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Bessels contain Continued Fractions of Progressions

Introduction

The January 2000 issue of this journal carried a nice article [1] on continued fractions by Shailesh Shirali. After discussing various continued fractions for numbers related to e, he left us with the intriguing question as to how one could possibly evaluate the continued fraction

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots\cdots\cdots.$$

The question is interesting because this continued fraction is simpler-looking than the ones which were studied in that article. We answer this question here and show that the discussion naturally involves the Bessel functions, thus explaining the title. However, we shall begin with some details about continued fractions which complement his discussion. One place where continued fractions are known to appear naturally is in the study of the so-erroneously-called Pell's equation.

In a series of very well-written articles [2], Amartya Kumar Dutta discussed various aspects of Mathematics in ancient India. In particular, he discussed Brahmagupta's work on *Samasabhavana* and the *Chakravala method* for finding solutions to 'Pell's equation'. In fact, it is amusing to recall what Andre Weil, one of the great mathematicians of the last century wrote once, while discussing Fermat's writings on the problem of finding integer solutions to $x^2 - Dy^2 = 1$:

> *What would have been Fermat's astonishment if some missionary, just back from India, had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier!*

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Continued fractions, Bessel functions.

The Chakravala method can be described in terms of continued fractions. Let us begin with some rather elementary things which were known so long back and have gone out of fashion to such an extent that they are not as widely known as they ought perhaps to be.

Linear Diophantine Equations with SCF's

As in Shirali's article, let us denote by

$$[a_0\,;\, a_1, a_2, a_3, \cdots] \tag{1}$$

the SCF (simple continued fraction)

$$a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\cdots. \tag{2}$$

Here the a_i are natural numbers. Evidently, any rational number has a finite SCF. For instance,

$$\frac{763}{396} = [1\,;\, 1, 12, 1, 1, 1, 9].$$

Its successive convergents are $\frac{1}{1}, \frac{2}{1}, \frac{25}{13}, \frac{27}{14}, \frac{52}{27}, \frac{79}{41}, \frac{763}{396}$. Note that if the n-th convergent is $\frac{p_n}{q_n}$, then $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$. This holds for any continued fraction, as can be seen by induction. This gives a method of finding all positive integral solutions (in particular, the smallest one) x, y to a Diophantine equation of the form $ax - by = c$. For instance, consider the equation

$$396x - 763y = 12.$$

Look at the SCF for $\frac{763}{396}$ and compute its penultimate convergent $\frac{79}{41}$. Now, if x, y are positive integers satisfying

$$396x - 763y = 12,$$

then combining with the fact that $396 \times 79 - 763 \times 41 = 1$, we get

$$x - (79 \times 12) = 763t, \quad y - (41 \times 12) = 396t$$

An eventually periodic SCF gives a quadratic irrational number.

for some integer t. This gives all solutions, and the smallest solution in natural numbers x, y is obtained by taking $t = -1$ and turns out to be $(185, 96)$.

The reader is left with deriving similarly the corresponding expression for any linear equation.

Quadratic Equations from SCF's

Evidently, finite CF's give only rational numbers. Given the fact that a periodic decimal expansion gives rational numbers too, a reader might be tempted to guess that a periodic CF gives rationals. After just a little thought, it becomes apparent that an eventually periodic SCF gives a quadratic irrational number. For example, $[1\,;\,1, 1, \cdots]$ is the 'golden ratio' $(1+\sqrt{5})/2$. This is because the value s satisfies $s = 1 + 1/s$ and is positive. Similarly, the SCF $[1\,;\,3, 2, 3, 2, \cdots] = \sqrt{5/3}$, as it gives the quadratic equation $s - 1 = (s + 1)/(3s + 4)$, and $[0\,;\,3, 2, 1, 3, 2, 1, \cdots] = (\sqrt{37} - 4)/7$ as it gives the equation $s = (3 + 2s)/(10 + 7s)$, etc.

Consider a quadratic Diophantine equation in two variables

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad (3)$$

where a, b, c, f, g, h are integers. Thinking of this as a polynomial in x and solving it, one obtains

$$ax+hy+g = \pm\sqrt{(h^2 - ab)y^2 + 2(hg - af)y + g^2 - ac)}.$$

For any integral solution, the expression inside the square root (which we write as $ry^2 + 2sy + t$ now) must be a perfect square, say v^2. Once again, solving this as a polynomial in y, we get

$$ry + s = \pm\sqrt{(s^2 - rt + rv^2)}.$$

Hence, $s^2 - rt + rv^2$ must be a perfect square u^2. In other words, the original equation does not have integral solutions unless the equation $u^2 - rv^2 = w$ has a solution, where w is a constant defined in terms of a, b, c, f, g, h.

$u^2 - rv^2 = 1$ can always be solved in positive integers using the SCF for $\sqrt{r}$.

An equation of the form $u^2 + rv^2 = w$ for r positive has only finitely many solutions. Therefore, let us discuss the equation $u^2 - rv^2 = \pm w$ where r, w are positive integers and r is not a perfect square. The SCF for $\sqrt{r}$ provides a way of obtaining infinitely many solutions of the special equation $u^2 - rv^2 = 1$. Consequently, for given r, w if we find one solution (u_0, v_0) of $u^2 - rv^2 = w$, one can find infinitely many by the *samasabhavana* (composition) $x = uu_0 + rvv_0$, $y = uv_0 + vu_0$ for any u, v with $u^2 - rv^2 = 1$. However, the method of CF's will provide even one solution only for certain w's; namely, those which appear as one of the denominators while expressing $\sqrt{r}$ as a continued fraction.

Let us now show how $u^2 - rv^2 = 1$ can always be solved in positive integers using the SCF for $\sqrt{r}$. It is a simple exercise to show that the SCF for $\sqrt{r}$ has the form

$$[a_1\,;\, b_1, b_2, \cdots, b_n, 2a_1, b_1, b_2, \cdots, b_n, 2a_1, \cdots]. \qquad (4)$$

If p/q is a penultimate convergent of a recurring period, then it is easy to check that $p^2 - rq^2 = \pm 1$. In fact, if the period is even, this is always 1. If the period is odd, then the penultimate convergents of the first, second, third period, ... alternately satisfy the equations

$$x^2 - ry^2 = -1, \qquad x^2 - ry^2 = 1.$$

For example,

$$\sqrt{13} = [3\,;\, 1, 1, 1, 1, 6, \cdots].$$

The period is 5 which is odd. The penultimate convergent to the first period is

$$3 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1} = \frac{18}{5}.$$

Therefore, $(18, 5)$ is a solution of $u^2 - 13v^2 = -1$.

The penultimate convergent to the second period is computed to be 649/180. Therefore, (649, 180) is a solution of $u^2 - 13v^2 = 1$.

SCF's in Arithmetic Progressions

In his discussion, Shirali showed that the following SCF's can be evaluated in terms of the exponential function; he showed

$$[2\,;\,6,10,14,\cdots] = \frac{e+1}{e-1}, \qquad [1\,;\,3,5,7,\cdots] = \frac{e^2+1}{e^2-1}.$$

The SCF's here involve terms in arithmetic progression. What about a general SCF of the form $[a\,;\,a+d, a+2d,\cdots]$? For example, can the SCF $[0\,;\,1,2,3,\cdots]$ be evaluated in terms of some 'known' numbers and functions? Shirali started with the differential equation $(1-x)y'' = 2y' + y$ which he remarked "does not seem to solvable in closed form".

We start with any arithmetic progression $a, a+d, a+2d,\cdots$ where a is *any real number* and d is *any non-zero real number*, and show how it can be evaluated.

Let us consider the differential equation

$$dxy'' + ay' = y. \tag{5}$$

Actually, heuristic reasons can be given as to why one looks at this differential equation but we directly start with it here and show its relation to our problem. Let $y = y(x)$ be a solution of the above differential equation satisfying $y(0) = ay'(0)$. Let us denote the r-th derivative of y by y_r for simplicity of notation. By repeated differentiation, we get $dxy_{r+2} + (a+rd)y_{r+1} = y_r$ for all $r \geq 0$ (with y_0 denoting y). Therefore, we have

$$\frac{y_0}{y_1} = a + \frac{dxy_2}{y_1} = a + \frac{dx}{a+d+}\frac{dx}{a+2d+}\cdots\cdots. \tag{6}$$

Observe that

$$[a\,;\,a+d, a+2d, a+3d,\cdots] \;=\; \frac{y(1/d)}{y'(1/d)}.$$

A solution function such as above can be very easily obtained as a series; we get

$$y = c_0 + c_0 \sum_{n \geq 1} \frac{x^{n+1}}{(n+1)!a(a+d)\cdots(a+nd)} \qquad (7)$$

for any c_0. This evaluates the SCF $[a\,;\, a+d, a+2d, a+3d, \cdots]$ in terms of these series. As we shall see now, these series are special values of modified Bessel functions and, for certain choices of a and d, the series are even expressible in terms of e, etc.

Before proceeding further, let us note that the SCF whose evaluation was asked for by Shirali is:

$$[0\,;\, 1, 2, 3, \cdots] = \frac{\sum 1/\left((n+1)!n!\right)}{\sum 1/(n!)^2}. \qquad (8)$$

Its approximate value is 0.7.

For general a, d as above, the solution function

$$y = y(x) = c_0 + c_0 \sum_{n \geq 1} \frac{x^{n+1}}{(n+1)!a(a+d)\cdots(a+nd)} \qquad (9)$$

is related to Bessel functions in the following manner. First, the Bessel differential equation $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ has certain solutions

$$J_\alpha(x) \;=\; \sum_{n \geq 0} \frac{(-1)^n (x/2)^{2n+\alpha}}{n!\Gamma(n+1+\alpha)}; \qquad (10)$$

these are usually referred to as Bessel functions of the first kind. Here $\Gamma(s)$ is the Gamma function. If α is not an integer, then $J_{-\alpha}$ (defined in the obvious manner) is another independent solution to the Bessel differential equation above. Closely related to the J_α is the so-called modified Bessel function of the first kind

$$I_\alpha(x) = \sum_{n \geq 0} \frac{(x/2)^{2n+\alpha}}{n!\Gamma(n+1+\alpha)}. \qquad (11)$$

Thus, we have

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots\cdots\cdots = \frac{I_1(2)}{I_0(2)}.$$

The function $I_\alpha(x)$ is a solution of the differential equation $x^2y'' + xy' - (x^2 + \alpha^2)y = 0$. Indeed, $I_\alpha(x) = i^{-\alpha}J_\alpha(ix)$ for each x. Using the relation $\Gamma(s+1) = s\Gamma(s)$ and the value $\Gamma(1/2) = \sqrt{\pi}$, it is easy to see that the solution function

$$y = c_0 + c_0 \sum_{n\geq 1} \frac{x^{n+1}}{(n+1)!a(a+d)\cdots(a+nd)} \qquad (12)$$

above, is related to the modified Bessel function of the first kind as:

$$y(x^2/d) = c_0\Gamma(a/d)(x/d)^{1-a/d}I_{a/d-1}(2x/d). \qquad (13)$$

In particular,

$$[a\,;\, a+d, a+2d, a+3d, \cdots] = \frac{y(1/d)}{y'(1/d)} = \frac{I_{a/d-1}(2/d)}{I_{a/d}(2/d)}. \qquad (14)$$

Conclusion

Before finishing, we recall some SCF's evaluated out by Shirali:

$$[2\,;\, 6, 10, 14, \cdots] = \frac{e+1}{e-1}, \qquad [1\,;\, 3, 5, 7, \cdots] = \frac{e^2+1}{e^2-1}.$$

Our formula above yields for the same SCF's the expressions:

$$[2\,;\, 6, 10, 14, \cdots] = \frac{I_{-1/2}(1/2)}{I_{1/2}(1/2)}, \qquad (15)$$

$$[1\,;\, 3, 5, 7, \cdots] = \frac{I_{-1/2}(1)}{I_{1/2}(1)}. \qquad (16)$$

It is clear from the definition that

$$I_{-1/2}(1) = \sqrt{\frac{2}{\pi}} \sum_{n\geq 0} \frac{1}{(2n)!} = \sqrt{\frac{2}{\pi}}\, \frac{e+e^{-1}}{2}, \quad (17)$$

$$I_{1/2}(1) = \sqrt{\frac{2}{\pi}} \sum_{n\geq 0} \frac{1}{(2n+1)!} = \sqrt{\frac{2}{\pi}}\, \frac{e-e^{-1}}{2}. \quad (18)$$

Therefore, for these special parameters, the value of the modified Bessel function is expressible in terms of e and one can recover Shirali's expressions.

Suggested Reading

[1] Shailesh A Shirali, *Resonance*, Vol.5, No.1, p.14, 2000.

[2] Amartya Kumar Dutta, *Resonance*, Vol.7, No.4, pp.4-19, No.10, pp.6-22, 2002; Vol.8, No.11, pp.10-24, 2003.

Burnside is hot stuff

P Kesava Menon discovered the beautiful identity

$$\sum_{(a,n)=1} (a-1, n) = \phi(n) d(n)$$

where ϕ is Euler's totient function and $d(n)$ denotes the number of divisors of n. It is a nice exercise to deduce this identity from an application of Burnside's 'counting' lemma to the group of units of the ring of integers modulo n. Recall that Burnside's lemma is the simple statement that when a finite group G acts on a finite set S, the numbers $\sum_{s\in S} |G_s|$ and $\sum_{g\in G} |S^g|$ match; here $|G_s|$ is the number of elements of G which fix a particular point s in S and $|S^g|$ is the number of elements of S fixed by a particular element g.

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