MA330
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Explanation of Landen's Paper

The paper by John Landen *(A New method of computing the sum of certain series Phil. Trans. 1759-60 (51) p. 553-565)* is being discussed in class. Here we give an outline of the proofs.

1. The notation hyp.log is equivalent to just the natural log in modern notation. Let $\mathbf{i} = \sqrt{-1}$. The paper declares values of its log. In modern notation, we may write $\log(\mathbf{i}) = \pi\mathbf{i}/2$ and $\log(-\mathbf{i}) = \log(1/\mathbf{i}) = -\log(\mathbf{i}) = -\pi\mathbf{i}/2$.

   It follows that $\log(-1) = \log(\mathbf{i}^2) = \pi\mathbf{i}$. However, $\log(-1)$, can be also viewed as $-\pi\mathbf{i}$ since $-1 = (-\mathbf{i})^2$.

   The paper names $a = \pi/2$ and hence determines $\log(1/\mathbf{i}) = -\log(\mathbf{i}) = -a\mathbf{i} = a/\mathbf{i}$.

   We shall simply use $\log(\mathbf{i}) = \pi\mathbf{i}/2$ and $\log(-1) = \pm\pi\mathbf{i}$.

2. The main formula of interest is the series expansion
$$\log(\frac{1}{1-x}) = x + x^2/2 + \cdots = \sum x^n/n.$$
   We shall denote it as $L_1(x)$. (It is not named in the paper.)

   In fact, we define: $L_r(x) = \sum x^n/(n^r)$.

3. The main aim of the paper is to determine the values of $L_2(1), L_4(1), \cdots$ for various even subscripts, by using a novel idea (in 1759).

4. We start with $L_1(1/x)$ and transform it thus:

   $L_1(1/x) = \log(\frac{1}{1-1/x}) = \log(\frac{x}{x-1}) = \log(x) - \log(x-1)$.

   Noticing that $\log(x-1) = \log(-1) + \log(1-x)$ we get:
$$\begin{aligned} L_1(1/x) &= \log(x) - \log(-1) - \log(1-x) \\ &= \log(x) - \log(-1) + \log(\tfrac{1}{1-x}) \\ &= \log(x) - \log(-1) + L_1(x) \end{aligned}$$

5. The main tool used in the paper is to calculate a certain transform for any desired function $f(x)$. We shall use the notation $\mathcal{LA}(f(x)) = \int f(x)/x\,dx$.

   Here are some properties of this transformation:

   - $\mathcal{LA}(L_1(x)) = L_2(x)$. More generally, $\mathcal{LA}(L_d(x)) = L_{d+1}(x)$.
   - $\mathcal{LA}(L_d(1/x)) = -L_d(1/x)$.
   - $\mathcal{LA}(c) = c\log(x)$ if c is a constant.
   - $\mathcal{LA}(\log(x)^m) = \frac{\log(x)^{m+1}}{m+1}$.

   We shall use these as needed.

6. We rearrange the formula in (4) above and transform it thus:

   $\mathcal{LA}(L_1(x)) - \mathcal{LA}(L_1(1/x)) = \mathcal{LA}(\log(-1) - \log(x))$.

   This gives us: $L_2(x) + L_2(1/x) = \log(-1)\log(x) - \log(x)^2/2 + C$

   where C is a constant to be determined. If we use $x = 1$, then we see that the LHS is $2L_2(1)$ while the RHS is $0 - 0 + C$. Hence $C = 2L_2(1)$.

   Thus, we get

   $L_2(x) + L_2(1/x) = 2L_2(1) + \log(-1)\log(x) - \log(x)^2/2$.

7. We deduce two equations from this, using $x = -1$ and $x = \mathbf{i}$. Then we deduce the value of $L_2(1)$.

   - For $x = -1$, we see that the LHS is
     $\sum \frac{(-1)^n + (-1)^{-n}}{n^2} = 2\sum(-1)^n/n^2$
     And this simplifies to: $2\left(-1/1^2 + 1/2^2 - 1/3^2 + \cdots\right)$.
     The RHS, on the other hand, simplifies to $2L_2(1) + \log(-1)^2 - \log(-1)^2/2 = 2L_2(1) - \pi^2/2$.
     Dividing by 2, we get:

     $$\text{Equation 1:} \quad -1/1^2 + 1/2^2 - 1/3^2 + \cdots = L_2(1) - \pi^2/4.$$

   - For $x = \mathbf{i}$, the LHS becomes $\sum \frac{\mathbf{i}^n + \mathbf{i}^n(-1)^n}{n^2}$. Clearly only even n terms survive and we get:

     $$-2/2^2 + 2/4^2 - 2/6^2 + \cdots.$$

     It is seen that all denominators have a factor of 4 and numerators have a factor of 2. So we get

     $$(1/2)\left(-1/1^2 + 1/2^2 - 1/3^2 + \cdots\right).$$

     The RHS evaluates to

     $$2L_2(1) - 3\pi^2/8.$$

     Multiplying both sides by 2, we get:

     $$\text{Equation 2:} \quad -1/1^2 + 1/2^2 - 1/3^2 + \cdots = 4L_2(1) - 3\pi^2/4.$$

8. The above two equations give:

   $$L_2(1) - \pi^2/4 = 4L_2(1) - 3\pi^2/4$$

   which gives $3L_2(1) = \pi^2/2$ or $L_2(1) = \pi^2/6$.