MA362
Sathaye Exam 1 Due 11/01/19

All answers require a careful reasoning to justify them, for a positive grade.

This is a take-home exam 1. It must be submitted on or before the due date. Each question **must** start on a new page and all pages should be stapled in correct order.

All rings shall be assumed to be commutative with unity, unless otherwise stated.

1. **Q.1 Definitions 20 pts.**

   - Define an irreducible element in a ring R. Give examples of irreducible and reducible elements in $Q[X]$ with justification.
   - Define a prime ideal in a ring R. Give examples of prime and non prime ideals in $Q[X, Y]$ with justification.
   - Define a maximal ideal in a ring R. Give examples of maximal and non maximal ideals in $Q[X, Y]$ with justification.
   - Let L be an extension of a field F. Define what is meant by $[L : F]$ (also termed the dimension of L over F). Give an example of an extension of Q of dimension 3. Justify your answer.

2. **Q.2 Homomorphisms 30 pts.** Let

$$g(X) = 3X^{12} + 4X^3 + 8X + 14 \in \boldsymbol{Z}[X].$$

   Let $Q[X]/(g(X)) = R$.

   Answer the following questions.

   (a) Is $g(X)$ irreducible?

   (b) Is R an integral domain? Is it a field?

   (c) What is the dimension of R over Q (i.e. $[R : Q]$)?

   (d) Give an example, with justification, of a field extension L of Q for which $[L : Q] = 12$.

3. **Q.3 Residue class rings 30 pts.** Let $R = \boldsymbol{Z}[X]/(3)$ and let $g(X)$ be as in question 2. Let $\overline{g}[X]$ be equal to $h(g(X))$, i.e. the image of $g(X)$ under the canonical homomorphism $h : \boldsymbol{Z}[X] \to R$.

   Let $S = R/(\overline{g}[X])$.

   (a) Write out $\overline{g}[X]$ after simplification.

   (b) Factor $\overline{g}[X]$ completely.

   (c) Determine, with proof, if S is a field.

   (d) Determine the dimension $[S : \boldsymbol{Z}_3]$.

4. **Q.4 Extension fields 20 pts.** Either prove or disprove the following statements. Be sure to state needed theorems or supporting arguments.

   (a) A finite extension of any field is a finite field.

   (b) If E is any extension of the complex field C and $w \in E$ then w is algebraic over C.

   (c) $C(x)$ is algebraically closed for any x in an extension field of C provided $x \notin C$.
   Note: C is the field of complex numbers $R(i)$.

   (d) Let $F \subset G$ be finite fields. Then F, G have the same characteristic,say p. Moreover, if $p > 0$ then $\log_p(|F|)$ divides $\log_p(|G|)$.

   (e) Suppose that F is a finite field and u, v are two elements in an extension field E of F. Suppose that $[F(u) : F] = n$ and $[F(v) : F] = n$ where $n > 1$ is an integer. Then $[F(u, v) : F] = n^2$.