MA361
Sathaye **Group problems**

1. Suppose that G is a group with 7 elements. Explain why G must be a cyclic group. **You will need Lagrange Theorem. Study it well.**

   Given any $x \in G$ argue that $o(x) = 1$ or 7.

2. Suppose that G is a cyclic group with 12 elements.

   Given any $x \in G$, is it still true that every $x \in G$ has order 1 or 12. Either prove this or give a counterexample.

3. Suppose we have groups G and H. Then we define a group structure on $G \times H$ as follows:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

   where the two terms use the operations in G and H respectively.

   Note: G may be equal to H, as a set or even as a group. Prove that this defines a group $G \times H$.

4. Consider $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. Explicitly list all 4 elements in K.

   Argue that an element of K has order 1 or 2, but no other orders are possible.

   Is this a contradiction to Lagrange Theorem?

   Explain why K is not a cyclic group.

   Prove or disprove that K is abelian.

5. We may denote above group K as $\mathbb{Z}_2^2$. Formulate a definition of $\mathbb{Z}_2^n$ for $n = 3, 4, \cdots$. Is the statement about orders still valid in these groups?

6. Define a binary operation on $\Re$ by $x * y = \lfloor x + y \rfloor$. Is $*$ associative?

   Prove your claim.

7. **Permutations** Given a set A, a permutation is a bijective map of A to A. These are a group under composition. The group may be denoted as S_A.

   We are particularly interested in finite A. If A has n elements, then we call the group S_n.

   If $A = \{1, 23\}$ then list all 6 elements of S_3.

   I recommend that the map $1 \to a, 2 \to b, 3 \to c$ be simply denoted as a triple $\sigma = (a, b, c)$.

   Compute the following compositions which are marked as $\cdot$ and even that may be dropped later.

   - $(1, 2, 3) \cdot (3, 2, 1)$.
     What element is the identity element?
   - If $\sigma = (2, 1, 3)$ then what is $\sigma \cdot \sigma$ What is the order of σ?
   - If $\tau = (2, 3, 1)$ then what is its order?