MA362
Sathaye Homework 2 Due 10/04/19

1. **Polynomials.** **8 pts.** Let $R = \boldsymbol{Z}[X]$, the polynomial ring in one variable X with integer coefficients.

   Answer the following:

   (a) Let $I = (5\boldsymbol{Z}[x])$, the ideal generated by 5 in R. Explain why $R/I = \boldsymbol{Z}_5[X]$.

   (b) Explain why R has infinitely many polynomials of degree 3. How many such polynomials are in R/I.

   (c) In the ring $\boldsymbol{Z}_5[X]$ determine $\phi_3(2X^{200} + X^{171} - 17X^35 + 73)$. (The answer must be an element of $\boldsymbol{Z}_5$)

   (d) Let F be a finite field and F^F denote the set of all functions $f : F \to F$. Let the elements of F be $\{a_1, a_2, \cdots, a_n\}$. Prove that there is a polynomial $f_j(X)$ of degree n such that $f_j(a_i) = 0$ when $i \neq j$ while $f_j(a_j) = 1$.
   Using this or otherwise, prove that $F^F = F[X]$.

2. **Factorization.** **12 pts.**

   Determine the complete factorization of the given polynomials over the given fields. Be sure to write down a clear reasoning.

   (a) $X^4 + 4 \in Q$.

   (b) $X^4 + 1 \in Q$.

   (c) $2X^3 + X^2 + 2X + 1$ in $\boldsymbol{Z}_5[X]$.

   (d) $2X^3 + X^2 + 2X + 1$ in $Q[X]$.

3. **Ring homomorphism.** **10 pts.**

   Answer the following questions.

   (a) Let $\psi : \boldsymbol{Z}_n \to \boldsymbol{Z}_m$ be a homomorphism. Let $\psi([1]_n) = [x]_m$.
   Let d be the $GCD(n, m)$. Explain why $[xn]_m = [0]_m$.
   Using this, or directly, prove that x is a multiple of m/d.

   (b) Using the above result, or directly, prove that if $GCD(n, m) = 1$ then ψ is the zero map.

   (c) Suppose $\psi : \boldsymbol{Z}_{15} \to \boldsymbol{Z}_{21}$. Determine all possible such homomorphisms, by listing all possible values of $\psi([1]_{15})$.

   (d) If $\psi : \boldsymbol{Z}_{15} \to \boldsymbol{Z}_{21}$ has $\psi([1]_{15}) = [7]_{21}$, then determine the Kernel and the image of ψ.

   (e) Let R be a commutative ring with 1 and having a prime characteristic p. Prove that the wellknown Frobenius map $F : R \to R$ defined by $F(x) = x^p$ is a ring homomorphism.