

Part 4: The Cubic and Quartic from Bombelli to Euler

Section 1 describes various *algebraic* methods used to tackle the cubic and quartic (the Trigonometric Method is elsewhere). Section 2 contains a detailed description, essentially due to Euler, of how to obtain all the roots of a cubic, in all cases. At one point in Section 2 we need to find the cube roots of an arbitrary complex number. A solution of this problem can be obtained by looking back on the Trigonometric Method, but by now we are a little tired of cubic equations. So in Section 3 we prove De Moivre's Formula, use it to find a trigonometric expression for the n -th roots of a complex number, and sketch the history of the formula.

1 Miscellaneous Algebraic Approaches to the Cubic and Quartic

For about 100 years after Cardano, “everybody” wanted to say something about the cubic and quartic, even the great Newton. There were many contributions, mostly of little value. After a while most workers turned away from algebra and went to the calculus, where the action was.

1.1 Bombelli on the Cubic and Quartic

Bombelli's *L'Algebra* first appeared (in Italian) in 1572. Parts were written in the 1550s, not long after Cardano's *Ars Magna*. Bombelli was an engineer, a busy man, and a perfectionist, hence the long delay. *L'Algebra*, in Latin translation, became the book from which many seventeenth century figures, even up to Leibniz, learned their algebra.

With Bombelli the slow improvement in notation continues. He redoes, more efficiently, Cardano's work on the cubic, and does the quartic(s) in detail. And there is a great deal else, with some beautiful ideas. Bombelli devised a continued fraction procedure for approximating square roots. He was also one of the first European mathematicians to look at the work of Diophantus. He fell in love with it and rewrote *L'Algebra*, adding many

Diophantine problems, thus contributing to the seventeenth century revival of number theory. But Bombelli is now mainly remembered for beginning to play with complex numbers in a serious way.

Recall that in the irreducible case Cardano's Formula asks us to find the cube root of a complex number. Cardano can't handle this case, and (almost) admits that he can't. Bombelli doesn't give up. He describes the basic rules for adding and multiplying complex numbers and verifies that, at least in some cases, the desired cube root is a complex number. Here is an example from Bombelli's work.

The equation $x^3 = 15x + 4$ has the obvious solution $x = 4$. But Cardano's Formula gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Bombelli looks for a cube root of $2 + \sqrt{-121}$ by hoping it will look like $a + \sqrt{-b}$, where a and b are real and b is positive. It is easy to verify that if this is the case, then $(a - \sqrt{-b})^3 = 2 - \sqrt{-121}$. "Multiply" together the equations

$$(a + \sqrt{-b})^3 = 2 + \sqrt{-121} \quad \text{and} \quad (a - \sqrt{-b})^3 = 2 - \sqrt{-121}.$$

We get $(a^2 + b)^3 = 125$, so $a^2 + b = 5$.

Now expand $(a + \sqrt{-b})^3$ and match its real part with the real part of $2 + \sqrt{-121}$. We get $a^3 - 3ab = 2$. Substitute $5 - a^2$ for b and simplify. This yields the equation $4a^3 - 15a - 2 = 0$, which has the obvious solution $a = 2$.

We conclude that $2 + \sqrt{-1}$ is a cube root of $2 + \sqrt{121}$.¹ There are two other cube roots, but Bombelli doesn't notice, and concludes that $\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-121}$. So he adds and obtains the obvious root 4 for the original equation.

The point here isn't finding a root of $x^3 = 15x + 4$; that can be done by inspection. What the calculation does is to show that Cardano's Formula makes *formal* sense for at least some instances of the *casus irreducibilis*. Mathematicians thus had clear reason to take complex numbers seriously.

Note that the calculation only goes through because $4a^3 - 15a - 2 = 0$ has an obvious solution. In fact $4a^3 - 15a - 2 = 0$, if we make the substitution $2a = x$, becomes the original equation! So the only point of the whole exercise is as an ad for complex numbers. And since Bombelli's procedure only works when the original equation has an obvious root, the ad is not entirely convincing.

¹Bombelli may of course have cheated, like most textbook writers, by *starting* with $2 + \sqrt{-1}$ and cubing it, then "discovering" the cube root.

Bombelli's idea took a long time to be carried out in the "general" case, largely because for many years acceptance of complex numbers was at best grudging. The understanding of complex numbers remained primitive. I can't resist jumping forward 100 years to Wallis, a very good mathematician.² Here is Wallis' hilarious 1673 attempt at an informal explanation of imaginary numbers. A landowner has a low-lying field by the sea, which is a square of area 1600 (square) perches. This field disappears under the sea, so his property has changed in area by -1600 perches. But this disappeared property is a square, so it has side $\sqrt{-1600}$!

1.2 A Suggestion of Viète

In 1591, Viète used a substitution equivalent to

$$x = y - \frac{p}{3y}$$

to solve $x^3 + px + q = 0$. Go through the substitution process and simplify. When the smoke clears, we end up with a quadratic equation in y^3 . Viète's method is essentially the same as Cardano's, stripped of motivation and geometric justification. But it points to the future. Viète uses parameters (these are positive, Cardano's types remain). And the work is done by formal manipulation of expressions: Viète has created the first system of algebraic notation that is recognizably modern. After an initial struggle, it would feel familiar to a current high school student.

1.3 Descartes' Solution of the Quartic (1637)

Descartes reduces the problem to $x^4 + px^2 + qx + r = 0$ by eliminating the x^3 term as usual. Then he tries to find t , u and v such that

$$x^4 + px^2 + qx + r = 0 = (x^2 - tx + u)(x^2 + tx + v) \quad (1)$$

identically in x . By comparing coefficients, he gets

$$u + v - t^2 = p, \quad t(u - v) = q, \quad \text{and} \quad uv = r. \quad (2)$$

Rewrite the first equation as $u + v = t^2 + p$ and the second as $u - v = q/t$. Add and subtract. We get expressions for $2u$ and $2v$. But by the last equation

²Wallis also wrote on the history of mathematics. He was mainly interested in "proving" that English Protestant mathematicians had been the pioneers in algebra, and that foreign Catholics had stolen their work.

of (2), $4uv = 4r$ and we arrive at

$$\left(t^2 + p + \frac{q}{t}\right) \left(t^2 + p - \frac{q}{t}\right) = 4r.$$

Simplify. We get a cubic equation in the variable t^2 . Take any solution of the cubic—Descartes can handle the irreducible case by a variant of the Trigonometric Method—determine a value of t , then find u and v . We are down to two quadratic equations and the rest is routine.

If we look back on Ferrari’s Method, we may think that Cardano/Ferrari also has expressed the quartic as a product of two quadratics. For once Ferrari has found his “ y ” (see the Cardano chapter) he can express the original quartic polynomial as a difference of squares, then factor in the obvious way.

This is a natural but incorrect modern misreading. The concept of polynomial occurs nowhere in Cardano: he always works with equations, and x is a particular unknown number, not a variable or “indeterminate.” Descartes is closer to the concept of polynomial, though he has the disconcerting habit of saying that he is multiplying equations. To the modern ear, that makes no sense, for an equation is an *assertion*, and one cannot multiply assertions. And Descartes writes that the equation $x^2 - 5x + 6 = 0$ is one in which x has the value 2 and “at the same time” the value 3. Descartes is a careful writer, so the remark reflects some remaining confusion. Many of the concepts we take for granted went through a long evolutionary phase. The major advances in mathematics are conceptual, not technical.³

1.4 Tschirnhaus Transformations

By the last quarter of the seventeenth century, mathematical activity was focused on the calculus, but there was sporadic work on the theory of equations. Tschirnhaus⁴ (1683) for a while thought that he could find a formula

³In 1770, Waring wrote that Sir John Wilson had observed that if p is prime, then p divides $(p - 1)! + 1$. This is true, and is now known as Wilson’s Theorem, though Wilson didn’t prove it and Leibniz had (almost) proved the result some 70 years earlier. Waring wrote that he could not prove the result because he had no “*notatio*,” meaning formula, for primes. In *Disquisitiones Arithmeticae*, Gauss gave a simple proof (Lagrange had given a complicated one) and acerbically wrote that what Waring needed was *notio* (an idea), not *notatio*.

⁴Tschirnhaus is sometimes called “the father of porcelain.” This cheerfully ignores the fact that porcelain was being made in China some 400 years earlier. Well before 1700, Chinese porcelain was being exported to Europe. But he *was* instrumental in developing the famous Dresden china.

for the roots of $P(x) = 0$, where $P(x)$ is a polynomial of degree n , by using appropriate substitutions of the shape $y = Q(x)$, where $Q(x)$ has degree $n - 1$. To find the appropriate $Q(x)$, Tschirnhaus devised a clever way of eliminating x from the system $P(x) = 0$, $Q(x) - y = 0$, thus starting a subject that would become known as *Elimination Theory*.

The procedure worked well for $n = 3$ and $n = 4$. But as Leibniz pointed out to Tschirnhaus, to find the appropriate $Q(x)$ for polynomials of degree 5 seems to require solving equations of degree bigger than 5.⁵

1.5 Euler's Solution of the Quartic

We sketch a variant due to Lagrange. To solve $x^4 + px^2 + qx + r = 0$, let $x = u + v + w$. Squaring, he obtains

$$x^2 = u^2 + v^2 + w^2 + 2(uv + uw + vw).$$

Squaring again, after some fooling with identities he gets

$$\begin{aligned} x^4 &= (u^2 + v^2 + w^2)^2 + 4(u^2 + v^2 + w^2)(uv + uw + vw) \\ &\quad + 4(u^2v^2 + u^2w^2 + v^2w^2) + 8uvw(u + v + w). \end{aligned}$$

Then he substitutes into the original equation and brings together the terms which have a factor of $u + v + w$, also the terms which have a factor of $uv + uw + vw$. The "coefficient" of $u + v + w$ is $8uvw + q$ and the coefficient of $uv + uw + vw$ is $4(u^2 + v^2 + w^2) + 2p$. He wants these to vanish, so he needs

$$8uvw + q = 0 \quad \text{and} \quad 4(u^2 + v^2 + w^2) + 2p = 0. \quad (3)$$

If these equations are satisfied, the original equation becomes

$$(u^2 + v^2 + w^2)^2 + p(u^2 + v^2 + w^2) + 4(u^2v^2 + u^2w^2 + v^2w^2) + r = 0. \quad (4)$$

From (3) we get

$$u^2 + v^2 + w^2 = -p/2. \quad (5)$$

Substituting into (4) we obtain

$$u^2v^2 + u^2w^2 + v^2w^2 = p^2/16 - r/4, \quad (6)$$

and again from (3)

$$u^2v^2w^2 = q^2/64. \quad (7)$$

⁵The substitutions Tschirnhaus used fell out of fashion, but have been revived by workers in computational algebra.

The last three equations say that u^2 , v^2 , and w^2 are the roots of

$$y^3 + (p/2)y^2 + (p^2/16 - r/4)y - q^2/64 = 0.$$

Finally, find these three roots, and calculate $u + v + w$.

It is clear that this method is horrendously more complicated than Ferrari's. What Euler and Lagrange were looking for is a *uniform* method for dealing with the cubic and the quartic, in the hope that it would generalize to higher degrees. And indeed another way of looking at the solutions of the cubic and quartic, pioneered by Vandermonde and (mainly) Lagrange was to be the key to the nineteenth century proofs that the quintic is not "solvable by radicals."⁶

2 Complex Numbers and Finding all Roots of the Cubic

In this section, we find all solutions of $x^3 + px + q = 0$, where p and q are arbitrary *complex* numbers.

2.1 The Equation $x^3 = z$

We first find all solutions of $x^3 = 1$. Note that

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

so the solutions of $x^3 = 1$ are 1, together with the roots of $x^2 + x + 1 = 0$, which are $(-1 + \sqrt{-3})/2$ and $(-1 - \sqrt{-3})/2$. The first of these is usually called ω . Note that the second is $\bar{\omega}$, the complex conjugate of ω . A short calculation shows that $\bar{\omega} = \omega^2$. The numbers 1, ω , and $\bar{\omega}$ are called the *cube roots of unity*.

Let z be a complex number other than 0, and let r be a solution of $x^3 = z$. If z is real, there is a real solution r , by the Intermediate Value Theorem. If z is not real it is not yet clear that there is an r such that $r^3 = z$ (there is).

Let $x = ry$. Then $x^3 = z$ iff $r^3y^3 = z$ iff $y^3 = 1$. Thus ωr and $\bar{\omega}r$ are the other solutions: after we have found one cube root, we can calculate them all.

⁶But the main contributor to the new understanding was Galois, a political radical.

2.2 The Cardano Formula Revisited

In the Cardano chapter, we used the identity

$$(u + v)^3 - 3uv(u + v) - (u^3 + v^3) = 0. \quad (8)$$

Note that (8) holds for all complex numbers u and v . Assume for now that every complex number has a cube root (that will be proved later). Let u be a cube root of $-q/2 + \sqrt{(q/2)^2 + (p/3)^3}$ ⁷ and v a cube root of $-q/2 - \sqrt{(q/2)^2 + (p/3)^3}$, with the additional proviso that $3uv = -p$.

Theorem 1. *The number $u + v$ is a solution of $x^3 + px + q = 0$. Moreover, all solutions can be obtained in this way.*

Proof. It is easy to see that if u and v are as described, then $3uv = -p$ and $u^3 + v^3 = -q$. The desired follows immediately from (8).

Conversely, let x be a root of our cubic. We may suppose that x has shape $u + v$ for some u and v such that $3uv = -p$. For look at the equation $y^2 - xy - p/3$. This has two solutions (say u and v). The sum of the solutions is x and their product is $-p/3$.

Because x is a root of the cubic and $x = u + v$, we have $(u + v)^3 + p(u + v) + q = 0$. Use identity (8) to rewrite this as

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

Since $3uv = -p$, it follows that $u^3 + v^3 = -q$. From $3uv = -p$ we obtain $u^3v^3 = -p^3/27$. So u^3 and v^3 are the roots of $z^2 + qz - p^3/27 = 0$. These roots are

$$-q/2 \pm \sqrt{(q/2)^2 + (p/3)^3},$$

and therefore x has the desired shape.

Here is a more concrete description of the roots. Let u be a cube root of $-q/2 + \sqrt{(q/2)^2 + (p/3)^3}$, and v_* any cube root of $-q/2 - \sqrt{(q/2)^2 + (p/3)^3}$. Then $u^3v_*^3 = -p^3/27$, and therefore uv_* is a cube root of $-p^3/27$. It follows that uv_* is "almost" equal to $-p/3$. More precisely, $uv_* = \epsilon(-p/3)$, where

⁷The meaning of $\sqrt{(q/2)^2 + (p/3)^3}$ is not entirely clear. If $(q/2)^2 + (p/3)^3$ is real and non-negative, its square root has by mathematical convention a definite meaning. But which of the two square roots of -5 should be called $\sqrt{-5}$? And if z is a non-zero complex number, which of the two numbers whose square is z should be called \sqrt{z} ? There are reasonable ways to answer the question, but it is probably best not to worry—seventeenth century mathematicians certainly didn't. They didn't worry about the analogous question for cube roots, but definitely should have.

ϵ is a cube root of unity (see Section 2.1). Let $v = \overline{\epsilon p/3}$. Then $u^3 = -q/2 + \sqrt{(q/2)^2 + (p/3)^3}$, $v^3 = -q/2 - \sqrt{(q/2)^2 + (p/3)^3}$, and $3uv = -p$.

Thus by Theorem 1 $u + v$ is a solution of our cubic. The other solutions have shape $s + t$, where s runs through the other two cube roots of u^3 , and t through the other two cube roots of v^3 , with the condition $3st = -p$. We have proved

Theorem 2. *Let p and q be complex numbers. If we have expressed one root of $x^3 + px + q = 0$ in standard form $u + v$ where $3uv = -p$, then the other roots are $\omega u + \overline{\omega}v$ and $\overline{\omega}u + \omega v$.*

We now have a Cardano Formula for all the roots of the cubic! Note how beautifully symmetric it is. The above method is in general outline due to Euler, in the second quarter of the eighteenth century, though Euler had p and q real.

Bombelli had drawn a connection between complex numbers and the cubic many years before. But the above calculations, though technically simple, require a high degree of comfort with complex numbers, and in particular with the notion that “cube root” is 3-valued.⁸

There is another less symmetric way of finding all the roots once we know one. Suppose we know a root c of $x^3 + px + q = 0$. The polynomial $x - c$ divides $x^3 + px + q$. Imagine doing the division. We get a quadratic in x , which we set equal to 0. Or else note that $c^3 + pc + q = 0$. So our equation can be rewritten as

$$(x^3 - c^3) + p(x - c) = 0, \quad \text{that is,} \quad (x - c)[x^2 + cx + c^2 + p] = 0,$$

and again we are down to solving a quadratic.

The above procedure is called *depressing the degree*. The quadratic equation may indeed be depressingly ugly. But once we have it—the calculation is mechanical—the rest is easy.⁹

3 De Moivre’s Formula

3.1 Introduction

Let n be a positive integer. Nowadays “De Moivre’s Formula” means the identity

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \quad (9)$$

⁸Notation conspires against understanding: when we write $\sqrt[3]{z}$, we are tacitly building in the assumption that it refers to only one object.

⁹Depressing the degree is part of the algorithm some computer programs use to find the roots of polynomials. It is the sort of thing best left to machines.

This formulation is due to Euler. We prove that the formula is correct, use it to take care of some unfinished business about cube roots of complex numbers, and sketch some of the history.

3.2 Proof of De Moivre's Formula

The proof is by induction. Suppose we know that the result holds when $n = k$. We show that the result holds when $n = k + 1$. Note that

$$(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)(\cos \theta + i \sin \theta)^k.$$

But by the induction hypothesis

$$(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta,$$

and therefore

$$(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)(\cos k\theta + i \sin k\theta).$$

Multiply out the right-hand side. We get

$$\cos \theta \cos k\theta - \sin \theta \sin k\theta + i(\sin \theta \cos k\theta + \cos \theta \sin k\theta). \quad (10)$$

By using the trigonometric identities $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and $\sin(x + y) = \sin x \cos y + \cos x \sin y$ we find that the expression in (10) is identically equal to $\cos(k + 1)\theta + i \sin(k + 1)\theta$.

3.3 Finding an n -th Root of a Complex Number

Let z be the non-zero complex number $a + bi$, where a and b are real. For simplicity, write $|z|$ for $\sqrt{a^2 + b^2}$. The real number $|z|$ is called the *norm* of z . We have

$$z = |z| \left(\frac{a}{|z|} + i \frac{b}{|z|} \right).$$

Note that $(a/|z|)^2 + (b/|z|)^2 = 1$. It follows that

$$\frac{a}{|z|} = \cos \phi \quad \text{and} \quad \frac{b}{|z|} = \sin \phi$$

for some number ϕ , and therefore $z = |z|(\cos \phi + i \sin \phi)$.

Denote by $|z|^{1/n}$ the ordinary n -th root of $|z|$. By De Moivre's Formula

$$\left[|z|^{1/n} \left(\cos \frac{\phi}{n} + i \sin \frac{\phi}{n} \right) \right]^n = |z|(\cos \phi + i \sin \phi) = z, \quad (11)$$

so we have found an n -th root of z .

3.4 Back to the *Casus Irreducibilis*

Let $n = 3$ in formula (11). We conclude that every complex number has a cube root. This completes the proof of the correctness of the Cardano Formula for the irreducible case.

Note that the fact that every complex number has a cube root was well known before Euler. It is enough to show that $\cos \phi + i \sin \phi$ has a cube root. Let $x = \cos(\phi/3)$. Then $4x^3 - 3x = \cos \phi$. This equation has a real root r , and it is easy to verify that $r + i\sqrt{1 - r^2}$ is a cube root of $\cos \phi + i \sin \phi$. But the De Moivre's Formula argument is much neater.

We now have a more sophisticated but more informative way of deriving the trigonometric solution for the irreducible case. The Cardano Formula yields an expression of the form

$$\sqrt[3]{a + ib} + \sqrt[3]{a - ib},$$

where a and b are real and $b \neq 0$. Express $a \pm ib$ in the form $|z|(\cos \phi \pm i \sin \phi)$. By De Moivre's Formula, $|z|^{1/3}(\cos(\phi/3) \pm i \sin(\phi/3))$ are cube roots of $a + ib$ and $a - ib$. Their product is $-p/3$, as desired. Add: we get that $2|z|^{1/3} \cos(\phi/3)$ is a root of the cubic.

By playing around with complex numbers, we can show that if the discriminant of the cubic is positive, two of the roots are non-real and one root is real, while if the discriminant is negative then all roots are real. Since the Cardano chapter already has a proof of these facts, there is little reason to prove them again (anyway, they make nice exercises).

3.5 Finding all the n -th Roots of a Complex Number

Suppose that we know one n -th root r of z . By imitating the argument of Section 2.1, we can see that *all* of the n -th roots of z can be obtained by multiplying r by an n -th root of 1. So we concentrate on the equation $x^n = 1$. If k is a positive integer, then by De Moivre's Formula

$$\left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right)^n = \cos(2k\pi) + i \sin(2k\pi) = 1.$$

As k ranges from 0 to $n - 1$, the numbers $\cos(2k\pi/n) + i \sin(2k\pi/n)$ are all different, so they give all the solutions of $x^n = 1$. These n numbers are called the n -th roots of unity.

3.6 The History of De Moivre's Formula

3.6.1 The Beginnings

The formula arises from consideration of two apparently quite different problems, one coming from algebra/trigonometry and the other from calculus. The first problem is to find a formula for $\cos(\phi/n)$ given that $\cos \phi = c$. The second problem is to find a decomposition of $a^n \pm x^n$ as a product of linear and/or quadratic polynomials with real coefficients. This decomposition is needed in order to integrate $1/(a^n \pm x^n)$ by the method of partial fractions.

By 1676, Newton had obtained formulas for $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos \theta$ and $\sin \theta$. With hindsight, one can splice together these formulas to obtain De Moivre's Formula. The thing that stood in the way, for Newton and others is that attention was focused on $\cos(n\theta)$ (usually) or $\sin(n\theta)$. It so happens that the object $\cos(n\theta) + i \sin(n\theta)$ behaves far more simply than its components, but of course it was natural to concentrate on the familiar trigonometric functions. On the calculus side, there is a significant false step in 1702. Leibniz argues incorrectly that $x^4 + a^4$ cannot be expressed as a product of two quadratics with real coefficients. The error is pointed out by one of the Bernoullis.

3.6.2 Cotes

In 1722¹⁰ a paper of Cotes appears giving complete factorizations over the reals of $a^{2m} \pm x^{2m}$ and $a^{2m+1} \pm x^{2m+1}$ for all positive integers m .

For example, Cotes asserts the equivalent of

$$a^{2m} - x^{2m} = (a - x)(a + x) \prod_{k=0}^{m-1} \left(a^2 - 2ax \cos(2k\pi/2m) + x^2 \right). \quad (12)$$

Cotes doesn't use radians—the modern version of the trigonometric functions was introduced by Euler. And unfortunately Cotes gives a clever geometric version of the above result, not the algebraic version. Cotes doesn't supply a proof.

To connect Cotes with De Moivre's Formula, multiply both sides of (12) by -1 and let $a = 1$. The roots of $x^2 - 2x \cos(2k\pi/2m) + 1 = 0$ are $\cos(2k\pi/2m) \pm i \sin(2k\pi/2m)$. We get a similar result by starting from $a^{2m+1} - x^{2m+1}$. So, though not in modern form, Cotes has, or could have had, expressions for the n -th roots of unity.

¹⁰The paper was presumably written at some earlier time: Cotes died in 1716.

3.6.3 De Moivre

What de Moivre knew and when he knew it is complicated by the fact that he looked at the problem repeatedly from 1707 to 1739.

Let $T_n(y)$ be the polynomial such that $2 \cos(n\phi) = T_n(2 \cos \phi)$ (the 2's are there because $2 \cos \phi$ behaves a bit more nicely than $\cos \phi$.) In 1707, de Moivre asserts that the equation $T_n(y) = 2a$ has the solution

$$y = \sqrt[n]{a + \sqrt{a^2 - 1}} + \sqrt[n]{a - \sqrt{a^2 - 1}}$$

if n is odd. He gives several variants of the assertion and explicitly remarks that his expression looks like Cardano's Formula. But he doesn't give a proof until 1722, when he uses a complicated argument based on the recurrence $T_{n+1}(y) = yT_n(y) - T_{n-1}(y)$. If in 1707 he had put $a = \cos(n\phi)$ (but he didn't) he would have obtained

$$\cos \phi = \sqrt[n]{\cos(n\phi) + i \sin(n\phi)} + \sqrt[n]{\cos(n\phi) - i \sin(n\phi)} \quad (13)$$

which leads to a method for finding n -th roots and is fairly close to the formula named after him. He finally writes down (13) explicitly in a book published in 1730. In 1739, he turns to the problem of finding all n -th roots, and gives a (slightly flawed) formulation from which Cotes' factorizations follow easily.

3.6.4 Euler

In 1748, Euler's two volume *Introductio in analysin infinitorum* appears. Volume 1 is arguably the best mathematics book ever written, with an incredible treasury of results derived by bold manipulation of infinite quantities.

De Moivre's Formula appears here in modern form. Since the conceptual base is the right one, the proof is very easy. Euler plays around formally with infinite series obtained from the binomial expansion of $(1 + z/n)^n$ for "infinite" n and obtains the beautiful relationship

$$\cos \phi + i \sin \phi = e^{i\phi}. \quad (14)$$

De Moivre's Formula then just says that $e^{in\phi} = (e^{i\phi})^n$! The complicated formulas for $\cos(n\phi)$ have disappeared, to be replaced by a natural property of the exponential function.

Substitute $-\phi$ for ϕ in (14). We get $\cos \phi - i \sin \phi = e^{-i\phi}$. Then by adding and subtracting we obtain

$$\cos \phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \quad \text{and} \quad \sin \phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}.$$

Thus \cos and \sin are “exponential-like.” The slightly messy classical addition laws for the cosine and sine now come easily, via (14), from the fact that $e^{i(x+y)} = e^{ix}e^{iy}$.

Why did all of this take so long? We should remember that (versions of) the cosine and sine functions were comfortably familiar, while the exponential, particularly the complex exponential, was not. We still see a residue of this in some formulas of classical applied mathematics, where cosines hang grimly on. Discomfort with complex numbers hasn’t entirely vanished.

Early workers in algebra were overly fixated on equation solving. They showed great virtuosity and technical skill, sometimes at the expense of structural insight. So for example they went after n -th roots rather than n -th powers, missing the simple version of de Moivre’s Formula due to Euler.

Students probably consider the exponential function familiar and the logarithm less so. But when Napier introduced his notion of logarithm in 1614, he had no concept of exponential function. Versions of the logarithm were repeatedly discovered later in the seventeenth century, with the discoverer sometimes not realizing there was any connection with the logarithm tables he routinely used to multiply! These logarithms were all devised to solve problems that we now think of as finding the area under a hyperbola, that is, integrating $1/x$.

Even after mathematicians were using the “natural logarithm” to integrate, they didn’t work with the exponential function. It wasn’t until Euler that the central role of the exponential function, even as a function of a real variable, was understood.