepsilon}{2}$), then **multiplied by 3** (3 times $\frac{3x\,\varepsilon}{2} = \frac{9x\,\varepsilon}{2}$, then **divided by** 0 ($\frac{9x\,\varepsilon}{2}$ times ∞ or $\frac{9x}{2} \cdot \varepsilon \cdot \infty = \frac{9x}{2}$) **equals** 63?

The answers: (1.1) 5,0,0,0,0, $5 \cdot \varepsilon = 0$ (since no further operations are pending!).
(1.2) $10 \cdot \infty = 10\infty$
(1.3) Solution to $\frac{9x}{2} = 63$, so **x=14.**

Problem 2.

कः खेन विहृतो राशिः कोट्या युक्तोऽथवोऽनितः।
kaḥ khena vihṛto rāśiḥ koṭyā yukto'thavo'nitaḥ ।

वर्गितः स्वपदेनाढ्यः खगुणो नवतिर्भवेत्॥ बीज. १२०
vargitaḥ svapadenāḍhyaḥ khaguṇo navatirbhavet ।। bīja. 120

What is the number (x), **divided** by 0 ($x \cdot \infty$) , **augmented or reduced** by 100,000,000 (still $x \cdot \infty$ by the **winning rule**), **squared** ($x \cdot \infty \cdot x \cdot \infty = x^2 \cdot \infty$) and then **augmented** by its own square root ($x^2 \cdot \infty + x \cdot \infty = (x^2+x) \cdot \infty$), **multiplied** by 0 ($(x^2+x) \cdot \infty \cdot \varepsilon = (x^2+x)$) **becomes** 90?
The answer: $(x^2+x) = 90$, so x=9. We discard the answer -10 since the question asks for a rāshī, which, by conventional wisdom is non negative!

Problem 3.

कः सार्धसहितो राशिः खगुणो वर्गितो युतः।
kaḥ sārdhasahito rāśiḥ khaguṇo vargito yutaḥ ।

स्वपदाभ्यां खभक्तश्च जातः पञ्चदशोच्यताम्॥ बीज. १२१
svapadābhyāṁ khabhaktaśca jātaḥ pañcadaśocyatām ।। bīja. 121
What is the number (x) , **combined with its own half** ($x + \frac{x}{2}$ or $\frac{3x}{2}$),

multiplied by 0 ($\frac{3x\,\varepsilon}{2}$), **squared** ($\frac{3x\,\varepsilon}{2} \cdot \frac{3x\,\varepsilon}{2} = \frac{9x^2}{4} \cdot \varepsilon$), **augmented by twice its square root**
($\left(\left(\frac{9x^2}{4}\right) + \left(\frac{6x}{2}\right)\right) \cdot \varepsilon$), **divided** by zero ($\frac{9x^2+12x}{4} \cdot \varepsilon \cdot \infty = \frac{9x^2+12x}{4}$) **becomes** 15.

Answer: Thus, we solve, after cross multiplying, $9x^2+12x-60=0$. The accepted solution is 2, the negative solution $-\frac{10}{3}$ being discarded.

4. Discussion of the exercises.

We now analyze what would happen if we were to analyze these questions using traditional thought processes and not invoke the idempotent properties on our symbols.

The first exercise can be carried out as a simple limiting calculation. Imagine that the zero mentioned in the problem may be interpreted as a small number ε and at the end we shall take limit as ε goes to zero.

For exercise 1, the final equation reads:

$\frac{9x \cdot \varepsilon}{2 \cdot \varepsilon} = 60$ and since ε actually cancels, the limit as ε goes to zero would give us the same equation that we ended up solving.

Bhāskarāchārya, himself, suggests that such a limiting process is of great use in astronomical calculations.

For exercise 2, if we attempt a similar plan, we get:

$$\left(\left(\frac{x^2\infty^2}{1}\right)+\left(\frac{x\infty}{1}\right)\right)\cdot\frac{1}{\infty}=15.$$

Now, if we were to think of ∞ as a variable tending to infinity, our left hand side itself goes to infinity leaving us no useful limit. He does not offer a limiting process in the explanation, in contrast with the Līlāvatī.

Thus, Bhāskarāchārya definitely could not have a simple minded limiting process in mind. He certainly knew limits. This should be evident from the fact that he is credited to having stated the result that the derivative of the sine function is the cosine function. This result requires the calculation of a most delicate limit.

Thus, he must have had a different scheme of calculations in mind. We are offereing a possible alternative.

For exercise 3, a similar attempt with limits leads to:

$$\left(\left(\frac{9x^2\varepsilon^2}{4}\right)+\left(\frac{6x\varepsilon}{2}\right)\right)\cdot\frac{1}{\varepsilon}=15$$

If we try to take the limit of the left hand side, then we get the equation

$\left(\frac{3x}{1}\right)=15$, which leads to $x=5$, not the recommended solution! But, at least we get an answer!

Thus, Bhāskarāchārya probably included this as a sample of a different solution process. It probably was meant to illustrate the parallel properties of the two idempotents ∞ and ε.

This gives additional support to our suggestion that Bhāskarāchārya must have a formal algebraic calculation in mind, rather than the routine limit. His comment about using the process in Astronomy only occurs in Līlāvatī where the limit indeed works! He makes no such comment in the Bījagaõita book.

5. Conclusion.

What should we deduce from the above discussion? I am proposing that Bhāskarāchārya might have toyed with the ideas of more sophisticated algebraic systems. He has demonstrated his creative ability in the solution of the so-called Pell's equation (which was systematically studied in India from Brahmagupta onwards.) The Pell's equation seeks a solution in integers to an equation $x^2-by^2=1$ where b is a positive non square integer. The problem was raised by Fermat (in the seventeenth century) and was subsequently solved by Lagrange and others, the name of Pell being mistakenly introduced by Euler. It turns out that Brahmagupta had initiated the solution of the equation and Bhāskarāchārya provided the necessary generalized algorithm for solution. Bhāskarāchārya's method gives a way of handling what in Modern Mathematics would be denoted by the numbers $p+q\sqrt{b}$ and developed usual algebraic operation of such numbers in terms of operations on the pairs (p, q). In this situation, the resulting numbers form a well behaved fields unlike the khahara and khaguõa numbers.

It is quite likely that the resulting algebraic systems turned out to be too far ahead of times and were abandoned by Bhāskarāchārya himself. We may never know, but it is interesting to wonder about it.