MA361
Sathaye **Permutation problems**

Some concrete problems and topics for investigation are presented below. Be sure to experiment with calculations. On Monday, there would be a quiz in class to calculate various quantities.

1. Evaluate the following permutations and present your answer as a product of disjoint cycles.

$$(1\ 2\ 3\ 4)(2\ 3\ 4),\ (3\ 4)(2\ 3\ 6\ 7)(4\ 3),\ (1\ 2\ 3)(3\ 2\ 1\ 4\ 5)(3\ 2\ 1)$$

   Answer:

$$(1\ 2\ 4\ 3),\ (2\ 4\ 6\ 7),\ (1\ 3\ 2\ 4\ 5).$$

2. A transposition of the form $(i\ j)$ is said to be adjacent if $j = i+1$. Show that every transposition can be written as a product of adjacent transpositions.

   Answer: **Hint:** $(1\ 3) = (2\ 3)(1\ 2)(2\ 3)$.

3. If σ is an $r - cycle$, and τ is any permutation, then $\sigma^\tau = \tau\sigma\tau^{-1}$ is also an r-cycle. Check this with examples.

   Generalize to the case of an arbitrary permutation σ. Again test some examples.

   Answer:

4. Let G be a group. Define a relation $g \sim h$ iff $g = uhu^{-1}$ for some $u \in G$.

   Prove that $\sim$ is an equivalence relation. The set of all elements $g \in G$ such that $g \sim h$ is called the conjugacy class of h in G.

   Determine all the differemt conjugacy classes in S_3. Similarly do this for S_4.

   Answer: S_3 has 3 classes and S_4 has 5 classes.

5. Prove that in an abelian group G, all conjugacy classes are singleton sets, so the number of classes is equal to the order of the group.

6. Let G be a group and for a given $g \in G$ we define the conjugation homomorphism ϕ_g defined by $\phi_g(h) = ghg^{-1}$.

   Show that the composition $\phi_a \circ \phi_b$ is equal to the conjugation homomorphism $\phi_a ab$.

   Verify these facts for elements in S_3 and S_4.

7. If G is abelian, then every ϕ_g is simply the identity map!

8. Kernel of any homomorphism F of a group G is defined as $Ker(F) = \{g \in G \mid F(g) = e_G\}$. Prove that $Ker(F)$ is always a subgroup of G.

   Determine the kernels of the conjugation homomorphisms in S_3 and S_4.

   Answer: