**36.12 Example** The Sylow 2-subgroups of $S_3$ have order 2. The subgroups of order 2 in $S_3$ in Example 8.7 are

$$\{\rho_0, \mu_1\}, \qquad \{\rho_0, \mu_2\}, \qquad \{\rho_0, \mu_3\}.$$

Note that there are three subgroups and that $3 \equiv 1 \pmod 2$. Also, 3 divides 6, the order of $S_3$. We can readily check that

$$i_{\rho_2}[\{\rho_0, \mu_1\}] = \{\rho_0, \mu_3\} \qquad \text{and} \qquad i_{\rho_1}[\{\rho_0, \mu_1\}] = \{\rho_0, \mu_2\}$$

where $i_{\rho_j}(x) = \rho_j x \rho_j^{-1}$, illustrating that they are all conjugate. ▲

**36.13 Example** Let us use the Sylow theorems to show that no group of order 15 is simple. Let $G$ have order 15. We claim that $G$ has a normal subgroup of order 5. By Theorem 36.8 $G$ has at least one subgroup of order 5, and by Theorem 36.11 the number of such subgroups is congruent to 1 modulo 5 and divides 15. Since 1, 6, and 11 are the only positive numbers less than 15 that are congruent to 1 modulo 5, and since among these only the number 1 divides 15, we see that $G$ has exactly one subgroup $P$ of order 5. But for each $g \in G$, the inner automorphism $i_g$ of $G$ with $i_g(x) = gxg^{-1}$ maps $P$ onto a subgroup $gPg^{-1}$, again of order 5. Hence we must have $gPg^{-1} = P$ for all $g \in G$, so $P$ is a normal subgroup of $G$. Therefore, $G$ is not simple. (Example 37.10 will show that $G$ must actually be abelian and therefore must be cyclic.) ▲

We trust that Example 36.13 gives some inkling of the power of Theorem 36.11. *Never underestimate a theorem that counts something, even modulo $p$.*

## EXERCISES 36

### Computations

Group 1

In Exercises 1 through 4, fill in the blanks.

1. A Sylow 3-subgroup of a group of order 12 has order ______.
2. A Sylow 3-subgroup of a group of order 54 has order ______.
3. A group of order 24 must have either ______ or ______ Sylow 2-subgroups. (Use only the information given in Theorem 36.11.)
4. A group of order $255 = (3)(5)(17)$ must have either ______ or ______ Sylow 3-subgroups and ______ or ______ Sylow 5-subgroups. (Use only the information given in Theorem 36.11.)
5. Find all Sylow 3-subgroups of $S_4$ and demonstrate that they are all conjugate.
6. Find two Sylow 2-subgroups of $S_4$ and show that they are conjugate.

### Concepts

In Exercises 7 through 9, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

7. Let $p$ be a prime. A *p-group* is a group with the property that every element has order $p$.
8. The *normalizer* $N[H]$ of a subgroup $H$ of a group $G$ is the set of all inner automorphisms that carry $H$ onto itself.

**9.** Let $G$ be a group whose order is divisible by a prime $p$. The *Sylow $p$-subgroup* of a group is the largest subgroup $P$ of $G$ with the property that $P$ has some power of $p$ as its order.

**10.** Mark each of the following true or false.

_____ **a.** Any two Sylow $p$-subgroups of a finite group are conjugate.
_____ **b.** Theorem 36.11 shows that a group of order 15 has only one Sylow 5-subgroup.
_____ **c.** Every Sylow $p$-subgroup of a finite group has order a power of $p$.
_____ **d.** Every $p$-subgroup of every finite group is a Sylow $p$-subgroup.
_____ **e.** Every finite abelian group has exactly one Sylow $p$-subgroup for each prime $p$ dividing the order of $G$.
_____ **f.** The normalizer in $G$ of a subgroup $H$ of $G$ is always a normal subgroup of $G$.
_____ **g.** If $H$ is a subgroup of $G$, then $H$ is always a normal subgroup of $N[H]$.
_____ **h.** A Sylow $p$-subgroup of a finite group $G$ is normal in $G$ if and only if it is the only Sylow $p$-subgroup of $G$.
_____ **i.** If $G$ is an abelian group and $H$ is a subgroup of $G$, then $N[H] = H$.
_____ **j.** A group of prime-power order $p^n$ has no Sylow $p$-subgroup.

Group 2

**Theory**

**11.** Let $H$ be a subgroup of a group $G$. Show that $G_H = \{g \in G \mid gHg^{-1} = H\}$ is a subgroup of $G$.

**12.** Let $G$ be a finite group and let primes $p$ and $q \neq p$ divide $|G|$. Prove that if $G$ has precisely one proper Sylow $p$-subgroup, it is a normal subgroup, so $G$ is not simple.

**13.** Show that every group of order 45 has a normal subgroup of order 9.

**14.** Prove Corollary 36.4.

**15.** Let $G$ be a finite group and let $p$ be a prime dividing $|G|$. Let $P$ be a Sylow $p$-subgroup of $G$. Show that $N[N[P]] = N[P]$. [*Hint:* Argue that $P$ is the only Sylow $p$-subgroup of $N[N[P]]$, and use Theorem 36.10.]

**16.** Let $G$ be a finite group and let a prime $p$ divide $|G|$. Let $P$ be a Sylow $p$-subgroup of $G$ and let $H$ be any $p$-subgroup of $G$. Show there exists $g \in G$ such that $gHg^{-1} \leq P$.

**17.** Show that every group of order $(35)^3$ has a normal subgroup of order 125.

**18.** Show that there are no simple groups of order $255 = (3)(5)(17)$.

**19.** Show that there are no simple groups of order $p^r m$, where $p$ is a prime, $r$ is a positive integer, and $m < p$.

**20.** Let $G$ be a finite group. Regard $G$ as a $G$-set where $G$ acts on itself by conjugation.

**a.** Show that $G_G$ is the center $Z(G)$ of $G$. (See Section 15.)

**b.** Use Theorem 36.1 to show that the center of a finite nontrivial $p$-group is nontrivial.

Group 3

**21.** Let $p$ be a prime. Show that a finite group of order $p^n$ contains *normal* subgroups $H_i$ for $0 \leq i \leq n$ such that $|H_i| = p^i$ and $H_i < H_{i+1}$ for $0 \leq i < n$. [*Hint:* See Exercise 20 and get an idea from Section 35.]

**22.** Let $G$ be a finite group and let $P$ be a normal $p$-subgroup of $G$. Show that $P$ is contained in every Sylow $p$-subgroup of $G$.

## SECTION 37 Applications of the Sylow Theory

In this section we give several applications of the Sylow theorems. It is intriguing to see how easily certain facts about groups of particular orders can be deduced. However, we should realize that we are working only with groups of finite order and really making be either 18 or 36. If the order is 18, the normalizer is then of index 2 and therefore is normal in $G$. If the order is 36, then $H \cap K$ is normal in $G$. ▲

**37.15 Example** Every group of order $255 = (3)(5)(17)$ is abelian (hence cyclic by the Fundamental Theorem 11.12 and not simple, since 255 is not a prime). By Theorem 36.11 such a group $G$ has only one subgroup $H$ of order 17. Then $G/H$ has order 15 and is abelian by Example 37.10. By Theorem 15.20, we see that the commutator subgroup $C$ of $G$ is contained in $H$. Thus as a subgroup of $H$, $C$ has either order 1 or 17. Theorem 36.11 also shows that $G$ has either 1 or 85 subgroups of order 3 and either 1 or 51 subgroups of order 5. However, 85 subgroups of order 3 would require 170 elements of order 3, and 51 subgroups of order 5 would require 204 elements of order 5 in $G$; both together would then require 375 elements in $G$, which is impossible. Hence there is a subgroup $K$ having either order 3 or order 5 and normal in $G$. Then $G/K$ has either order (5)(17) or order (3)(17), and in either case Theorem 37.7 shows that $G/K$ is abelian. Thus $C \leq K$ and has order either 3, 5, or 1. Since $C \leq H$ showed that $C$ has order 17 or 1, we conclude that $C$ has order 1. Hence $C = \{e\}$, and $G/C \simeq G$ is abelian. The Fundamental Theorem 11.12 then shows that $G$ is cyclic. ▲

## EXERCISES 37

### Computations

1. Let $D_4$ be the group of symmetries of the square in Example 8.10.
   **a.** Find the decomposition of $D_4$ into conjugate classes.
   **b.** Write the class equation for $D_4$.
2. By arguments similar to those used in the examples of this section, convince yourself that every nontrivial group of order not a prime and less than 60 contains a nontrivial proper normal subgroup and hence is not simple. You need not write out the details. (The hardest cases were discussed in the examples.)

### Concepts

3. Mark each of the following true or false.

_____ **a.** Every group of order 159 is cyclic.
_____ **b.** Every group of order 102 has a nontrivial proper normal subgroup.
_____ **c.** Every solvable group is of prime-power order.
_____ **d.** Every group of prime-power order is solvable.
_____ **e.** It would become quite tedious to show that no group of nonprime order between 60 and 168 is simple by the methods illustrated in the text.
_____ **f.** No group of order 21 is simple.
_____ **g.** Every group of 125 elements has at least 5 elements that commute with every element in the group.
_____ **h.** Every group of order 42 has a normal subgroup of order 7.
_____ **i.** Every group of order 42 has a normal subgroup of order 8.
_____ **j.** The only simple groups are the groups $\mathbb{Z}_p$ and $A_n$ where $p$ is a prime and $n \neq 4$.

Group 4

**Theory**

**4.** Prove that every group of order (5)(7)(47) is abelian and cyclic.

**5.** Prove that no group of order 96 is simple.

**6.** Prove that no group of order 160 is simple.

**7.** Show that every group of order 30 contains a subgroup of order 15. [*Hint:* Use the last sentence in Example 37.12, and go to the factor group.]

**8.** This exercise determines the conjugate classes of $S_n$ for every integer $n \geq 1$.

**a.** Show that if $\sigma = (a_1, a_2, \cdots, a_m)$ is a cycle in $S_n$ and $\tau$ is any element of $S_n$ then $\tau\sigma\tau^{-1} = (\tau a_1, \tau a_2, \cdots, \tau a_m)$.

**b.** Argue from (a) that any two cycles in $S_n$ of the same length are conjugate.

**c.** Argue from (a) and (b) that a product of $s$ disjoint cycles in $S_n$ of lengths $r_i$ for $i = 1, 2, \cdots, s$ is conjugate to every other product of $s$ disjoint cycles of lengths $r_i$ in $S_n$.

**d.** Show that the number of conjugate classes in $S_n$ is $p(n)$, where $p(n)$ is the number of ways, neglecting the order of the summands, that $n$ can be expressed as a sum of positive integers. The number $p(n)$ is the **number of partitions of** $n$.

**e.** Compute $p(n)$ for $n = 1, 2, 3, 4, 5, 6, 7$.

**9.** Find the conjugate classes and the class equation for $S_4$. [*Hint:* Use Exercise 8.]

**10.** Find the class equation for $S_5$ and $S_6$. [*Hint:* Use Exercise 8.]

**11.** Show that the number of conjugate classes in $S_n$ is also the number of different abelian groups (up to isomorphism) of order $p^n$, where $p$ is a prime number. [*Hint:* Use Exercise 8.]

**12.** Show that if $n > 2$, the center of $S_n$ is the subgroup consisting of the identity permutation only. [*Hint:* Use Exercise 8.]

## SECTION 38 FREE ABELIAN GROUPS

In this section we introduce the concept of free abelian groups and prove some results concerning them. The section concludes with a demonstration of the Fundamental Theorem of finitely generated abelian groups (Theorem 11.12).

### Free Abelian Groups

We should review the notions of a generating set for a group $G$ and a finitely generated group, as given in Section 7. In this section we shall deal exclusively with abelian groups and use additive notations as follows:

$$0 \text{ for the identity, } + \text{ for the operation,}$$

$$\left.\begin{aligned} na &= \underbrace{a + a + \cdots + a}_{n \text{ summands}} \\ -na &= \underbrace{(-a) + (-a) + \cdots + (-a)}_{n \text{ summands}} \end{aligned}\right\} \text{ for } n \in \mathbb{Z}^+ \text{ and } a \in G.$$

$$0a = 0 \text{ for the first } 0 \text{ in } \mathbb{Z} \text{ and the second in } G.$$

We shall continue to use the symbol $\times$ for direct product of groups rather than change to direct sum notation.