MA361
Sathaye [1] **Sections 1,2,3**

Due on Jan. 18, 2019.

Section 1

1. Evaluate the following complex numbers in the form $a + bi$.

$$i^5, i^{35}, (-i)^{28}, \frac{2+3i}{3+4i}, |5+12i|.$$

Answer: Note that $i^4 = 1$, so all powers of i can be reduced.

The answers are $i, -i, 1, \dfrac{18+i}{25}, \sqrt{25+144} = 13$.

2. Rewrite the following complex numbers in the polar form $re^{i\theta}$.

$$\frac{1+i\sqrt{3}}{2}, (\frac{1+i\sqrt{3}}{2})^5, (\frac{1+i\sqrt{3}}{2})^{12}.$$

Answer: The number $\frac{1+i\sqrt{3}}{2}$ is $e^{i\pi/3}$. So the answers are:

$$e^{i5\pi/3} = (1)e^{-i\pi/3} = \frac{1-i\sqrt{3}}{2}$$

3. Find all complex solutions of

$$z^4 = 1, z^6 = -64.$$

Answer: For the first equation, note that $|z|^4 = 1$, so $|z| = 1$ or $z = e^{i\theta}$. We need to check $e^{4i\theta} = 1$, i.e. $4i\theta = 2n\pi$. So $\theta = n\pi/4$ and $n = 0, 1, 2, 3$ give four values $1, -1, i, -i$ for z.

For the second equation, set $z = 2w$, so we need to solve $w^6 = -1 = e^{i\pi}$.. If we set $w = e^{i\theta}$ then we have to solve $6\theta = \pi + 2n\pi/6$ for $n = 0, 1, 2, 3, 4, 5.z$ is found by multiplying by 2.

4. Calculate the given expressions using the given modular addition (Notation: $a +_n b$).

$$10 +_1 718, 20.5 +_{25} 6.7, (3\pi)/4 +_{2\pi} (3\pi)/2.$$

Answer: Modulo 1, all integers are zero, so the first answer is 0. The second answer is 27.2 modulo 25, 2.2.

The third answer is $3\pi(1/2 + 1/4) = 3\pi(3/4) = \pi(9/4)$. Modulo 2π this becomes $\pi/4$.

Section 2

Class practice only

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Section 3

Determine if the given given function ϕ gives an isomorphism of the first binary structure with the second. Explain your conclusion.

[1] 2 points each, except 4 for the last problem.

1. $(\mathbb{Z}, +)$ with $(\mathbb{Z}, +)$ where $\phi(n) = -n$.

   Answer: ϕ is clearly 1-1 and onto. The homomorphism condition needs

   $$\phi(n+m) - (n+m) = (-n) + (-m) = \phi(n) + \phi(m).$$

2. $(\mathbb{Z}, +)$ with $(\mathbb{Z}, +)$ where $\phi(n) = 3n$.

   Answer: The map is not onto, since $\phi(n)3n \neq 1$ for any integer n.

3. $(\mathbb{Q}\ , +)$ with $(\mathbb{Q}\ , +)$ where $\phi(x) = 3x$.

   Answer: 1-1 and onto is easy to check since $\phi^{-1}(x) = x/3$ exists. Homomorphism condition is routine.

4. $(M_2(R), \cdot)$ with $(\Re, \cdot)$ where $\phi(A) = \det(A)$.

   Answer: The map is not injective since $\phi(M) = 0$ for $M = \begin{pmatrix} 1 & t \\ 1 & t \end{pmatrix}$ for all values of t.

5. Show that $\phi(n) = 1 - n$ is a one-to-one and onto function from $\mathbb{Z}$ to $\mathbb{Z}$. Determine a new binary operation $*$ on $\mathbb{Z}$ such that ϕ gives an isomorphism

   $$(\mathbb{Z}, +) \rightarrow (\mathbb{Z}, *).$$

   Answer: We wish to define ϕ such that

   $$1 - n - m = \phi(n+m) = \phi(n) * \phi(m).$$

   If we define $a * b = a + b - 1$, then

   $\phi(n) * \phi(m) = (1-n) * (1-m) = 2 - n - m - 1 = 1 - n - m = \phi(n+m)$ as desired.