Notes on Affine Geometry Learned from Abhyankar

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1 Introduction.

The aim of this essay is to communicate what I learned from my Guru Shreeram Shankar Abhyankar and also some work inspired from it. As is legendary, Abhyankar's written work was always extremely precise, composed like a computer program, yet ornamented like classic Indian poetry. He used to say that when the arguments flow naturally and fit together without depending on the reader's imagination, then the theorems are true. He took great pains to create a symmetric structure to all his sections, never abusing or reusing symbols and trying to supply as many details as possible.

I, however, will take inspiration from one of his most popular book "Algebraic Geometry for Scientists and Engineers " , where his informal style of motivating the topic without technical complications is clearly visible. It reflects his natural style of talking and lecturing.

I will restrict to the discussion of affine varieties over a field, or, to put it algebraically, the study of ideals in finite dimensional polynomial rings over a field. As will be clear, there are already a large number of outstanding problems even for plane affine curves and surfaces in three space.

2 Basic Ideas.

Let R be any commutative ring with unity. A **polynomial ring in n variables** over R is a ring isomorphic to $R[X_1, \cdots, X_n]$ where $X_1, \cdots, X_n$ are algebraically independent over R.

We note that if we let $n = 0$ then we simply get the ring R.

Definition: Variable in a Polynomial Ring

Any sequence of polynomials $(F_1, \cdots, F_n)$ which generate the polynomial ring, i.e. satisfy $R[F_1, \cdots, F_n]$ is said to be a complete set of variables.

A subsequence $(F_1, \cdots, F_r)$ is said to be a partial set of variables or a block of r variables. When $r = 1$, F_1 is simply said to be a variable.

Sometimes the word "coordinate" is used in place of "variable" to emphasize the geometric connection.

We shall be mainly interested in polynomial rings over a field k. The characteristic of k shall be denoted by π.

Let $n \geq 1$ and consider $A = k[X_1, \cdots, X_n] \approx k^{[n]}$. A sequences of polynomials $(F_1, \cdots, F_n)$ in A is said to be a coordinate system if the ring $k[F_1, \cdots, F_n]$ generated by $F_1, \cdots, F_n$ is equal to A.

If I is any ideal in $A = k^{[n]}$, then the residue class ring $B = A/I$ is said to be an affine algebra over k. For example, if $A = k[X, Y]$ and $I = (Y^2 - X^3)$ an ideal in A, then $B = A/I = k[x, y]$ if x, y denote the images of X, Y in B. It can be seen that the affine algebra is isomorphic to $k[t^2, t^3]$, the subring of

$k[t] \approx k^{[1]}$.

Usually, by a point of the curve defined by $Y^2 - X^3 = 0$ we mean elements $a, b \in k$ such that $b^2 - a^3 = 0$. Such a point corresponds to a maximal ideal $(X - a, Y - b)$ in A/I. The ideal may be alternatively described as the set of all elements of A vanishing at the point (a, b).

We turn this around to say that **a point** of the affine algebra A/I is any maximal ideal in it. .

In general, this gives us a notion of affine varieties defined by such ideals I. Classically, the word variety was restricted to the so-called irreducible varieties. In algebraic terms, this corresponds to ideals I which are prime (or, in a more general viewpoint, their radicals are prime.). In modern terminology, one may use the word "scheme", but it comes equipped with additional structures which we don't wish to be entangled with.

Definition: Coordinate Algebra The affine algebra A/I is said to be the coordinate algebra of the ideal (or variety) I.

Geometrically, a polynomial ring $k^{[n]}$ corresponds to an affine n-space over k and prime ideals I in it define irreducible varieties in n-space with coordinate algebra $k^[[n]]/I$.

We note that equations like $aX + bY + cZ = d$ define a plane in the affine 3-space, if (a, b, c) is not equal to $(0, 0, 0)$. We also note that the coordinate algebra of $(aX + bY + cZ - d)$ is isomorphic to $k^{[2]}$. [1]

We define **an abstract hyperplane** in n-space to be any F in $k^{[n]}$ such that the coordinate algebra $k^{[n]}/(F)$ is isomorphic to $k^{[n-1]}$.

For $n = 2, 3$ we use the usual terms: line and plane, respectively.

Some **natural questions** present themselves:

NQ1 Given two ideals I, J in $A = k^{[n]}$ how to tell if their coordinate algebras are isomorphic?

NQ2 If the coordinate algebras are isomorphic (i.e. $A/I \approx A/J$), then is it possible that there is an automorphism σ of A, such that $\sigma(I) = J$?

NQ3 Given an affine algebra A/I what are possible polynomial rings A^* such that $A/I \approx A^*/I^*$?

This complicated sounding question really amounts to asking, what is the minimum number of elements $u_1, \cdots, u_r$ such that $A/I = k[u_1, \cdots, u_r]$?

NQ4 Given an affine algebra $k[x_1, \cdots, x_n] \approx k[X_1, \cdots, X_n]/I$, can we determine the minimum number of generators of I in terms of the properties of the affine algebra.

[1] For proof, note that if $a \neq 0$, then it will be the polynomial ring $k[y, z]$ where y, z are respectively images of Y, Z.

We will study several instances of these questions and refer to these at that time.

3 Polynomial Rings over a field

Let $n \geq 1$ and $A = k[X_1 \cdots, X_n] \approx k^{[n]}$. Let $\Phi : A \to B$ be a surjective k-homomorphism.

Then B is a k-algebra and is finitely generated by $x_1, \cdots, x_n$, if we set $x_i = \Phi(X_i)$ for $i = 1, \cdots, n$.

A set of elements $b_1, \cdots, b_r$ in B are said to be algebraically dependent over k if there is a non zero polynomial $F(X_1, \cdots, X_r)$ such that $F(b_1, \cdots, b_r) = 0$. The largest number of algebraically independent elements in B is defined to be **the transcendence degree** $trdeg_k(B)$.

The usual dimension theory of affine algebras shows that

- $trdeg_k(k^{[n]}) = n$.
- $trdeg_k(B) \leq n$ and it is equal to n if and only if Φ has trivial kernel, and hence is an isomorphism.
- In particular, if $B \approx k^{[m]}$, then $m \geq n$ and $m = n$ if and only if Φ is an isomorphism.

Special Cases related to the Natural Question NQ4: Now, we assume that Φ, A, B are as above and further $B = k[Y_1, \cdots, Y_m] \approx k^{[m]}$. Set $\Phi(X_i) = x_i = F_i(Y_1, \cdots, Y_m)$ for $i = 1, \cdots, n$.

First consider the case when $n = m+1$ where $m \geq 0$. Let P be the prime ideal $ker(\Phi)$. We wish to study P.

It is not too difficult to prove that P is a principal ideal, say generated by an irreducible polynomial $f \in A$. [2]

1. In case $m = 0$, we easily see that P is necessarily generated by a non constant linear polynomial and hence it is generated by a variable in $A = k^{[1]}$.
2. In general, for $m \geq 1$, the ideal P is still a principal prime ideal, say $P = (f)$ and clearly, f is an abstract hyperplane in n-space.

[2]To see this, take any irreducible $f \in P$ and note that the coordinate ring $B^* = A/(f)$ of the ideal (f) has transcendence degree $n - 1 = m$. Without loss of generality, we may assume that X_n is present in F and then the images of $X_1, \cdots, X_{n-1}$ in B^* are seen to form a transcendence basis of B^*. Since B has the same transcendence degree, these must remain algebraically independent in B. Thus, P cannot contain any non zero polynomial in $X_1, \cdots, X_{n-1}$.

On the other hand, if P has any polynomial g not divisible by f, then it is seen that the X_n-resultant of f, g is exactly such an element in the ideal P.

3. This raises:

   Natural Question NQ5 When is an abstract hyperplane in n-space a variable. Expressed differently, given two embeddings Φ, Ψ of the $(n-1)$-space B into the n-space A, when is there an automorphism σ of A, such that $\Phi = \Psi \circ \sigma$?

   The story of this question is interesting.

4. First, if k has positive characteristic π, then there are examples (attributed to B. Segre) which give a family of abstract lines which are not variables.

   One such family of examples is given by $\Phi : k[X,Y] \to k[T]$ defined by $\Phi(X) = T^{\pi q} + T, \Phi(Y) = T^{\pi^2}$ where π, q are coprime. It is easy to show that Φ is surjective and has kernel generated by $f = (X^\pi - Y^q)^\pi - Y$.

   Given a pair of integers (a,b) we shall define that **the pair is principal** if a divides b or b divides a. If neither divides the other, then it is said to be non principal.

   Moreover, to see that f is not a variable, we notice that $(\deg_Y(f), \deg_X(f)) = (pq, p^2)$ and this is non principal.

   On the other hand, if f were to be a variable, then $(\deg_Y(f), \deg_X(f))$ is principal. Indeed, this is one of the equivalent formulations of the well known automorphism theorem (Jung and van der Kulk) on the k-Automorphism group of $k[X,Y]$.

5. The celebrated **Epimorphism Theorem of Abhyankar and Moh** states that if $f \in k[X,Y]$ is an abstract line and if at least one of the two degrees $\deg_Y(f), \deg_X(f)$ is not divisible by p, the characteristic of k, then f is a variable.

   The theorem that an abstract line is a variable was independently proved by **Suzuki** for $k = \mathbb{C}$ using function theory techniques.

   The Abhyankar-Moh proof leads to extensive theory of plane curves with one place at infinity. Moreover, it directly establishes that $(\deg_Y(f), \deg_X(f))$ is principal for an abstract line f and then it is easy to established by a sequence of well defined automorphisms of the form

   $$\sigma(X,Y) = (X, Y + u(X)) \text{ or } \sigma(X,Y) = (X + u(Y), Y)$$

   we can reduce f to a linear polynomial.

   This, in turn, also gives a proof of the automorphism theorem at least in characteristic zero. Indeed, even in characteristic π, the proof works as long as one of $(\deg_Y(f), \deg_X(f))$ is not divisible by π.

One of our aims in these notes is to present the basic ideas of the Abhyankar-Moh proof.

6. The conjecture known as **Abhyankar-Sathaye Conjecture** states that for k of characteristic zero, every abstract hyperplane in n-space is a variable for all $n \geq 1$.

   We note that in case $n = 2$, the well known k-Automorphism structure of $k[X, Y]$ was heavily used. For $n \geq 3$ this structure is not well understood.

7. For any polynomial ring S in n-variables over a ring R, let $Aut_R(S)$ denote the R-automorphisms of S.

   Let A be a polynomial ring in n variables $(F_1, \cdots, F_n)$ over k. Let $(F_1, \cdots, F_r)$ be a block of r variables in A where $1 \leq r < n$. Note that A is then naturally a polynomial ring in the remaining $(n - r)$ variables $F_{r+1}, \cdots, F_n$ over the subring $R = k[F_1, \cdots, F_r]$. It is clear that elements of $Aut_R(A)$ form a subgroup of $Aut_k(A)$.

   Definition: Block automorphisms Let us define an r-block automorphism of A to be any member of $Aut_R(A)$ where R is the k-algebra generated by some r-block.

   Definition: Tame Block automorphisms A block automorphism in $Aut_R(A)$ is said to be tame if its action on its the non block variables $F_{r+1}, \cdots, F_n$ is of the form $F_i \to c_i F_i + d_4$ where $0 \neq c_i \in k$ and $d_i \in R$. We may describe the automorphism being linear in its non block variables.

   Definition: Mild (Block) Automorphisms We propose that in general $Aut_k(A)$ is generated by the linear k-automorphism $GL_k(A)$ and the set of r-block automorphisms of A as r varies over $1 \leq r < n$. [3]

   Let us call such automorphisms "mild". In general, the concept of mild block automorphisms is more extensive than the usual notion of "tame" automorphisms (corresponding to what we called block tame).

   Moreover, the automorphism theorem of Jung and van der Kulk shows that for $n = 2$ the concept of "tame" and "mild" coincides.

   Indeed the well known Nagata automorphism which was conjectured by Nagata to be non tame, is a 1-block mild automorphism. The

[3] The linear automorphisms $GL_k(A)$ are only mentioned for convenience. In fact, they can be easily seen to be generated by tame block automorphisms which only use linear terms, by the theory of the Gauss-Jordan forms.

conjecture was established by Shestakov and Umirbaev in 2002 after nearly 25 years.

The Nagata automorphism is:

$$\sigma(X) = X, \sigma(Y) = X(XY - Z^2) + Z, \sigma(Z) = X(XY - Z^2)^2 + 2Z(XY - Z^2) + Y$$

8. The next case of the Abhyankar-Sathaye conjecture is $n = 3$ and asks if every abstract plane is a variable. This remains unsolved, except in situations where the needed automorphism can be identified as 1-block automorphism and then the force of the Abhyankar-Moh theorem can be used effectively.

   To explain this, let us set up the notation. Let $\Phi : k^{[3]} \approx k[X, Y, Z] \to k[U, V] \approx k^{[2]}$ be surjective.

   - Let (F) be the kernel of Φ and assume that $(F) \bigcap k[X, Y] = (0)$. Then, by a suitable identification, we may assume that $\Phi(k[X, Y]) \subset \Phi(k[X, Y, Z]) = k[U, V]$.
   - Further, assume the condition that $k[X, Y] \bigcap k[U] \neq k$. Then the we have Russell-Sathaye Theorem asserting that F is a variable.
   - The proof proceeds by showing that $k[X, Y] \bigcap k[U]$ must be of the form $k[p(U)]$ for some polynomial $p(U)$ in U. Moreover, we may assume after an automorphism that $X = p(U)$.
   - In case $X = U$, we can set $R = k[X] = k[U]$ and note that our homomorphism Φ may be thought of as an R-epimorphism of $R[Y, Z]$ onto $R[V]$.

     Suitable modifications of the Abhyankar-Moh theorem then yield the result.
   - In case X is a higher degree polynomial in U, we proceed by a Chinese remainder type argument.

4 Some Basic Curve Theory

4.1 Notations

Let k denote a ground field. We will generally assume that k is algebraically closed and characteristic 0, unless otherwise declared.

An affine curve is a finitely generated k-algebra A of transcendence degree 1 over k. Unless otherwise declared, we will generally assume that A is an integral domain (geometrically this means that the curve is irreducible).

A maximal ideal of A corresponds to a point of the curve.

An embedding of a curve in an affine n-space is determined by a choice of generators $(x) = (x_1, x_2, \cdots, x_n)$, so that $A = k[x_1, \cdots, x_n]$. Two embeddings $(x), (y)$ are said to be equivalent if there is an automorphism σ of $k[X_1, \cdots, X_n] \approx k^{[n]}$ such that $y_i = \sigma_i(x_1, \cdots, x_n)$.

Let $A = k[x_1, \cdots, x_n]$ be an affine irreducible curve and let $K = k(x_1, \cdots, x_n)$ be its quotient field. K is called the function field of the curve A.

A discrete valuation ring (DVR) V of K/k is a DVR such that $k \subset V \subset K = qt(V)$. Let us denote by $v(\,)$ the valuation defined by V. Let $M(V)$ the maximal ideal of V. Let f_V be the canonical map from V onto $V/M(V)$ which we denote by k_V and note that k can be canonically identified with its image contained in k_V.

We say that V is at finite distance for A if $A \subset V$. It can be seen that the ideal $M(V) \bigcap A$ is a maximal ideal of A, say $\mathfrak{m}$. We say that $\mathfrak{m}$ is the center of V on A and the local ring $A_{\mathfrak{m}}$ is said to be the local ring of the point $\mathfrak{m}$.

The valuation V is said to be at infinity for A if $A \not\subset V$. If this is the case, then there is some x_i such that

$$0 > \mathrm{ord}_V(x_i) = \min\{\mathrm{ord}_V(x_j) \mid j = 1, \cdots, n.\}$$

If we set A_i to be the k-algebra generated by x_j/x_i for all $j \in \{1, \cdots, n\} \setminus \{i\}$. It is easy to see that then V is at finite distance for A_i. Thus, every valuation V has a center on at least one of the A_i. The set of A_i as i varies from $1, \cdots, n$ and $x_i \neq 0$ is said to form a projective model of the curve A. It can be shown that the local ring of the center of V on A_i is the same for all i for which V is at finite distance.

There are two concrete methods to identify a valuation. For convenience, we consider a plane curve $A = k[x, y]$. Suppose that we wish to identify all valuations of $K = qt(A)$ centered at origin (i.e. (x, y).) Let $f(X, Y)$ be such that $A = k[X, Y]/(f(X, Y))$.

- **Valuations at finite distance:** Write $f = f_d + f_{d+1} + \cdots$ where f_i are homogeneous expressions of degree i in X, Y and $f_d \neq 0$. We call d the multiplicity of f at (x, y). We say f is regular in Y if $f_d(0, y) \neq 0$.

Then by the Weierstrass' Preparation Theorem, we can write $f = \epsilon F$ where $\epsilon \in k[[X,Y]]$ is a unit and $F = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X)$ where $a_i(X) \in k[[X]]$. We factor F as a polynomial in $k[[X]][Y]$ as $F = F_1 \cdot F_2 \cdot F_s$ where each F_i is irreducible in $k[[X]][Y]$.

We assume that f has no multiple factors and then it can be deduced that F has no multiple factors. Each F_i defines a valuation v_i on A as follows.

For any $0 \neq h(x,y) \in A$ define

$$v_i(h(x,y)) = \ \mathrm{ord}_{\mathrm{X}}\mathrm{Resultant}(\mathrm{F_i(X,Y), h(X,Y); Y})$$

The valuation is naturally extended to the quotient field $qt(A)$ by $v_i(h_1/h_2) = v_i(h_1) - v_i(h_2)$.

- **Valuations at infinity:** We can consider $k[X]$ as naturally a subring of $k((X^{-1}))$ (the field of power series in X^{-1} over k.)

  Without loss of generality, we may assume f to be monic in Y. [4]

  Then we can factor $f = f_1 \cdot f_2 \cdot f_s$ where each $f_i \in k((X^{-1}))[Y]$ is irreducible.

  As above, for any $0 \neq h(x,y) \in A$ define

$$v_i(h(x,y)) = \ \mathrm{ord}_{\mathrm{X^{-1}}}\mathrm{Resultant}(\mathrm{f_i(X,Y), h(X,Y); Y}).$$

  The valuation is naturally extended to the quotient field $qt(A)$ by $v_i(h_1/h_2) = v_i(h_1) - v_i(h_2)$.

5 The Expansions Paper.

5.1 Places of a curve.

Consider an irreducible polynomial $F(X,Y) \in k[X,Y]$. Let A be its coordinate ring and $K = \mathrm{qt}(A)$ its function field. Then the places of A correspond to valuations of K/k. The places at infinity are those valuations whose valuation rings do not contain A. [5]

We say that A has one place at infinity if there is exactly one place at infinity and moreover the residue field of the valuation ring coincides with k, in other words, the valuation is "residually rational".

[4] This is the condition equivalent to the "regular in Y" described at finite distance.

[5] Technically, in this case, place is the canonical map of the valuation ring to the residue field, and valuation is the map to the ring of integers augmented by infinity; but we do not need this distinction for this short exposition.

A more direct definition can be given which is valid even for a reducible F. Arrange F to be pre-monic in Y, [6] then **the number of places at infinity** of F can be determined to be the number of irreducible factors of F in the ring $k((X^{-1}))[Y]$. In characteristic zero, each such factor gives a Newton-Puiseux expansion defined by $X = \tau^{-n}, Y = \eta(\tau)$ where $\eta(\tau) \in k^*((\tau))$ where k^* is a finite algebraic extension of k generated by the coefficients of $\eta(\tau)$ over k. Moreover, it is assumed that the GCD of n and the support of $\eta(\tau)$ is 1. Such an expansion defines a place at infinity of degree $[k^* : k]$.

If F is irreducible in $k((X^{-1}))[Y]$, then we say that F has one place at infinity of degree $[k^* : k]$. If the degree is 1, then we say that F has one place (or branch) at infinity (sometimes shortened to just "F is a one place curve"). Note that then F is already an irreducible polynomial in $k[Y, X]$.

In the case of a one place curve, it can be shown that F is already pre-monic in any choice of variables and so the definition is not dependent on our choice of coordinates.

One way to make this definition intrinsic is as follows.

We first define the number of places at any point P of the curve F. If P is defined by a maximal ideal $(X - a, Y - b)$, then F is in the maximal ideal $(U, V) \subset k[Y, X]$ where $U = X - a, V = Y - b$. Then the number of branches of F at P are defined as the number of irreducible factors of F when viewed as a member of the complete ring $k[[U, V]]$. If k is not algebraically closed, then the definition is refined by going to the completion of the corresponding maximal ideal. Now, to define the number of branches at infinity, we take a projective completion of the curve and count the number of branches of the curve at points at infinity.

5.2 Characteristic sequences.

The basic tool is a certain clever reorganization of the classic characteristic sequences invented by Abhyankar.[7] We reproduce these definitions, since they are still not that well known. Following [A] 6.4 we first define various characteristic sequences.

Let $\nu \neq 0$ be a given integer and J a subset of of integers bounded below.

This set J will be replaced by the support of the Newton-Puiseux expansion for Y and ν will be the order of X. Here, the definitions are given in

[6] Which means F times a non zero element of k becomes monic in Y.

[7] Abhyankar used to relate that he invented several ideas about plane curves in his personal notes of Zariski's lectures on curves but always presumed that he had simply learned them in Zariski's course. he discovered that they were his own inventions only when he found out that Zariski was not aware of them. His idea of introducing the q-sequence made the complicated formulas involved in manipulating Newton-Puiseux expansions into much simpler statements of invariance. He wrote two separate papers entitled inversion and invariance of characteristic pairs exploring the power of these techniques.

the abstract setup.

We inductively define an integer $h(\nu, J) = h$ and two sequences $m(\nu, J) = (m_1(\nu, J), \cdots, m_h(\nu, J))$ and $d(\nu, J) = (d_1(\nu, J), \cdots, d_{h+1}(\nu, J))$ as follows.

Convention: To keep our display simple, we shall drop the augment (ν, J) when it is fixed.

(1) If J is empty, then $h = 0$, $d_1 = |\nu|$ and $m(\nu, J)$ is empty. Actually, the original also sets $m_0 = -\infty$ but we drop it for simplifying the setup.

(2) If J is nonempty, set $D = \gcd(J \bigcup \{\nu\})$, the greatest common divisor of the set J together with ν , $d_1 = |\nu|$, $m_1 = \min J$ and $d_2 = \gcd\{m_1, d_1\}$.

If $d_2 = D$, then put $h = 1$ and stop.

(3) If $d_1, \cdots, d_{r+1}$ and $m_1, \cdots, m_r$ are defined and $D = d_{r+1}$, then put $h = r$ and stop; otherwise, define $m_{r+1} = \min\{p \in J \mid p \not\equiv 0 \mod d_{r+1}\}$, $d_{r+2} = \gcd\{m_1, \cdots, m_{r+1}\} = \gcd\{d_{r+1}, m_{r+1}\}$

Note that $D = d_{h+1}$ and d_{i+1} divides d_i for $i = 1, \cdots, h$. Some natural expressions in these numbers are also useful to define. Naturally, these notations should also be augmented by (ν, J), for precision.

(1) **The q-sequence.** $q_1 = m_1$ and $q_i = m_i - m_{i-1}$ for $i = 2, \cdots, h$. For convenience, we also set $q_{h+1} = m_{h+1} = \infty$.

(2) **The s-sequence.** $s_i = \sum_1^i q_j d_j$ for $i = 1, \cdots, h$.

(3) **The r-sequence.**$r_i = s_i/d_i$ and $\delta_i = -r_i$ for $i = 1, \cdots, h$.

(4) **The n-sequence.** $n_i = d_i/d_{i+1}$ for $i = 1, \cdots, h$.

5.3 Connection with a place of a curve.

Assume that $F(X, Y)$ defines a curve with one rational place at infinity and assume the characteristic of k is $\pi = 0$.[8] We consider the Newton-Puiseux expansion given by a power series parametrization: $X = \tau^{-n}, Y = \eta(\tau) \in k((\tau))$, where $n = \deg_Y(F(X, Y))$, and it is assumed that the support of $\eta(\tau)$ and n and have GCD 1. In this case, we get the well known induced factorization: $F(\tau^{-n}, Y) = \prod_{j=1}^n (Y - \eta(\omega^j \tau))$ where ω is a primitive n-th root of unity.[9]

There are three essential ingredients of the Abhyankar-Moh theory, which are responsible for most of its successes.

[8] It is enough to assume that π does not divide n.

[9] This can also be made to work if $\pi > 0$ provided π does not divide n.

1. **The Irreducibility Criterion of Abhyankar and Moh** *Assume that* $n = \deg_Y f(X,Y) \not\equiv 0 \mod \pi$ *and* $f(X,Y)$ *is monic in* Y*. Then* $f(X,Y)$ *has one place at infinity iff there is a "test series"* $u(\tau) \in k((\tau))$ *such that* $\operatorname{ord}_\tau \mathrm{f}(\tau^{-\mathrm{n}}), \ulcorner(\tau) > \mathrm{s_h}(-\mathrm{n}, \mathrm{u}(\tau))$

   Moreover, given any series $u(\tau)$ passing this test, there is a "root" $y(\tau)$ satisfying: $f(\tau^{-n}, y(\tau)) = 0$ and $\operatorname{ord}_\tau(\mathrm{y}(\tau) - \mathrm{u}(\tau)) > \mathrm{m_h}(-\mathrm{n}, \mathrm{u}(\tau))$

   This Lemma, originally in [AM2], was later reproved by Abhyankar in greater detail in [A2].

2. **The innovation of the approximate roots.** For convenience of notation, assume that $f(X,Y)$ is monic in Y of degree n. If we construct the characteristic sequences using the support of $\eta(\tau)$ ($\mathrm{Supp}(\eta(\tau))$ and $\nu = -n$, then for each $d_i = d_i(-n, \eta(\tau))$ for $i = 1, 2, \cdots, h = h(-n, \eta(\tau))$, we get approximate roots $g_i(X,Y)$ defined by

   - For $i = 1$, $g_1 = Y$ and for $i > 1$, $g_i(X,Y)$ is monic in Y of degree n/d_i and
   - $\deg_Y(f - g_i^{d_i}) < n - n/d_i$.

   It is shown that moreover such polynomials are uniquely defined by f for any factors of n, but for the d_i chosen from the characteristic sequence, it is also shown that each $g_i(X,Y)$ is a curve with one place at infinity.

3. **The One Place Theorem on Translates of one place curves.** *If* $\pi = 0$ *and* F *has one place at infinity, then* $F + \lambda$ *also has one place at infinity for any* $\lambda \in k$*.*

   Moreover, all translates have the same Newton-Puiseux expansion through the last characteristic term. In geometric terms, this means that F *and* $F + \lambda$ *go through each other at infinity through all the singular points in a sequence of quadratic transforms.*

   This is deduced from the irreducibilty lemma and the explicit calculation of the initial forms in terms terms of the approximate roots.

5.4 The value semigroup of a one place curve.

Equipped with the above results and elegant numerical manipulations of the various associated numbers defined from the support of the Newton-Puiseux expansion, the Expansions Paper established the basic structure of the coordinate ring $A = K[X,Y]/(F(X,Y))$ for a curve defined by $F(X,Y)$ and having one place at infinity. Let $\alpha : k[X,Y] \to k[x,y]$ be the canonical homomorphism with $\alpha(X) = x, \alpha(Y) = y$.

As described above, let $\eta(\tau)$ be a Newton-Puiseux expansion. We have described a valuation V on the quotient field of A by $V(h(x,y)) = \text{ ord}_\tau(\mathrm{h}(\tau^{-\mathrm{n}}, \eta(\tau)))$ with the natural convention that $\text{ord}_\tau(0) = \infty$.

We assume, without loss of generality, that $F(X,Y)$ is monic of degree n in Y and construct, as explained above, the approximate roots $G_1(X,Y) = Y, G_2(X,Y), \cdots, G_h(X,Y)$, which are curves with one place at infinity themselves. Then we set $\Gamma_F = \{V(h) \mid 0 \neq h \in A, \}$.

We deduce the sequences $m_i, q_i, d_i, r_r, s_i, n_i$ as described above, which are determined using $\nu = -n$ and $J = \text{Supp}\eta(\tau)$. We get the following detailed structure theorem for A.

1. Set $g_i = \alpha(G_i(X,Y))$ for $i = 1, \cdots, h$. Then $V(g_i) = r_i$. Define $g_0 = x$. Also define $r_0 = -n$ and $\delta_0 = n$. Call a monomial $g^a = \prod_{i=0}^h g_i^{a_i}$ a standard monomial in $g_0, \cdots, g_h$, if it satisfies:

   - $a_0 \geq 0$
   - and for $1 \leq i \leq h$, we have $0 \leq a_i < n_i$.

   Let $S(g)$ be the set of all standard monomials in $g_0, \cdots, g_h$.

2. Then $S(g)$ is a basis of A as a k-vector space. The set of all monomials g^a for which $a = (0, a_1, \cdots, a_h)$ can be seen to have cardinality n and it gives a free basis for A as a module over $k[g_0] = k[x]$.

3. The semigroup $\Gamma_F = \{\sum_0^h a_i r_i \mid \text{ where } g^a \in S(g)\}$. It has the important property that if $\sum_0^h a_i r_i$ is divisible, by d_j, then $a_j = a_{j+1} = \cdots a_h = 0$.

We remark that that expansion techniques can also be applied to irreducible elements $F(U,V)$ of the power series ring $k[[U,V]]$, in particular when it is the completion of of the local ring at the point at infinity in $\mathbb{P}^2$ of a one-place curve. If one does not insist that the expansion be of Newton-Puiseux type (with one of the variables a power of the parameter τ), the basic definitions can be made without reference to the characteristic π, see [Ru3].

5.5 Further Developments from the Expansions Paper.

One of the most important case of one place curves is the **polynomial curve** which may be defined as a curve defined by a polynomial parametrization so that the coordinate ring A is a subring of $k[t]$ where t is an indeterminate over k. Using a modified Lüroth theorem, we can assume that A has quotient field $k(t)$. In this case, the valuation V can be described as $V(h(x,y)) =$

$-\deg_t(h(x,y))$ and it is more convenient to work with non negative integers $\delta_i = -r_i$ for $i = 0, 1, \cdots, h$.

Indeed, this δ-sequence can be thus introduced for any one place curve as well and Sathaye termed the resulting semi-groups as **planar semigroups**.

A sequence of positive integers $(\delta_0, \cdots, \delta_h)$ is said to be **a characteristic** $\delta -$ *sequence* if it satisfies the following three axioms:

(1) Set $d_i = \gcd\{\delta_0, \cdots, \delta_{i-1}\}$ for $1 \le i \le h+1$. Set $n_i = \dfrac{d_i}{d_{i+1}}$ for $1 \le i \le h$. Then $d_{h+1} = 1$ and $n_i > 1$ for all $i \ge 2$.

(2) $\delta_i n_i \in \{\delta_0, \cdots, \delta_{i-1}\}\mathbb{N}$ = the semigroup generated by $\{\delta_0, \cdots, \delta_{i-1}\}$.

(3) $\delta_i < \delta_{i-1} n_{i-1}$ for $i \ge 2$. Set $\delta_i = \delta_{i-1} n_{i-1} - q_i$, so that $q_i > 0$ for $i \ge 2$.

This definition codifies the properties of the r-sequence in the Abhyankar-Moh theory and Sathaye proved the converse that given such a semigroup, it is given by a one place curve. [SS]. (Also see [A2]).

5.5.1 Proof of the Main Theorem via semigroup structure.

The explicit description of the δ-sequence can be used to prove a

Generation Lemma. *Any integer a can be uniquely written as*

$$a = a_0\delta_0 + a_1\delta_1 + \cdots a_h\delta_h$$

where $\{a_i\}_{i=0}^h$ are integers and $0 \le a_i < d_i/d_{i+1}$ for all $i = 1, 2, \cdots, h$. Moreover

1. *If a is divisible by d_i, then $a_{i+1} = \cdots = a_h = 0$.*
2. *An integer a belongs to the semigroup generated by $\delta_0, \delta_1, \cdots, \delta_h$ if and only if $a_0 \ge 0$.*
3. *If d_2 belongs to the semigroup generated by $\delta_0, \delta_1, \cdots, \delta_h$, then (δ_0, δ_1) is principal, i.e. one of them divides the other.*

It is now easy to see that the hypothesis of the main theorem implies that if we take $F(Y, X)$ to be the kernel of the canonical epimorphism $\gamma : k[Y, X] \to k[u, v] = k[Z]$, then its degree semigroup is $\{0, 1, 2, \cdots\}$ and $(\delta_0, \delta_1) = (\deg_Z(u), \deg_Z(v))$ and hence this pair of degrees is principal.

5.5.2 Degree semigroups of various curves with one place.

We can similarly define the degree semigroup for any affine curve with one place at infinity, namely the semigroup generated by the degrees of non zero elements of its coordinate rings, where the degrees are defined to be negatives of their corresponding values in the valuation at infinity. This gives rise to three sets of degree-semigroups:

- $S_{ci} = \{$semigroups of complete intersection space curves with one place at infinity$\}$,
- $S_{pl} = \{$semigroups of plane curves with one place at infinity$\}$ and
- $S_{pol} = \{$ semigroups of plane polynomial curves $\}$.

Clearly we have $S_{ci} \supseteq S_{pl} \supseteq S_{pol}$. Examples showing that these are strict containments were discussed in [Sa2], [SS].

One of the most important questions, originally raised by Abhyankar himself, was to characterize the degree semigroups of plane polynomial curves. This remains unsolved to date. The examples of δ-sequences which give a planar semi group but not the semi group of a polynomial curve are relatively easy to construct. The simplest was the sequence $(6, 8, 3)$, which was discovered by Moh and Sathaye as an exercise in using computers. But, the semigroup itself is also generated by $(3, 8)$ and hence is in S_{pol}. Sathaye had conjectured that the semi group $< 6, 22, 17 > \notin S_{pol}$, but its proof came after many years by M. Fujimoto, M. Suzuki and K. Yokoyama [Sfy]. They also came up with a smaller example $< 6, 21, 4 >$.

But all these are only initial calculations. There is no theory or conjecture for the restrictions imposed by a polynomial curve on its degree semigroup. Recently L. Makar-Limanov has started the investigation of determining the smallest possible element in a polynomial semigroup and has analyzed the case when 2 is in the semigroup. His conjecture states the only possible d-sequences are either $(2, 2m + 1)$ or $(6, 9, 2)$, where it is known that the first two numbers of a d-sequence can be always interchanged [ML].

5.5.3 Finiteness of Embeddings of One Place Curves.

Another way of formulating the epimorphism theorem is to say that any two embeddings of the affine line in the affine plane are equivalent by an automorphism and so we may say that there is only one equivalence class of embeddings of an affine line in the affine plane. Right after establishing the structure of the "value-semigroup" (generated by the r-sequence) of a curve with one rational place at infinity, Abhyankar raised the corresponding question for such curves:

Suppose that α, β are two epimorphisms from $k[Y, X]$ onto the coordinate ring A of a plane curve with one rational place at infinity. Does it follow that α and β are equivalent?

The answer to this question is, of course no. However, Abhyankar and Singh established that there are only finitely many equivalence classes of such epimorphisms. [A-SI]. In fact, they proved the following strong result.

Note that each embedding corresponds to the choice of two ring generators x, y such that $k[x, y]$. It is also clear that the induced r sequence has r_0, r_1 given by values of x, y at the valuation at infinity and by modifying by an elementary automorphism, we may arrange (r_0, r_1) to be non principal. This gives a number $d_2 = \gcd(r_0, r_1)$ which is clearly a number such that $-d_2$ is not in the value semigroup (by the non principal condition).[10] They proved that two embedding are equivalent if and only if the corresponding d_2 is equal! This, combined with the fact that there are only finitely many negative numbers not in the value-semigroup, we get the finiteness of embeddings with a very explicit bound on the number. We remark, that the number of possible d_2s is smaller than the number of missing values and the corresponding set has not been studied.

5.6 The Jacobian Problem

Inspired by their own new machinery, Abhyankar and Moh produced several papers attacking the famous **Jacobian Problem** which asks if given n polynomials $f_1, f_2, \cdots, f_n$ in the polynomial ring $k[X_1, \cdots, X_n]$ in n-variables over a field k of characteristic zero, such that their jacobian $J_{(X_1,\cdots,X_n)}(f_1, \cdots, f_n)$ is a non zero constant in k, is it true that $k[f_1, \cdots, f_n] = k[X_1, \cdots, X_n]$?

Indeed, this particular problem was rejuvenated and popularized by Abhyankar along with several other problems in Affine Geometry of two and three dimensions as a way to attract new students to important but accessible problems in Algebraic Geometry.

Abhyankar and Moh concentrated on the two dimensional problem where f_1, f_2 can be considered to be defining a polynomial curve over $k(X_1)$ with X_2 serving as the parameter t. Without loss of generality, we may assume that f_1 is monic of degree $n > 0$ in X_2 and f_2 is monic of degree $m >$ in X_2; since if one of the degrees is zero, then the Jacobian Problem is easily seen to have an affirmative answer.

It was quickly established that the Jacobian condition on the jacobian is equivalent to the following two conditions:

1. The Newton-Puiseux expansion can then be assumed to be defined by $f_1 = \tau^{-n}, f_2 = \eta(\tau) \in k(X_1)((\tau))$ where the characteristic m-sequence

[10] In case of the line, this would cause $r_0 = -\infty$ and we already know the result.

has $m_1 = -m, m_2, \cdots, m_h$ where $m_h \leq n - 1$.

2. Moreover, all the coefficients of terms before $n-1$ are actually in k while the coefficient of τ^{n-1} is a non constant linear expression in X_1.

Two striking results were deduced from this and the Abhyankar-Moh theory:

1. **A case of the Jacobian Problem.** If $n-2$ is **not a characteristic exponent** then the Jacobian problem has an affirmative answer (i.e. $k[f_1, f_2] = k[X_1, X_2]$).

2. **Two Point Theorem.** Using the above result, it can be shown that the Jacobian condition implies that f_1, f_2 have at most two points at infinity, i.e. their top degree forms in X_1, X_2 have at most two non associate factors.

   Moreover, **if it can be deduced that** the Jacobian condition implies f_1, f_2 have at most one point at infinity, then the Jacobian problem has an affirmative answer.

Several particular cases of the theorem have been resolved by investigations using these techniques. We only list a few of these technical results: The Jacobian Problem has an affirmative answer in case (i) $h \leq 2$, (ii) $h = 3$ with small d_h, (iii) degree at most 100, (iv) if we can show that the Newton diagram of f_1 is always a triangle with corners at the origin, and on X_1 and X_2 axes, in case f_1 has degree at least 2.

It is also established that in case the diagram is not a triangle, then it is contained in a box joining $(0,0)$ with some (a,b) with $a \neq b$. It is enough to show that this cannot occur.

Since, the analysis of the Jacobian problem can be reduced to the study of the polynomial curve (f_1, f_2) with X_2 as the parameter, Abhyankar proposed the problem of analyzing the polynomial curves, especially ones with coefficients in $k[X_1]$.

5.7 An unexpected Generalization.

The main epimorphism theorem was generalized by Sathaye in the following form [Sa3]:

Generalized Main Theorem. *Let k have characteristic zero and let $U(W_1, \cdots, W_p), V(W_1, \cdots, W_p) \in k[W_1][[W_2, \cdots W_p]]$ such that $n = \deg_{W_1}(U(W_1, 0, \cdots, 0)) > 0$ and $m = \deg_{W_1}(V(W_1, 0, \cdots, 0)) > 0$. Assume that there is $\Psi \in k[U,V][[W_2, \cdots, W_p]]$ such that the lowest $W_2, \cdots, W_p$-adic term in $\Psi(U, V, W_2, \cdots, W_p)$ is of the form $L(W_1) W_2^{q_2} \cdots W_p^{q_3}$ where $L(W_1)$ is a polynomial of degree 1 in W_1. Then m divides n or n divides m.*

Of course, if $p = 1$, then this is simply the main theorem. The generalization was a needed technical result to establish generalizations of the epimorphism theorem when the base k is taken to be a polynomial ring in one variable over a field. [Sa4].

6 Suzuki's Proof.

A proof of the Embedding Theorem contemporary with that of Abhyankar and Moh was given by M. Suzuki [Su]. It is very different in spirit, and the Embedding Theorem nowadays is often cited as the AMS-theorem. Suzuki's paper also was very influential. It uses methods of complex analysis, in particular the theory of pluri-subharmonic functions, to study polynomial maps $F : \mathbb{C}^2 \to \mathbb{C}$, where F is an irreducible polynomial. A key result, now usually referred to as Suzuki's formula, is that the topological Euler characteristic of any singular (special) fiber is at least as big as that of a regular (general) fiber. (This generalizes a fact well known in the case of proper maps.) In case $F_0 = F^{-1}(0) \simeq \mathbb{C}$ Suzuki then goes on to show that F_0 is in fact a regular fiber. His methods apply to morphisms $\Phi : X \to \mathbb{C}$ for surfaces more general than $\mathbb{C}^2$, in particular all smooth affine surfaces. His results have been extended and sharpened by M. Zaidenberg [Za], and a proof of Suzuki's formula relying on geometric methods rather than complex analysis, or, let us say, more accessible to algebraic geometers, has been given by R. Gurjar [Gu2].

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To be continued ...