MA362
Sathaye **Homework 1** **Due 9/13/19**

1. **Power series ring over a field.** Let F be a field and let $R = \{a(X) = \sum_0^\infty a_i x^i$ where $a_i \in F\}$. Here X is an indeterminate and we will write $R = F[[X]]$.

   Prove the following:

   (a) **1 pt**
   What are the additive identity (0) and unity (1) in R.

   (b) **1 pt**
   Prove that for any $a(X), b(X)$ in R, either $a(X)b(X) \neq 0$ or one of $a(X), b(X)$ is 0. Deduce that R is an integral domain.

   (c) **2 pt**
   Let $v(x) = v_1 X + v_2 X^2 + \cdots \in R$. Explain why $v(X)$ is not a unit.
   Prove that $1 + v(x)$ is a unit. **Hint:** Argue that $p(X) = 1 - v(x) + v(X)^2 + \cdots$ is the desired inverse. Be sure to explain why $p(X) \in R$.

   (d) **2 pt**
   Using the above, explain why $b(X) = \sum_0^\infty b_i X^i$ is a unit iff $b_0 \neq 0$.

2. **Boolean Rings.** A ring B is said to be Boolean if $b^2 = b$ for all $b \in B$.

   Prove the following:

   (a) **2 pt**
   Prove that for each $b \in B$ we have $b + b = 0$. **Hint.** Apply the hypothesis to $h = b + b$.

   (b) **2 pt**
   By a similar argument, deduce that $ab = ba$ for all $a, b \in B$.

   (c) **1 pt**
   Using the above or otherwise prove that $(a + b)^2 = a^2 + b^2$ for all $a, b \in B$.

3. **5 pt**
   Define " a characteristic of a ring R". Determine the characteristic for each of the following rings:

$$\boldsymbol{Z}_7, \quad \boldsymbol{Z}_8, \quad \boldsymbol{Z}_4 \times \boldsymbol{Z}_6, \quad \boldsymbol{Z}.$$

4. Suppose that R is a ring such that for every **non zero** $a \in R$ there is a unique $b \in R$ such that $aba = a$.

   Prove that R is a division ring using the following steps.

   (a) **3 pt**
   Show that R has no zero divisors. **Hint:** Argue that if this is not true, and $ac = 0$ (or $ca = 0$) then b and $b + c$ both satisfy the given condition. Now use uniqueness of b given a.
   In particular, cancellation holds in R.

   (b) **2 pt**
   Show that $bab = b$. **Hint:** Consider the equation $abab = ab$ and use above.

   (c) **2 pt**
   Show that ab is the identity in R.
   Also, note that it follows that b is the multiplicative inverse of a.

   (d) **2 pt**
   Explain why we are done.

5. **5 pt**
   Let m, n be coprime positive integers. Let $A = \boldsymbol{Z}_{mn}$ and let $R = \boldsymbol{Z}_m \times \boldsymbol{Z}_n$.
   Define a homomorphism $\psi : A \to R$ by

$$\psi([x]_{mn}) = ([x]_m, [x]_n).$$

   Using the class discussions, argue that ψ is an isomorphism.