Given below is a convenient version of the well known Euclidean algorithm which is easy to use and gives several useful conclusions fast.

This version is from a Mathematical section of a work on Astronomy by an astronomer Āryabhaṭa from India in the fifth century. The suggested working presented here differs from the traditional commentaries, but seems consistent with the original (very cryptic) description.

Let us say we have to calculate the GCD of the numbers $a = 414$ and $b = 189$. We also wish to solve the equation $ax - by = s$ in integers.

The work consists of three steps represented by the three matrices given below. I will explain the work further down.

$$A = \begin{pmatrix} 414 & * \\ 189 & 2 \\ 36 & 5 \\ 9 & 4 \\ 0 & * \end{pmatrix} \quad B = \begin{pmatrix} 11 & 414 & * \\ 5 & 189 & 2 \\ 1 & 36 & 5 \\ 0 & 9 & 4 \\ 1 & 0 & * \end{pmatrix} \quad C = \begin{pmatrix} 46 & 11 & 414 & * \\ 21 & 5 & 189 & 2 \\ 4 & 1 & 36 & 5 \\ 1 & 0 & 9 & 4 \\ 0 & 1 & 0 & * \end{pmatrix}$$

Here are the details:

1. The matrix $A$ represents the basic algorithm. You start with the two numbers on top. Call them $a, b$ in order. Divide the upper number by the one on the lower and write the remainder below. The quotient is recorded to the right. You continue until you get a 0.

   **Notation:** Let $n$ be the number of entries in the first column - including the zero entry. Here $n = 5$.

2. The number in $A$, just above 0 is the GCD of the top two numbers $414, 189$. Indeed, it is the GCD of any two successive entries above. Thus, the GCD in the example is $A(n-1, 1) = A(4, 1) = 9$. Let $d = 9$.

3. The matrix $B$ is obtained from $A$ by starting with a 1 at the bottom of the first column (inserted into $A$ at left). You write a 0 just above.

   Here is the formula to build the first column of $B$. To produce the next number above, multiply the current entry by the quotients' column entry and add the lower entry to it. In symbols:

$$B_{i,1} = B_{i+1,1}B_{i+1,3} + B_{i+2,1}.$$

4. When the first column of $B$ is thus filled, we can write the GCD of the original numbers as a combination.

   Specifically, the formula is:

$$B(1,1)B(2,2) - B(2,1)B(1,2) = \pm d = B(1,1)b - B(2,1)a.$$

   Thus, our example gives $(11)(189) - (5)414 = \pm 9$. The sign of the last number is $+$, which is easily verified by the unit digit. If you wish, you may also memorize the rule that sign is $+$ exactly when $n$ is odd, but it safer to check.

5. Sometimes we need to find all ways of expressing our GCD as a combination.

   This is where the matrix $C$ is useful.

   The matrix $C$ is obtained by a process similar to the one in $B$, except you start with 0 at the bottom of the new first column and 1 just above. Follow the same lifting procedure.

   Now, the above expression of the GCD can be updated to the general solution:

$$(C_{1,1}t + C_{1,2})b - (C_{2,1}t + C_{2,2})a = \pm d$$

   where $t$ is any integer.

   In our example, this becomes

$$(46t + 11)189 - (21t + 5)414 = 9.$$

   One of the main uses occurs when we take $t = -1$. Thus, for our example, we get

$$(-46 + 11)189 - (-21 + 5)414 = 9 \text{ or } (16)414 - (35)189 = 9.$$

   For specific problems this reverse order is important!

6. The first two numbers in the first column of $C$ are also useful. They give $a/d = 414/9 = 46$ and $b/d = 189/9 = 21$ respectively. Thus the $\mathrm{LCM}(a, b) = [a, b]$ is given by $(ab)/d = 46 \cdot 21 \cdot 9$.

   In some sens, it is not necessary to "build " $C$. We can just calculate these two numbers as $a/d, b/d$ and skip the work for $C$. If GCD is 1, then the first column of $C$ will come out simply as the first column of $A$, and nothing needs to be written!

7. To get any desired number $s$ expressed as a combination of $a, b$ we can proceed as follows:

   **Method 1** Write $s = md$ for an integer $m$.

   **(If this cannot be done, then there is no solution.)**

   Then use the above expression for $d$ and multiply it by $m$ to get a general solution. Choose a convenient $t$ as desired. Thus, we get:

$$(46t + 11m)189 - (21t + 5m)414 = 9m = s.$$

   Thus for $s = 54$ we take $m = 6$ and

$$(46t + 66)189 - (21t + 30m)414 = 54.$$

   For convenience, we could take $t = -1$ to yield $(20)189 - (9)414 = 54$.

   **Method 2** Build a new version of $B$ by creating a new first column in the following way. Start with any desired level and create a starting pair at that level and lift the numbers by the rule used for $B$.

   For instance, for $s = 54$, we start by writing $2, -1$ next to $36, 9$ respectively, so that $54 = (2)9 - (-1)36$. Thus we have a starting version of a new $B$ which we call $B^*$.

$$B^* = \begin{pmatrix} & 414 & * \\ & 189 & 2 \\ 2 & 36 & 5 \\ -1 & 9 & 4 \\ & 0 & * \end{pmatrix}$$

Then lifting process goes as follows:

$$B^* = \begin{pmatrix} & 414 & * \\ & 189 & 2 \\ 2 & 36 & 5 \\ -1 & 9 & 4 \\ & 0 & * \end{pmatrix} \to B^{**} = \begin{pmatrix} & 414 & * \\ 9 & 189 & 2 \\ 2 & 36 & 5 \\ -1 & 9 & 4 \\ & 0 & * \end{pmatrix} \to B^{***} = \begin{pmatrix} 20 & 414 & * \\ 9 & 189 & 2 \\ 2 & 36 & 5 \\ -1 & 9 & 4 \\ & 0 & * \end{pmatrix}$$

Note that this does not reach the bottom and it is not necessary (though we could fill it, if we wanted it!)

Then $(20)189 - (9)414 = 54$ as desired. This will always work, except we may need to multiply by $-1$ to fix the signs.

You should practice with other examples given below.

8. *In the traditional Indian style, I will refrain from giving the proof. However, in the modern age, you must try to make up the proof. Think about what it could be.*

Having studied the process of solving ax-by=c you should solve the following problems. First five are routine problems, while the last two illustrate the original applications.

1. Find a number (and all such positive numbers) which is divisible by 29 after adding 2 to it and also divisible by 45 after subtracting 7 from it.

   Do determine the smallest positive such number.

   **Hint: You need to solve the equation $29a - 45b = 9$. Do analyze how this will help you answer the question.**

2. Find a number divisible by 3 after subtracting 5 and divisible by 31 after subtracting 7. ( A small variation from Bhāskara I of 600 A.D.)

   Do determine the smallest positive such number.

3. Find a number which has remainder 5 when divided by 8, remainder 4 when divide by 9 and remainder 1 when divided by 7.

   Do determine the smallest positive such number.

   ( A small variation from Bhāskara I of 600 A.D. Bhāskara I 600 A.D.)

4. Find a number which leaves a remainder 1 when divided by $2, 3, 4, 5$ or 6 but is divisible by 7.

   Do determine the smallest positive such number.

   (Āryabhaṭīya bhāṣya 600 A.D. , Ibn-al-Haitam 1000 A.D., Fibonacci 1202 A.D.)

5. Suppose that if we count a set of objects by threes, then 2 are left; if we count by fives, then 3 are left and if we count by sevens, then 2 are left. How many objects are there? (It is implied that you should find the smallest such positive number. (Sunzi fifth century - the origin of the term "Chinese Remainder Theorem!")

6. The residue of the revolutions of Saturn is 24. Find the day number. (laghubhāskarīya 600 A.D.) Explanation: Let $x$ be the day number or the number of days elapsed since the beginning of the yuga (actually the current Kali yuga). Yuga is a long period consisting of $1,577,917,500$ civil days. The planet Saturn makes $146,564$ revolutions in a yuga, so makes $\frac{146{,}564}{1{,}577{,}917{,}500}$ revolutions in a day. This number equals $\frac{36{,}641}{394{,}479{,}375}$ . The statement that the "residue of revolutions is 24" means that the fractional number of revolutions is $\frac{24}{394{,}479{,}375}$. So, your task is to find (smallest) number of days elapsed and corresponding number of revolutions from the beginning of this yuga!

   By the way, this number of days in a yuga, which formally has $432,000$ years gives the average year as 365.259 days in length. The equation you get is

$$36,641x - 24 = 394,479,375y$$

   where $y$ is the number of complete revolutions.

7. The Sun's mean position was obtained as longitude 148 deg. 20 min. Calculate the day number. (Mahābhāskarīya 600 A.D.) and the number of revolutions of the Sun.

   For Sun, the reduced number of daily revolutions comes out to be $\frac{576}{210{,}389}$. The fractional number of revolutions is calculated to be $\frac{86{,}688}{210{,}389}$ after throwing away fractional part (less than 0.01) from the numerator. Verify this from the longitude.

## Note on the time scale in Indian tradition and astronomy.

It was customary to do astronomical calculations on a grand scale. One use of these calculations is that by axiom, the planetary positions were 0 in the beginning and their current values can be established by knowing their periods. Of course, we know that corrections are always needed, to both the formulas and observations. Usually, these are handled by special theories giving additional small variations in the answers.

Usually, the calculations start with the day number from the beginning of the year with a given knowledge of the starting positions from record.

Here are some more amusing data. The current yuga (Kali) has $432,000$ years and the previous three (Kṛta, Tretā and Dvāpāra ) were respectively $4, 3, 2$ times as long! These four together are often called a (mahā)yuga. The current one has only $5,104$ years elapsed.

The creator Brahmā was approximately 8 and a half of *his* years old at the beginning of the current yuga, which is declared as $26,782,530,123,179$ of human years.

There are other variants of this huge number, though they all agree in the top significant digits.

According to another school of thought, he is however, about 50 years old.

How does a day of Brahmā compare with the human scale of time?

A yuga (or mahāyuga = the four together) is $4,320,000$ years. A thousand of these is a kalpa. A day in the life of the creator is 2 kalpas (one for the day and one for the night!) or 8.64 billion years!

Thus, the current estimate of the age of earth matches a day's work for Brahmā.

During religious rituals, it is customary to recite the current year of the yuga, the approximate age of Brahmā, also the geographic position of the site, the current positions of various planets etc.

It is unfortunate that the same care was rarely taken by the authors of the books! So, Indian history has to be very vague about the relative dates of various events and often it is hard to separate facts from fiction!