MA330
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Note on the Brouncker-Wallis algorithm

We present a convenient method to compute continued fraction expansion of a number which satisifies a quadratic equation. The calculations are similar to the euclidean algorithm but is organized as a change of basis calculation for a quadratic form or equivalently a symmetric matrix.

People with experience in Linear algebra should investigate the connections.

1. We start with the root of a quadratic equation $aX^2 + 2bX + c = 0$. We shall associate a 2×2 symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. We will augment a 2×2 identity matrix to it on the left to help us keep track of the change of basis.

   We will illustrate the calculations and explain the method at the same time. So, let us consider the root $\sqrt{7}$ which satisfies the quadratic equation $x^2 - 7 = 0$.

   Thus our starting matrix shall be:

$$M_0 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -7 \end{array} \right).$$

2. The second part of this matrix is related to the quadratic equation and we change it to a quadratic form in x, y as $f(x, y) = x^2 - 7y^2$. We calculate $f(1, 1) = 1 - 7 = -6$.

   When the evaluation is negative, evaluate $f(t, 1)$ for $t = 1, 2, \cdots$ until the last value which keeps the same sign, i.e. keeps the answer negative in this case.

   Thus we have $f(1, 1) = -6, f(2, 1) = 4-7 = -3, f(3, 1) = 9-7 = 2 > 0$. So we set $t = 2$ and make the transformation $x \to x + 2y, y \to y$.

   We can do it by hand as $f(x+2y, y) = x^2+4xy-3y^2$. But we want to do it mechanically and also keep track of the changes. So here are the steps.

   We perform the row operation $R_2 + 2R_1$ i.e. add 2 times the first row to the second. At the same time to keep the last part symmetric we make similar transformation on it, which is $C_4 + 2C_3$.

   Here are the steps:

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -7 \end{array} \right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & -7 \end{array} \right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & -3 \end{array} \right) = M_1.$$

3. Read off the new quadratic form $f(x, y) = x^2+4xy-3y^2$. (Note that it is the exact same answers that we got by hand.) Now $f(1, 1) = 1 + 4 - 3 = 2 > 1$.

   In this case, the rule is to try $f(1, t)$ for $t = 1, 2, \cdots$ as long as the answer stays positive.

   Thus we have $f(1, 1) = 2, f(1, 2) = 1 + 8 - 12 = -3 < 0$. So we set $t = 1$ and make the transformation $x \to x, y \to y + x$.

   Done by hand, this makes $x^2 + 4x(y + x) - 3(y + x)^2 = 2x^2 - 2xy - 3y^2$.

   The matrix transformations are similar but in reverse, namely $R_1 + R_2$ and $C_3 + C_4$.

We get:

$$M_1 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 3 & 1 & 3 & -1 \\ 2 & 1 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 3 & 1 & 2 & -1 \\ 2 & 1 & -1 & -3 \end{array}\right) = M_2$$

4. We show the next few steps without detailed explanations.

$$\text{Using } t = 1 \text{ we get } M_3 = \left(\begin{array}{cc|cc} 3 & 1 & 2 & 1 \\ 5 & 2 & 1 & -3 \end{array}\right)$$

$$\text{Using } t = 1 \text{ again, we get } M_4 = \left(\begin{array}{cc|cc} 8 & 3 & 1 & -2 \\ 5 & 2 & -2 & -3 \end{array}\right)$$

5. The quadratic form is now $f(x, y) = x^2 - 4xy - 3y^2$. Also note that $8^2 - 7(3^2) = 64 - 63 = 1$. Similarly $5^2 - 7(2^2) = -3$. Thus the evaluations of the form $x^2 - 7y^2$ for the pairs in the first two columns are always recorded in the diagonal of the second matrix. The reader should verify this. Indeed, this is the point of keeping the first part of our display!

   We thus have our solution of $x^2 - 7y^2 = 1$.

   If we continue our process, something nice happens.

6. The next step is the transformation $x \to x + 4y, y \to y$. since for $f(x, y) = x^2 - 4xy - 3y^2$ we get $f(1, 1) = -6, f(2, 1) = -7, f(3, 1) = -6, f(4, 1) = -3, f(5, 1) = 2 > 0$. So we set $t = 4$.

   The changed matrix is:

$$M_5 = \left(\begin{array}{cc|cc} 8 & 3 & 1 & 2 \\ 37 & 14 & 2 & -3 \end{array}\right)$$

7. Note that M_5 and M_1 have the same second parts, so the quadratic form has repeated! Thus any further steps shall be periodic from now on.

   Also, if we keep track of the values of t in each step, we get the continued fraction expansion of $\sqrt{7}$. In our case, the values are: $2, 1, 1, 1, 4$ with the last 4 being repeated! verify this by what you already know.

   Indeed, we could have just saved these values and could reconstruct the first part of our matrix starting from the identity. Study this!