MA361 (Sathaye)
(190206) **Exam 1** **Name**________________________

Outline of solutions

You must submit solutions to these problems on the blank sheets provided. Be sure to start each question on a new page and be sure to staple the pages together in correct order at the end.

All answers must carry explanations in words. You should first write what you claim to prove, and then proceed with the proof. Any formulas or theorems you use must be stated clearly, before using.

In the answers, I have included the symbol ‡ to indicate statements which need proof/discussion. These will be taken up in class.

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1. Let $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ be defined by $f([1]) = [5]$. Prove that f is an isomorphism (meaning f is a $1-1$ and onto homomorphism).

   If we define a map $g : \mathbb{Z}_6 \to \mathbb{Z}_6$ by $g([1]) = [3]$, then prove that this map is no longer an isomorphism.

   Answer: Since f is defined by $f([1]) = [5]$ we claim that $f(n) = [5n]$ for all $[n]$.‡.

   The property of homomorphism is now easy to prove using this formula. ‡.

   The properties of 1-1 and onto are deduced by showing that the six images of $\mathbb{Z}_6$ elements give six images which fill up $\mathbb{Z}_6$.

   If we consider g, then it obeys the formula $g([n]) = [3n]$. Thus $g([2]) = [3 \cdot 2] = [6] = [0]$. But $[2] \neq [0]$ and hence g is not 1-1.

2. Let $G = \mathbb{Z}_5$ and $H = \mathbb{Z}_3$. Suppose that $h : G \to H$ is a group homomorphism. Prove that h is the zero map, i.e. $h(x) = 0$ for every $x \in \mathbb{Z}_5$.

   Answer: Let $h([1]) = [t] \in \mathbb{Z}_3$. Then as above, we get a homomorphism with formula $h([n]) = [n \cdot t]$.

   Thus $h([5]) = [5t]$. Since $[5] = [0]$ in G, we should have $[5t] = [0]$. ‡.

   But this equation implies $5t \equiv 0 \mod 3$. Deduce that $t = 0$. ‡.

   This shows the claim. ‡.

3. If G_1, G_2 are groups, then we defined a product group $G_1 \times G_2 = \{(a, b) \mid a \in G_1, b \in G_2\}$.

   List all elements of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ as pairs of elements of $\mathbb{Z}_3$.

   Answer: List all the 9 elements from $([0], [0]), ([0], [1]), \cdots ([2], [2])$.

   Prove that every such element is of order 1 or 3.

   Answer: Consider any element $([i], [j])$. Note that $3([i], [j]) = ([3i], [3j]) = ([0], [0])$. This means the order of $([i], [j])$ is 1 or 3. ‡, ‡.

4. State Lagrange's Theorem on the order of a subgroup completely.

   Explain what is wrong with the following argument:

   Suppose that a group G has 9 elements. Since 9 divides $|G| = 9$, by Lagrange's Theorem, there must be an element in G of order 9.

   Answer: **Theorem: Lagrange.** Suppose that G is a finite group of order n and H is a subgroup. Then H is finite with, say, d elements. ‡.

   Moreover the order $|H|$ of H divides the order n of G.

   This theorem says nothing about existence of elements with given orders (or even subgroups of given orders.) It only gives a statement about the order of an H known to exist!

   The above example provides a clear counterexample. ‡.