# TRANSLATES OF POLYNOMIALS

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**Abstract**. We undertook this study of affine pencils especially to celebrate the 70th birthday of our good friend C. S. Seshadri. The first named author met Seshadri in Paris in 1958 and had the pleasure of seeing him frequently ever since. We are very happy to say to him: JEEVEMA SHARADAH SHATAM.

## 1. Introduction

Let $f = 0$ be a hypersurface in the $n$-dimensional affine space over a field $k$ with $n > 1$, i.e., $f \in R \setminus k$ where $R$ is the polynomial ring $k[X_1, \ldots, X_n]$. We want to consider the pencil of hypersurfaces $f - c = 0$ with $c$ varying over $k$, and wish to consider the sets $\mathrm{singset}(f) = \{c \in k : f-c \text{ is singular}\}$ and $\mathrm{redset}(f) = \{c \in k : f-c \text{ is reducible}\}$. In the first case $f-c$ is singular means the local ring $R_P/((f-c)R_P)$ is nonregular for some $P \in \mathrm{spec}(R)$ with $f-c \in P$, and in the second case $f-c$ is reducible means $f-c = gh$ with $g, h$ in $R \setminus k$.

As consequences of the two famous theorems of Bertini, which may be called Bertini I or Singular Bertini, and Bertini II or Reducible Bertini, it can be shown that, under suitable conditions, $\mathrm{singset}(f)$ and $\mathrm{redset}(f)$ are finite. One of our aims is to give short direct proofs of these consequences. We shall do this in Sections 3 and 4. In Section 2 we shall recall the Theorems of Bertini and also the Theorem of Lüroth. Before outlining the contents of the rest of the paper, let us fix some notation.

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By $|S|$ we denote the cardinality of a set $S$. By $U(S)$ we denote the multiplicative group of all units in a ring $S$, and by $S^\times$ we denote the set of nonzero elements in it. By $\mathrm{QF}(S)$ we denote the quotient field of a domain $S$. By $A = R/(fR)$ we denote the affine coordinate ring of $f = 0$, and we identify $k$ with its image under the residue class epimorphism $\phi : R \to A$; if $f$ is irreducible in $R$ then by $L = \mathrm{QF}(A)$ we denote the function field of $f = 0$. By an affine domain over a field $k'$ we mean an overdomain of $k'$ which is a finitely generated ring extension of $k'$. By a DVR we mean a real discrete valuation ring; if the said ring has quotient field $L'$ then we call it a DVR of $L'$; if it also contains a subfield $k'$ then we call it a DVR of $L'/k'$. Note that a finitely generated free abelian group is isomorphic to $\mathbb{Z}^r$ for a unique nonnegative integer $r$ which is called its rank; we shall apply this to the multiplicative group $U(A')/U(k')$ for an overdomain $A'$ of a field $k'$.

Given polynomials $g_1, \dots, g_m$ in one or more variables with coefficients in some field, we write $\gcd(g_1, \dots, g_m) = 1$ or $\neq 1$ to mean that they do not or do have a nonconstant common factor. In particular we shall apply this to the partial derivatives $f_{X_1}, \dots, f_{X_n}$ of $f$.

Let $R^* = k^*[X_1, \dots, X_n]$ where $k^*$ is an algebraic closure of $k$. Let $\mathrm{singset}(f)^* = \{c \in k^* : R^*_P/((f-c)R^*_P)$ is nonregular for some $P \in \mathrm{spec}(R^*)$ with $f - c \in P\}$, and $\mathrm{redset}(f)^* = \{c \in k^* : f - c = gh$ for some $g, h$ in $R^* \setminus k^*\}$. We shall also consider the multiple set and the primary set of $f$ defined by putting $\mathrm{multset}(f)^* = \{c \in k^* : f - c = gh^2$ for some $g \in R^* \setminus \{0\}$ and $h \in R^* \setminus k^*\}$, and $\mathrm{primset}(f) = \{c \in k : f - c = gh^\mu$ for some $g \in k^\times$ and $h \in R \setminus k$ and integer $\mu > 1\}$.

In Section 5 we shall discuss the notion of composite pencils, and we shall find some bounds for $|\mathrm{redset}(f)|$. A series of Examples, to be outlined in Remark 5, will illustrate the spread of the various values which $|\mathrm{redset}(f)|$ can take.

In Section 6 we shall extend our study to more general pencils $f - cw$ where $w \in R^\times$ with $\gcd(f, w) = 1$. In Remark 8 we shall link-up the redset of a general pencil to Klein's parametrization of a special rational surface, and in Question 4 we shall pose a related problem.

In Section 7 we shall employ a refined version of $\mathrm{redset}(f)$ to give necessary conditions for the ring $R[1/f]$ to be isomorphic to the ring $R[1/f']$ where $f' = 0$ is another hypersurface. This is when $f$ and $f'$ are irreducible. In the reducible case we shall give necessary conditions in terms of the quotient group $U(R[1/f])/U(k)$ and the function fields of the irreducible components of $f = 0$. Now an isomorphism of the rings $R[1/f]$ and $R[1/f']$ can be geometrically paraphrased as biregular equivalence of the complements of the hypersurfaces $f = 0$ and $f' = 0$ in affine $n$-space; moreover, an automorphism of $R$ sending $f$ to $f'$ induces such an isomorphism. In Questions 5 to 8 of Section 7, these facts provide a link-up of the results of that section to: Abhyankar's theorem on exceptional nonruled varieties, the birational invariance of the arithmetic genus of a nonsingular projective variety via the domination part of Abhyankar's desingularization theory, the epimorphism theorems and problems discussed by Abhyankar in his Kyoto Notes, and the work of Zariski, Fan, Teicher, and others on the topology of complements. We were originally motivated to consider the said isomorphisms by a question which Roger Wiegand asked us in 1988.

In Section 8 we shall deduce the finiteness of the redset of a hypersurface from that of a plane curve by the intervention of Zariski's famous Lemma 5. It is by means of this Lemma that Abhyankar proved the Galois case of the Jacobian Problem. More precisely he deduced the Galois case from the birational case which was itself proved by using Zariski's Main Theorem.

In Section 9, we shall find a bound for the singset of plane curve $f$ in terms of its deficiency set defset$(f)$ which is the set of all constants $c$ for which the algebraic rank $\rho_a(f - c)$ is different from the pencil-rank $\rho_\pi(f)$. In the complex case, $\rho_a(f)$ of an irreducible $f$ coincides with its first homology rank. In the general case, $\rho_a(f)$ is defined in terms of the genera and numbers of branches of the various irreducible components of $f$, and the pencil-rank $\rho_\pi(f)$ is the general value of $\rho_a(f - c)$ taken over all constants $c$. We shall show how the pencil rank is related to the Zeuthen-Segre invariant. We shall also discuss Jung's formula which relates the rank to the Zeuthen-Segre invariant.

In Section 10 we shall extend our study of the deficiency set to that of a general pencil.

To put things in proper perspective, in Section 11 we shall briefly talk about linear systems and pencils on normal varieties, and say a few words about the Zeuthen-Segre invariant of a nonsingular projective algebraic surface.

## 2. Theorems of Bertini and Lüroth

Considering a linear systems of codimension one subvarieties of an algebraic variety, and calling it irreducible if its generic member is irreducible, in Mantra form, Bertini's Theorems may be stated thus:

Bertini I or Singular Bertini. Outside the singularities of the variety and outside its base points, members of an irreducible linear system do not have variable singularities.

Bertini II or Reducible Bertini. If a linear system, without fixed components, is not composite with a pencil, then it is irreducible.

These were obtained by Bertini in his 1882 paper [**Ber**]. They were revisited by Zariski in [**Za1**] and [**Za2**].

We shall also use the equally hoary:

Theorem of Lüroth. If a curve has a rational parametrization then it has a faithful rational parametrization.

This is in his 1875 paper [**Lur**]. We need the refined version given by Abhyankar-Eakin-Heinzer in [**AEH**]. Also see Igusa [**Igu**] and Nagata [**Na2**].

## 3. Singset

Let us now prove our:

Singset Theorem. If $k$ is of characteristic zero then singset$(f)$ is finite.

**Proof**. Let $I$ be the ideal in $R^*$ generated by $f_{X_1}, \dots, f_{X_n}$. For any $P \in \operatorname{spec}(R^*)$ with $I \subset P$, consider the residue class map $\Phi_P : R^* \to R^*/P$. Since all the partials of $f$ belong to $P$, it follows that $D(\Phi_P(f)) = 0$ for every $\Phi_P(k^*)$-derivation of $\mathrm{QF}(R^*/P)$. Therefore, since $k^*$ is of characteristic zero, we have $\Phi_P(f) = \Phi_P(\kappa(P))$ for a unique $\kappa(P) \in k^*$. Clearly $P + (f-c)R^* = R^*$ whenever $\kappa(P) \neq c \in k^*$. Let $P_1, \dots, P_s$ be the minimal primes of $I$ in $R^*$, where $s = 0 \Leftrightarrow I = R^*$. Then for all $c \in k^* \setminus \{\kappa(P_1), \dots, \kappa(P_s)\}$ we have

$I + (f - c)R^* = R^*$. Since $k$ is of characteristic zero, it follows that $\mathrm{singset}(f) \subset \{\kappa(P_1), \ldots, \kappa(P_s)\}$, and hence $\mathrm{singset}(f)$ is finite.

## 4. Redset

Next we prove our:

Redset Theorem. If $f$ is irreducible in $R$ and $k$ is relatively algebraically closed in $L$, then $\mathrm{redset}(f)$ is finite.

**Proof**. By the following Lemma we can find a finite number of DVRs $V_1, \ldots, V_t$ of $L/k$ such that $A \cap V_1 \cap \cdots \cap V_t = k$. For every $z \in L^\times$ let $W_i(z) = \mathrm{ord}_{V_i}(z)$, and let $W : L^\times \to \mathbb{Z}^t$ be the map given by putting $W(z) = (W_1(z), \ldots, W_t(z))$. Let $G$ be the set of all $g \in R \setminus k$ such that $gh = f - c$ for some $h \in R \setminus k$ and $c \in k^\times$. Since the degree of $g$ is clearly smaller than the degree of $f$, the set $G$ is contained in a finite dimensional $k$-vector-subspace of $R$. Therefore for every $i \in \{1, \ldots, t\}$, the set $W_i(\phi(g))_{g \in G}$ is bounded from below. Since $h$ also belongs to $G$ and clearly $W_i(\phi(g)) = -W_i(\phi(h))$, it follows that the set $W_i(\phi(g))_{g \in G}$ is also bounded from above. Since, for $1 \le i \le t$, the set $W_i(\phi(g))_{g \in G}$ is bounded from both sides, it follows that $W(\phi(G))$ is a finite set. Also clearly $\phi(G) \subset U(A)$. Let $g_1 h_1 = f - c_1$ and $g_2 h_2 = f - c_2$ with $g_1, h_1, g_2, h_2$ in $R \setminus k$ and $c_1, c_2$ in $k^\times$ be such that $W(\phi(g_1)) = W(\phi(g_2))$. Then $\phi(g_1)/\phi(g_2) \in A \cap V_1 \cap \cdots \cap V_t = k$ and hence $\phi(g_2) = c\phi(g_1)$ for some $c \in k^\times$. Consequently $g_2 - cg_1$ is divisible by $f$ in $R$ and hence, because $\deg(g_2 - cg_1) < \deg(f)$, we must have $g_2 = cg_1$. Therefore, by subtracting the equation $g_2 h_2 = f - c_2$ from the equation $g_1 h_1 = f - c_1$ we get $c_2 - c_1 = g_1 h_1 - g_2 h_2 = g_1(h_1 - ch_2)$ which implies that $c_2 - c_1 \in k$ is divisible in $R$ by the positive degree polynomial $g_1$. Consequently we must have $c_2 = c_1$. Therefore, because the set $W(\phi(G))$ is finite, we conclude that $\mathrm{redset}(f)$ is finite.

Lemma. Given any affine domain $A'$ over a field $k'$, let $k''$ be the algebraic closure of $k'$ in $L' = \mathrm{QF}(A')$, and let $A''$ be the integral closure of $A'$ in $L'$. Then there exists a finite number of DVRs $V_1, \ldots, V_t$ of $L'/k'$ such that $A'' \cap V_1 \cap \cdots \cap V_t = k''$. Moreover, if $k'' = k'$ then $A' \cap V_1 \cap \cdots \cap V_t = k'$ and $U(A')/U(k')$ is a finitely generated free abelian group of rank $r$ with $r \le \max(0, t - 1)$.

**Proof**. Let $m$ be the transcendence degree of $L'$ over $k'$. If $m = 0$ then we can take $t = 0$, and if also $k' = k''$ then clearly $U(A')/U(k') = 1$ and hence $r = 0$. So assume $m > 0$. Then by Noether Normalization $A'$ is integral over a polynomial ring $\widehat{A} = k'[Y_1, \dots, Y_m] \subset L'$. Let $\widehat{L} = k'(Y_1, \dots, Y_m)$ and $\widehat{V} = \{g/h : g, h \in \widehat{A}$ with $h \neq 0$ and $\deg(g) \leq \deg(h)\}$. Then $\widehat{V}$ is a DVR of $\widehat{L}/k'$ with $\widehat{A} \cap \widehat{V} = k'$. Let $V''$ be the integral closure of $\widehat{V}$ in $L'$. Then $V'' = V_1 \cap \dots \cap V_t$ where $V_1, \dots, V_t$ are DVRs of $L'/k'$ with $t > 0$, and by the following Sublemma we have $A'' \cap V'' = k''$. It follows that $A'' \cap V_1 \cap \dots \cap V_t = k''$, and if $k'' = k'$ then $A' \cap V_1 \cap \dots \cap V_t = k'$. Now assuming $k'' = k'$, let $W : U(L') \to \mathbb{Z}^t$ be the homomorphism (from a multiplicative group to an additive group) given by $W(z) = (W_1(z), \dots, W_t(z))$ with $W_i(z) = \mathrm{ord}_{V_i}(z)$. Then $U(A') \cap \ker(W) = U(k')$ and hence we get a monomorphism $\overline{W} : U(A')/U(k') \to \mathbb{Z}^t$. Therefore $U(A')/U(k')$ is a finitely generated free abelian group of rank $r$ with $r \leq t$. Suppose if possible that $r = t$. Then we can find $z_1, \dots, z_t$ in $U(A')$ such that the $t \times t$ matrix $W_i(z_j)$ has a nonzero determinant. Now the column vectors of this matrix are $\mathbb{Q}$-linearly independent vectors in $\mathbb{Q}^t$ and hence the column vector $(1, 0, \dots, 0)$ can be expressed as a $\mathbb{Q}$-linear combination of them, i.e., we can find rational numbers $a_1, \dots, a_t$ such that $\sum_{1 \leq j \leq t} W_i(z_j) a_j = 1$ or $0$ according as $i = 1$ or $2 \leq i \leq t$. Next we can find integers $a, b_1, \dots, b_t$ with $a > 0$ such that $a a_j = b_j$ for $1 \leq j \leq t$. Let $z = \prod_{1 \leq j \leq t} z_j^{b_j}$. Then $z \in U(A')$ with $W_1(z) = a > 0 = W_i(z)$ for $2 \leq i \leq t$. Consequently $z \in A' \cap V_1 \dots \cap V_t$ but $z \notin k'$ which is a contradiction. Therefore we must have $r \leq t - 1$.

SUBLEMMA. Let $\widehat{A}$ and $\widehat{V}$ be normal domains with a common quotient field $\widehat{L}$. Let $A''$ and $V''$ be the respective integral closures of $\widehat{A}$ and $\widehat{V}$ in an algebraic field extension $L'$ of $\widehat{L}$. Then $A'' \cap V''$ is the integral closure of $\widehat{A} \cap \widehat{V}$ in $L'$.

**Proof**. Clearly the integral closure of $\widehat{A} \cap \widehat{V}$ in $L'$ is contained in $A'' \cap V''$. Conversely, given any $z \in L'$ let $H(Z)$ be the minimal monic polynomial of $z$ over $\widehat{L}$. By Kronecker's Theorem (or obviously), if $z \in A''$ then $H(Z) \in \widehat{A}[Z]$, and if $z \in V''$ then $H(Z) \in \widehat{V}[Z]$. Consequently, if $z \in A'' \cap V''$ then $H(Z) \in (\widehat{A} \cap \widehat{V})[Z]$. Thus $A'' \cap V''$ is integral over $\widehat{A} \cap \widehat{V}$. Therefore $A'' \cap V''$ is the integral closure of $\widehat{A} \cap \widehat{V}$ in $L$.

REMARK 1. With notation as in the proof of the Redset Theorem, in the $n = 2$ case, the Lemma also follows by taking $V_1, \dots, V_t$ to be the valuation rings of the places at infinity of the plane curve $f = 0$. Moreover, in this case, in the proof of the Redset Theorem, the finiteness of $W(\phi(G))$ can be deduced from the boundedness from below of the sets $W_1(\phi(G)), \dots, W_t(\phi(G))$ by invoking the fact that the number of zeros of a rational function on the curve $f = 0$ equals the number of its poles.

REMARK 2. Assuming $k$ to be of characteristic zero, the $n > 2$ case of the Redset Theorem can be deduced from the $n = 2$ case by invoking the famous Lemma 5 of Zariski's paper [**Za1**] thus. By applying a linear $k$-automorphism to $R$ and multiplying $f$ by an element in $k^\times$, we can arrange $f$ to be a monic polynomial of positive degree in $X_1$ with coefficients in $S = k[X_2, \dots, X_n]$. Now $\phi(X_2), \dots, \phi(X_n)$ is a transcendence basis of $L/k$ and hence, invoking Zariski's Lemma 5 and applying a $k$-linear automorphism to $S$, we can arrange the field $k(\phi(X_3), \dots, \phi(X_n))$ to be relatively algebraically closed in $L$. Let $\widetilde{R} = \widetilde{k}[X_3, \dots, X_n]$ with $\widetilde{k} = k(X_3, \dots, X_n)$, and let $\widetilde{L} = \mathrm{QF}(\widetilde{A})$ with $\widetilde{A} = \widetilde{R}/(f\widetilde{R})$. By Gauss Lemma, $f$ is irreducible in $\widetilde{R}$ and hence, by the $n = 2$ case of the Redset Theorem, the set $\{c \in \widetilde{k} : f - c \text{ is reducible in } \widetilde{R}\}$ is finite. Therefore, again by Gauss Lemma, the set $\{c \in k : f - c \text{ is reducible in } R\}$ is finite.

REMARK 3. Assume $f$ is irreducible in $R$. Now the Redset Theorem says that if the ground field $k$ is relatively algebraically closed in the function field $L$ then redset$(f)$, i.e., the set of all constants $c$ for which the polynomial $f - c$ factors, is finite. We want to observe that this set, although determined by $f$, is not determined by the affine coordinate ring $A$. Indeed if redset$(f)$ is nonempty, say it contains a constant $c$ (which is necessarily nonzero because $f$ is irreducible), then every constant $\gamma$ in $A$ factors. Namely, since $c$ is in redset$(f)$, we can write $f - c = gh$ with $g, h$ in $R \setminus k$, and multiplying both sides by $-\gamma/c$ we get $(-\gamma/c)f + \gamma = g'h'$ with $g' = g$ and $h' = (-\gamma/c)h$. Thus we have factored $\gamma$ not only in $A$, i.e., modulo the ideal $fR$, but also "modulo" the constant multiples $kf$ of $f$. Note that if $\gamma \neq 0$ then both $g'$ and $h'$ belong to $R \setminus k$, but if $\gamma = 0$ then $h'$ does not. To take care of this extra "desire" and to make sure that the "multiplier $e = -(\gamma/c)$" of $f$ is also nonzero, by taking $e = ug - (\gamma/c)$ with any $u \in R^\times$ we get $ef + \gamma = g'h'$

with $g' = g \in R \setminus k$ and $h' = uf - (\gamma/c)h \in R \setminus k$. As example, $f = X_1X_2 + 1$ is irreducible with $\text{redset}(f) = \{1\}$, and for any $\gamma \in k$ we have $(X_1X_2 - \gamma)f + \gamma = X_1[X_1X_2^2 + (1-\gamma)X_2]$.

## 5. Generic Members and Composite Pencils

Let $R^\sharp = k(Z)[X_1, \ldots, X_n]$ where $Z$ is an indeterminate over $R$. By the generic member of the pencil $(f - c)_{c \in k}$ we mean the hypersurface $f^\sharp = 0$ with $f^\sharp = f - Z \in R^\sharp$. Let $\text{singset}(f^\sharp)$ be the set of all $c \in k(Z)$ such that $R^\sharp_P/((f^\sharp - c)R^\sharp_P)$ is nonregular for some $P \in \text{spec}(R^\sharp)$ with $f^\sharp - c \in P$, and let $\text{redset}(f^\sharp)$ be the set of all $c \in k(Z)$ such that $f^\sharp - c$ is reducible in $R^\sharp$.

By Gauss Lemma $f^\sharp$ is irreducible in $R^\sharp$, i.e., $0 \notin \text{redset}(f^\sharp)$. Let $\phi^\sharp : R^\sharp \to R^\sharp/(f^\sharp R^\sharp)$ be the residue class epimorphism. Clearly $R \cap \ker(\phi^\sharp) = \{0\}$ and $\phi^\sharp(f) = \phi^\sharp(Z)$, and hence there exits a unique isomorphism $\psi^\sharp : k(X_1, \ldots, X_n) \to \text{QF}(R^\sharp/(f^\sharp R^\sharp))$ such that for all $r \in R$ we have $\psi^\sharp(r) = \phi^\sharp(r)$. Thus the triple $\phi^\sharp(k(Z)) \subset \phi^\sharp(R^\sharp) \subset \text{QF}(\phi^\sharp(R^\sharp))$ is isomorphic to the triple

$$k^\sharp = k(f) \subset A^\sharp = k(f)[X_1, \ldots, X_n] \subset L^\sharp = k(X_1, \ldots, X_n)$$

and hence we may regard the above three displayed sets as the ground field, the affine coordinate ring, and the function field of $f^\sharp = 0$.

The ring $A^\sharp$ is regular because it is the localization of the regular ring $R$ at the multiplicative set $k[f]^\times$. Thus we have the:

Generic Singset Theorem. $0 \notin \text{singset}(f^\sharp)$.

In view of the above isomorphism of triples, by the Redset Theorem we get the:

Generic Redset Theorem. If $k^\sharp$ is relatively algebraically closed in $L^\sharp$ then $\text{redset}(f^\sharp)$ is finite.

In [**AEH**] we proved the following:

Refined Lüroth Theorem. Assume that $k^\sharp$ is not relatively algebraically closed in $L^\sharp$. Let $\widehat{B}^\sharp$ be the integral closure of $k[f]$ in $L^\sharp$, let $\widehat{k}^\sharp$ be the algebraic closure of $k^\sharp$ in $L^\sharp$, and let $\nu = [\widehat{k}^\sharp : k^\sharp]$. Then $\nu$ is an integer with $\nu > 1$, and there exist $\widehat{f} \in R \setminus k$ and $\Lambda \in k[Z] \setminus k$ with $\widehat{B}^\sharp = k[F]$ and $\widehat{k}^\sharp = k(F)$ such that $f = \Lambda(\widehat{f})$ and $\deg_Z \Lambda = \nu$.

REMARK 4. In the above situation, every member of the pencil $(f-c)_{c\in k^*}$ consists of $\nu$ members (counted properly) of the pencil $(\widehat{f}-\widehat{c})_{\widehat{c}\in k^*}$, and so we say that **the pencil** $(f-c)_{c\in k^*}$ **is composite with the pencil** $(\widehat{f}-\widehat{c})_{\widehat{c}\in k^*}$. In greater detail, for any $c\in k^*$ we have $\Lambda(Z)-c=\widehat{c}_0\prod_{1\le i\le\nu}(Z-\widehat{c}_i)$ where $\widehat{c}_0,\widehat{c}_1,\dots,\widehat{c}_\nu$ in $k^*$ with $\widehat{c}_0\neq 0$, and by substituting $\widehat{f}$ for $Z$ we get $f-c=\widehat{c}_0\prod_{1\le i\le\nu}(\widehat{f}-\widehat{c}_i)$ and so the hypersurface $f=c$ is the union of the hypersurfaces $\widehat{f}=\widehat{c}_1,\dots,\widehat{f}=\widehat{c}_\nu$. Since $\deg_Z\Lambda=\nu>1$, if $k$ is of characteristic 0 then by taking a root $\zeta$ of the $Z$-derivative of $\Lambda(Z)$ in $k^*$ we see that $\Lambda(Z)-\Lambda(\zeta)$ has a multiple root in $k^*$ and hence $f-\Lambda(\zeta)$ has a nonconstant multiple factor in $R^*$, and therefore $\mathrm{multset}(f)^*\neq\emptyset$. Note that if $k$ is of characteristic $p>0$ then this does not work as can be seen by taking $\Lambda(Z)=Z^p+Z$. Without assuming any condition on the pair $(k^\sharp,L^\sharp)$ we see that, for any $c\in k^*$, every multiple factor of $f-c$ in $R^*$ divides $f_{X_j}$ for $1\le j\le n$, and hence: $\mathrm{multset}(f)^*\neq\emptyset\Rightarrow\gcd(f_{X_1},\dots,f_{X_n})\neq 1$. Again without assuming any condition on the pair $(k^\sharp,L^\sharp)$ we see that $c\mapsto c-Z$ gives an injection of $\mathrm{redset}(f)$ into $\mathrm{redset}(f^\sharp)$, and hence $|\mathrm{redset}(f)|\le|\mathrm{redset}(f^\sharp)|$. Thus, in view of the above two Theorems, we get the:

COMPOSITE PENCIL THEOREM. We have the following:

(I) $\gcd(f_{X_1},\dots,f_{X_n})=1\Rightarrow\mathrm{multset}(f)^*=\emptyset$.

(II) characteristic of $k$ is 0 and $\mathrm{multset}(f)^*=\emptyset\Rightarrow k^\sharp$ is relatively algebraically closed in $L^\sharp$.

(III) $k^\sharp$ is relatively algebraically closed in $L^\sharp\Rightarrow\mathrm{redset}(f)$ and $\mathrm{redset}(f^\sharp)$ are finite with $|\mathrm{redset}(f)|\le|\mathrm{redset}(f^\sharp)|$.

(IV) $k^\sharp$ is not relatively algebraically closed in $L^\sharp\Rightarrow\mathrm{redset}(f)^*=k^*$.

Next we prove the:

MIXED PRIMSET THEOREM. If $k^\sharp$ is relatively algebraically closed in $L^\sharp$ then $\mathrm{primset}(f)=\emptyset$.

**Proof**. If $\mathrm{primset}(f)\neq\emptyset$ then for some $c\in k$ and integer $\mu>1$ we have $f-c=gh^\mu$ with $g\in k^\times$ and $h\in R\setminus k$, and this implies $[k(h):k(f)]=\mu$ and hence $k^\sharp$ is not relatively algebraically closed in $L^\sharp$.

Let us now prove the following theorem which is some kind of a mixture of the Redset Theorem and the Generic Redset Theorem.

Mixed Redset Theorem. If $k^\sharp$ is relatively algebraically closed in $L^\sharp$ then $U(A^\sharp)/U(k^\sharp)$ is a finitely generated free abelian group of rank $r$ with $|\mathrm{redset}(f)| \le r$.

**Proof**. Assuming that $k^\sharp$ is relatively algebraically closed in $L^\sharp$, by the Lemma we see that $U(A^\sharp)/U(k^\sharp)$ is a finitely generated free abelian group of some rank $r$. Suppose if possible that $|\mathrm{redset}(f)| > r$ and take distinct elements $c_1, \dots, c_s$ in redset$(f)$ with integer $s > r$. By the Mixed Primset Theorem, for $1 \le i \le s$ we have

$$f - c_i = g_i h_i \quad \text{where} \quad g_i, h_i \text{ in } R \setminus k \quad \text{with} \quad \gcd(g_i, h_i) = 1.$$

Since $s > r$, there exist integers $a_1, \dots, a_s$ at least one of which is nonzero, say $a_\iota \neq 0$, such that

$$\prod_{1 \le i \le s} g_i^{a_i} \in U(k(f)). \tag{2}$$

For the rest of the proof we shall give two alternative arguments.

First Argument. Clearly the images of $g_i$ and $h_i$ in $U(A^\sharp)/U(k^\sharp)$ are inverses of each other and hence replacing $g_i$ by $h_i$ for those $i$ for which $a_i < 0$, we can arrange matters so that $a_i \ge 0$ for all $i$, and hence in particular $a_\iota > 0$. Again since the images of $g_i$ and $h_i$ in $U(A^\sharp)/U(k^\sharp)$ are inverses of each other, by (2) we get

$$\prod_{1 \le i \le s} h_i^{a_i} \in U(k(f)). \tag{3'}$$

Any element in $k(f)^\times$ can be written as $u(f)/v(f)$ where $u(Z), v(Z)$ in $k[Z]^\times$ with $u(Z)\widehat{u}(Z) + v(Z)\widehat{v}(Z) = 1$ for some $\widehat{u}(Z), \widehat{v}(Z)$ in $k[Z]$, and substituting $Z = f$ we get $u(f)\widehat{u}(f) + v(f)\widehat{v}(f) = 1$; it follows that if $u(f)/v(f) \in R$ then $v(f) \in k^\times$ and hence $u(f)/v(f) \in k[f]$. Thus $R \cap k(f) = k[f]$ and therefore by (2) and (3′) we get

$$\prod_{1 \le i \le s} g_i^{a_i} = g(f) \quad \text{and} \quad \prod_{1 \le i \le s} h_i^{a_i} = h(f) \tag{4'}$$

with $g(Z), h(Z)$ in $k[Z]^\times$. Multiplying the two equations in (4′) we obtain

$$\prod_{1 \le i \le s} (f - c_i)^{a_i} = g(f)h(f)$$

and hence by the $k$-isomorphism $k[Z] \to k[f]$ with $Z \mapsto f$ we get

$$\prod_{1 \le i \le s} (Z - c_i)^{a_i} = g(Z)h(Z)$$

and therefore we have

$$(5') \qquad g(Z) = \overline{g} \prod_{1 \le i \le s} (Z - c_i)^{\alpha_i} \quad \text{and} \quad h(Z) = \overline{h} \prod_{1 \le i \le s} (Z - c_i)^{\beta_i}$$

with $\overline{g}, \overline{h}$ in $k^\times$ and nonnegative integers $\alpha_i, \beta_i$ such that $\alpha_i + \beta_i = a_i$. Substituting $Z = f$ and $f - c_i = g_i h_i$ in $(5')$ and then comparing it with $(4')$ we get

$$(6') \quad \prod_{1 \le i \le s} g_i^{a_i} = \overline{g} \prod_{1 \le i \le s} (g_i h_i)^{\alpha_i} \quad \text{and} \quad \prod_{1 \le i \le s} h_i^{a_i} = \overline{h} \prod_{1 \le i \le s} (g_i h_i)^{\beta_i}.$$

Clearly $\gcd(f - c_i, f - c_j) = 1$ for all $i \neq j$, and hence by (1) and $(6')$ we get

$$(7') \qquad a_i = \alpha_i \text{ and } a_i = \beta_i \text{ for } 1 \le i \le s.$$

Since $\alpha_\iota + \beta_\iota = a_\iota > 0$ with $\alpha_\iota \ge 0$ and $\beta_\iota \ge 0$, we must have either $\alpha_\iota < a_\iota$ or $\beta_\iota < a_\iota$ which contradicts $(7')$. Therefore $|\mathrm{redset}(f)| \le r$.

Second Argument. We shall make this more condensed. By (2) we get

$$(3'') \qquad \prod_{1 \le i \le s} g_i^{a_i} = \overline{u} \prod_{1 \le i \le \sigma} u_i(f)^{\alpha_i} \quad \text{with} \quad \overline{u} \in k^\times$$

where $\alpha_1, \dots, \alpha_\sigma$ are nonzero integers and $u_1(Z), \dots, u_\sigma(Z)$ are pairwise coprime monic members of $k[Z] \setminus k$. As in the First Argument, for any $u, v$ in $k[Z] \setminus k$ with $\gcd(u, v) = 1$ we have $u(f), v(f)$ in $R \setminus k$ with $\gcd(u(f), v(f)) = 1$. Therefore, by looking at divisibility by a nonconstant irreducible factor of $g_i$ in $R$, in view of (1) and $(3'')$, we see that for each $i \in \{1, \dots, s\}$ with $a_i \neq 0$ there is some $\theta(i) \in \{1, \dots, \sigma\}$ with $u_{\theta(i)}(Z) = Z - c_i$ and $\alpha_{\theta(i)} = a_i$. In particular we get $u_{\theta(\iota)}(Z) = Z - c_\iota$ and $\alpha_{\theta(\iota)} = a_\iota$. Again in view of (1) and $(3'')$, by looking at divisibility by a nonconstant irreducible factor of $h_\iota$ in $R$, we get a contradiction. Therefore $|\mathrm{redset}(f)| \le r$.

Remark 5. Assume $n = 2$. Let $t$ be the number of places at infinity of the irreducible plane curve $f^\sharp = 0$ and let $V_1, \dots, V_t$ be their valuation rings. Then $t$ is a positive integer and by the Lemma and the Mixed Redset Theorem we see that if $k^\sharp$ is relatively algebraically closed in $L^\sharp$, then $R^\sharp \cap V_1 \cap \dots \cap V_t = k^\sharp$ and

$|\mathrm{redset}(f)| \leq t-1$. We shall show that this bound is the best possible. In fact, in the following Examples 1 to 5, assuming $k$ to contain sufficiently many elements, we shall show that, for any $t > 0$, $|\mathrm{redset}(f)|$ can be equal to any integer $\mu$ with $0 \leq \mu \leq t-1$; Example 1 works for $0 \leq \mu = t-1$ and $t > 0$; Example 2 works for $1 \leq \mu < t-1$ and $t > 2$; Example 3 works for $\mu = 0$ and $t > 3$; Example 4 works for $\mu = 0$ and $t > 0$; Example 5 works for $\mu = 1$ and $t > 1$. As common notation for these examples, let $n = 2$, let $m \geq 0$ be an integer, and let $a(X_1) = (X_1 - a_1)\dots(X_1 - a_m)$ where $a_1, \dots, a_m$ are pairwise distinct elements in $k$. Moreover, for the irreducible $f \in R \setminus k$ to be constructed, let $\tau(f)$ denote the number of places at infinity of the plane curve $f = 0$, and let $\tau(f^\sharp)$ denote the number of places at infinity of the plane curve $f^\sharp = 0$.

EXAMPLE 1. Let $f = a(X_1)X_2 + X_1 - z$ with $z \in k \setminus \{a_1, \dots, a_m\}$. Then $f$ is irreducible in $R$ and

$\mathrm{redset}(f) = \mathrm{redset}(f)^* = \{a_1 - z, \dots, a_m - z\}$.

Moreover, $k^\sharp$ is relatively algebraically closed in $L^\sharp$ and we have $|\mathrm{redset}(f)| = |\mathrm{redset}(f)^*| = m$ with $\tau(f) = \tau(f^\sharp) = m+1$.

**Proof**. Clearly $(X_1 - a_i)$ divides $f - (a_i - z)$ in $R$ for $1 \leq i \leq m$, and hence $\{a_1 - z, \dots, a_m - z\} \subset \mathrm{redset}(f)$. By Gauss Lemma we also see that $\mathrm{redset}(f)^* \subset \{a_1 - z, \dots, a_m - z\}$. Therefore $f$ is irreducible in $R$, and we have $\mathrm{redset}(f) = \mathrm{redset}(f)^* = \{a_1 - z, \dots, a_m - z\}$. In particular $|\mathrm{redset}(f)| = |\mathrm{redset}(f)^*| = m < \infty$, and hence by the Composite Pencil Theorem we see that $k^\sharp$ is relatively algebraically closed in $L^\sharp$. For the degree form (which consists of the highest degree terms) of $f$ we have $\mathrm{defo}(f) = X_1^m X_2$, and hence the points at infinity of the curve $f = 0$ are $(X_0, X_1, X_2) = (0,1,0)$ and $(X_0, X_1, X_2) = (0,0,1)$. Homogenizing $f$ we obtain

$$(X_1 - a_1X_0)\dots(X_1 - a_mX_0)X_2 + X_0^m X_1 - zX_0^{m+1}$$

Putting $X_1 = 1$ we get

$$f_1 = (1 - a_1X_0)\dots(1 - a_mX_0)X_2 + X_0^m - zX_0^{m+1}$$

and hence $(0,1,0)$ is a simple point. Putting $X_2 = 1$ we get

$$f_2 = (X_1 - a_1X_0)\dots(X_1 - a_mX_0) + X_0^m X_1 - zX_0^{m+1}$$

and hence $(0,0,1)$ is an $m$-fold point with $m$ distinct tangents. Therefore $\tau(f) = m+1$, and hence by taking $(z + Z, k^\sharp)$ for $(z, k)$ we get $\tau(f^\sharp) = m+1$.

EXAMPLE 2. Assume $m \geq 1$ and let $\mu$ be any integer with $1 \leq \mu \leq m$. Let $b(X_1) = (X_1 - b_1) \dots (X_1 - b_m)$ where $b_1, \dots, b_m$ be pairwise distinct elements in $k$ such that $b_i = a_i$ for $1 \leq i \leq \mu$, and $b_i \notin \{a_1, \dots, a_m\}$ for $\mu < i \leq m$. If $m \neq 1$ then let $\gamma$ be any nonzero element of $k$, and if $m = 1$ then let $\gamma$ be any nonzero element of $k$ such that $Z^2 + \gamma Z + 1 = (Z - \gamma_1)(Z - \gamma_2)$ with $\gamma_1 \neq \gamma_2$ in $k$. Let $f = a(X_1)X_2^2 + \gamma b(X_1)X_2 + X_1 - z$ with $z \in k \setminus \{a_1, \dots, a_\mu\}$. Then $f$ is irreducible in $R$ and $\mathrm{redset}(f) = \mathrm{redset}(f)^* = \{a_1 - z, \dots, a_\mu - z\}$. Moreover, $k^\sharp$ is relatively algebraically closed in $L^\sharp$ and we have $|\mathrm{redset}(f)| = |\mathrm{redset}(f)^*| = \mu$ with $\tau(f) = \tau(f^\sharp) = m + 2$.

**Proof**. Clearly $(X_1 - a_i)$ divides $f - (a_i - z)$ in $R$ for $1 \leq i \leq \mu$, and hence $\{a_1 - z, \dots, a_\mu - z\} \subset \mathrm{redset}(f)$. Again by Gauss Lemma we see that if for some $c \in k^* \setminus \{a_1 - z, \dots, a_\mu - z\}$ we have $c \in \mathrm{redset}(f)$ then there exists $\eta \in (k^*)^\times$ and a disjoint partition $\{1, \dots, m\} = U \coprod V$ such that upon letting

$$u(X_1) = \prod_{i \in U} (X_1 - a_i) \quad \text{and} \quad v(X_1) = \prod_{i \in V} (X_1 - a_i)$$

we have

$$f - z - c = [u(X_1)X_2 + \eta]\,[v(X_1)X_2 + (1/\eta)(X_1 - z - c]\,.$$

Equating the coefficients of $X_2$ we get

$$(*) \qquad \gamma b(X_1) = (1/\eta)(X_1 - z - c)u(X_1) + \eta v(X_1).$$

This gives a contradiction because the LHS is divisible by $(X_1 - a_1)$, but exactly one term in the RHS is so divisible. Therefore $\mathrm{redset}(f)^* \subset \{a_1 - z, \dots, a_m - z\}$. Consequently $f$ is irreducible in $R$, and we have $\mathrm{redset}(f) = \mathrm{redset}(f)^* = \{a_1 - z, \dots, a_m - z\}$. In particular $|\mathrm{redset}(f)| = |\mathrm{redset}(f)^*| = m < \infty$, and hence by the Composite Pencil Theorem we see that $k^\sharp$ is relatively algebraically closed in $L^\sharp$. Now $\mathrm{defo}(f) = X_1^m X_2^2$, and hence the points at infinity of the curve $f = 0$ are $(X_0, X_1, X_2) = (0, 1, 0)$ and $(X_0, X_1, X_2) = (0, 0, 1)$. Homogenizing $f$ we obtain

$$\begin{aligned}&(X_1 - a_1X_0) \dots (X_1 - a_mX_0)X_2^2 \\ &+ \gamma(X_1 - b_1X_0) \dots (X_1 - b_mX_0)X_2X_0 + X_1X_0^{m+1} - zX_0^{m+2}.\end{aligned}$$

Putting $X_1 = 1$ we get

$$\begin{aligned} f_1 =& (1 - a_1X_0)\dots(1 - a_mX_0)X_2^2 \\ &+ \gamma(1 - b_1X_0)\dots(1 - b_mX_0)X_2X_0 + X_0^{m+1} - zX_0^{m+2} \end{aligned}$$

and hence $(0, 1, 0)$ is a double point with two distinct tangents. Putting $X_2 = 1$ we get

$$\begin{aligned} f_2 =& (X_1 - a_1X_0)\dots(X_1 - a_mX_0) \\ &+ \gamma(X_1 - b_1X_0)\dots(X_1 - b_mX_0)X_0 + X_1X_0^{m+1} - zX_0^{m+2} \end{aligned}$$

and hence $(0, 0, 1)$ is an $m$-fold point with $m$ distinct tangents. Therefore $\tau(f) = m + 2$, and hence by taking $(z + Z, k^\sharp)$ for $(z, k)$ we get $\tau(f^\sharp) = m + 2$.

Example 3. Assume $m \geq 2$. Let $\alpha_i$ and $\alpha_{ij}$ be the elements in $k$ such that

$$a(X_1) = X_1^m + \sum_{1\leq j\leq m} \alpha_j X_1^{m-j}$$

and

$$a(X_1)/(X_1 - a_i) = X_1^{m-1} + \sum_{1\leq j\leq m-1} \alpha_{ij} X_1^{m-1-j} \quad \text{for} \quad 1 \leq i \leq m.$$

Let $b(X_1) = X_1^m + \sum_{i\in\{1,2,m\}} \beta_i X_1^{m-i}$ where $(\beta_i)_{i\in\{1,2,m\}}$ are elements $k$ such that:

(i) $b(a_i) \neq 0$ for $1 \leq i \leq m$;

(ii) if $m > 3$ then $\beta_1 \neq \alpha_1$, and $\beta_2 \neq \beta_1\alpha_{i1} - \alpha_{i1}^2 + \alpha_{i2}$ for $1 \leq i \leq m$;

(iii) if $m = 3$ then $\beta_1 \neq \alpha_1$, and $\beta_2 \neq \beta_1\alpha_{i1} - \alpha_{i1}^2 + \alpha_{i2} + 1$ for $1 \leq i \leq m$;

(iv) if $m = 2$ then $\beta_1 \neq 1 + \alpha_1$, and $\beta_2 \neq \beta_1\alpha_{i1} - \alpha_{i1}^2 - \alpha_{i1} - a_i$.

Let $f = a(X_1)X_2^2 + b(X_1)X_2 + X_1 - z$ with $z \in k$. Then $f$ is irreducible in $R$ and $\operatorname{redset}(f) = \operatorname{redset}(f)^* = \emptyset$. Moreover, $k^\sharp$ is relatively algebraically closed in $L^\sharp$ and we have $|\operatorname{redset}(f)| = |\operatorname{redset}(f)^*| = 0$ with $\tau(f) = \tau(f^\sharp) = m + 2$.

**Proof**. Let $\mu = 0$. By (i) we have $\gcd(a(X_1), b(X_1)) = 1$ and hence the proof is identical with the proof of Example 1 except we have to get a contradiction to (*) where $\gamma = 1$ and $c$ is any element of $k^*$. So assume (*) and let $y = z + c$. Then comparing degrees and coefficients of $X_1^m$ in the LHS and RHS of (*) we see that $\eta = 1$ and either $(u(X_1), v(X_1)) = (1, a(X_1))$ or

$$(u(X_1), v(X_1)) = (a(X_1)/a_i(X_1), a_i(X_1)) \text{ for some } i \in \{1, \ldots, m\}.$$

Let us expand the said LHS and RHS and compare the coefficients of $X_1^{m-1}$ and $X_1^{m-2}$ in them. Then in case of $m > 3$ we have

$$\begin{aligned}
&(u(X_1), v(X_1)) = (1, a(X_1)) \\
&\Rightarrow X_1^m + \sum_{j \in \{1,2,m\}} \beta_j X_1^{m-j} = (X_1 - y) + \left[ X_1^m + \sum_{1 \le j \le m} \alpha_j X_1^{m-j} \right] \\
&\Rightarrow \beta_1 = \alpha_1
\end{aligned}$$

and

$$\begin{aligned}
&(u(X_1), v(X_1)) = (a(X_1)/a_i(X_1), a_i(X_1)) \\
&\Rightarrow X_1^m + \sum_{j \in \{1,2,m\}} \beta_j X_1^{m-j} = (X_1 - y) \left[ X_1^{m-1} + \sum_{1 \le j \le m} \alpha_{ij} X_1^{m-1-j} \right] \\
&\qquad\qquad + (X_1 - a_i) \\
&\Rightarrow \beta_1 = \alpha_{i1} - y \text{ and } \beta_2 = -y\alpha_{i1} + \alpha_{i2} \\
&\Rightarrow \beta_2 = \beta_1 \alpha_{i1} - \alpha_{i1}^2 + \alpha_{i2}
\end{aligned}$$

and hence we get a contradiction by (ii). Likewise in case of $m = 3$ we have

$$\begin{aligned}
&(u(X_1), v(X_1)) = (1, a(X_1)) \\
&\Rightarrow X_1^3 + \beta_1 X_1^2 + \beta_2 X_1 + \beta_3 = (X_1 - y) + (X_1^3 + \alpha_1 X_1^2 + \alpha_2 X_1 + \alpha_3) \\
&\Rightarrow \beta_1 = \alpha_1
\end{aligned}$$

and

$$\begin{aligned}
&(u(X_1), v(X_1)) = (a(X_1)/a_i(X_1), a_i(X_1)) \\
&\Rightarrow X_1^3 + \beta_1 X_1^2 + \beta_2 X_1 + \beta_3 = (X_1 - y)(X_1^2 + \alpha_{i1} X_1 + \alpha_{12}) \\
&\qquad\qquad + (X_1 - a_i) \\
&\Rightarrow \beta_1 = \alpha_{i1} - y \text{ and } \beta_2 = -y\alpha_{i1} + \alpha_{i2} + 1 \\
&\Rightarrow \beta_2 = \beta_1 \alpha_{i1} - \alpha_{i1}^2 + \alpha_{i2} + 1
\end{aligned}$$

and hence we get a contradiction by (iii). Finally in case of $m = 2$ we have

$$\begin{aligned}
&(u(X_1), v(X_1)) = (1, a(X_1)) \\
&\Rightarrow X_1^2 + \beta_1 X_1 + \beta_2 = (X_1 - y) + (X_1^2 + \alpha_1 X_1 + \alpha_2) \\
&\Rightarrow \beta_1 = 1 + \alpha_1
\end{aligned}$$

and

$$\begin{aligned}
&(u(X_1), v(X_1)) = (a(X_1)/a_i(X_1), a_i(X_1)) \\
&\Rightarrow X_1^2 + \beta_1 X_1 + \beta_2 = (X_1 - y)(X_1 + \alpha_{i1}) + (X_1 - a_i) \\
&\Rightarrow \beta_1 = \alpha_{i1} - y + 1 \text{ and } \beta_2 = -y\alpha_{i1} - a_i \\
&\Rightarrow \beta_2 = \beta_1 \alpha_{i1} - \alpha_{i1}^2 - \alpha_{i1} - a_i
\end{aligned}$$

and hence we get a contradiction by (iv).

EXAMPLE 4. Let $f = a(X_1)X_2^2 + X_2 + X_1^{m+1} - z$ with $z \in k$. Then $f$ is irreducible in $R$ and $\text{redset}(f) = \text{redset}(f)^* = \emptyset$. Moreover, $k^\sharp$ is relatively algebraically closed in $L^\sharp$ and we have $|\text{redset}(f)| = |\text{redset}(f)^*| = 0$ with $\tau(f) = \tau(f^\sharp) = m + 1$.

**Proof**. By Gauss Lemma we see that if $c \in \text{redset}(f)^*$ then for some $u(X_1)$, $v(X_1)$, $p(X_1)$, $q(X_1)$ in $R^\times$ we have

$$f - c = [u(X_1)X_2 + p(X_1)][v(X_1)X_2 + q(X_1)]$$

and equating coefficients of powers of $X_2$ we get

$$(4_1) \qquad u(X_1)v(X_1) = a(X_1)$$

and

$$(4_2) \qquad p(X_1)q(X_1) = r(X_1) \quad \text{with} \quad r(X_1) = X_1^{m+1} - z - c$$

and

$$(4_3) \qquad u(X_1)q(X_1) + v(X_1)p(X_1) = 1.$$

For a while suppress the variable $X_1$. Note that then $\deg(a) = m$ and $\deg(r) = m + 1$. First suppose $m$ is even. Then $m + 1$ is odd. Multiplying $(4_3)$ by $u$ and using $(4_1)$ we get

$$(4_4') \qquad u^2 q + ap = u.$$

By $(4_2)$ we know that $\deg(pq)$ is odd and hence $(\deg(q), \deg(p)) =$ (even, odd) or (odd, even). Therefore $(\deg(u^2q), \deg(ap)) =$ (even, odd) or (odd, even), and hence $\deg(u^2q+ap) = \max(\deg(u^2q), \deg(ap)) > \deg(u)$ which contradicts $(4'_4)$. Next suppose $m$ is odd. Then $m+1$ is even. Multiplying $(4_3)$ by $p$ and using $(4_2)$ we get

$$ur + vp^2 = p. \tag{$4''_4$}$$

By $(4_1)$ we know that $\deg(uv)$ is odd and hence $(\deg(u), \deg(v)) =$ (even, odd) or (odd, even).

Therefore $(\deg(ur), \deg(vp^2)) =$ (even, odd) or (odd, even), and hence $\deg(ur + vp^2) = \max(\deg(ur), \deg(vp^2)) > \deg(p)$ which contradicts $(4''_4)$. Thus we have shown that $\text{redset}(f)^* = \emptyset$. Consequently $f$ is irreducible in $R$ and $\text{redset}(f) = \emptyset$. In particular $|\text{redset}(f)| = |\text{redset}(f)^*| = 0 < \infty$, and hence by the Composite Pencil Theorem we see that $k^\sharp$ is relatively algebraically closed in $L^\sharp$. Now $\text{defo}(f) = X_1^m X_2^2$, and hence the points at infinity of the curve $f = 0$ are $(X_0, X_1, X_2) = (0, 1, 0)$ and $(X_0, X_1, X_2) = (0, 0, 1)$. Homogenizing $f$ we obtain

$$(X_1 - a_1X_0)\dots(X_1 - a_mX_0)X_2^2 + X_2X_0^{m+1} + X_1^{m+1}X_0 - zX_0^{m+2}.$$

Putting $X_1 = 1$ we get

$$f_1 = (1 - a_1X_0)\dots(1 - a_mX_0)X_2^2 + X_2X_0^{m+1} + X_1^{m+1}X_0 - zX_0^{m+2}$$

and hence $(0, 1, 0)$ is a simple point. Putting $X_2 = 1$ we get

$$f_2 = (X_1 - a_1X_0)\dots(X_1 - a_mX_0) + X_0 + X_1^{m+1}X_0 - zX_0^{m+2}$$

and hence $(0, 0, 1)$ is an $m$-fold point with $m$ distinct tangents. Therefore $\tau(f) = m + 1$, and hence by taking $(z + Z, k^\sharp)$ for $(z, k)$ we get $\tau(f^\sharp) = m + 1$.

Example 5. Assume $m \geq 2$. Let $f = X_2^m a(X_1/X_2) + z$ with $z \in k^\times$. Then $f$ is irreducible in $R$ and $\text{redset}(f) = \text{redsset}(f)^* = \{z\}$. Moreover, $k^\sharp$ is relatively algebraically closed in $L^\sharp$ and we have $|\text{redset}(f)| = |\text{redset}(f)^*| = 1$ with $\tau(f) = \tau(f^\sharp) = m$.

**Proof**. Obviously $z \in \text{redset}(f)$. Moreover, for any $c \neq z$, homogenizing $f - c$ and then putting $X_2 = 1$ we get the polynomial $f_2 = a(X_1) + (z - c)X_0^m$ which is clearly irreducible and hence so in $f - c$. Also obviously $f - c = 0$ has $m$ places at infinity.

## 6. More General Pencils

Let us now study more general pencils of hypersurfaces $f-cw=0$ with $c$ varying over $k$, where $w \in R^\times$ is such that $\gcd(f,w)=1$. Let $R^\flat = R^\sharp = k(Z)[X_1,\dots,X_m]$.

The assumption $\gcd(f,w)=1$ says that the pencil $(f-cw)_{c\in k}$ is without fixed components. However, it may have base loci of dimension $< n-1$, i.e., there may be primes of height $>1$ in $R$ which contain $f$ and $w$ both, and nothing can be said about the singularities of $f-cw=0$ at a base point. To indicate these primes, for any $g_1,\dots,g_m$ in $R$ we let $\mathcal{V}(g_1,\dots,g_m)$ denote the variety defined by $g_1=\dots=g_m=0$, i.e. we put

$\mathcal{V}(g_1,\dots,g_m) = \{P \in \operatorname{spec}(R) \text{ with } (g_1,\dots,g_m)R \subset P\}$;

now $\mathcal{V}(f,w)$ is the set of all base loci of our pencil; for $n=2$ the set $\mathcal{V}(f,w)$ is finite and its members are the base points of the pencil. Likewise for any $g_1,\dots,g_m$ in $R^*$ we put

$\mathcal{V}(g_1,\dots,g_m)^* = \{P \in \operatorname{spec}(R^*) \text{ with } (g_1,\dots,g_m)R^* \subset P\}$,

and for any $g_1,\dots,g_m$ in $R^\flat$ we put

$\mathcal{V}(g_1,\dots,g_m)^\flat = \{P \in \operatorname{spec}(R^\flat) \text{ with } (g_1,\dots,g_m)R^\flat \subset P\}$.

Let $\operatorname{singset}(f,w) = \{c \in k : R_P/((f-cw)R_P)$ is nonregular for some $P \in \operatorname{spec}(R)\setminus\mathcal{V}(f,w)$ with $f-cw \in P\}$, and let $\operatorname{singset}(f,w)^* = \{c \in k^* : R^*_P/((f-cw)R^*_P)$ is nonregular for some $P \in \operatorname{spec}(R^*)\setminus\mathcal{V}(f,w)^*$ with $f-cw \in P\}$. Also let $\operatorname{redset}(f,w) = \{c \in k :$ $f-cw = gh$ for some $g,h$ in $R\setminus k\}$, and let $\operatorname{redset}(f,w)^* = \{c \in k^* :$ $f-cw = gh$ for some $g,h$ in $R^*\setminus k^*\}$. Finally let $\operatorname{multset}(f,w)^* = \{c \in k^* : f-cw = gh^2$ for some $g \in (R^*)^\times$ and $h \in R^*\setminus k^*\}$, and let $\operatorname{primset}(f,w) = \{c \in k : f-cw = gh^\mu$ for some $g \in k^\times$ and $h \in R\setminus k$ with integer $\mu > 1\}$.

Sometimes we need to projectivise the pencil $f-cw=0$ by allowing $c$ to vary over $k\cup\{\infty\}$ and declaring that $f-\infty w$ means $w$. To take care of this we put $\operatorname{redset}(f,w)_+ = \operatorname{redset}(f,w)$ or $\operatorname{redset}(f,w)\cup\{\infty\}$ according as we cannot or can write $w=gh$ with $g,h$ in $R\setminus k$. Likewise we put $\operatorname{redset}(f,w)^*_+ = \operatorname{redset}(f,w)^*$ or $\operatorname{redset}(f,w)^*\cup\{\infty\}$ according as we cannot or can write $w=gh$ with $g,h$ in $R^*\setminus k^*$. Similarly we put $\operatorname{multset}(f,w)^*_+ = \operatorname{multset}(f,w)^*$ or $\operatorname{multset}(f,w)^*\cup\{\infty\}$ according as we cannot or can write $w = gh^2$ for some $g \in (R^*)^\times$ and $h \in R^*\setminus k$. Finally we put $\operatorname{primset}(f,w)_+ = \operatorname{primset}(f,w)$ or $\operatorname{primset}(f,w)\cup\{\infty\}$ according as we cannot or can write $w = gh^\mu$ with $g \in k^\times$ and $h \in R\setminus k$ with integer $\mu>1$. Let

$$d = \max(\deg(f), \deg(w))$$

and observe that, without assuming $\deg(f) > 0$ but assuming $d > 0$, the condition $\gcd(f, w) = 1$ implies that

$$(') \qquad \deg(f - cw) < d \text{ for at most one } c \in k^* \cup \{\infty\}$$

and

$$('') \qquad \gcd(f - c_1 w, f - c_2 w) = 1 \text{ for all } c_1 \neq c_2 \text{ in } k^* \cup \{\infty\}.$$

By the generic member of the pencil $(f - cw)_{c\in k}$ we mean the hypersurface $(f, w)^\flat = 0$ with $(f, w)^\flat = f - Zw \in R^\flat$. Let $\operatorname{singset}(f, w)^\flat = \{c \in k(Z) : R^\flat_P/(f - (Z + c)w)R^\flat_P)$ is nonregular for some $P \in \operatorname{spec}(R^\flat) \setminus \mathcal{V}(f, w)^\flat$ with $f - (Z + c)w \in P\}$. Let $\operatorname{redset}(f, w)^\flat = \{c \in k(Z)$ such that $f - (Z + c)w = gh$ for some in $g, h$ in $R^\flat \setminus k(Z)\}$.

By Gauss Lemma $(f, w)^\flat$ is irreducible in $R^\flat$, i.e., $0 \notin \operatorname{redset}(f, w)^\flat$. Let $\phi^\flat : R^\flat \to R^\flat/((f, w)^\flat R^\flat)$ be the residue class epimorphism. Clearly $R \cap \ker(\phi^\flat) = \{0\}$ and $\phi^\flat(f) = \phi^\flat(Z)\phi^\flat(w)$, and hence there exists a unique isomorphism $\psi^\flat : k(X_1, \ldots, X_n) \to \mathrm{QF}(R^\flat/((f, w)^\flat R^\flat))$ such that for all $r \in R$ we have $\psi^\flat(r) = \phi^\flat(r)$. Thus the triple $\phi^\flat(k(Z)) \subset \phi^\flat(R^\flat) \subset \mathrm{QF}(\phi^\flat(R^\flat))$ is isomorphic to the triple

$$k^\flat = k(f/w) \subset A^\flat = k(f/w)[X_1, \ldots, X_n] \subset L^\flat = k(X_1, \ldots, X_n)$$

and hence we may regard the above three displayed sets as the ground field, the affine coordinate ring, and the function field of $(f, w)^\flat = 0$.

Let us reiterate that $\operatorname{singset}(f, w)$, $\operatorname{singset}(f, w)^*$, and $\operatorname{singset}(f, w)^\flat$ denote singularities outside the base points. For any $P \in \operatorname{spec}(A^\flat)$ with $f - (Z + c)w \in P$ and $(f, w)A^\flat \not\subset P$, upon letting $Q = R \cap P$, we see that the local ring $A^\flat_P$ is regular because it equals the localization of the regular local ring $R_Q$ at its multiplicative subset $k[f/w]^\times$ or $k[w/f]^\times$ depending on whether $w \notin P$ or $f \notin P$. Thus we have the:

General Generic Singset Theorem. $0 \notin \operatorname{singset}(f, w)^\flat$.

In view of the above isomorphism of triples, by the General Redset Theorem which we shall state and prove in a moment, we get the:

General Generic Redset Theorem. If $k^\flat$ is relatively algebraically closed in $L^\flat$, then $\operatorname{redset}(f, w)^\flat$ is finite.

General Redset Theorem. If $f$ is irreducible in $R$ with $\deg(f) \geq \deg(w)$ and $k$ is relatively algebraically closed in $L$, then $\mathrm{redset}(f, w)$ is finite.

**Proof**. By the Lemma we can find a finite number of DVRs $V_1, \dots, V_t$ of $L/k$ such that $A[1/\phi(w)] \cap V_1 \cap \cdots \cap V_t = k$. For every $z \in L^\times$ let $W_i(z) = \mathrm{ord}_{V_i}(z)$, and let $W : L^\times \to \mathbb{Z}^t$ be the map given by putting $W(z) = (W_1(z), \dots, W_t(z))$. Let $G$ be the set of all $g \in R \setminus k$ such that $gh = f - cw$ for some $h \in R \setminus k$ and $c \in k^\times$. Since the degree of $g$ is clearly smaller than the degree of $f$, the set $G$ is contained in a finite dimensional $k$-vector-subspace of $R$. Therefore for every $i \in \{1, \dots, t\}$, the set $W_i(\phi(g))_{g\in G}$ is bounded from below. Since $h$ also belongs to $G$ and clearly $W_i(\phi(g)) = W_i(\phi(w)) - W_i(\phi(h))$ with integer $W_i(\phi(w))$ which depends only on $i$ and $w$, it follows that the set $W_i(\phi(g))_{g\in G}$ is also bounded from above. Since, for $1 \leq i \leq t$, the set $W_i(\phi(g))_{g\in G}$ is bounded from both sides, it follows that $W(\phi(G))$ is a finite set. Also clearly $\phi(G) \subset U(A[1/\phi(w)])$. Let $g_1h_1 = f - c_1w$ and $g_2h_2 = f - c_2w$ with $g_1, h_1, g_2, h_2$ in $R \setminus k$ and $c_1, c_2$ in $k^\times$ be such that $W(\phi(g_1)) = W(\phi(g_2))$. Then $\phi(g_1)/\phi(g_2) \in A[1/\phi(w)] \cap V_1 \cap \cdots \cap V_t = k$ and hence $\phi(g_2) = c\phi(g_1)$ for some $c \in k^\times$. Consequently $g_2 - cg_1$ is divisible by $f$ in $R$ and hence, because $\deg(g_2 - cg_1) < \deg(f)$, we must have $g_2 = cg_1$. Therefore, by subtracting the equation $g_2h_2 = f - c_2w$ from the equation $g_1h_1 = f - c_1w$ we get $(c_2 - c_1)w = g_1h_1 - g_2h_2 = g_1(h_1 - ch_2)$ which implies that $(c_2 - c_1)w$ is divisible in $R$ by the positive degree polynomial $g_1$ with $\gcd(w, g_1) = 1$. Consequently we must have $c_2 = c_1$. Therefore, because the set $W(\phi(G))$ is finite, we conclude that $\mathrm{redset}(f, w)$ is finite.

The following version of Lüroth's Theorem was first proved by Igusa [**Igu**], and then it was deduced by Nagata [**Na2**] as a consequence of Abhyankar's paper [**A01**].

General Refined Lüroth Theorem. Assume that $k^\flat$ is not relatively algebraically closed in $L^\flat$. Let $\widehat{k^\flat}$ be the algebraic closure of $k^\flat$ in $L^\flat$, and let $\nu = [\widehat{k^\flat} : k^\flat]$. Then $\nu$ is an integer with $\nu > 1$, and there exist $\Gamma \in k[Z]^\times$, $\Omega \in k[Z]^\times$, $\widehat{f} \in R \setminus k$, and $\widehat{w} \in R \setminus k$, with $\gcd(\Gamma, \Omega) = 1$, $\max(\deg(\Gamma), \deg(\Omega)) = \nu$, $\gcd(\widehat{f}, \widehat{w}) = 1$, $\max(\deg(\widehat{f}), \deg(\widehat{w})) > 0$, and $\deg(\widehat{w}) = \min\{\deg(\widehat{f} - \widehat{c}\widehat{w}) : \widehat{c} \in k^* \cup \infty\}$, such that $\widehat{k^\flat} = k(\widehat{f}/\widehat{w})$ and $f/w = \Gamma(\widehat{f}/\widehat{w})/\Omega(\widehat{f}/\widehat{w})$.

Remark 6. In the above situation, every member of the pencil $(f - cw)_{c\in k^*}$ consists of $\nu$ members (counted properly) of the pencil $(\widehat{f} - \widehat{c}\widehat{w})_{\widehat{c}\in k^*}$, and so we say that **the pencil** $(f - cw)_{c\in k^*}$ **is composite with the pencil** $(\widehat{f} - \widehat{c}\widehat{w})_{\widehat{c}\in k^*}$. In greater detail, upon letting $\Lambda(Y, Z) = \Gamma(Z) - Y\Omega(Z)$ we get irreducible $\Lambda(Y, Z)$ in $k[Y, Z]$ of $Y$-degree 1 and $Z$-degree $\nu$ such that $\Lambda(f/w, \widehat{f}/\widehat{w}) = 0$. For most $c \in k^*$ we have $\Lambda(c, Z) = \widehat{c}_0 \prod_{1\le i\le\nu}(Z - \widehat{c}_i)$ where $\widehat{c}_0, \widehat{c}_1, \dots, \widehat{c}_\nu$ in $k^*$ with $\widehat{c}_0 \neq 0$ and so the "object" $f/w = c$ is the union of the "objects" $\widehat{f}/\widehat{w} = \widehat{c}_1, \dots, \widehat{f}/\widehat{w} = \widehat{c}_\nu$. Alternatively, by letting $\gamma(Z, T) = T^\nu\Gamma(Z/T)$, $\omega(Z, T) = T^\nu\Omega(Z/T)$, and $\lambda(Y, Z, T) = T^\nu\Lambda(Y, Z/T)$, we get polynomials which are homogeneous of degree $\nu$ in $(Z, T)$, and for which we have $f/w = \gamma(\widehat{f}, \widehat{w})/\omega(\widehat{f}, \widehat{w})$ and $\lambda(Y, Z, T)/\omega(Z, T) = (\gamma(Z, T)/\omega(Z, T)) - Y$. Now $\lambda(c, Z, T) = \widehat{c}_0 \prod_{1\le i\le\nu}(Z - \widehat{c}_iT)$, and so the hypersurface $f - cw = 0$ is the union of the hypersurfaces $\widehat{f} - \widehat{c}_1\widehat{w} = 0, \dots, \widehat{f} - \widehat{c}_\nu\widehat{w} = 0$. In the above phrase "most $c \in k^*$" we were referring to the tacit assumption that $\lambda(c, Z, T)$ is not divisible by $T$; in the contrary case, if $T^\mu$ is the highest power of $T$ which divides $\lambda(c, Z, T)$ then $\mu$ of the roots, say $c_1, \dots, c_\mu$, have "gone to infinity" and the hypersurface $f - cw = 0$ is composed of the hypersurfaces $\widehat{w} = 0, \dots, \widehat{w} = 0, \widehat{f} - \widehat{c}_{\mu+1}\widehat{w} = 0, \dots, \widehat{f} - \widehat{c}_\nu\widehat{w} = 0$ with $\widehat{w} = 0$ occurring $\mu$ times. By factoring $\omega(Z, T)$ we get $\omega(Z, T) = \widetilde{c}_0T^\epsilon \prod_{\epsilon+1\le i\le\nu}(Z - \widetilde{c}_iT)$ where $\widetilde{c}_0, \widetilde{c}_{\epsilon+1} \dots, \widetilde{c}_\nu$ are elements in $k^*$ with $\widetilde{c}_0 \neq 0$. The assumption $\gcd(\Gamma, \Lambda) = 1$ implies that $\widetilde{c}_i \neq \widehat{c}_j$ for $\epsilon + 1 \le i \neq \nu$ and $\mu + 1 \le j \le \nu$, and if $\epsilon > 0$ then the phrase "most $c \in k^*$" can be changed to the phrase "all $c \in k^*$." In other words, for two different members of the "old pencil" $(f - cw)_{c\in k^*\cup\{\infty\}}$ the corresponding $\nu$-tuples in the "new pencil" $(\widehat{f} - \widehat{c}\widehat{w})_{\widehat{c}\in k^*\cup\{\infty\}}$ are disjoint. Upon multiplying the polynomials $\Gamma$ and $\Omega$ by suitable constants it can be arranged that $f = \gamma(\widehat{f}, \widehat{w})$ and $w = \omega(\widehat{f}, \widehat{w})$; now for all $c \in k^*$ we have $f - cw = \lambda(c, \widehat{f}, \widehat{w})$. The picture is completed by noting that every member of the projectivised pencil $(f - cw)_{c\in k^*\cup\{\infty\}}$ consists of $\nu$ members of the projectivised pencil $(\widehat{f} - \widehat{c}\widehat{w})_{\widehat{c}\in k^*\cup\{\infty\}}$, counted with multiplicities.

The above correspondence $(\widehat{c}_1, \dots, \widehat{c}_\nu) \mapsto c$ can be elucidated by noting that the $(Z - \widehat{c}_1)$-adic, $\dots, (Z - \widehat{c}_\nu)$-adic valuations of $k^*(Z)$ are the extensions of the $(Y - c)$-adic valuation of $k^*(Y)$ with $Y = \Gamma(Z)/\Omega(Z)$. Here the $(Z - \infty)$-adic and the $(Y - \infty)$-adic valuations are the negative degree functions, and the multiplicity of the root $\widehat{c}_i$

is the ramification exponent of the $(Z-\widehat{c}_i)$-adic valuation. As said above, since $\gcd(\Gamma,\Omega)=1$, for any two different $c$'s in $k^*\cup\{\infty\}$ the two corresponding sets $(\widehat{c}_1,\dots,\widehat{c}_\nu)$ are disjoint. By $(')$ we see that $\deg(\widehat{f}-\widehat{c}\widehat{w})<\widehat{d}$ for at most one $\widehat{c}\in k^*\cup\{\infty\}$ and this can equal a $\widehat{c}_i$ for at most one $c\in k^*\cup\{\infty\}$.

Since $[k^*(Z):k^*(Y)]=\nu>1$, if the characteristic of $k$ is 0 then, for some $c\in k^*$, there exits $\widehat{c}\in k^*$ such that $(Z-\widehat{c})$-adic valuation lies above the $(Y-c)$-adic valuation and has ramification exponent greater than one, and hence $f-cw$ has a nonconstant multiple factor in $R$, and therefore $\mathrm{multset}(f,w)^*\neq\emptyset$. To find $c$ explicitly, upon letting $\Theta(Y)$ be the $Z$-resultant of $\Lambda(Y,Z)$ and its $Z$-derivative $\lambda_Z(Y,Z)$ we can show that $\Theta(Y)$ has a root in $k^*$ which is not a root of the coefficient of $Z^\nu$ in $\Lambda(Y,Z)$; now for $c$ we can take such a root.

Without assuming any condition on the pair $(k^\flat,L^\flat)$, if $c\in k^*$ is such that $f-cw=gh^2$ with $g\in R^*\setminus\{0\}$ and $h\in R^*\setminus k^*$ then $(f/w)-c=(g/w)h^2$ and hence taking the partials of both sides relative to $X_i$ and multiplying by $w^2$ we get $fw_{X_i}-wf_{X_i}=w^2[(g/w)_{X_i}h^2+2hh_{X_i}(g/w)]=h[(gw_{X_i}-wg_{X_i})h+2gwh_{X_i}]$; therefore $\mathrm{multset}(f,w)^*\neq\emptyset\Rightarrow\gcd(fw_{X_1}-wf_{X_1},\dots,fw_{X_n}-wf_{X_n})\neq 1$.

Again without assuming any condition on the pair $(k^\flat,L^\flat)$ we see that $c\mapsto c-Z$ gives an injection of $\mathrm{redset}(f,w)$ into $\mathrm{redset}(f,w)^\flat$, and hence $|\mathrm{redset}(f,w)|\leq|\mathrm{redset}(f,w)^\flat|$.

Thus, in view of the General Generic Redset Theorem, the General Refined Lüroth Theorem, and the above two observations $(')$ and $('')$, we get the:

General Composite Pencil Theorem. We have the following:

(I) $\gcd(fw_{X_1}-wf_{X_1},\dots,fw_{X_n}-wf_{X_n})=1\Rightarrow\mathrm{multset}(f,w)^*=\emptyset$.

(II) characteristic of $k$ is 0 and $\mathrm{multset}(f,w)^*=\emptyset\Rightarrow k^\flat$ is relatively algebraically closed in $L^\flat$.

(III) $k^\flat$ is relatively algebraically closed in $L^\flat\Rightarrow\mathrm{redset}(f,w)$ and $\mathrm{redset}(f,w)^\flat$ are finite with $|\mathrm{redset}(f,w)|\leq|\mathrm{redset}(f,w)^\flat|$.

(IV) $k^\flat$ is not relatively algebraically closed in $L^\flat\Rightarrow$
$|(k^*\cup\{\infty\})\setminus\mathrm{redset}(f,w)^*_+|\leq 1$ and $\deg(f-cw)<d$ for any
$c\in(k^*\cup\{\infty\})\setminus\mathrm{redset}(f,w)^*_+$.

By slightly changing the proof of the Singset Theorem we shall now prove the:

General Singset Theorem. If $k$ is of characteristic zero then singset$(f, w)$ is finite.

**Proof**. Let $I$ be the ideal in $R^*$ generated by the $n$ elements $fw_{X_i} - wf_{X_i}$ with $1 \le i \le n$, and note that these elements when divided by $w^2$ give us the partials $(f/w)_{X_i}$. For any $P \in \mathrm{spec}(R^*)$ with $I \subset P$ and $w \notin P$, consider the residue class map $\Phi_P : R^* \to R^*/P$. Since all the partials of $f/w$ when multiplied by $w^2$ belong to $P$, it follows that $D(\Phi_P(f)/\Phi_P(w)) = 0$ for every $\Phi_P(k^*)$-derivation of $\mathrm{QF}(R^*/P)$. Therefore, since $k^*$ is of characteristic zero, we have $\Phi_P(f)/\Phi_P(w) = \Phi_P(\kappa(P))$ for a unique $\kappa(P) \in k^*$. For any $c \in k^*$ we clearly have: $f - cw \in P \Leftrightarrow c = \kappa(P)$. Let $P_1, \dots, P_s$ be the minimal primes of $I$ in $R^*$ which do not contain $w$. Then for any $c \in k^* \setminus \{\kappa(P_1), \dots, \kappa(P_s)\}$ and $Q \in \mathrm{spec}(R^*)$ with $f - cw \in Q$ and $(f, w)R^* \not\subset Q$, we must have $I \not\subset Q$. Since $k$ is of characteristic zero, it follows that singset$(f, w) \subset \{\kappa(P_1), \dots, \kappa(P_s)\}$, and hence singset$(f, w)$ is finite.

Remark 7. To explain the ideas behind the proofs of the Singset and General Singset Theorems, suppose $k$ to be of characteristic zero. If the equations $f_{X_1} = \cdots = f_{X_n} = 0$ have only a finite number of common solutions $(a_{i1}, \dots, a_{in})_{1 \le i \le s}$ in $(k^*)^n$ then upon letting $c_i = f(a_{i1}, \dots, a_{in})$ we clearly get singset$(f) = \{c_1, \dots, c_s\}$ provided $k = k^*$ and hence singset$(f) \subset \{c_1, \dots, c_s\}$ without that proviso. Without assuming the common solutions to be finite, this is generalized by letting $P_1, \dots, P_s$ to be the minimal primes of $(f_{X_1}, \dots, f_{X_n})R^*$, i.e., upon letting $\mathcal{V}(P_1)^*, \dots, \mathcal{V}(P_s)^*$ to be the irreducible components of the affine variety $f_{X_1} = \cdots = f_{X_n} = 0$, then showing that $f$ is constant on $\mathcal{V}(P_i)^*$ and letting $c_i$ be that constant value. To apply this to get the singset of the more general pencil $f - cw$ outside its base points we take only those irreducible components of the variety $fw_{X_1} - wf_{X_1} = \cdots = fw_{X_n} - wf_{X_n} = 0$ which are not contained in the hypersurface $w = 0$. Note that, since in these two theorems the characteristic is assumed to be zero, the ideal $I$ is a nonzero ideal. Observe that here the characteristic assumption is essential as is shown by taking $f = X_1^p + \cdots + X_n^p$ with characteristic $p > 0$ and $k = k^*$ and noting that then singset$(f) = k$. However, no assumption on the characteristic is required in the Generic Singset Theorem and the General Generic Singset Theorem.

In the Mixed Primset Theorem we showed that if the special pencil $f-c$ is noncomposite, i.e., if $k^\sharp$ is relatively algebraically closed in $L^\sharp$, then primset$(f)$ is empty. However, if the general pencil $f-cw$ is noncomposite, i.e., if $k^\flat$ is relatively algebraically closed in $L^\flat$, then primset$(f,w)$ may be nonempty. More precisely, we prove the:

General Mixed Primset Theorem Assume that $k$ is perfect and $k^\flat$ is relatively algebraically closed in $L^\flat$. For any $c \in k \cup \{\infty\}$ let $d(c) = \deg(f-cw)$. For any $c \in \text{primset}(f,w)_+$ by definition we have $f - cw = g(c)h(c)^{\mu(c)}$ with $g(c) \in k^\times$, $h(c) \in R \setminus k$, and integer $\mu(c) > 1$; we shall assume that the triple $(g(c), h(c), \mu(c))$ is so chosen that $\mu(c)$ is maximal. Call $c' \in k \cup \{\infty\}$ inseparable if $(f - c'w)_{X_i} = 0$ for $1 \le i \le n$; otherwise call $c'$ separable. Then we have the following:

(I) primset$(f,w)_+$ has at most one inseparable member.

(II) If primset$(f,w)_+$ has an inseparable $c'$ then it has at most one $c \neq c'$ with $d(c) = d$. If primset$(f,w)_+$ has an inseparable member then $|\text{primset}(f,w)_+| \le 3$.

(III) primset$(f,w)_+$ has at most three members $c$ with $d(c) = d$.

(IV) If primset$(f,w)_+$ has three distinct members $c_i$ with $d(c_i) = d$ for $1 \le i \le 3$ then $\{\mu(c_1), \mu(c_2), \mu(c_3)\} = \{2,3,5\}$.

(V) $|\text{primset}(f,w)| \le |\text{primset}(f,w)_+| \le 4$.

**Proof**. To prove (I) and (II), assuming primset$(f,w)_+$ to have an inseparable member, and replacing $f$ and $w$ by suitable $k$-linear combinations of them, we may suppose $\infty$ to be that inseparable member. Note that now $k$ must be of characteristic $p > 0$ and $w = (w')^p$ with $w' \in R \setminus k$. If primset$(f,w)_+$ has another inseparable member then replacing it by a suitable $k$-linear combination of it and $w$ we may suppose the other inseparable member to be 0; now $f = (f')^p$ with $f' \in R \setminus k$ and $k(f/w) \subset k(f'/w')$ with $[k(f'/w') : k(f/w)] = p$ contradicting the noncompositness of our pencil. This proves (I). In particular, we must have $f_{X_j} \neq 0$ for some $j \in \{1, \dots, n\}$. If primset$(f,w)_+$ has two separable members $c_1 \neq c_2$ with $d(c_1) = d = d(c_2)$ then for $1 \le i \le 2$ we get $f_{X_j} = (f - c_i w)_{X_j} = g(c_i)h(c_i)_{X_j} h(c_i)^{\mu(c_i)-1}$ and hence, because clearly $\gcd(h(c_1)^{\mu(c_1)-1}, h(c_2)^{\mu(c_2)-1}) = 1$, we see that

$h(c_1)^{\mu(c_1)-1}h(c_2)^{\mu(c_2)-1}$ divides $f_{X_j}$ in $R$, and therefore

$$\begin{aligned} d-1 &\geq \deg(f_{X_j}) \\ &\geq \deg(h(c_1)^{\mu(c_1)-1}) + \deg(h(c_2)^{\mu(c_2)-1}) \\ &= d - (d/\mu(c_1)) + d - (d/\mu(c_2)) \end{aligned}$$

and dividing the first and the last expressions by $d$ and rearranging terms suitably we obtain

$$(1/\mu(c_1)) + (1/\mu(c_2)) \geq 1 + (1/d).$$

Since our pencil is noncomposite, we must have $\gcd(\mu(c_1), \mu(c_2)) = 1$, and hence the left hand side is at most $5/6$ which is less than the right hand side, giving a contradiction. This proves (II).

To prove (III) to (V), let if possible $c_i$ in $\text{primset}(f,w)_+$ with $d(c_i) = d$ for $1 \leq i \leq u$ be distinct members where $u = 3$ or $4$. If $(f/w)_{X_j} = 0$ for $1 \leq j \leq n$ then $k$ must be of characteristic $p > 0$ and we must have $f/w = (f'/w')^p$ for some $f', w'$ in $R^\times$ contradicting the noncompositness of our pencil. Therefore $(f/w)_{X_j} \neq 0$ for some $j \in \{1, \dots, n\}$. By the equation $f - c_i w = g(c_i)h(c_i)^{\mu(c_i)}$ we see that $f - c_i w$ and $(f - c_i w)_{X_j}$ are both divisible by $h(c_i)^{\mu(c_i)-1}$ in $R$. Clearly $w^2(f/w)_{X_j} = fw_{X_j} - wf_{X_j}$ and hence if $c_i = \infty$ then $w^2(f/w)_{X_j}$ is divisible by $h(c_i)^{\mu(c_i)-1}$ in $R$. If $c_i \in k$ then $w^2(f/w)_{X_j} =$ $w^2((f - c_i w)/w)_{X_j} = (f - c_i w)w_{X_j} - w(f - c_i w)_{X_j}$ and hence again $w^2(f/w)_{X_j}$ is divisible by $h(c_i)^{\mu(c_i)-1}$ in $R$. Clearly $\gcd(h(c_i), h(c_{i'}) =$ $1$ for all $i \neq i'$, and hence $w^2(f/w)_{X_j}$ is divisible by $\prod_{1 \leq i \leq u} h(c_i)^{\mu(c_i)-1}$ in $R$. Therefore

$$\begin{aligned} 2d-1 &\geq \deg(w^2(f/w)_{X_j}) \\ &\geq \sum_{1 \leq u} \deg(h(c_i)^{\mu(c_i)-1}) \\ &= \sum_{1 \leq i \leq u} [d - (d/\mu(c_i))] \end{aligned}$$

and dividing the first and the last expressions by $d$ and rearranging terms suitably we obtain

$$\sum_{1 \leq i \leq u} (1/\mu(c_1)) \geq (u-2) + (1/d).$$

Again as above, two different $\mu_i$ must have gcd 1, for otherwise, the system is easily seen to be composite. In case of $u = 4$, the LHS of the above inequality has each term at most $1/2$ and at most one term equal to $1/2$, leading to a maximum value less than $2 = (u-2)$, which is a contradiction in view of the RHS. This proves (III) and hence also (V). In case of $u = 3$, we see that $2, 3, 5$ are the only choices for mutually coprime $\mu(c_i)$ giving a sum bigger than $1 = (u-2)$ for the LHS, which proves (IV).

REMARK 8. Referring to (IV) above, for $n = 2$ and $k$ any field of characteristic zero, there is indeed an example of a pencil with three such members in the primset. Namely we use the fact that the surface $X^2 + Y^3 + Z^5 = 0$ is rational. The specific parametrization is in Klein's Lectures on the Icosahedron [**Kle**]. To quote it explicitly, from Article 13 of Chapter I of [**Kle**], let

$$f = H_1^2 \quad \text{and} \quad w = H_2^3 \quad \text{and} \quad v = 1728H_3^5$$

where

$$\begin{aligned} H_1 = X_1^{30} + X_2^{30} - 10005X_1^{10}X_2^{10}(X_1^{10} + X_2^{10}) \\ +522X_1^5X_2^5(X_1^{20} - X_2^{20}) \end{aligned}$$

and

$$H_2 = -[X_1^{20} + X_2^{20} + 494X_1^{10}X_2^{10}] + 228X_1^5X_2^5(X_1^{10} - X_2^{10})$$

and

$$H_3 = X_1X_2[X_1^{10} - X_2^{10} + 11X_1^5X_2^5].$$

Let us put

$$P = X_1^{10} - X_2^{10} \quad \text{and} \quad Q = X_1^5X_2^5.$$

Then

$$\begin{aligned} H_1 &= (X_1^{10} + X_2^{10})^3 - 10008X_1^{10}X_2^{10}(X_1^{10} + X_2^{10}) \\ &\qquad + 522X_1^5X_2^5(X_1^{20} - X_2^{20}) \\ &= (X_1^{10} + X_2^{10})(P^2 - 10004Q^2 + 522PQ) \end{aligned}$$

and hence

$$\begin{aligned} f &= (P^2 + 4Q^2)(P^2 - 10004Q^2 + 522PQ)^2 \\ &= (P^2 + 4Q^2)[P^4 + 1044P^3Q + ((522)^2 - 20008)P^2Q^2 \\ &\qquad - (1044)(10004)PQ^3 + (10004)^2Q^4] \\ &= P^6 + 1044P^5Q + [(522)^2 - 20004]P^4Q^2 - (1044)(10000)P^3Q^3 \\ &\quad + [(10004)^2 + (1044)^2 - 4(20008)]P^2Q^4 \\ &\quad - 4(1044)(10004)PQ^5 + (20008)^2Q^6. \end{aligned}$$

Also

$$H_2 = -P^2 - 496Q^2 + 228PQ$$

and hence

$$\begin{aligned}
w &= -[P^6 + 3(496)P^4Q^2 + 3(496)^2P^2Q^4 + (496)^3Q^6] \\
&\quad + 3(228)PQ[P^4 + 2(496)P^2Q^2 + (496)^2Q^4] \\
&\quad - 3(228)^2P^2Q^2[P^2 + 496Q^2] + (228)^3P^3Q^3 \\
&= -P^6 + 684P^5Q - 3[496 + (228)^2]P^4Q^2 \\
&\quad + [6(228)(496) + (228)^3]P^3Q^3 - 3[(496)^2 + (228)^2(496)]P^2Q^4 \\
&\quad + 3(228)(496)^2PQ^5 - (496)^3Q^6.
\end{aligned}$$

Moreover

$$H_3 = X_1X_2(P + 11Q)$$

and hence

$$\begin{aligned}
H_3^5 &= Q(P + 11Q)^5 \\
&= P^5Q + 5(11)P^4Q^2 + 10(11)^2P^3Q^3 + 10(11)^3P^2Q^4 \\
&\qquad + 5(11)^4PQ^5 + (11)^5Q^6.
\end{aligned}$$

Adding the coefficients of $P^6, P^5Q, \dots, Q^6$ in $f$ and $w$, and comparing the sums to the corresponding coefficients in $H_3^5$, we get $f + w = 1728H_3^5 = v$.

Now we shall prove the:

General Mixed Redset Theorem. Assume that $k$ is perfect and $k^\flat$ is relatively algebraically closed in $L^\flat$. Note that now by the Lemma there exists a finite number of DVRs $V_1, \dots, V_t$ of $L^\flat/k^\flat$ with $A^\flat[1/w] \cap V_1 \cap \dots \cap V_t = k^\flat$ and, for any such $t$, we have that $t$ is a positive integer and $U(A^\flat[1/w])/U(k^\flat)$ is a finitely generated free abelian group of rank $r \le t - 1$. By the General Mixed Primset Theorem, upon letting $\rho = |\text{primset}(f, w)|$, we get an integer $\rho$ with $0 \le \rho \le 4$. We claim that $|\text{redset}(f, w)| \le r + \rho$.

**Proof**. As in the proof of the Mixed Redset Theorem, suppose if possible that we have distinct elements $c_1, \dots, c_s$ in $k$ with integer $s > r + \rho$ such that

$$(1^*) \qquad f - c_iw = g_ih_i \quad \text{where} \quad g_i, h_i \text{ in } R \setminus k.$$

Let $\tau = s - \rho$. Then $\tau$ is an integer with $\tau > r$, and upon relabelling $c_1, \dots, c_s$ we can arrange matters so that

$$\text{(2*)} \qquad \gcd(g_i, h_i) = 1 \quad \text{for} \quad 1 \le i \le \tau.$$

Since $\tau > r$, there exist integers $a_1, \dots, a_\tau$ at least one of which is nonzero, say $a_\iota \neq 0$, such that

$$\text{(3*)} \qquad \prod_{1 \le i \le \tau} g_i^{a_i} \in U(k(f/w)).$$

As in the Second Argument of the proof of the Mixed Redset Theorem, by (3*) we get

$$\text{(4*)} \qquad \prod_{1 \le i \le \tau} g_i^{a_i} = \overline{u} \prod_{1 \le i \le \sigma} u_i(f/w)^{\alpha_i} \quad \text{with} \quad \overline{u} \in k^\times$$

where $\alpha_1, \dots, \alpha_\sigma$ are nonzero integers and $u_1(Z), \dots, u_\sigma(Z)$ are pairwise coprime monic members of $k[Z] \setminus k$. For any $u, v$ in $k[Z] \setminus k$ with $\gcd(u, v) = 1$ we have $w^{\deg(u)} u(f/w), w^{\deg(v)} v(f/w)$ in $R \setminus k$ with $\gcd(w^{\deg(u)} u(f/w),\ w^{\deg(v)} v(f/w)) = 1$. Therefore, by looking at divisibility by a nonconstant irreducible factor of $g_i$ in $R$, in view of (1*), (2*), and (4*), we see that for each $i \in \{1, \dots, \tau\}$ with $a_i \neq 0$ there is some $\theta(i) \in \{1, \dots, \sigma\}$ with $u_{\theta(i)}(Z) = Z - c_i$ and $\alpha_{\theta(i)} = a_i$. In particular we get $u_{\theta(\iota)}(Z) = Z - c_\iota$ and $\alpha_{\theta(\iota)} = a_\iota$. Again in view of (1*), (2*), and (4*), by looking at divisibility by a nonconstant irreducible factor of $h_\iota$ in $R$, we get a contradiction.

Remark 9. In the $n = 2$ case of the above theorem we can take $V_1, \dots, V_t$ to be valuation rings of the places at infinity of the curve $f^\flat = 0$ together with the valuation rings of its places at finite distance centered at points where it meets the curve $w = 0$.

Remark 10. To draw a deduction chart for the various incarnations of the Redset Theorem, as already observed, we have:

Lemma $\Rightarrow$ Redset Theorem $\Rightarrow$ Generic Redset Theorem
$\Rightarrow$ Composite Pencil Theorem

and

Lemma $\Rightarrow$ General Redset Theorem
$\Rightarrow$ General Generic Redset Theorem
$\Rightarrow$ General Composite Pencil Theorem

and we have:

$$\text{Lemma} \Rightarrow \text{Mixed Redset Theorem}$$

and

$$\text{Lemma} \Rightarrow \text{General Mixed Redset Theorem}$$

where A $\Rightarrow$ B means B can be deduced from A. Also clearly:

$$\text{Composite Pencil Theorem} \Rightarrow \text{Redset Theorem when } k = k^*$$

and

$$\text{General Composite Pencil Theorem} \Rightarrow \text{General Redset Theorem when } k = k^*.$$

REMARK 11. The condition $\deg(f) \geq \deg(w)$ in the General Redset Theorem is necessary as well as reasonable. It is reasonable because if $\deg(f) < \deg(w)$ and we pass to the projective $n$-space by homogenizing $f$ and $w$, then $f$ becomes reducible by acquiring the hyperplane at infinity counted $\deg(w) - \deg(f)$ times, and we have not yet obtained any control over the rest of the pencil. To see that it is necessary, take $w$ to be a polynomial in $f$ of degree $\nu > 1$ and note that then, for every $c \in k^*$, $f - cw$ is clearly a product of $\nu$ members of $R^* \setminus k^*$.

It is time to whet the appetite of the reader by raising a few questions.

QUESTION 1. As the bounds found in the Mixed Redset Theorem were sharpened in Remark 5 and Examples 1 to 5, can you do similar things to sharpen the bounds found in the General Mixed Redset Theorem?

QUESTION 2. In the Mixed Redset Theorem it was shown that $k^\sharp$ relatively algebraically closed in $L^\sharp \Rightarrow |\text{redset}(f)| \leq \text{rank}(U(A^\sharp)/U(k^\sharp))$. Can you similarly obtain a good bound in the context of the Redset Theorem by showing that, assuming $f$ to be irreducible in $R$, $k$ relatively algebraically closed in $L \Rightarrow |\text{redset}(f)| \leq \text{rank}(U(A)/U(k))$? Can you relate the conditions "$k$ relatively algebraically closed in $L$" and "$k^\sharp$ relatively algebraically closed in $L^\sharp$"? In case of $n = 2$, how far can you relate the numbers of places at infinity of the curves $f = 0$ and $f^\sharp = 0$?

QUESTION 3. Can you, in a manner similar to Question 2, relate the hypotheses and conclusions of the General Redset Theorem and the General Mixed Redset Theorem?

QUESTION 4. Let the unique component set of $(f, w)$ be defined by putting $\mathrm{uniset}(f, w) = \{c \in k : f - cw = gh^{\mu}$ for some $g \in k^{\times}$ and irreducible $h \in R \setminus k$ with integer $\mu > 1\}$, and let us put $\mathrm{uniset}(f, w)_{+} = \mathrm{uniset}(f, w)$ or $\mathrm{uniset}(f, w) \cup \{\infty\}$ according as we cannot or can write $w = gh^{\mu}$ with $g \in k^{\times}$ and irreducible $h \in R \setminus k$ with integer $\mu > 1$. Note that clearly $\mathrm{uniset}(f, w) \subset \mathrm{primset}(f, w)$ and $\mathrm{uniset}(f, w)_{+} \subset \mathrm{primset}(f, w)_{+}$. Also note that the Generic Mixed Redset Theorem and its proof remain valid if we let $\rho = |\mathrm{uniset}(f, w)|$. In this manner we get a possibly stronger from of the Generic Mixed Redset Theorem. Can you show that this is indeed a stronger from? In other words, can you show that $|\mathrm{uniset}(f, w)_{+}| \le 3$ or $|\mathrm{uniset}(f, w)_{+}| \le 2$? Hint: vis-a-vis Remark 8, study all parametrizations of the Klein surface. You may also redo Question 3 by putting in the above stronger form of the Generic Mixed Redset Theorem.

## 7. Complements of Hypersurfaces

We shall now give several criteria, i.e., necessary conditions, for the ring $R[1/f]$ to be isomorphic to the ring $R[1/f']$ where $f'$ is another member of $R \setminus k$. In geometric terms, this amounts to giving criteria for the complements of the hypersurfaces $f = 0$ and $f' = 0$ in the affine $n$-space to be biregularly equivalent to each other. Note that $R[1/f]$ may be viewed as the affine coordinate ring of the hypersurface $f(X_1, \dots, X_n)X_{n+1} - 1 = 0$ in the affine $(n + 1)$-space. Also note that if the hypersurfaces $f = 0$ and $f' = 0$ are automorphic, i.e., if there exists an automorphism of $R$ which sends $f$ to $f'$, then the rings $R[1/f]$ and $R[1/f']$ must be isomorphic, and hence our criteria also provide necessary conditions for the hypersurfaces $f = 0$ and $f' = 0$ to be automorphic.

For any $c \in k$ let us write $f - c = f_{c,0} f_{c,1}^{e(c,1)} \dots f_{c,q(c)}^{e(c,q(c))}$ with $f_{c,0} \in k^{\times}$, integers $1 \le e(c, 1) \le \dots \le e(c, q(c))$, and pairwise coprime irreducible elements $f_{c,1}, \dots, f_{c,q(c)}$ in $R \setminus k$. Note that then $c \in \mathrm{redset}(f) \Leftrightarrow e(c, 1) + \dots + e(c, q(c)) > 1$. Let $e(c)$ denote the sequence $(e(c, 1), \dots, e(c, q(c)))$. Let $\mathrm{refset}(f) =$ the refined redset of $f$ be defined to be the family of sequences $e(c)_{c \in \mathrm{redset}(f)}$.

Let $f'-c = f'_{c,0}{f'}_{c,1}^{e'(c,1)} \dots {f'}_{c,q'(c)}^{e'(c,q'(c))}$ and $e'(c) = (e'(c,1), \dots, e'(c,q'(c)))$ be the corresponding factorization and the corresponding sequence for $f'-c$. Let us write $\mathrm{refset}(f) = \mathrm{refset}(f')$ to mean that there exists a bijection $\theta : \mathrm{redset}(f) \to \mathrm{redset}(f')$ such that for all $c \in \mathrm{redset}(f)$ we have $e(c) = e'(\theta(c))$, i.e., $q(c) = q'(c)$ and $e(c,i) = e'(\theta(c),i)$ for $1 \le i \le q(c)$. Thus, geometrically speaking, $\mathrm{refset}(f) = \mathrm{refset}(f')$ means there is a multiplicities preserving bijection between the reducible members of the pencils $f - c = 0$ and $f' - c = 0$.

Let us write $R[1/f] \approx R[1/f']$ to mean that there exists a ring isomorphism of $R[1/f]$ onto $R[1/f']$, and let us write $R[1/f] \approx_k R[1/f']$ to mean that there exists a ring $k$-isomorphism of $R[1/f]$ onto $R[1/f']$. As our first criterion we want to show that (i) if $R[1/f] \approx_k R[1/f']$ then $q(0) = q'(0)$, and (ii) if $f$ and $f'$ are irreducible in $R$ and $R[1/f] \approx_k R[1/f']$ then $\mathrm{refset}(f) = \mathrm{refset}(f')$. But since it takes only a little more effort, we might as well prove the somewhat stronger:

Refset Criterion. (i) If $R[1/f] \approx R[1/f']$ then $q(0) = q'(0)$, and (ii) if $f$ and $f'$ are irreducible in $R$ and $R[1/f] \approx R[1/f']$ then $\mathrm{refset}(f) = \mathrm{refset}(f')$.

Since $R$ is a UFD and clearly $k = \{0\} \cup U(R)$ is relatively algebraically closed in the quotient field $k(X_1, \dots, X_n)$ of $R$, by applying the following Corollary of Lemma with $(k', A') = (k, R[1/f])$, this follows by taking $S = R$ and changing capital letters to lower case letter1/s in the even stronger:

UFD Criterion. Let $S$ be a UFD, and let $F, F'$ be in $S \setminus U(S)$. For any $C \in \{0\} \cup U(S)$ let us write $F - C = F_{C,0} F_{C,1}^{E(C,1)} \dots F_{C,Q(C)}^{E(C,Q(C))}$ with $F_{C,0} \in U(S)$, integers $1 \le E(C,1) \le \dots \le E(C,Q(C))$, and pairwise coprime irreducible elements $F_{C,1}, \dots, F_{C,Q(C)}$ in $S \setminus U(S)$. Let $F' - C = F'_{C,0}{F'}_{C,1}^{E'(C,1)} \dots {F'}_{C,Q'(C)}^{E'(C,Q'(C))}$ be the corresponding factorization of $F' - C$. Assume that there exists a ring isomorphism $\Delta : S[1/F] \to S[1/F']$ such that $\Delta(U(S)) = U(S)$. Then (i) $Q(0) = Q'(0)$. Moreover (ii) if $F$ and $F'$ are irreducible in $S$ then there exists a bijection $\Theta : U(S) \to U(S)$ such that for all $C \in U(S)$ we have $Q(C) = Q'(\Theta(C))$ and $E(C,i) = E'(\Theta(C),i)$ for $1 \le i \le Q(C)$.

**Proof**. Clearly $S[1/F]$ is UFD in which the irreducible nonunits are associates of the irreducible nonunits of $S$ except $F_{0,1}, \dots, F_{0,Q(0)}$

which have become units. Also clearly $U(S[1/F])/U(S)$ is a free abelian group of rank $Q(0)$ generated by $F_{0,1},\ldots,F_{0,Q(0)}$. Similarly $U(S[1/F'])/U(S)$ is a free abelian group of rank $Q'(0)$ generated by $F'_{0,1},\ldots,F'_{0,Q(0)}$. For any ring isomorphism $\Delta : S[1/F] \to S[1/F']$ we obviously have $\Delta(U(S[1/F]) = U(S[1/F'])$. Therefore, since we are assuming $\Delta(U(S)) = U(S)$, we get an induced isomorphism $U(S[1/F])/U(S) \to U(S[1/F'])/U(S)$ and hence $Q(0) = Q'(0)$. This proves (i). Now assume that $F$ and $F'$ are irreducible in $S$. Then because of the said induced isomorphism we must have either $\Delta(F) = \alpha F'$ with $\alpha \in U(S)$ or $\Delta(F) = \alpha/F'$ with $\alpha \in U(S)$. If $\Delta(F) = \alpha F'$ then, letting $\Theta : U(S) \to U(S)$ be the bijection given by $C \mapsto \Delta(C)/\alpha$, we have

$$\begin{aligned}\Delta(F_{C,0}) \prod_{1\le i\le Q(C)} \Delta(F_{C,i}^{E(C,i)}) &= \Delta(F - C) = \alpha(F' - \Theta(C)) \\ &= \alpha F'_{\Theta(C),0} \prod_{1\le i\le Q'(\Theta(C))} {F'}_{\Theta(C),i}^{E'(\Theta(C),i)}\end{aligned}$$

and hence we get $Q(C) = Q'(\Theta(C))$ and $E(C,i) = E'(\Theta(C),i)$ for $1 \le i \le Q(C)$. If $\Delta(F) = \alpha/F'$ then, letting $\Theta : U(S) \to U(S)$ be the bijection given by $C \mapsto \alpha/\Delta(C)$, we have

$$\begin{aligned}&\Delta(F_{C,0}) \prod_{1\le i\le Q(C)} \Delta(F_{C,i}^{E(C,i)}) \\ &= \Delta(F - C) = (-\Delta(C)/F')(F' - \Theta(C)) \\ &= (-\Delta(C)/F')F'_{\Theta(C),0} \prod_{1\le i\le Q'(\Theta(C))} {F'}_{\Theta(C),i}^{E'(\Theta(C),i)}\end{aligned}$$

and hence again we get $Q(C) = Q'(\Theta(C))$ and $E(C,i) = E'(\Theta(C),i)$ for $1 \le i \le Q(C)$. In the last two sentences we have used the obvious facts that the elements $\Delta(F_{C,1}),\ldots,\Delta(F_{C,Q(C)})$ are pairwise coprime irreducible nonunits in $S[1/F']$ and so are the elements $F'_{\Theta(C),1},\ldots,F'_{\Theta(C),Q'(C)}$; moreover, the elements $\Delta(F_{C,0})$, $\alpha F'_{\Theta(C),0}$, and $(-\Delta(C)/F')F'_{\Theta(C),0}$ are units in $S[1/F']$.

Corollary of Lemma. Let $A'$ be an affine domain over a field $k'$, let $k''$ be the algebraic closure of $k'$ in $L' = \mathrm{QF}(A')$, and let $\overline{k} = k'' \cap A'$. Then $\overline{k}$ can be located only using the ring structure of $A'$ by noting that it is only subfield of $A'$ which equals the intersection of $A'$ with a finite number of DVRs of $L'$.

**Proof**. Clearly $\overline{k}$ is a subfield of $A'$ and by the Lemma there exists a finite number of DVRs $V_1, \dots, V_t$ of $L'/k'$ with $A' \cap V_1 \cap \dots \cap V_t = \overline{k}$. Let $V'_1, \dots, V'_{t'}$ be any finite number of DVRs of $L'$ such that $A' \cap V'_1 \cap \dots \cap V'_{t'}$ is a subfield $\widetilde{k}$ of $A'$. Let $\widehat{k} = \overline{k} \cap \widetilde{k}$. We want to show that then $\widehat{k} = \overline{k}$. Clearly $\widehat{k}$ is a subfield of $\overline{k}$ and we have $\widehat{k} = \overline{k} \cap V'_1 \cap \dots \cap V'_{t'}$. For any $i$ the intersection $\widehat{k} \cap V'_i$ is either a DVR of $\overline{k}$ or equals $\overline{k}$. Let $i_1 < \dots < i_s$ be those values of $i$ for which the said intersection is a DVR, and let $W_j = \overline{k} \cap V'_{i_j}$. Now $W_1, \dots, W_s$ are a finite number of DVRs of the field $\overline{k}$ and their intersection is the field $\widehat{k}$. Now it is a well-known fact that if $W_1, \dots, W_s$ are any finite number of valuation rings with a common quotient field $\overline{k}$ then $\overline{k}$ is the quotient field of their intersection, and hence in our situation we must have $\widehat{k} = \overline{k}$. A proof of the said fact can be found in Theorem (11,11) on page 38 of [**Na1**]. Since our valuations are real, we can deduce the fact from the Approximation Theorem (see Theorem 18 on page 45 of volume II of [**ZaS**]) thus. Since the statement is obvious when $s = 0$, suppose $s > 0$. Let $M(W_i)$ denote the maximal ideal of $W_i$. By the said theorem we can find $u \in W_1 \setminus M(W_1)$ such that $u \in M(W_i)$ for all $i > 1$. Now $u \in W_1 \cap \dots \cap W_s$ and, since the valuations are real, for any $v \in W_1$ we can find an integer $m > 0$ such that $vu^m \in W_1 \cap \dots \cap W_s$ and hence $v \in \mathrm{QF}(W_1 \cap \dots \cap W_s)$. Therefore $\mathrm{QF}(W_1 \cap \dots \cap W_s) = \mathrm{QF}(W_1) = \overline{k}$.

Example 6. Take $f = X_1^{a_1} \dots X_n^{a_n} - 1$ and $f' = X_1^{a'_1} \dots X_n^{a'_n} - 1$ where $1 \le a_1 \le \dots \le a_n$ and $1 \le a'_1 \le \dots \le a'_n$ are integers such that $\gcd(a_1, a_j) = 1$ and $\gcd(a'_1, a'_{j'}) = 1$ for some $j \in \{2, \dots, n\}$ and $j' \in \{2, \dots, n\}$, and $a_i \neq a_{i'}$ for some $i \in \{1, \dots, n\}$ and $i' \in \{1, \dots, n\}$. Then clearly $\mathrm{redset}(f) = \{-1\}$ and $\mathrm{redset}(f') = \{-1\}$ with $\mathrm{refset}(f) = \{(a_1, \dots, a_n)\}$ and $\mathrm{refset}(f') = \{(a'_1, \dots, a'_n)\}$. Consequently $\mathrm{refset}(f) \neq \mathrm{refset}(f')$. It follows that for all $b, b'$ in $k^\times$ we have $\mathrm{refset}(bf) \neq \mathrm{refset}(b'f')$ and therefore by the Refset Criterion we get $R[1/(bf)] \not\approx R[1/(b'f')]$. Hence in particular no automorphism of $R$ can send the ideal $fR$ to the ideal $f'R$.

Now suppose $n = 2$. Let $\epsilon : R \to k[T, T^{-1}]$ be the $k$-homomorphism given by $(X_1, X_2) \mapsto (T^{-a_2}, T^{a_1})$. Then clearly $\ker(\epsilon) = fR$. Since $\gcd(a_1, a_2) = 1$, for some integers $b_1, b_2$ we have $b_1 a_1 - b_2 a_2 = 1$; by adding a positive multiple of $a_2$ to $b_1$ and adding the same multiple of $a_1$ to $b_2$ we can arrange $b_1$ to be positive and then automatically $b_2$ will also become positive; clearly $\epsilon(X_1^{b_2} X_2^{b_1}) = T$. Also we have $\beta_1 a_1 - \beta_2 a_2 = -1$ where $\beta_1 = -b_1$ and $\beta_2 = -b_2$; by adding a positive multiple of $a_2$ to $\beta_1$ and adding the same multiple of $a_1$ to $\beta_2$ we can arrange $\beta_1$ to be positive and then automatically $\beta_2$ will also become positive; clearly $\epsilon(X_1^{\beta_2}X_2^{\beta_1}) = T^{-1}$. Thus $\epsilon$ is surjective. Similarly we find a surjective $k$-homomorphism $\epsilon' : R \to k[T, T^{-1}]$ with $\ker(\epsilon') = f'R$. However, as shown above, $\epsilon$ and $\epsilon'$ do not differ from each other by an automorphism of $R$. Thus $f = 0$ and $f' = 0$ are "hyperbolas" which are not automorphic to each other.

Next we come to the:

Nonruled Criterion. Recall that an irreducible $\overline{f} \in R \setminus k$ is said to be ruled if, after identifying $k$ with a subfield of $R/(\overline{f}R)$, there exits a subfield $\widetilde{L}$ of $\overline{L} = \mathrm{QF}(R/(\overline{f}))$ such that $k \subset \widetilde{L} \subset \widetilde{L}(t) = \overline{L}$ where $t$ is transcendental over $\widetilde{L}$. Let us relabel $f_{0,1}, \dots, f_{0,q(0)}$ so that $f_{0,i}$ is nonruled or ruled according as $1 \le i \le r$ or $r+1 \le i \le q(0)$, and let us relabel $f'_{0,1}, \dots, f_{0,q'(0)}$ so that $f'_{0,i}$ is nonruled or ruled according as $1 \le i \le r'$ or $r'+1 \le i \le q'(0)$; note that now the exponent sequences $e(0,1), \dots, e(0, q(0))$ and $e'(0,1), \dots, e'(0, q'(0))$ need not be nondecreasing. Assume that $R[1/f] \approx_k R[1/f']$. Then $r = r'$ and, after relabelling $f_{0,1}, \dots, f_{0,r}$ suitably, we have $\mathrm{QF}(R/(f_{0,i}R)) \approx_k \mathrm{QF}(R/(f'_{0,i}R))$ for $1 \le i \le r$.

**Proof**. Recall that $\mathfrak{V}(R) = \{R_P : P \in \mathrm{spec}(R)\}$. For $0 \le i \le n$ let $R_i = k[X_0/X_i, \dots, X_n/X_i]$ where $X_0 = 1$, and let the projective $n$-space $\mathfrak{P}$ over $k$ be defined to be the nonsingular projective model $\cup_{0 \le i \le n}\mathfrak{V}(R_i)$ of $K = k(X_1, \dots, X_n)$ over $k$. For the language of models see Abhyankar's books [**A02, A06, A09**]. In particular note that for any valuation ring $V$ of $K/k$, i.e., valuation ring with quotient field $K$ and having $k$ as a subfield, the center of $V$ on $\mathfrak{P}$ is the unique member of $\mathfrak{P}$ dominated by $V$; we identify $k$ with a subfield of $V/M(V)$ where $M(V)$ is the maximal ideal of $V$, and we let $\mathrm{restrdeg}_k(V)$ denote the transcendence degree of $V/M(V)$ over $k$. By a prime divisor of $K/k$ we mean a DVR $V$ of $K/k$ such that $\mathrm{restrdeg}_k(V) = n-1$; we call the prime divisor ruled if there exits a subfield $\widetilde{L}$ of $V/M(V)$ such that $k \subset \widetilde{L} \subset \widetilde{L}(t) = V/M(V)$ where $t$ is transcendental over $\widetilde{L}$. Upon letting $V_i = R_{f_{0,i}R}$ and $V'_i = R_{f'_{0,i}R}$ we get nonruled prime divisors $V_1, \dots, V_r$ and $V'_1, \dots, V'_{r'}$ of $K/k$. Note that $\mathfrak{V}(R[1/f]) \subset \mathfrak{P}$ and $\mathfrak{V}(R[1/f']) \subset \mathfrak{P}$. By Proposition 3 on page 336 of [**A01**] we see that $V_1, \dots, V_r$ are exactly all the nonruled prime divisors of $K/k$ whose center on $\mathfrak{P}$ is not in $\mathfrak{V}(R[1/f])$, and

$V_1', \dots, V_{r'}'$ are exactly all the nonruled prime divisors of $K/k$ whose center on $\mathfrak{P}$ is not in $\mathfrak{V}(R[1/f'])$. Since $R[1/f] \approx_k R[1/f']$, it follows that $r = r'$ and, after relabelling $f_{0,1}, \dots, f_{0,r}$ suitably, we have $\mathrm{QF}(R/(f_{0,i}R)) \approx_k \mathrm{QF}(R/(f_{0,i}'R))$ for $1 \le i \le r$.

EXAMPLE 7. Take $f = X_1^a + \cdots + X_n^a - 1$ and $f' = X_1^{a'} + \cdots + X_n^{a'} - 1$ where $a$ and $a'$ are positive integers with $a > a'$ and $a > n > 1$; assume that $a$ and $a'$ are nondivisible by the characteristic of $k$ in case the latter is nonzero. The polynomials $f$ and $f'$ are clearly irreducible in $R$. It is expected that:

(7*) $f$ is nonruled and $\mathrm{QF}(R/(fR)) \not\approx_k \mathrm{QF}(R/(f'R))$.

Assuming (7*), by the Nonruled Criterion we would get $R[1/f] \not\approx_k R[1/f']$.

QUESTION 5. Can you prove the above statement (7*)? Hint for $n = 2$: show that the genus of the nonsingular plane curve $f = 0$ is $(a-1)(a-2)/2$ and that of $f' = 0$ is $(a'-1)(a'-2)/2$; see [**A06**]. Hint for $n > 2$: show that the arithmetic genus of the nonsingular hypersurface $f = 0$ is $(a-1)\dots(a-n)/n!$ and that of $f' = 0$ is $(a'-1)\dots(a'-n)/n!$; see [**A06**]. Further hint for $n > 2$: by using the domination part of the desingularization theory, show that the arithmetic genus of a nonsingular projective model is a birational invariant; see [**A09**].

To avoid the problem of showing that $f$ is nonruled in the above Question 5, let us establish the:

GENERIC CRITERION. Assume that $f$ and $f'$ are irreducible in $R$. Also assume that $R[1/f] \approx_k R[1/f']$. Then the generic members of the pencils $R^\sharp((f-Z)R^\sharp) \approx_{k(Z)} R^\sharp((f'-Z)R^\sharp)$ where we recall that $Z$ is an indeterminate over $R$ and $R^\sharp = k(Z)[X_1, \dots, X_n]$. Hence in particular the said generic members are birationally equivalent to each other, i.e., $\mathrm{QF}(R^\sharp((f-Z)R^\sharp)) \approx_{k(Z)} \mathrm{QF}(R^\sharp((f'-Z)R^\sharp))$.

**Proof**. Given a $k$-isomorphism $\delta : R[1/f] \to R[1/f']$, as in the proof of the UFD Criterion, for some $\alpha \in k^\times$ we have either $\delta(f) = \alpha f'$ or $\delta(f) = \alpha/f'$. Consequently we have either $\delta(k[f]^\times) = \delta(k[f']^\times)$ or $\delta(k[f]^\times) = \delta(k[1/f']^\times)$ respectively. Recall that $k^\sharp = k(f)$ and $A^\sharp = k(f)[X_1, \dots, X_n]$. Let $k'^\sharp = k(f')$ and $A'^\sharp = k(f')[X_1, \dots, X_n]$.

Clearly $A^\sharp$ is the localization of $R[1/f]$ at $k[f]^\sharp$, and $A'^\sharp$ is the localization of $R[1/f']$ at $k[f']^\sharp$ as well as at $k[1/f']^\times$. Therefore $\delta$ has a unique extension to an isomorphism $\delta^\sharp : A^\sharp \to A'^\sharp$. As noted before, the pair $(k(Z), R^\sharp((f-Z)R^\sharp))$ is isomorphic to the pair $(k^\sharp, A^\sharp)$, and similarly the pair $(k(Z), R^\sharp((f'-Z)R^\sharp))$ is isomorphic to the pair $(k'^\sharp, A'^\sharp)$. It follows that $R^\sharp((f-Z)R^\sharp) \approx_{k(Z)} R^\sharp((f'-Z)R^\sharp)$ and hence $\mathrm{QF}(R^\sharp((f-Z)R^\sharp)) \approx_{k(Z)} \mathrm{QF}(R^\sharp((f'-Z)R^\sharp))$.

Question 6. In connection with Example 7, can you show that $\mathrm{QF}(R^\sharp((f-Z)R^\sharp)) \not\approx_{k(Z)} \mathrm{QF}(R^\sharp((f'-Z)R^\sharp))$? Assuming this, by the Generic Criterion, we would get $R[1/f] \not\approx_k R[1/f']$.

Remark 12. In geometric terms, considering the birational equivalence of the complements of two hypersurfaces $f=0$ and $f'=0$ in the affine $n$-space, and relating it to their refined redsets as well as to the birational equivalence of their irreducible components and the biregular equivalence of the generic members of their associated pencils $f-c=0$ and $f'-c=0$, we have the following:

(I) The first part of the Redset Criterion says that if the affine complements of two hypersurfaces are biregularly equivalent then they have the same number of irreducible components.

(II) The second part of the Redset Theorem says that if the affine complements of two irreducible hypersurfaces are biregularly equivalent then they have the same refined redsets.

(III) the Nonruled Criterion says that if the affine complements of two hypersurfaces are biregularly equivalent then their nonruled irreducible components are birationally equivalent.

(IV) the Generic Criterion says that if the affine complements of two irreducible hypersurfaces are biregularly equivalent then the generic members of their associated pencils are biregularly equivalent and hence birationally equivalent.

Question 7. In the complex case, can you topologize the conclusions of Remark 12 by replacing biregular equivalence by some kind of topological type? Can you somehow relate this to the topology of complements as exemplified by the work of Zariski, Fan, Teicher, and others as discussed in [**GaT**]? Can you also tie it to Abhyankar's algebraization of the tame fundamental groups of complements as described in [**A03**]?

QUESTION 8. We have already noted the fact that the hypersurfaces $f = 0$ and $f' = 0$ being automorphic implies the biregular equivalence of their complements. Can you exploit this fact to link-up the results of Remark 12 with the epimorphism theorems and problems discussed by Abhyankar in his Kyoto Notes [**A04**]

## 8. Redset of a Plane Curve and Zariski's Lemma

We shall now show how, in case of characteristic zero, the finiteness of the redset of a hypersurface can be deduced from that of a plane curve via the famous Lemma 5 of Zariski's Bertini II paper [**Za1**]. It may be noted that, Abhyankar [**A05, A06, A08**] reduced the Galois case of the Jacobian Problem to the birational case by means of Zariski's Lemma and then settled the birational case by using Zariski's Main Theorem for which reference may be made to [**A09**].

REMARK 13. As pointed out in Remark 1, for $n = 2$, the Lemma next to the Redset Theorem follows by taking $V_1, \ldots, V_t$ to be the valuation rings of the places at infinity of the irreducible plane curve $f = 0$, and then the argument in the proof of the Redset Theorem shows that if $k$ is relatively algebraically closed in $L$ then redset$(f)$ is finite. To see how, for characteristic zero, the $n = 2$ case of the Redset Theorem implies the $n > 2$ case we can proceed thus. Let $\iota$ be the smallest positive integer $\leq n$ such that $f \in k[X_1, \ldots, X_\iota]$. If $\iota = 1$ then by doing nothing, and if $\iota > 1$ then by applying a $k-$automorphism to $R$ of the form $X_j \mapsto X_j$ for all $j \in \{1, \ldots, n\} \setminus \{\iota\}$ and $X_\iota \mapsto X_\iota + X_1^l$ with $l >$ twice the total degree of $f$, we can arrange matters so that $f$ is essentially monic in $X_1$, i.e., for some $e' \in k^\times$ and integer $e > 0$ we have $f = e'X_1^e +$ terms of $X_1$-degree $< e$. For any positive integer $i \leq n$ let $R_i = k_i[X_1, \ldots, X_i]$ with $k_i = k(X_{i+1}, \ldots, X_n)$. Now clearly $f \in R_\iota \setminus k_\iota$ and by Gauss Lemma, $f$ is irreducible in $R_\iota$. Let us identify $k_\iota$ with a subfield of $L_\iota = \mathrm{QF}(R_\iota/(fR_\iota))$. If $\iota = 2$ and $k_\iota$ were relatively algebraically closed in $L_\iota$ then by the $n = 2$ case of the Redset Theorem we would see that $\{c \in \widetilde{k} : f - c = gh$ with $g, h$ in $R_\iota \setminus k_\iota\}$ is finite and hence so is redset$(f)$. In the next Example 8 we shall show that $f$ irreducible in $R_\iota$ does not imply $k_\iota$ relatively algebraically closed in $L_\iota$. However, in case of characteristic zero, by taking $(L', Z_1, \ldots, Z_s) = (L, \phi(X_2), \ldots, \phi(X_n))$ in the Second Version of Zariski's Lemma given below, we can arrange $k_2$ to be relatively algebraically closed in $L_2$.

EXAMPLE 8. The geometric significance of the condition that the ground field $k$ of the irreducible polynomial $f$ be relatively algebraically closed in its function field $L$ is due to the well-known fact that, in case of characteristic zero, it is equivalent to assuming $f$ to be absolutely irreducible, i.e., irreducible in $R^*$. To illustrate this take $n = 2$ and $f = X_1^2 + X_2^u$ with integer $u > 0$. Assume that $-1$ is not a square in $k$; for instance $k$ could be the field of real numbers. Let $R_1 = k_1[X_1]$ and $R_1^* = k_1^*[X_1]$ where $k_1^*$ is an algebraic closure of $k_1 = k(X_2)$. Now clearly $f$ is always irreducible in $R_1$, but it is irreducible in $R_1^* \Leftrightarrow u$ is odd. Let $\phi_1 : R_1 \to R_1/(fR_1)$ be the canonical epimorphism, and let us identify $k_1$ with a subfield of $L_1 = \mathrm{QF}(R_1/(fR_1))$. Note that now $L_1 = k(\overline{X}_1, X_2)$ with $\overline{X}_1 = \phi_1(X_1)$. If $u = 2v$ is even then $(\overline{X}_1/X_2^v)^2 = -1$ and hence $k_1$ is not relatively algebraically closed in $L_1$. If $u = 2v + 1$ is odd then $(\overline{X}_1/X_2^v)^2 = -X_2$ and hence $L_1 = k(\overline{X}_1/X_2^v)$ and therefore $k_1$ is relatively algebraically closed in $L_1$.

REMARK 14. Before coming to Zariski's Lemma, let us note that he calls maximally algebraic (m.a. for short) what we have called relatively algebraically closed., i.e., a field is m.a. in an overfield if every element of the overfield which is algebraic over the field belongs to the field; likewise he calls a field quasi-maximally algebraic (q.m.a. for short) in an overfield if every element of the overfield which is separable algebraic over the field belongs to the field; this is sometimes called relatively separably closed. Recall that an overfield is said to be regular over (or a regular extension of) a field if the overfield is a finitely generated extension of the field such that the field is m.a. in the overfield and the overfield is separably generated over the field.

The well-known fact about an irreducible polynomial mentioned in Example 8, also applies to any prime ideal $P$ in $R$, after identifying $k$ with a subfield of the function field $\mathrm{QF}(R/P)$ of the variety $\mathcal{V}(P)$. Namely, $P$ is absolutely prime $\Leftrightarrow$ $\mathrm{QF}(R/P)$ is regular over $k$. Recall that $P$ is absolutely prime means $PR^*$ is prime, and note that then: $P$ is absolutely prime $\Leftrightarrow P\overline{k}[X_1, \ldots, X_n]$ is prime for every field extension $\overline{k}$ of $k \Leftrightarrow P\overline{k}[X_1, \ldots, X_n]$ is prime for every algebraic field extension $\overline{k}$ of $k \Leftrightarrow P\overline{k}[X_1, \ldots, X_n]$ is prime for every finite algebraic field extension $\overline{k}$ of $K$. Let us call an ideal $Q$ in $R$ quasiprime if it is primary; note that this implies $Q \neq R$; also note $fR$ is quasiprime $\Leftrightarrow$ $f = gh^{\mu}$ for some $g \in k^{\times}$ and irreducible $h \in R \setminus k$ with integer $\mu > 0$. Let us call $P$ absolutely quasiprime to mean that $PR^*$ is quasiprime, and note that then: $P$ is absolutely quasiprime $\Leftrightarrow$ $P\overline{k}[X_1, \dots, X_n]$ is quasiprime for every field extension $\overline{k}$ of $k$ $\Leftrightarrow$ $P\overline{k}[X_1, \dots, X_n]$ is quasiprime for every algebraic field extension $\overline{k}$ of $k$ $\Leftrightarrow$ $P\overline{k}[X_1, \dots, X_n]$ is quasiprime for every finite algebraic field extension $\overline{k}$ of $k$. As a well-known variation of the above well-known fact we have that $P$ is absolutely quasiprime $\Leftrightarrow$ $k$ is q.m.a. in $\mathrm{QF}(R/P)$. Proofs of all these assertions can be found in [**ZaS**].

Out of the following four versions of Zariski's Lemma, the first two constitute Lemma 5 of [**Za1**], the third is Proposition I.6.1 of [**Za3**], and the fourth is Theorem 2.4 of [**Mat**] or Proposition 9.31 of [**FrJ**].

Zariski's Lemma. For any finitely generated field extension $L'$ of $k$ we have the following.

First Version. Assume that $k$ is of characteristic zero and m.a. in $L'$. Let elements $Z_1, Z_2$ in $L'$ be algebraically independent over $k$. Then for all except a finite number of $c$ in $k$ we have that $k(Z_1 + cZ_2)$ is m.a. in $L'$.

Second Version. Assume that $k$ is of characteristic zero and m.a. in $L'$. Let elements $Z_1, \dots, Z_s$ in $L'$, with $s > 1$, be algebraically independent over $k$. Then by applying a $k$-linear automorphism to $k[Z_1, \dots, Z_s]$ it can be arranged that $k(Z_1, \dots, Z_{s-1})$ is m.a. in $L'$. In other words, there exists a nonsingular $n \times n$ matrix $C = (C_{ij})$ over $k$ such that upon letting $Z_i^C = \sum_{1 \le j \le s} C_{ij} Z_j$ we have that $k(Z_1^C, \dots, Z_{s-1}^C)$ is m.a. in $L'$. Moreover, the constants $C_{ij}$ are nonspecial in the sense that for every $\Lambda \in k$ there is $H_i(\Lambda) \subset k$ with $|k \setminus H_i(\Lambda)| < \infty$ for $1 \le i < s$ such that: if $C_{ij} = 1$ for $1 \le i \le s$, $C_{ij} = 0$ for $1 < i + 1 < j \le s$, $C_{12} \in H_1(1)$, and $C_{i.i+1} \in H_i(C_{i-1,i})$ for $1 < i < s$. then $k(Z_1^C, \dots, Z_{s-1}^C)$ is m.a. in $L'$.

Third Version. Assume that $k$ is q.m.a. in $L'$. Let $z_1, \dots, z_r$ be a finite number of elements in $L'$ such that $\mathrm{trdeg}_k k(z_1, \dots, z_r) > 1$. Let $u_1, \dots, u_r$ be indeterminates over $L'$.
Then $k(u_1, \dots, u_r, u_1 z_1 + \dots + u_r z_r)$ is q.m.a. in $L'(u_1, \dots, u_r)$.

Fourth Version. Assume that $L'$ is regular over $k$. Let elements $Z_1, Z_2$ in $L'$ be algebraically independent over $k$, and assume that $D(Z_2) \neq 0$ for some derivation $D$ of $L'/k$. Then for all except a finite number of $c$ in $k$ we have that $L'$ is regular over $k(Z_1 + cZ_2)$.

## 9. Singset of a Plane Curve and the Zeuthen-Segre Invariant

We have found bounds on redset and primset for the special pencil $(f-c)_{c\in k}$ but in case of the singset we have only stated that it is finite under appropriate conditions. Actually, from the proof of the Singset Theorem it does follow that, in case of characteristic zero, $|\text{singset}(f)| \le$ the number of irreducible components of the variety of partials $\mathcal{V}(f_{X_1}, \ldots, f_{X_n})^*$.

Assuming $n = 2$ with $(X_1, X_2) = (X, Y)$, and $k$ is of characteristic zero with $k =$ its algebraic closure $k^*$, it is possible to give a more quantitative estimate of the singset using the rank $\rho(f)$ of $f$ as introduced in Section 11 of [**AbA**]. As we shall see, this rank $\rho(f)$ is related to the Zeuthen-Segre invariant. For the convenience of the reader let us review the definition of $\rho(f)$.

Let $g, h$ in $R$. For $Q = (u, v)$ in the affine plane $\mathcal{A} = k^2$ we define the intersection multiplicity $I(g, h; Q)$ to be the $k$-vector-space dimension of $S/(g, h)S$ where $S$ is the localization of $R$ at the maximal ideal $(X - u, Y - v)R$. Note that: if $(g, h)R \not\subset (X - u, Y - v)R$ then $I(g, h; Q) = 0$; if $(g, h)R \subset qR$ for some $q \in R \setminus k$ with $q(u, v) = 0$ then $I(g, h; Q) = \infty$; otherwise $I(g, h; Q)$ is a positive integer. We define further intersection multiplicities by putting

$$I(g, h; \mathcal{A}) = \sum_{Q\in\mathcal{A}} I(g, h; Q) \text{ and } I(g, h; f) = \sum_{\{Q\in\mathcal{A}: f(u,v)=0\}} I(g, h; Q)$$

and

$$I(g, h; \mathcal{A} \setminus f) = \sum_{\{Q\in\mathcal{A}: f(u,v)\neq 0\}} I(g, h; Q)$$

with the usual conventions about infinity, and we note that these are nonnegative integers or $\infty$. We also put

$$\widehat{I}(g, h; \mathcal{A}) = \max_{\mu\in k} I(g, h - \mu; \mathcal{A})$$

and we note that $\widehat{I}(g, h; \mathcal{A})$ is a nonnegative integer or $\infty$, and: $\widehat{I}(g, h; \mathcal{A}) = \infty \Leftrightarrow \gcd(g, h - c) \neq 1$ for some $c \in k$. Next we put

$$\alpha(g, h; \mathcal{A}) = \{\lambda \in k : I(g, h - \lambda; \mathcal{A}) < \widehat{I}(g, h; \mathcal{A})\}$$

and

$$\beta(g, h; \mathcal{A}) = \sum_{0\neq\lambda\in\alpha(g,h;\mathcal{A})} [\widehat{I}(g, h; \mathcal{A}) - I(g, h - \lambda; \mathcal{A})]$$

and we note that then $\beta(g, h; \mathcal{A})$ is a nonnegative integer or $\infty$. Finally we define

$$\rho(f) = I(f_X, f_Y; \mathcal{A} \setminus f) + \beta(f_Y, f; \mathcal{A})$$

and we note that this is a nonnegative integer or $\infty$.

By augmenting $\mathcal{A}$ by points on the line at infinity we get the projective plane $\mathcal{P}$ over $k$. For any $Q \in \mathcal{P} \setminus \mathcal{A}$ we define $I(g, h; Q)$ in an obvious manner and we put

$$I(g, h; \mathcal{P}) = \sum_{Q \in \mathcal{P}} I(g, h; Q).$$

For any $Q = (u, v) \in \mathcal{A}$ we put

$$\overline{\chi}(f; Q) = (\text{number of branches of } f \text{ at } Q) - 1$$

and we note that if $f(u, v) = 0$ then $f(X + u, Y + v)$ is a product of $\overline{\chi}(f; Q) + 1$ irreducible nonconstant power series in $k[[X, Y]]$ in case $f(u, v) = 0$, and if $f(u, v) \neq 0$ then $\overline{\chi}(f; Q) = -1$. We let

$$\overline{\chi}(f; \mathcal{A}) = \sum_{Q=(u,v) \in \mathcal{A} \text{ with } f(u,v)=0} \overline{\chi}(f; Q)$$

and we note that this is a nonnegative integer. We define the integer $\overline{\chi}(f; \mathcal{P}) \geq \overline{\chi}(f; \mathcal{A})$ in an analogous manner and we put

$$\overline{\chi}(f; \infty) = \overline{\chi}(f; \mathcal{P}) - \overline{\chi}(f; \mathcal{A}).$$

If $f$ is irreducible then by $\gamma(f)$ we denote its genus, and we note that by the genus formula for $f$, given any $g \in R$ with $g - c \notin fR$ for all $c \in k$, we have

$$2\gamma(f) - 2 = \deg(d\phi(g)) = \sum_{V \in \mathfrak{R}(f, \mathcal{P})} \operatorname{ord}_V(d(\phi(g))$$

where $\deg(\phi(g))$ is the degree of the divisor of the differential of $\phi(g)$ in the function field $L/k$ of $f$, with canonical epimorphism $\phi : R \to A = R/(fR)$ and $L = \mathrm{QF}(A)$, and where $\mathfrak{R}(f, \mathcal{P})$ is the set of all DVRs of $L/k$; we also put $\mathfrak{R}(f, \mathcal{A}) = \{V \in \mathfrak{R}(f, \mathcal{P}) : A \subset V\}$ and $\mathfrak{R}(f, \infty) = \mathfrak{R}(f, \mathcal{P}) \setminus \mathfrak{R}(f, \mathcal{A})$; moreover, for any $Q \in \mathcal{P}$ we let $\mathfrak{R}(f, Q)$ denote the set of $V \in \mathfrak{R}(f, \mathcal{P})$ having center $Q$ on $f$, and we note that then $\mathfrak{R}(f, Q)$ is a finite set which is empty if and only if "$Q$ does not lie on $f$."

In the general case, by writing $f = f_1 \dots f_s$ with irreducible $f_1, \dots, f_s$, we generalize the definition of $\gamma(f)$ by putting

$$\gamma(f) = 1 + \sum_{1 \leq i \leq s} (\gamma(f_i) - 1).$$

Relabelling $f_1, \dots, f_s$ suitably we can arrange that the first $r$ of them are pairwise nonassociates, and every $f_i$ is an associate of $f_j$ for some $j \leq r$; now we put $\mathrm{rad}(f) = f_0 f_1 \dots f_r$ where $0 \neq f_0 \in k$ is chosen so that the coefficients of the highest lexicographic terms of $f$ and $\mathrm{rad}(f)$ are equal; the lexicographic order is such that $\deg(X^a Y^b) \geq \deg(X^{a'} Y^{b'}) \Leftrightarrow$ either $b = b'$ and $a \geq a'$ or $b > b'$. Now we define the algebraic rank of $f$ by putting

$$\rho_a(f) = 2\gamma(\mathrm{rad}(f)) + \overline{\chi}(\mathrm{rad}(f); \mathcal{P}).$$

If $k = \mathbb{C}$ then by $\rho_t(f)$ we denote the rank of the first homology group of $f$, i.e., of the point-set $\{(u, v) \in \mathbb{C}^2 : f(u, v) = 0\}$.

Consider the condition:

$$(*) \qquad f \text{ is } Y\text{-monic of } Y\text{-degree } N > 0,$$

i.e., $\deg_{X,Y}(f - f_0 Y^N) < N$ with $f_0 \in k^\times$ and integer $N > 0$. Also consider the conditions:

$$(1*) \qquad \gcd(f_Y, f - c) = 1 \text{ for all } c \in k,$$

and

$$(2*) \qquad \gcd(f_Y, f - c) = 1 \text{ for all } c \in k, \text{ and } f \text{ is irreducible},$$

and

$$(3*) \qquad \gcd(f_Y, f - c) = 1 \text{ for all } c \in k,\ f \text{ is irreducible, and } k = \mathbb{C},$$

and

$$(4*) \qquad \mathrm{rad}(f) = f.$$

In (11.2) of [**AbA**] it is shown that:

$$(9.1) \qquad (*) + (1*) \Rightarrow \rho(f) = (1 - N) + I(f, f_Y; \mathcal{A}) - I(f_X, f_Y; f)$$

where all the terms are integers. In (11.5) of [**AbA**] it is shown that:

$$(9.2) \qquad (*) + (2*) \Rightarrow \rho(f) = \rho_a(f)$$

and just after (11.5) it is asserted that:

$$(9.3) \qquad (*) + (3*) \Rightarrow \rho(f) = \rho_t(f).$$

In a moment we shall generalize (9.1) by showing that:

$$(9.4) \qquad (*) + (4*) \Rightarrow \rho_a(f) = (1 - N) + I(f, f_Y; \mathcal{A}) - I(f_X, f_Y; f)$$

where obviously all the terms are integers. From this we shall deduce that:

(9.5)

$$(*) \Rightarrow \rho_a(f) = (1 - N) + \deg_Y[f] + I(f, f_Y/\widehat{[f]}; \mathcal{A}) - I(f_X, f_Y/\widehat{[f]}; f)$$

with all terms integers, where

$$[f]=\mathrm{gcd}(f,f_X,f_Y) \quad \text{and} \quad \widehat{[f]}=\mathrm{gcd}(f_X,f_Y)$$

with the gcds made unique by requiring them to be $Y$-monic. While proving (9.5) we shall also show that:

$$(9.6) \qquad (*)\Rightarrow \begin{cases} f=[f]\mathrm{rad}(f) \\ \text{and } \widetilde{[f]}=[f]\widetilde{f} \text{ where } \widetilde{f}\in R \text{ with } \mathcal{V}(f,\widetilde{f})=\emptyset \\ \text{and } \widehat{[f]}=\prod_{c\in\mathrm{multset}(f)^*}[f-c] \\ \text{and hence } |\mathrm{multset}(f)^*|\le \deg_Y \widehat{[f]}. \end{cases}$$

As a consequence of (9.5) we shall show that:

$$(9.7) \qquad \begin{cases} \text{there exists a unique integer } \rho_\pi(f) \text{ together with} \\ \text{a unique finite subset } \mathrm{defset}(f) \text{ of } k \text{ such that} \\ \rho_a(f-c)=\rho_\pi(f) \text{ for all } c\in k\setminus\mathrm{defset}(f) \text{ and} \\ \rho_a(f-c)\neq\rho_\pi(f) \text{ for all } c\in\mathrm{defset}(f). \end{cases}$$

In reference to (9.7) we put

$$\rho_\pi(f)=\text{the pencil-rank of the pencil } (f-c)_{c\in k}$$

and

$$\mathrm{defset}(f)=\text{the deficiency set of } f.$$

From (9.7) we shall deduce that:

$$(9.8) \qquad (*)\Rightarrow \rho_\pi(f)=(1-N)+\widehat{I}(f,f_Y/\widehat{[f]};\mathcal{A}).$$

From (9.8) we shall deduce that:

$$(9.9) \qquad (*)\Rightarrow \rho_\pi(f)\ge\rho_a(f-c) \text{ for all } c\in k\setminus\mathrm{multset}(f)^*.$$

From (9.8) we shall deduce the jungian formula for the pencil-rank saying that:

$$(9.10) \qquad \rho_\pi(f)=1-|\mathcal{V}_\infty(f)|+\sum_{c\in\mathrm{defset}(f)}[\rho_\pi(f)-\rho_a(f-c)]$$

with the points at infinity of $f$ defined by

$$\mathcal{V}_\infty(f)=\text{set of height-one members of } \mathcal{V}(f^+)$$

where $f^+$ is the degree form of $f$ consisting of its highest degree terms. We are using the adjective jungian in view of the fundamental contribution of Jung [**Jun**] to the theory of algebraic rank. The Zeuthen-Segre invariant of $f$ is the integer $\zeta(f)$ defined by putting

$$\zeta(f) = -|\mathcal{V}_\infty(f)| - \rho_\pi(f) + \sum_{c\in\mathrm{defset}(f)} [\rho_\pi(f) - \rho_a(f-c)]$$

and from (9.10) it immediately follows that:

(9.11) $$\zeta(f) = -1.$$

Finally from (9.11) we shall deduce the:

(9.12) Defset Theorem. If (*) then $\mathrm{singset}(f) \setminus \mathrm{multset}(f)^* \subset \mathrm{defset}(f)$, and $|\mathrm{defset}(f)| \le 1 + \rho_a(f) + \deg_Y \widehat{[f]}$ with $|\mathrm{singset}(f)| \le 1 + \rho_a(f) + 2\deg_Y \widehat{[f]}$.

Before turning to the proof of items (9.4) to (9.12), let us establish some common

Notation and Calculation. Given any $f' \in R^\times$, write $f' = \theta f'_1 \dots f'_{s'}$ where $f'_1, \dots, f'_{s'}$ are irreducible members of $R \setminus k$ and $\theta \in k^\times$, and let $\phi'_i : R \to A'_i = R/(f'_iR)$ be the canonical epimorphism and identify $k$ with a subfield of $L'_i = \mathrm{QF}(A'_i)$. Let $Z_i = \{V \in \mathfrak{R}(f'_i, \infty) : \mathrm{ord}_V \phi'_i(f) \ge 0\}$ and $P_i = \{V \in \mathfrak{R}(f'_i, \infty) : \mathrm{ord}_V \phi'_i(f) < 0\}$, where the letters $Z$ and $P$ are meant to suggest zeros and poles, and note that these are obviously finite sets. Let

$$Z(f, f') = \sum_{1\le i\le s'} \sum_{V\in Z_i} \mathrm{ord}_V \phi'_i(f) \text{ and } P(f, f') = \sum_{1\le i\le s'} \sum_{V\in P_i} \mathrm{ord}_V \phi'_i(f)$$

and note that $Z(f, f')$ is a nonnegative integer or $\infty$ according as $\gcd(f, f') = 1$ or $\gcd(f, f') \ne 1$, and $P(f, f')$ is always a nonpositive integer. Clearly for each $V \in Z_i$ there is a unique $c_i(V) \in k$ such that $\mathrm{ord}_V \phi'_i(f - c_i(V)) > 0$. Let $D(f, f') = \cup_{1\le i\le t}\{c_i(V) : V \in S_i\}$ and Let $E(f, f') = D(f, f') \cup \mathrm{singset}(f)$. Note that clearly $D(f, f')$ is a finite subset of $k$ and hence by the Singset Theorem so is $E(f, f')$. Assuming (*), write $f = f_1 \dots f_s$ where $f_1, \dots, f_s$ are irreducible $Y$-monic members of $R \setminus k$, and let $\phi_i : R \to A_i = R/(f_iR)$ be the canonical epimorphism and identify $k$ with a subfield of $L_i = \mathrm{QF}(A_i)$.

Without assuming (*), consider the conditions:
(1′)

$$\begin{cases} \text{for a given } Q=(u,v)\in\mathcal{A} \text{ with } f(u,v)=0, \text{ and} \\ \text{for every irreducible factor } g \text{ of } f' \text{ in } R\setminus k \text{ with } g(u,v)=0 \\ \text{we have } f_Y\in gR \text{ with } f\notin gR \text{ and } f_X\notin gR, \end{cases}$$

(2′) $$f'=f_Y \text{ and } Q=(u,v)\in\mathcal{A} \text{ with } f(u,v)=0$$

and

(3′) $$\gcd(f,f')=1.$$

Then we have (I) to (III) stated below.

(I) If (*)+(1′) then

$$I(f,f';Q)-I(f_X,f';Q)=I(X-u,f';Q)$$

where all the terms are integers.

(II) If (*)+(2′) then

$$I(X-u,f';Q)=\overline{\chi}(f;Q)+\sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,Q)}\operatorname{ord}_V(d\phi_i(X))$$

where all the terms are integers.

(III) If (*)+(3′) then for all $\lambda\in k$ we have

$$I(f,f';\mathcal{A})=-Z(f,f')-P(f-\lambda,f')$$

where all the terms are integers.

**Proof of (I)**. If (*)+(1′) then we have, with all terms integers,

$$\begin{aligned}
\text{LHS of (I)} &= \sum_{1\le i\le s'} [I(f,f'_i;Q) - I(f_X,f'_i;Q)] \\
&= \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,Q)} [\mathrm{ord}_V(\phi'_i(f)) - \mathrm{ord}_V(\phi'_i(f_X))] \\
&= \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,Q)} \mathrm{ord}_V(\phi'_i(X-u)) \\
&= \sum_{1\le i\le s'} I(X-u,f'_i;Q) \\
&= \text{RHS of (I)}.
\end{aligned}$$

**Proof of (II)**. If (*)+(2′) then we have, with all terms integers,

$$\begin{aligned}
\text{LHS of (II)} &= -1 + I(X-u,f;Q) \\
&= -1 + \sum_{1\le i\le s} \sum_{V\in\mathfrak{R}(f_i,Q)} \mathrm{ord}_V(\phi_i(X-u)) \\
&= -1 + \sum_{1\le i\le s} \sum_{V\in\mathfrak{R}(f_i,Q)} [\mathrm{ord}_V(d\phi_i(X)) + 1] \\
&= \text{RHS of (II)}.
\end{aligned}$$

**Proof of (III)**. If (*)+(3′) then we have, with all terms integers,

$$\begin{aligned}
\text{LHS of (III)} &= -Z(f,f') - P(f,f') \\
&\qquad \text{(since number of zeros of a function equals} \\
&\qquad\qquad \text{number of its poles)} \\
&= \text{RHS of (III)} \\
&\qquad (\text{since } \mathrm{ord}_V\phi'_i(f) = \mathrm{ord}_V\phi'_i(f-\lambda) \text{ for all } V\in P_i \\
&\qquad\qquad \text{and } \lambda\in k.)
\end{aligned}$$

**Proof of (9.4)**. Assume (*)+(4*). Write $f = f_1 \dots f_r$ with pairwise distinct irreducible $f_1,\dots,f_r$ in $R\setminus k$. Let $\phi_i : R \to A_i = R/(f_i)R$ be the canonical epimorphism and identify $k$ with a subfield of

$L_i = \mathrm{QF}(A_i)$. Letting $\overline{\sum}$ stand for summation over $\{Q = (u, v) \in \mathcal{A} : f(u, v) = 0\}$ we get, with all terms integers,

$$\begin{aligned}
&\text{RHS of (9.4)}\\
&= 1 - N + \overline{\sum}[I(f, f_Y; Q) - I(f_X, f_Y; Q)]\\
&= 1 - N + \overline{\sum}\left[\overline{\chi}(f; Q) + \sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,Q)} \mathrm{ord}_V(d\phi_i(X))\right]\\
&\qquad\qquad\qquad\text{(by taking } f' = f_Y \text{ in (I) and (II))}\\
&= 1 - N + \overline{\chi}(f; \mathcal{A}) + \sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,\mathcal{A})} \mathrm{ord}_V(d\phi_i(X))
\end{aligned}$$

$$\begin{aligned}
&= 1 - N + \overline{\chi}(f; \mathcal{A}) + (2\gamma(f) - 2)\\
&\qquad\qquad - \sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,\infty)} \mathrm{ord}_V(d\phi_i(X))\\
&\qquad\text{(by genus formula for } f_i \text{ and definition of } \gamma(f))\\
&= 2\gamma(f) + \overline{\chi}(f; \mathcal{A}) - N - 1\\
&\qquad\qquad - \sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,\infty)} [\mathrm{ord}_V(\phi_i(X)) - 1]\\
&\qquad\qquad\qquad\text{(because } \mathrm{ord}_V\phi_i(X) \neq 0)\\
&= 2\gamma(f) + \overline{\chi}(f; \mathcal{A}) - N - 1 - [-N - (\overline{\chi}(f; \infty) + 1)]\\
&= 2\gamma(f) + \overline{\chi}(f; \mathcal{P})\\
&= \text{LHS of (9.4).}
\end{aligned}$$

**Proof of (9.5) and (9.6)**. Assume (*). Let $\overline{f} = \mathrm{rad}(f)$. Then by the argument in the proof of the Singset Theorem we see that

(i) $\quad [f] = f/\overline{f} \quad$ and $\quad \widehat{[f]} = [f]h$ where $h \in R$ with $\mathcal{V}(f, h) = \emptyset$.

and

$$\widehat{[f]} = \prod_{c\in\mathrm{multset}(f)^*} [f - c] \quad \text{and hence} \quad |\mathrm{multset}(f)^*| \le \deg_Y \widehat{[f]}.$$

which proves (9.6). Applying (9.4) to $\overline{f}$ we get

$$\rho_a(f) = 1 - \deg_Y \overline{f} + I(\overline{f}, \overline{f}_Y; \mathcal{A}) - I(\overline{f}_X, \overline{f}_Y; \overline{f}).$$

The first two terms of the above RHS combine to give $1-N+\deg_Y[f]$ which are the first three terms of the RHS of (9.5). It remains to compare the intersection multiplicity terms, i.e., the proof of (9.5) will be completed by showing that

(i*) $I(\overline{f},\overline{f}_Y;\mathcal{A})-I(\overline{f}_X,\overline{f}_Y;\overline{f})=I(f,f_Y/\widehat{[f]};\mathcal{A})-I(f_X,f_Y/\widehat{[f]};f)$.

For any $Q=(u,v)\in\mathcal{A}$ with $f(u,v)=0$, by taking $(\overline{f},\overline{f}_Y)$ for $(f,f')$ in (I) we get

(ii) $$I(\overline{f},\overline{f}_Y;Q)-I(\overline{f}_X,\overline{f}_Y;Q)=I(X-u,\overline{f}_Y;Q),$$

and, in view of (i), by taking $(f,f_Y/\widehat{[f]})$ for $(f,f')$ in (I) we get

(iii) $$I(f,f_Y/\widehat{[f]};Q)-I(f_X,f_Y/\widehat{[f]};Q)=I(X-u,f_Y/\widehat{[f]};Q).$$

By summing over $Q=(u,v)\in\mathcal{A}$ with $f(u,v)=0$, the LHS of (ii) gives the LHS of (i*), and the RHS of (iii) gives the RHS of (i*). Therefore the proof of (9.5) will be complete by proving that, for any $Q=(u,v)\in\mathcal{A}$ with $f(u,v)=0$, the RHS of (ii) equals the RHS of (iii). Let $\psi:R\to R/(X-u)R$ be the canonical epimorphism, and let $W$ be the localization of $\psi(R)$ at $\psi((Y-v)R)$. Now

$$\begin{aligned}
\text{RHS of (ii)} &= \mathrm{ord}_W\psi(\overline{f}_Y)\\
&= -1+\mathrm{ord}_W\psi(\overline{f}) \quad (\text{because } \overline{f}(u,v)=0)\\
&= -1+\mathrm{ord}_W\psi(f/\widehat{[f]}) \quad (\text{by (i)})\\
&= [-1+\mathrm{ord}_W\psi(f)]-\mathrm{ord}_W\psi(\widehat{[f]})\\
&= \mathrm{ord}_W\psi(f_Y)-\mathrm{ord}_W\psi(\widehat{[f]}) \quad (\text{because } f(u,v)=0)\\
&= \mathrm{ord}_W\psi(f_Y/\widehat{[f]})\\
&= \text{RHS of (iii)}
\end{aligned}$$

and this completes the proof.

**Proof of (9.7) to (9.12)**. For any $k$-automorphism $\sigma$ of $R$ we clearly have $\rho_a(f-c)-\rho_a(\sigma(f)-c)=|\mathcal{V}_\infty(\sigma(f)-c)|-|\mathcal{V}_\infty(f-c)|$ for all $c\in k$, and hence: (9.7) is true for $f\Leftrightarrow$ it is true for $\sigma(f)$. After having defined $\rho_\pi(f)$ and defset$(f)$, for any $k$-automorphism $\sigma$ of $R$ we clearly have $\jmath(\sigma(f))=\jmath(f)$, and hence: (9.9) is true for $f\Leftrightarrow$ it is true for $\sigma(f)$. Also it is well-known that $\sigma(f)$ satisfies (*) for some $k$-automorphism $\sigma$ of $R$. Therefore in proving (9.7) to (9.12), without loss of generality we may and **we shall assume that $f$ satisfies (*).** Let $f'=f_Y/\widehat{[f]}$. Clearly $f_X$ and $f_Y$ are unchanged if we replace

$f$ by $f-c$ with $c \in k$, and hence so are $\widehat{[f]}$ and $f'$. Given any $c \in k$, by taking $f-c$ for $f$ in (9.5) we get

(1) $\rho_a(f-c) = 1-N+\deg_Y[f-c]+I(f-c,f';\mathcal{A})-I(f_X,f';f-c)$

with all terms integers. By (9.6) we have $\gcd(f-c,f')=1$ and hence by taking $(0,f-c)$ for $(\lambda,f)$ in (III) we get

(2) $$I(f-c,f';\mathcal{A}) = -Z(f-c,f') - P(f,f')$$

where all the terms are integers. In view of (9.6), by the definitions of the sets $D(f,f')$, $\mathrm{multset}(f)^*$, $\mathrm{singset}(f)$ and $E(f,f')$ we see that

(3) $$\begin{cases} Z(f-c,f') > 0 \text{ or } Z(f-c,f') = 0 \\ \text{according as } c \in D(f,f') \text{ or } c \notin D(f,f'), \end{cases}$$

and

(4) $$\deg_Y[f-c] > 0 \text{ or } = 0 \text{ according as } c \in \mathrm{multset}(f)^* \text{ or } c \notin \mathrm{multset}(f)^*,$$

and

(5)
$$I(f_X,f';f-c) \geq 0 \text{ and if } c \notin \mathrm{singset}(f) \text{ then } I(f_X,f';f-c) = 0,$$

and

(6) $E(f,f') = D(f,f') \cup \mathrm{singset}(f)$ with $\mathrm{multset}(f)^* \subset \mathrm{singset}(f)$.

By (1) to (6) we get

(7) $$c \notin E(f,f') \Rightarrow \rho_a(f-c) = 1-N-P(f,f')$$

with all terms integers. Since $E(f,f')$ is a finite set and the above RHS is independent of $c$, this proves (9.7) and establishes the existence of $\rho_\pi(f)$ and $\mathrm{defset}(f)$. Now by (1) to (7) we see that

(8) $$\rho_\pi(f) = 1-N-P(f,f')$$

and

(9) $$\mathrm{defset}(f) \subset E(f,f')$$

and

(10) $$\rho_\pi(f) = (1-N) + \widehat{I}(f,f';\mathcal{A})$$

with all terms integers, which proves (9.8). By (9.5) and (9.8) we get

(11) $$\begin{cases} \rho_\pi(f) - \rho_a(f-c) = & -\deg_Y[f-c] + I(f_X,f';f-c) \\ & + \left(\widehat{I}(f,f';\mathcal{A}) - I(f-c,f';\mathcal{A})\right). \end{cases}$$

Since the second parenthesis term and the third big parenthesis term in the above RHS are obviously nonnegative, in view of (4) we see that

$$\rho_\pi(f) - \rho_a(f-c) \geq -\deg_Y[f-c] \tag{12}$$

and

$$\rho_\pi(f) \geq \rho_a(f-c) \text{ for all } c \in k \setminus \text{multset}(f)^* \tag{13}$$

which proves (9.9). By (1), (2) and (8) we get

$$\rho_\pi(f) - \rho_a(f-c) = -\deg_Y[f-c] + I(f_X, f'; f-c) + Z(f-c, f') \tag{14}$$

with all terms integers. Now, with all terms integers, we have

$$\begin{aligned}
&\text{RHS of (9.10)}\\
&= \sum_{c\in E(f,f')} [\rho_\pi(f) - \rho_a(f-c)]\\
&\qquad\qquad\qquad \text{(by (9))}\\
&= -\deg_Y[\widehat{f}] + \sum_{c\in E(f,f')} [I(f_X, f'; f-c) + Z(f-c, f')]\\
&\qquad\qquad\qquad \text{(by (9.6), (4), (6) and (14))}\\
&= -\deg_Y[\widehat{f}] + I(f_X, f'; \mathcal{A}) + \sum_{c\in E(f,f')} Z(f-c, f')\\
&\qquad\qquad\qquad \text{(by (5) and (6))}\\
&= -\deg_Y[\widehat{f}] + I(f_X, f'; \mathcal{A}) + \sum_{1\leq i\leq s'} \sum_{V\in Z_i} \text{ord}_V \phi_i'(f - c_i(V))\\
&\qquad\qquad\qquad \text{(by (7) and (9))}\\
&= -\deg_Y[\widehat{f}] + I(f_X, f'; \mathcal{A}) + \sum_{1\leq i\leq s'} \sum_{V\in Z_i} [\text{ord}_V \phi_i'(f_X) + \text{ord}_V \phi_i'(X)]\\
&\qquad \text{(because } f_Y \in f_i'R \text{ and } \text{ord}_V \phi_i'(f - c_i(V)) \neq 0 \neq \text{ord}_V \phi_i'(X))\\
&= -\deg_Y[\widehat{f}] + I(f_X, f'; \mathcal{A}) + Z(f_X, f') + Z(X, f')\\
&= -\deg_Y[\widehat{f}] + [-Z(f_X, f') - P(f_X, f')] + Z(f_X, f') + Z(X, f')\\
&\qquad\qquad\qquad \text{(by taking } (0, f_X) \text{ for } (\lambda, f) \text{ in (III))}\\
&= -\deg_Y[\widehat{f}] + Z(X, f') - P(f_X, f'))
\end{aligned}$$

and we have

$$
\begin{aligned}
&-\deg_Y \widehat{[f]} + Z(X,f') - P(f_X,f')) \\
&= -\deg_Y \widehat{[f]} + Z(X,f') - \sum_{1\le i\le s'} \sum_{V\in P_i} \mathrm{ord}_V \phi'_i(f_X) \\
&= -\deg_Y \widehat{[f]} + Z(X,f') - \sum_{1\le i\le s'} \sum_{V\in P_i} [-\mathrm{ord}_V \phi'_i(X) + \mathrm{ord}_V \phi'_i(f)] \\
&\qquad\qquad (\text{because } f_Y \in f'_i R \text{ and } \mathrm{ord}_V \phi'_i(f) \neq 0 \neq \mathrm{ord}_V \phi'_i(X))
\end{aligned}
$$

$$
\begin{aligned}
&= -\deg_Y \widehat{[f]} + Z(X,f') + P(X,f') - P(f,f') \\
&= -\deg_Y \widehat{[f]} + Z(X,f') + P(X,f') - P(f,f') \\
&= -\deg_Y \widehat{[f]} - \deg_Y f' - P(f,f') \\
&= -\deg_Y f_Y - P(f,f') \\
&= 1 - N - P(f,f') \\
&= \rho_\pi(f) \\
&\quad (\text{by } (8))
\end{aligned}
$$

and so we get

$$
\rho_\pi(f) = 1 - |\mathcal{V}_\infty(f)| + \sum_{c\in \mathrm{defset}(f)} [\rho_\pi(f) - \rho_a(f-c)] \tag{15}
$$

and hence

$$
\zeta(f) = -1 \tag{16}
$$

which proves (9.10) and (9.11). By (9.6) we see that

$$
c \notin \mathrm{multset}(f)^* \Rightarrow I(f_X, f'; f-c) = I(f_X, f_Y; f-c)
$$

and hence

$$
c \in \mathrm{singset}(f) \setminus \mathrm{multset}(f)^* \Rightarrow I(f_X, f'; f-c) > 0
$$

and therefore, because the third big parenthesis term in the RHS of (11) is nonnegative, in view of (4) and (11) we conclude that

$$
\mathrm{singset}(f) \setminus \mathrm{multset}(f)^* \subset \mathrm{defset}(f). \tag{17}
$$

Rewriting the jungian excess formula (9.9) we get

$$
\rho_\pi(f) = \sum_{c\in \mathrm{defset}(f)} [\rho_\pi(f) - \rho_a(f-c)] \tag{18}
$$

and adding $\rho_a(f) - \rho_\pi(f)$ to both sides we obtain

$$\rho_a(f) = \sum_{c \in \mathrm{defset}(f) \setminus \{0\}} [\rho_\pi(f) - \rho_a(f-c)]. \tag{19}$$

By (4), (12) and (19) we get

$$\begin{aligned}\rho_a(f) + &\sum_{0 \neq c \in \mathrm{multset}(f)^*} \deg_Y[f-c] \\ &\geq \sum_{0 \neq c \in \mathrm{defset}(f) \setminus \mathrm{multset}(f)^*} [\rho_\pi(f) - \rho_a(f-c)]\end{aligned}$$

and hence by (9.6) we have

$$\rho_a(f) + \deg_Y \widehat{[f]} \geq \sum_{0 \neq c \in \mathrm{defset}(f) \setminus \mathrm{multset}(f)^*} [\rho_\pi(f) - \rho_a(f-c)]$$

and therefore, since every term in the above RHS is nonnegative, we obtain

$$|\mathrm{defset}(f) \setminus \{0\}| \leq \rho_a(f) + \deg_Y \widehat{[f]}$$

and hence

$$|\mathrm{defset}(f)| \leq 1 + \rho_a(f) + \deg_Y \widehat{[f]} \tag{20}$$

By (17) we have

$$|\mathrm{singset}(f)| \leq |\mathrm{defset}(f)| + |\mathrm{multset}(f)^*|$$

and hence, in view of (9.6), by (20) we get

$$|\mathrm{singset}(f)| \leq 1 + \rho_a(f) + 2\deg_Y \widehat{[f]}. \tag{21}$$

The proof of (9.12) is completed by (17), (20) and (21).

Question 9. Is it always true that $\mathrm{redset}(f) \subset \mathrm{defset}(f)$?

Question 10. More explicitly, is it possible to find a nonsingular reducible $f \in R \setminus k$ satisfying (*) such that $[f] = 1$ and $\rho_a(f) = \rho_\pi(f)$. If such an $f$ is found then $0 \in \mathrm{redset}(f) \setminus \mathrm{defset}(f)$ and we have a negative answer to Question 9. In this connection it would be worthwhile to study the famous examples of Artal-Bartolo, Cassou-Nogues, and Luengo-Velasco [**ACL**] in which they construct curves whose redsets and singsets are empty and whose ranks are arbitrarily high.

Question 11. In connection with the Defset Theorem (9.12), consider the Example given by the pencil $(h(Y) - c)_{c \in k}$ where

$f = h(Y) \in k[Y]$ is monic of degree $N > 1$; using (9.5) we can easily show that: (i) $\rho_\pi(f) = 1-N$, (ii) $\rho_a(f-c) = 1-N+\deg_Y[f-c]$ for all $c \in k$, and (iii) $\rho_\pi(f) = \sum_{c\in k}[\rho_\pi(f)-\rho_a(f-c)]$; thus $\rho_a(f-c) \geq \rho_\pi(f)$ for all $c \in k$, and $\rho_a(f-c) > \rho_\pi(f)$ for precisely those $c$ for which $c = h(c')$ with $h_Y(c') = 0$. It is easy to make similar examples for pencils obtained by replacing $Y$ by any nonconstant polynomial in $R$. However we ask: is it true that if the pencil $(f-c)_{c\in k}$ is noncomposite, i.e., if $k(f)$ is m.a. in $k(X,Y)$, then $\rho_\pi(f) \geq \rho_a(f-c)$ for all $c \in k$? If this has a positive answer then, for noncomposite pencils, the term $2\deg_Y\widehat{[f]}$ may be dropped from the estimate given in (9.12).

QUESTION 12. Can rank and defset be generalized to $n > 2$?

**MORAL.** Thus it is all a matter of measuring change in a quantity relative to the corresponding change in another quantity on which the previous quantity depends. This after all is the Newton-Leibnitz idea of derivative. By avoiding limits, which are algebraically awkward to take, it also gives rise to the twisted derivative introduced in [**A07**] and exploited in numerous succeeding papers summarized in [**A10**] for calculating Galois groups and hence fundamental groups. The same principle of measuring change gives rise to the invariants $I, \beta, \gamma, \rho, \rho_a, \rho_\pi, \jmath, \zeta$. Amongst these, $\beta$ measures the change in $I$, the quantities $\beta, \gamma, \rho, \rho_a, \rho_\pi$ are different incarnations of the same underlying reality, and the almost identical quantities $\jmath$ and $\zeta$ measure the change in $\rho$. Or, as may be easier to remember, the genus $\gamma$ measures the change in the intersection multiplicity $I$, while the Zeuthen-Segre invariant $\zeta$ measures the change in the genus $\gamma$.

## 10. Defset of a General Pencil

In this section we continue to assume $n = 2$ with $(X_1, X_2) = (X, Y)$, and $k$ is of characteristic zero with $k =$ its algebraic closure $k^*$. We shall now extend our study of rank and defset to a general pencil $(f - cw)_{c\in k\cup\{\infty\}}$ where $w \in R^\times = R \setminus \{0\}$ with $\gcd(f, w) = 1$ and $d = \max(\deg(f), \deg(w))$. Recall that by convention $f - \infty w = w$.

Let $\mathcal{L}_\infty = \mathcal{P} \setminus \mathcal{A} = \{(\infty, v) : v \in k \cup \{\infty\}\}$ and call this the line at infinity. In homogeneous coordinates $(X, Y, Z)$ we think of $\mathcal{L}_\infty$ as given by $Z = 0$, and also as the $(Y/Z)$-axis. As abbreviations, for the two special points of this line, i.e., for the two infinite points of the fundamental triangle, we put $(\infty) = (\infty, 0)$ and $((\infty)) = (\infty, \infty)$ and note that in homogeneous coordinates they are $(1, 0, 0)$ and $(0, 1, 0)$.

Let $R_0 = k[[X,Y]]$ and note that $k(X,Y) \cap R_0$ is the localization of $R$ at the maximal ideal $(X,Y)R$ and $R_0$ is its completion. Let $e \geq 0$ be an integer. Define the $e$-th homogenization of any $g \in R$ by putting $g_{[e]} = g_{[e]}(X,Y,Z) = Z^{e+\deg(g)} g(X/Z, Y/Z)$ in case $g \neq 0$, and $g_{[e]} = g_{[e]}(X,Y,Z) = 0$ in case $g = 0$. For any $Q = (u,v) \in \mathcal{P}$, define the $k$-monomorphism $T^{[Q,e]} : R \to R_0$ with $g \mapsto g^{[Q,e]} = g^{[Q,e]}(X,Y)$ by putting

$$(10.1) \qquad g^{[Q,e]}(X,Y) = \begin{cases} g(X+u, Y+v) & \text{if } Q \in \mathcal{A} \\ g_{[e]}(1, Y+v, X) & \text{if } ((\infty)) \neq Q \in \mathcal{L}_\infty \\ g_{[e]}(X, 1, Y) & \text{if } Q = ((\infty)) \end{cases}$$

and call this the Taylor map at $[Q,e]$.

Maclaurin and Taylor were two disciples of Newton of calculus-fame, the former expanding things around the origin and the latter around other points! Newton having accomplished many other things, the adjective "of calculus-fame" is clearly directed more towards the disciples!!.

For any $Q \in \mathcal{P}$ let $T^{[Q]} : R \to R_0$ be given by $g \mapsto g^{[Q]} = g^{[Q,0]}$. Note that if $g^{[Q]}(0,0) \neq 0$ then $\overline{\chi}(g;Q) = -1$, and if $g^{[Q]}(0,0) = 0 \neq g$ then $g^{[Q]}$ is product of $1 + \overline{\chi}(g;Q)$ irreducibles in $R_0$. Also note that for any $g,h$ in $R$ we have $I(g,h;Q) = [R_0/(g^{[Q]}, h^{[Q]})R_0 : k]$. For any $g, h, \overline{g}, \overline{h}$ in $R$ we put

$$(10.2) \qquad I(g,h;\overline{g} \setminus \overline{h}) = \sum_{\{Q \in \mathcal{A} : \overline{g}^{[Q]}(0,0) = 0 \neq \overline{h}^{[Q]}(0,0)\}} I(g,h;Q).$$

Let

$$t = |\mathcal{V}(f,w)|$$

and let $Q_1 = (u_1, v_1), \ldots, Q_t = (u_t, v_t)$ in $\mathcal{A}$ be the $t$ points of $\mathcal{V}(f,w)$ called the finite base points of the pencil. For any $g, h, \overline{g}$ in $R$ we put

$$(10.3) \qquad I(g,h;\overline{g} \setminus \mathcal{V}(f,w)) = \sum_{\{Q \in \mathcal{A} \setminus \{Q_1, \ldots, Q_t\} : \overline{g}^{[Q]}(0,0) = 0\}} I(g,h;Q).$$

To avoid repeating the considerations of the previous section and for simplicity of calculation, most of the time we shall suppose that:

(r*) the elements $f, w, 1$ are linearly independent over $k$,

i.e., by replacing $f, w$ by suitable $k$-linear combinations of them, our pencil cannot be converted into the special pencil considered in the previous section.

From the previous section recall condition

(*) $\qquad f$ is $Y$-monic of $Y$-degree $N > 0$

and consider condition

(**) $\qquad$ (*) and $w$ is $Y$-monic of $Y$-degree $0 < M < N$.

We claim that:
(10.4)

$$\text{(r*)} \Rightarrow \begin{cases} \text{there exists a } k\text{-automorphism } \sigma \text{ of } R \\ \text{together with } c_i^* \neq c_2^* \text{ in } k \cup \{\infty\} \text{ such that} \\ \text{if we let the pair } (\sigma(f - c_1^* w), \sigma(f - c_2^* w)) \text{ be called } (f, w) \\ \text{then condition (**) is satisfied.} \end{cases}$$

Namely, by a well-known argument there exists a $k$-automorphism $\sigma_1$ of $R$ such that $\sigma_1(f), \sigma_1(w)$ are both Y-monic. If their degrees are distinct, then we already have (**). If not, then we can find $c_1^*, c_2^* \in k$ such that $\sigma_1(f - c_1^* w)$ has bigger degree than $\sigma_1(f - c_2^* w)$, which must be positive because of (r*). By applying a suitable $k$-automorphism $\sigma_2$ of $R$ to these two polynomials, we can arrange that condition (**) is satisfied. Now take $\sigma$ to be the composition of $\sigma_2$ with $\sigma_1$.

Analogous to (9.5) we shall prove that:

$$\text{(10.5)} \qquad (^{**}) \Rightarrow \begin{cases} \text{for any } c \in k \text{ we have:} \\ \rho_a(f - cw) \\ = (1 - N) \\ \quad + \deg_Y[f - cw] \\ \quad + I\left(f - cw, \frac{f_Y w - f w_Y}{\widehat{[f,w]}}; (f - cw) \setminus w\right) \\ \quad - I\left(f_X w - f w_X, \frac{f_Y w - f w_Y}{\widehat{[f,w]}}; (f - cw) \setminus w\right) \\ \quad + \sum_{1 \le i \le t} \left(I(X - u_i, \frac{f - cw}{[f - cw]}; Q_i) - 1\right) \end{cases}$$

with all terms integers, where

$$\widehat{[f,w]} = \gcd(f_X w - f w_X, f_Y w - f w_Y)$$

with, as before, the gcd made unique by requiring it to be $Y$-monic. While proving (10.5) we shall show that:
(10.6)

$$(^{**}) \Rightarrow \text{for any } c \in k \cup \{\infty\} \text{ we have } [f - cw] = \gcd(f - cw, \widehat{[f,w]}).$$

As a consequence of (10.5) we shall show that:
(10.7)

$$(\mathrm{r}^*) \Rightarrow \begin{cases} \text{there exists a unique integer } \rho_\pi(f,w) \text{ together with} \\ \text{a unique finite subset } \mathrm{defset}(f,w) \text{ of } k \cup \{\infty\} \text{ such that} \\ \rho_a(f-cw) = \rho_\pi(f,w) \text{ for all } c \in (k\cup\{\infty\}) \setminus \mathrm{defset}(f,w) \\ \text{and } \rho_a(f-cw) \neq \rho_\pi(f,w) \text{ for all } c \in \mathrm{defset}(f,w). \end{cases}$$

In reference to (10.7), we put

$$\rho_\pi(f,w) = \text{the pencil-rank of the pencil } (f-cw)_{c\in k\cup\{\infty\}}$$

and

$$\mathrm{defset}(f,w) = \text{the deficiency set of } (f,w).$$

From (10.7) we shall deduce that:
(10.8)

$$(^{**}) \Rightarrow \begin{cases} \rho_\pi(f,w) \\ = (1-N) \\ \quad +\max_{c\in k} I\left(f-cw, \frac{f_Y w - f w_Y}{\widehat{[f,w]}}; (f-cw)\setminus w\right) \\ \quad + \sum_{1\le i\le t} \min_{c\in k\setminus \mathrm{multset}(f,w)^*} \left(I(X-u_i, \frac{f-cw}{[f-cw]}; Q_i) - 1\right) \end{cases}$$

with all terms integers. From (10.8) we shall deduce that:

$$(10.9) \quad (^{**}) \Rightarrow \begin{cases} \rho_\pi(f,w) \ge \rho_a(f-cw) \\ \text{for all } c \in k \setminus (\mathrm{multset}(f,w)^* \cup \mathrm{conset}(f,w)^*) \end{cases}$$

where, without assuming $k = k^*$, the contact set of $(f,w)$ is defined by putting

$$\mathrm{conset}(f,w)^* = \cup_{Q=(u,v)\in\mathcal{V}(f,w)^*} \mathrm{conset}(f,w;Q)^*$$

with the set $\mathrm{conset}(f,w;Q)^*$ of size at most one defined by

$$\begin{cases} \mathrm{conset}(f,w;Q)^* \\ = \{c \in k^* : I(X-u, f-cw; Q) \\ \qquad > I(X-u, f-c'w; Q) \text{ for some } c' \in k^* \setminus \mathrm{multset}(f,w)^*\}. \end{cases}$$

From (10.8) we shall deduce that:
(10.10)

$$(^{**}) \Rightarrow \begin{cases} \rho_a(f) + \rho_a(w) \\ = 1 - t + \sum_{c\in \mathrm{defset}(f,w)\setminus\{0,\infty\}} [\rho_\pi(f,w) - \rho_a(f-cw)]. \end{cases}$$

Clearly (10.10) is equivalent to saying that:

$$(10.11) \qquad (^{**}) \Rightarrow \zeta(f,w) = -1$$

where

$$\zeta(f,w) = -t - 2\rho_\pi(f,w) + \sum_{c\in \operatorname{defset}(f,w)} [\rho_\pi(f,w) - \rho_a(f-cw)].$$

Finally, from (10.11) we shall deduce the:

(10.12) General Defset Theorem. If (**) then we have:

(i) $(a(f,w)\setminus b(f,w)) \subset \operatorname{defset}(f,w)$ where $a(f,w) = k \cap \operatorname{singset}(f,w)$ and $b(f,w) = \operatorname{multset}(f,w)^* \cup \operatorname{conset}(f,w)^*$.

(ii) $|\operatorname{defset}(f,w)| \le \rho_a(f) + \rho_a(w) + \deg_Y \widehat{[f,w]} + 2t + 1$.

(iii) $|\operatorname{singset}(f,w)| \le \rho_a(f) + \rho_a(w) + 2\deg_Y \widehat{[f,w]} + 3t + 2$.

Before turning to the proof of items (10.5) to (10.12), let us establish some common

Notation and Calculation. Given any $f' \in R^\times$, write $f' = \theta f'_1 \dots f'_{s'}$ where $f'_1,\dots,f'_{s'}$ are irreducible members of $R\setminus k$ and $\theta \in k^\times$, and let $\phi'_i : R \to A'_i = R/(f'_iR)$ be the canonical epimorphism and identify $k$ with a subfield of $L'_i = \mathrm{QF}(A'_i)$. Let $Z_i = \{V \in \mathfrak{R}(f'_i,\infty) : \operatorname{ord}_V \phi'_i(f) \ge \operatorname{ord}_V\phi'_i(w)\}$ and $P_i = \{V \in \mathfrak{R}(f'_i,\infty) : \operatorname{ord}_V\phi'_i(f) < \operatorname{ord}_V\phi'_i(w)\}$, where the letters $Z$ and $P$ are meant to suggest zeros and poles, and note that these are obviously finite sets.

Consider the conditions:

$$(0') \qquad \gcd(w,f') = 1,$$

(1')

$$\begin{cases} \text{for a given } Q=(u,v)\in\mathcal{A} \text{ with } f(u,v)=0, \text{ and} \\ \text{for every irreducible factor } g \text{ of } f' \text{ in } R\setminus k \text{ with } g(u,v)=0 \\ \neq w(u,v) \text{ we have } w^2(f/w)_Y \in gR \text{ with } f\notin gR \text{ and} \\ w^2(f/w)_X \notin gR, \end{cases}$$

(2') $f' = w^2(f/w)_Y$ and $Q=(u,v)\in\mathcal{A}$ with $f(u,v) = 0 \neq w(u,v)$

and

$$(3') \qquad \gcd(f,f') = 1.$$

Assuming (0'): Let

$$Z((f,w),f') = \sum_{1\le i\le s'} \sum_{V\in Z_i} [\operatorname{ord}_V\phi'_i(f) - \operatorname{ord}_V\phi'_i(w)]$$

and

$$P((f,w),f') = \sum_{1\le i\le s'} \sum_{V\in P_i} [\mathrm{ord}_V \phi_i'(f) - \mathrm{ord}_V \phi_i'(w)]$$

and note that $Z((f,w),f')$ is a nonnegative integer or $\infty$ according as $\gcd(f,f')=1$ or $\gcd(f,f')\neq 1$, and $P((f,w),f')$ is always a nonpositive integer. Clearly for each $V\in Z_i$ there is a unique $c_i(V)\in k$ such that $\mathrm{ord}_V \phi_i'(f-c_i(V)w) > \mathrm{ord}_V \phi_i'(w)$. Let $D((f,w),f') = \cup_{1\le i\le t}\{c_i(V) : V \in Z_i\}$ and let $E((f,w),f') = D((f,w),f')\cup \mathrm{singset}((f,w))$. Note that clearly $D((f,w),f')$ is a finite subset of $k$ and hence by the General Singset Theorem so is $E((f,w),f')$.

Without assuming $(0')$ but assuming (**): Write $f=f_1\dots f_s$ where $f_1,\dots,f_s$ are irreducible $Y$-monic members of $R\setminus k$, and let $\phi_i : R\to A_i = R/(f_iR)$ be the canonical epimorphism and identify $k$ with a subfield of $L_i = \mathrm{QF}(A_i)$.

With these conditions in mind, we have (I) to (IV) stated below.

(I) If (**)+$(1')$ then

$$I(f,f';Q) - I(w^2(f/w)_X, f';Q) = I(X-u,f';Q)$$

where all the terms are integers.

(II) If (**)+$(2')$ then

$$I(X-u,f';Q) = \overline{\chi}(f;Q) + \sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,Q)} \mathrm{ord}_V d\phi_i(X)$$

where all the terms are integers.

(III) If (**)+$(0')+(3')$ then for all $\lambda\in k$ we have

$$I(f,f';\mathcal{A}) - I(w,f';\mathcal{A}) = -Z((f,w),f') - P((f-\lambda w,w),f')$$

where all the terms are integers.

(IV) If (*) then

$$1-\deg_Y f = 2+\overline{\chi}(f;\infty) + \sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,\infty)} \mathrm{ord}_V d\phi_i(X)$$

with all terms integers.

**Proof of (I)**. If (**)+($1'$) then we have, with all terms integers,

$$\begin{aligned}
\text{LHS of (I)} &= \sum_{1\le i\le s'} \big[(I(f,f'_i;Q) - I(w^2(f/w)_X, f'_i;Q)\big] \\
&= \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,Q)} \big[\mathrm{ord}_V \phi'_i(f) - \mathrm{ord}_V \phi'_i(w^2(f/w)_X)\big] \\
&= \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,Q)} \mathrm{ord}_V \phi'_i(X-u) \\
&= \sum_{1\le i\le s'} I(X-u, f'_i;Q) \\
&= \text{RHS of (I)}.
\end{aligned}$$

**Proof of (II)**. If (**)+($2'$) then we have, with all terms integers,

$$\begin{aligned}
\text{LHS of (II)} &= -1 + I(X-u, f;Q) \\
&= -1 + \sum_{1\le i\le s} \sum_{V\in\mathfrak{R}(f_i,Q)} \mathrm{ord}_V \phi_i(X-u) \\
&= -1 + \sum_{1\le i\le s} \sum_{V\in\mathfrak{R}(f_i,Q)} [\mathrm{ord}_V d\phi_i(X) + 1] \\
&= \text{RHS of (II)}.
\end{aligned}$$

**Proof of (III)**. If (**)+($0'$) + ($3'$) then we have, with all terms integers,

LHS of (III)

$= -Z(f,f') - P(f,f') + Z(w,f') + P(w,f')$

(since number of zeros of a function equals number of its poles)

$=$ RHS of (III)

(since $\mathrm{ord}_V \phi'_i(f) = \mathrm{ord}_V \phi'_i(f-\lambda w)$ for all $V \in P_i$ and $\lambda \in k$.)

**Proof of (IV).** If (*) then we have, with all terms integers,

$$\begin{aligned}
\text{RHS of (IV)} &= 2+\overline{\chi}(f;\infty)+\sum_{1\le i\le s}\sum_{V\in\mathfrak{R}(f_i,\infty)}(\mathrm{ord}_V\phi_i(X)-1)\\
&\qquad(\text{because } \mathrm{ord}_V\phi_i(X)<0)\\
&= 2+\overline{\chi}(f;\infty)-\deg_Y f-\sum_{1\le i\le s}|\mathfrak{R}(f_i,\infty)|\\
&= 1-\deg_Y f\\
&= \text{LHS of (IV)}.
\end{aligned}$$

**Proof of (10.5) and (10.6).** Assume (**). Then by the argument in the proof of the Singset Theorem we see that
(i)
for any $c\in k\cup\{\infty\}$ we have $f-cw=[f-cw]\overline{f}$ where $\overline{f}=\mathrm{rad}(f-cw)$

and

$$\text{for any } c\in k\cup\{\infty\} \text{ we have } [f-cw]=\gcd(f-cw,\widehat{[f,w]})$$

which proves (10.6). Take $f'=\frac{f_Yw-fw_Y}{\widehat{[f,w]}}$. For any $c\in k$ and $Q=(u,v)\in\mathcal{A}$ with $f(u,v)-cw(u,v)=0\neq w(u,v)$, by using (I) we get

(ii) $\quad I(f-cw,f';Q)-I(f_Xw-fw_X,f';Q)=I(X-u,f';Q)$

with all terms integers. Let $\overline{f}=\mathrm{rad}(f-cw)$. Let $\psi:R\to R/(X-u)R$ be the canonical epimorphism, and let $W$ be the localization of $\psi(R)$ at $\psi((Y-v)R)$. Then we have, with all terms integers,

$$\begin{aligned}
&I(X-u,w^2(\overline{f}/w)_Y;Q)\\
&\quad=\mathrm{ord}_W\psi(w)+\mathrm{ord}_W\psi(\overline{f})-1\\
&\quad=\mathrm{ord}_W\psi(\overline{f})-1\\
&\qquad\quad(\text{since } w(u,v)\neq 0)\\
&\quad=\mathrm{ord}_W\psi(f-cw)-\mathrm{ord}_W\psi([f-cw])-1\\
&\quad=\mathrm{ord}_W\psi(w^2((f-cw)/w)_Y)-\mathrm{ord}_W\psi([f-cw])\\
&\quad=\mathrm{ord}_W\psi(w^2((f-cw)/w)_Y)-\mathrm{ord}_W\psi(\widehat{[f-w]})\\
&\qquad\quad(\text{since } \mathcal{V}(f-cw)\cap\mathcal{V}(\widehat{[f,w]}/[f-cw])=\emptyset\\
&\quad=\mathrm{ord}_W\psi(f')\\
&\quad=I(X-u,f';Q)
\end{aligned}$$

and hence, with all terms integers, we have

(iii) $$I(X-u, w^2(\overline{f}/w)_Y; Q) = I(X-u, f'; Q).$$

Now, upon letting $\widetilde{\sum}$ and $\widehat{\sum}$ stand for summations over $\{Q=(u,v)\in \mathcal{A} : f(u,v)-cw(u,v)=0\neq w(u,v)\}$ and $\{Q=(u,v)\in\mathcal{A} : f(u,v)-cw(u,v)=0\}$ respectively, we have, with all terms integers,

$$\begin{aligned}
&\text{RHS of (10.5)}\\
&= 1-N+\deg_Y[f-cw]+\sum_{1\le i\le t}\big(I(X-u_i,\overline{f};Q_i)-1\big)\\
&\qquad +\widetilde{\sum} I(X-u,w^2(\overline{f}/w)_Y;Q)\\
&\qquad\qquad\qquad \text{(by (i), (ii) and (iii))}\\
&= 1-\deg_Y\overline{f}+\widehat{\sum}\left[\overline{\chi}(\overline{f};Q)+\sum_{1\le i\le \overline{s}}\ \sum_{V\in\mathfrak{R}(\overline{f}_i,Q)}\operatorname{ord}_V d\overline{\phi}_i(X)\right]\\
&\qquad \text{(by taking } \overline{f} \text{ for } f \text{ in (II) and writing } \overline{f}=\overline{f}_1\dots\overline{f}_{\overline{s}} \text{ with}\\
&\qquad\quad \text{irreducible } Y\text{-monic members } \overline{f}_1,\dots,\overline{f}_{\overline{s}} \text{ of } R\setminus k \text{ and}\\
&\qquad\quad \text{letting } \overline{\phi}_i : R\to R/(\overline{f}_iR) \text{ be the canonical epimorphism)}\\
&= \rho_a(\overline{f})\\
&\quad \text{(by (IV))}\\
&= \text{LHS of (10.5).}
\end{aligned}$$

**Proof of (10.7) to (10.12)**. In view of (10.4), as in the proof of (9.7) to (9.12), while proving (10.7) to (10.12) we may and **we shall assume that** $(f,w)$ **satisfies (**).** Take

$$f'=\frac{f_Yw-fw_Y}{\widehat{[f,w]}}\qquad\text{and}\qquad f''=f_Xw-fw_X.$$

Let

$a(f,w)=k\cap\operatorname{singset}(f,w)$ and $b(f,w)=\operatorname{multset}(f,w)^*\cup\operatorname{conset}(f,w)^*$.

Now in the RHS of (10.5), for all $c\in k\setminus \operatorname{multset}(f,w)^*$ the second line is zero, for $c\in k\setminus D((f,w),f')$ the third line attains a maximum, for $c\in\operatorname{singset}(f,w)$ the fourth line is zero, and for $c\in k\setminus b(f,w)$ the fifth line attains a minimum. This proves that

(1) $\quad \operatorname{defset}(f,w)\subset \big(D((f,w),f')\cup \operatorname{singset}(f,w)\cup\operatorname{conset}(f,w)^*\big)$

and establishes (10.7) and (10.8). Clearly (10.9) is evident from (10.8). Also note that if $c \in \text{singset}(f,w) \setminus b(f,w)$ then the above consideration of the RHS lines of (10.5) establishes that $c \in \text{defset}(f,w)$, and hence

$$(a(f,w) \setminus b(f,w)) \subset \text{defset}(f,w) \tag{2}$$

which proves part (i) of (10.12). Clearly $|\text{conset}(f,w)^*| \le t$ and hence by (10.9) and assuming (10.10) we get

(3)
$$|\text{defset}(f,w)| \le \rho_a(f)+\rho_a(w)+t-1+|\text{multset}(f,w)^*|+|\text{conset}(f.w)^*|+2$$

where the last number 2 is added for possible $0,\infty$ in $\text{defset}(f,w)$ not accounted by (10.10). This gives

$$|\text{defset}(f,w)| \le \rho_a(f)+\rho_a(w)+\deg_Y \widehat{[f,w]}+2t+1 \tag{4}$$

showing that (10.10) $\Rightarrow$ part (ii) of (10.12). By (2) we see that we get

$$|\text{singset}(f,w)| \le |\text{defset}(f,w)| + \deg_Y \widehat{[f,w]}+t+1$$

and hence by (4) we get

$$|\text{singset}(f,w)| \le \rho_a(f)+\rho_a(w)+2\deg_Y \widehat{[f,w]}+3t+2 \tag{5}$$

showing that (10.10) $\Rightarrow$ part (iii) of (10.12). Thus, it only remains to prove (10.10). We shall do this in STEPS (6) to (12).

STEP (6). Using the proof of (10.5) we shall now show that:

$$(10.5^*)\qquad (^{***}) \Rightarrow \begin{cases} \rho_a(w) & = (1-M)+\deg_Y[w] \\ & \quad +I\left(w,f';w\setminus f\right)-I\left(f'',f';w\setminus f\right) \\ & \quad +\sum_{1\le i\le t}\left(I\left(X-u_i,\frac{w}{[w]};Q_i\right)-1\right) \end{cases}$$

where condition

(***) $\qquad f$ and $w$ are $Y$-monic of positive $Y$-degree $N \neq M$.

is obviously weaker than condition (**). So let $\overline{w}=\text{rad}(w)$. Then by the argument in the proof of the Singset Theorem we see that: $[w]=\gcd(w,\widehat{[f,w]})$ and $w=[w]\overline{w}$.

The idea of the proof is to redo the calculations in (10.5) reversing the role of $f,w$ and for this purpose, let us note that under our current notation, we have:

(i) $\qquad w^2(f/w)_Y = -f^2(w/f)_Y$ and $w^2(f/w)_X = -f^2(w/f)_X$

Note that our arguments in (I) thru (IV) remain valid under exchange of $f,w$, if we change the degree condition (**) by the weaker condition (***). Indeed, the degrees, $N, M$ never enter the calculations in (I) to (IV).

For $Q = (u, v) \in \mathcal{A}$ with $w(u, v) = 0 \neq f(u, v)$, by using calculations of (I) we get

$$\text{(ii)} \qquad I(w, f'; Q) - I(f'', f'; Q) = I(X - u, f'; Q)$$

with all terms integers. Let $\psi : R \to R/(X - u)R$ be the canonical epimorphism, and let $W$ be the localization of $\psi(R)$ at $\psi((Y - v)R)$. Then we have, in view of (i) and with all terms integers,

$$\begin{aligned}
I(X - u, f^2(\overline{w}/f)_Y; Q) &= \mathrm{ord}_W \psi(f) + \mathrm{ord}_W \psi(\overline{w}) - 1 \\
&= \mathrm{ord}_W \psi(\overline{w}) - 1 \\
&\qquad (\text{since } f(u, v) \neq 0) \\
&= \mathrm{ord}_W \psi(w) - \mathrm{ord}_W \psi([w]) - 1 \\
&= \mathrm{ord}_W \psi(f^2(w/f)_Y) - \mathrm{ord}_W \psi([w]) \\
&= \mathrm{ord}_W \psi(f^2(w/f)_Y) - \mathrm{ord}_W \psi(\widehat{[f, w]}) \\
&\qquad (\text{since } \mathcal{V}(w) \cap \mathcal{V}(\widehat{[f, w]}/[w]) = \emptyset \\
&= \mathrm{ord}_W \psi(f') \\
&= I(X - u, f'; Q)
\end{aligned}$$

and hence, with all terms integers, we have

$$\text{(iii)} \qquad I(X - u, f^2(\overline{w}/f)_Y; Q) = I(X - u, f'; Q).$$

Now, upon letting $\widetilde{\sum}$ and $\widehat{\sum}$ stand for summations over

$$\begin{aligned}
&\{Q = (u, v) \in \mathcal{A} : w(u, v) = 0 \neq f(u, v)\} \quad \text{and} \\
&\{Q = (u, v) \in \mathcal{A} : w(u, v) = 0\}
\end{aligned}$$

respectively, we have, with all terms integers,

RHS of (10.5*)

$$= 1 - M + \deg_Y[w] + \sum_{1\le i\le t} (I(X-u_i, \overline{w}; Q_i) - 1)$$
$$+ \widetilde{\sum} I(X-u, f^2(\overline{w}/f)_Y; Q)$$
(by (ii) and (iii))
$$= 1 - \deg_Y \overline{w} + \widehat{\sum}\left[\overline{\chi}(\overline{w};Q) + \sum_{1\le i\le \overline{s}} \sum_{V\in\mathfrak{R}(\overline{w}_i,Q)} \mathrm{ord}_V d\overline{\phi}_i(X)\right]$$
(by taking $\overline{w}$ for $w$ in (II) and writing $\overline{w} = \overline{w}_1 \dots \overline{w}_{\overline{s}}$ with irreducible $Y$-monic members $\overline{w}_1, \dots, \overline{w}_{\overline{s}}$ of $R \setminus k$ and letting $\overline{\phi}_i : R \to R/(\overline{w}_i R)$ be the canonical epimorphism)
$$= \rho_a(\overline{w})$$
(by (IV))
$$= \text{RHS of } (10.5)^*.$$

STEP (7). Combining (10.5*) with (10.5) we see that for any $c \in k$ we have:

(7*) $$\rho_a(f - cw) + \rho_a(w) + t - 1 = \sum_{1\le j\le 4} F_j(c)$$

where

$$F_1(c) = (1 - N - M) + \deg_Y[f - cw] + \deg_Y[w]$$

and

$$F_2(c) = I\left((f-cw)w, f'; (f-cw)w\right) - I\left(f'', f'; (f-cw)w\right)$$

and

$$F_3(c) = \sum_{1\le l\le t} \left(I(f'', f'; Q_l) - I((f-cw)w, f'; Q_l)\right)$$

and

$$F_4(c) = \sum_{1\le l\le t} \left(I\left(X - u_l, \frac{(f-cw)w}{[f-cw][w]}; Q_l\right) - 1\right).$$

Fix some $c_\pi \in k$ such that $\rho_a(f - c_\pi w) = \rho_\pi(f, w)$ and such that $c_\pi$ gives the various extremal values as described in (10.8). Explicitly, we assume that $c_\pi$ is chosen so that it satisfies the following additional conditions (i) to (iv).

(i) For each $V \in \mathfrak{R}(f_i', \infty)$, with $1 \leq i \leq s'$, we have $\mathrm{ord}_V(\phi_i'(f - c_\pi w)) = \min(\mathrm{ord}_V(\phi_i'(f), \mathrm{ord}_V(\phi_i'(w))$.

(ii) For each $V \in \mathfrak{R}(f_i', Q_l)$, with $1 \leq i \leq s'$ and $1 \leq l \leq t$, we have $\mathrm{ord}_V(\phi_i'(f - c_\pi w) = \min(\mathrm{ord}_V(\phi_i'(f), \mathrm{ord}_V(\phi_i'(w))$.

(iii) For $1 \leq l \leq t$ we have $I(X - u_i, f - c_\pi w; Q_i) = \min(I(X - u_i, f; Q_i), I(X - u_i, w; Q_i))$.

(iv) $c_\pi \notin \mathrm{defset}(f, w)$ and hence in particular $I\,(f'', f';\;\; (f - c_\pi w) \setminus w) = 0$ by the General Singset Theorem.

For various numerical functions $F(c)$ to be considered, with $c$ varying in $k$, let $H(F(c))$ denote the variation $\sum_{c \in D}(F(c_\pi) - F(c))$ where $D$ is a finite subset of $k$ which is defined below and which is a large enough "defset" to be applicable to all the relevant $F$'s.

(i*) For each $V \in \mathfrak{R}(f_i', \infty)$, with $1 \leq i \leq s'$, we define $c_V \in k$ thus. In case $\mathrm{ord}_V \phi_i'(f) < \mathrm{ord}_V \phi_i'(w)$, we take $c_V = c_\pi$. In case $\mathrm{ord}_V \phi_i'(f) \geq \mathrm{ord}_V \phi_i'(w)$, we take $c_V$ to be the unique element of $k$ such that $\mathrm{ord}_V \phi_i'(f - c_V w) > \mathrm{ord}_V \phi_i'(f - c_\pi w)$. Let $D_{\mathrm{i}}$ be the set of all these elements $c_V$.

(ii*) For each $V \in \mathfrak{R}(f_i', Q_l)$, with $1 \leq i \leq s'$ and $1 \leq l \leq t$, we define $c_V \in k$ thus. In case $\mathrm{ord}_V \phi_i'(f) < \mathrm{ord}_V \phi_i'(w)$, we take $c_V = c_\pi$. In case $\mathrm{ord}_V \phi_i'(f) \geq \mathrm{ord}_V \phi_i'(w)$, we take $c_V$ to be the unique element of $k$ such that $\mathrm{ord}_V \phi_i'(f - c_V w) > \mathrm{ord}_V \phi_i'(f - c_\pi w)$. Let $D_{\mathrm{ii}}$ be the set of all these elements $c_V$.

(iii*) For $1 \leq l \leq t$ we define $c_l \in k$ thus.
In case $I(X - u_l, f - c_\pi w; Q_l) = \max\{I(X - u_l, f - cw; Q_l) : c \in k\}$, we take $c_l = c_\pi$.
In case $I(X - u_l, f - c_\pi w; Q_l) < \max\{I(X - u_l, f - cw; Q_l) : c \in k\}$, we take $c_l$ to be the unique element of $k$ such that $I(X - u_l, f - c_l w; Q_l) = \max\{I(X - u_l, f - cw; Q_l) : c \in k\}$. Let $D_{\mathrm{iii}} = \{c_1, \ldots, c_t\}$.

(iv*) Let $D_{\mathrm{iv}}$ be the union of $\mathrm{defset}(f, w) \cap k$ and $\mathrm{singset}(f, w)$.

Let $D = D_{\mathrm{i}} \cup D_{\mathrm{ii}} \cup D_{\mathrm{iii}} \cup D_{\mathrm{iv}}$.

Now, in view of (7*), equation (10.10) is equivalent to the equation

$$\rho_a(f - c_\pi w) + \rho_a(w) + t - 1 = \sum_{1 \leq j \leq 4} H(F_j(c))$$

and hence to the equation

$$\sum_{1 \leq j \leq 4} H(F_j(c)) = \sum_{1 \leq j \leq 4} F_j(c_\pi). \tag{10.10*}$$

*In words, (10.10*) says that the function* $\sum_{1 \leq j \leq 4} F_j(c)$, *or equivalently the function* $\rho_a(f - cw) + \rho_a(w) + t - 1$, *replicates itself, i.e.,* *it has a constant value at most points and that value equals the total variation of the function.*

STEP (8). In view of (9.6) and (10.6) we see that

$$(8^*) \qquad H(F_1(c))) = -\deg_Y \widehat{[f,w]} + \deg_Y[w].$$

STEP (9). For each $V \in \mathfrak{R}(f'_i, \infty)$, with $1 \le i \le s'$, we clearly have

$$(9a) \qquad -\mathrm{ord}_V \phi'_i(f'') = \mathrm{ord}_V \phi'_i(X) - \mathrm{ord}_V \phi'_i((f - c_V w)w).$$

Now

$$\begin{cases} \mathrm{ord}_V \phi'_i(f) < \mathrm{ord}_V \phi'_i(w) \\ \Rightarrow \mathrm{ord}_V \phi'_i((f-cw)w) = \mathrm{ord}_V \phi'_i(fw) \text{ for all } c \in k \end{cases}$$

and thus in this case, using (9a), we get

$$\begin{aligned} &- H\left(\mathrm{ord}_V \phi'_i((f-cw)w)\right) - \mathrm{ord}_V \phi'_i(f'') \\ &= 0 - \mathrm{ord}_V \phi'_i(f'') \\ &= \mathrm{ord}_V \phi'_i(X) - \mathrm{ord}_V \phi'_i((f - c_\pi w)w). \end{aligned}$$

Likewise

$$\begin{cases} \mathrm{ord}_V \phi'_i(f) \ge \mathrm{ord}_V \phi'_i(w) \\ \Rightarrow \mathrm{ord}_V \phi'_i((f-cw)w) \\ = \mathrm{ord}_V \phi'_i(fw) \text{ except for exactly one } c = c_V \in k \end{cases}$$

and thus in this case, again using (9a), we get

$$\begin{aligned} &- H\left(\mathrm{ord}_V \phi'_i((f-cw)w)\right) - \mathrm{ord}_V \phi'_i(f'') \\ &= -\mathrm{ord}_V(\phi'_i((f-c_\pi w)w)) + \mathrm{ord}_V(\phi'_i((f-c_V w)w)) - \mathrm{ord}_V \phi'_i(f'') \\ &= \ \mathrm{ord}_V \phi'_i(X) - \mathrm{ord}_V \phi'_i((f - c_\pi w)w). \end{aligned}$$

Consequently we always have

$$(9b) \qquad \begin{cases} -H\left(\mathrm{ord}_V \phi'_i((f-cw)w)\right) - \mathrm{ord}_V \phi'_i(f'') \\ = \mathrm{ord}_V \phi'_i(X) - \mathrm{ord}_V \phi'_i((f - c_\pi w)w). \end{cases}$$

Clearly

$$(9c) \qquad \begin{aligned} H(F_2(c)) = H(I\left((f-cw)w, f'; (f-cw)w\right)) \\ -H(I\left(f'', f'; (f-cw)w\right)). \end{aligned}$$

Since $I\left(f'', f'; (f - c_\pi w)w \setminus w\right) = 0$ by our choice of $c_\pi$, we also have

$$(9d) \qquad I\left(f'', f'; (f - c_\pi w)w\right) = I\left(f'', f'; w\right).$$

Now

$$\begin{aligned}
&H(I(f'', f'; (f - cw)w)) \\
&= \sum_{c\in D}(I(f'', f'; (f - c_\pi w)w) - I(f'', f'; (f - cw)w)) \\
&= -\sum_{c\in D} I(f'', f'; (f - cw)w \setminus w) \quad \text{ by (9d)} \\
&= -I((f'', f'; \mathcal{A} \setminus w)) \\
&= I(f'', f'; w) - I(f'', f'; \mathcal{A}) \\
&= I(f'', f'; w) + \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,\infty)} \mathrm{ord}_V \phi'_i(f'')
\end{aligned}$$

and

$$\begin{aligned}
&H(I\left((f - cw)w, f'; (f - cw)w\right)) \\
&= -\sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,\infty)} H(\mathrm{ord}_V \phi'_i((f - cw)w) \\
&= \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,\infty)} [\mathrm{ord}_V \phi'_i(f'') + \mathrm{ord}_V \phi'_i(X) - \mathrm{ord}_V \phi'_i((f - c_\pi w)w)] \\
&\qquad \text{by (9b)}
\end{aligned}$$

and hence by (9c) we get

$$\begin{aligned}
H(F_2(c)) = &-\deg_Y(f') - I(f'', f'; w) \\
&- \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,\infty)} \mathrm{ord}_V \phi'_i((f - c_\pi w)w).
\end{aligned}$$

Clearly have $\deg_Y(f') = \deg_Y(f_Y w - f w_Y) - \deg_Y(\widehat{[f, w]})$ and, since $N \neq M$, we also have

$$\deg_Y(f') = N + M - 1 - \deg_Y(\widehat{[f, w]}).$$

Combining the above two displayed equations we conclude that

$$\begin{aligned}
H(F_2(c)) =& 1 - N - M + \deg_Y \widehat{[f, w]} - I(f'', f'; w) \\
&- \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,\infty)} \mathrm{ord}_V \phi'_i((f - c_\pi w)w)
\end{aligned}$$

where the last line is clearly equal to $I((f-c_\pi w)w, f', \mathcal{A})$ and hence by using (8*) we get
(9e)

$$\begin{cases} H(F_1(c)) + H(F_2(c)) \\ = 1 - N - M + \deg_Y[w] - I(f'', f'; w) + I((f-c_\pi w)w, f'; \mathcal{A}). \end{cases}$$

Clearly $\deg_Y[f - c_\pi w] = 0$ and hence by (9d) and (9e) we conclude that

$$(9^*) \qquad H(F_1(c)) + H(F_2(c)) = F_1(c_\pi) + F_2(c_\pi).$$

Step (10). Upon letting

$$F_{3,l}(c) = I(f'', f'; Q_i) - I((f-cw)w, f'; Q_l)$$

we get

$$\begin{aligned} &H(F_{3,l}(c)) \\ &= -H(I((f-cw)w, f'; Q_l)) \\ &= -\sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,Q_l)} H(\mathrm{ord}_V \phi'_i((f-cw)w) \\ &= \sum_{1\le i\le s'} \sum_{V\in\mathfrak{R}(f'_i,Q_l)} \big[\mathrm{ord}_V \phi'_i(X-u_l) - \mathrm{ord}_V \phi'_i((f-c_\pi w)w)) \\ &\qquad\qquad\qquad + \mathrm{ord}_V \phi'_i(f'')\big] \qquad \text{as in (9a)} \\ &= I(X-u_l, f'; Q_l) - I((f-c_\pi w)w, f'; Q_l) + I(f'', f'; Q_l) \end{aligned}$$

and hence

$$(10^*) \qquad \begin{cases} F_3(c) = \sum_{1\le l\le t} F_{3,l}(c) \text{ with} \\ H(F_{3,l}(c)) = F_{3,l}(c_\pi) + I(X-u_l, f'; Q_l). \end{cases}$$

Step (11). Upon letting

$$F_{4,l}(c) = \left( I\left( X - u_l, \frac{(f-cw)w}{[f-cw][w]}; Q_l \right) - 1 \right)$$

we clearly have

$$(11a) \quad \begin{cases} F_{4,l}(c) = & I(X-u_l, f-cw; Q_l) + I(X-u_l, w; Q_l) - 1 \\ & -I(X-u_l, [f-cw]; Q_l) - I(X-u_l, [w]; Q_l). \end{cases}$$

Let $\mu_l = I(X - u_l, f - c_\pi w; Q_l)$ and $\theta_l = \max\{I(X - u_l, f - cw; Q_l) : c \in k\}$ and $\nu_l = I(X - u_l, w; Q_l)$. Then

$$\begin{aligned}
&H(F_{4,l}(c)) \\
&= H(I(X - u_l, f - cw; Q_l)) - H(I(X - u_l, [f - cw]; Q_l)) \\
&= \mu_l - \theta_l - H(I(X - u_l, [f - cw]; Q_l)) \\
&= \mu_l - \theta_l - I(X - u_l, [w]; Q_l) + I(X - u_l, \widehat{[f, w]}; Q_l) \\
&= \mu_l - \theta_l - I(X - u_l, [w]; Q_l) + [\nu_l - 1 + \theta_l - I(X - u_l, f'; Q_l)] \\
&= I(X - u_l, (f - c_\pi w)w; Q_l) - 1 - I(X - u_l, f'; Q_l) \\
&\qquad\qquad\qquad\qquad - I(X - u_l, [w]; Q_l)
\end{aligned}$$

and hence, because of (11a) and the obvious fact that $I(X - u_l, [f - c_\pi w]; Q_l) = 0$, we get

$$\text{(11b)} \qquad \begin{cases} F_4(c) = \sum_{1 \le l \le t} F_{4,l}(c) \text{ with} \\ H(F_{4,l}(c)) = F_{4,l}(c_\pi) - I(X - u_l, f'; Q_l). \end{cases}$$

By (10*) and (11b) we see that

$$\text{(11*)} \qquad H(F_3(c)) + H(F_4(c)) = F_3(c_\pi) + F_4(c_\pi).$$

STEP (12). By (9*) and (11*) we get (10.10*).

**Conclusion.** Genus plus excess branch number is the rank of a curve. The total variation of the rank as a curve moves thru a pencil is independent of the pencil. A suitable modification of this variation is the Zeuthen-Segre invariant of the surface. For the plane it equals minus one. The defset of a polynomial is the set of translation constants which produce nongeneral rank. The defset gives a bound on the singset, i.e., the set of translation constants which produce singular curves or more generally singular hypersurfaces. The redset of a polynomial is the set of translation constants which produce reducible hypersurfaces. Bounds for the redset are found in terms of the group of units of the affine coordinate ring.

## 11. Linear Systems and Pencils on Normal Varieties

To say a word about the Zeuthen-Segre invariant of a surface, let us very briefly talk about linear systems and pencils on normal varieties.

So assume $k = k^*$, let $E$ be an irreducible $n$-dimensional normal algebraic variety over $k$, for $0 \le i \le n$ let $E_i$ be the set of all irreducible $i$-dimensional subvarieties of $E$, for any $C \in \cup_{0 \le i \le n} E_i$ let

$k(C)$ be the local ring of $C$ on $E$, let $\widehat{E}_i = \{k(C) : C \in E_{n-i}\}$, and finally let $\widehat{E} = \cup_{0\le i\le n}\widehat{E}_i$. Then, in the language of models (see [**A09**]), $\widehat{E}_i$ is the set of all $i$-dimensional members of the $n$-dimensional normal model $\widehat{E}$ of $k(E)/k$.

Recall that a premodel $\widehat{E}$ of a finitely generated field extension $K/k$ is a collection of local domains whose quotient field is $K$ and which have $k$ as a subring; $\widehat{E}$ is irredundant (resp: complete) means any valuation ring of $K/k$ dominates at most (resp: at least) one of its members; $\widehat{E}$ is a model if it is an irredundant premodel which can be expressed as a finite union $\widehat{E} = \cup_{0\le j\le m}\mathfrak{V}(B_j)$ where $B_j = k[x_{j0}, \ldots, x_{jm}]$ = an affine domain over $k$ and where $\mathfrak{V}(B_j)$ = the set of all localizations $(B_j)_P$ with $P$ varying over spec$(B_j)$. The normality assumption says that there is an injection $R \to B_j$ such that $B_j$ is the integral closure of the image in a finite algebraic field extension of the quotient field of the image, or equivalently that every member of $\widehat{E}$ is normal. The normality assumption implies that $\widehat{E}_1$ is a set of DVRs of $k(E)/k = K/k$. Recall that $E$ or $\widehat{E}$ is nonsingular means every member of $\widehat{E}$ is a regular local ring, and so nonsingular $\Rightarrow$ normal. If $m$ can be taken to be 0 then we call $\widehat{E}$ (resp: $E$) to be an affine model (resp: affine variety). If we can find nonzero elements $z_0, \ldots, z_m$ in $k(E)$ such that $x_{ji} = z_i/z_j$ for all $i, j$ in $\{0, \ldots, m\}$ then we call $\widehat{E}$ (resp: $E$) to be a projective model (resp: projective variety).

Now any $C \in E_{n-i}$ can be recovered from $k(C) \in \widehat{E}_i$ by observing that (closed) points $P$ in $C$ are characterized by saying that $k(P)$ are those members of $\widehat{E}_n$ for which $k(C)$ belongs to $\mathfrak{V}(k(P))$. Thus we may dispense with the geometric beginning of commencing with an algebraic variety, and start (and end) with a model. This economy of thought is the beauty of the language of models.

Let $\mathcal{D}$ be the group of all divisors on $E$, i.e., the set of all maps $E_{n-1} \to \mathbb{Z}$ with finite support. Let $\mathcal{D}_+$ be the set of all effective divisors, i.e., nonzero divisors $D$ with $D(E_{n-1}) \subset \mathbb{N}$ = the set of all nonnegative integers. The degree deg$(D)$ of any divisor $D$ is defined to be $\sum D(C)$ taken over all $C$ in $E_{n-1}$. The divisor $(z)$ of any $z \in k(S)^\times$ is defined by the equation $(z)(C) = \mathrm{ord}_{k(C)} z$. For any $k$-vector subspace $H$ of $k(E)$, by $\mathbb{P}(H)$ we denote the associated projective space, i.e., the set of all 1-dimensional subspaces of $H$; for any nonzero $z, z'$ in any $y \in \mathbb{P}(H)$ we clearly have $(z) = (z')$ and this common divisor is denoted by $(y)$. By a linear system on $E$ we mean a subset

$\mathcal{C}$ of $\mathcal{D}_+$ for which there exists a finite dimensional $k$-vector-subspace $H$ of $k(E)$ together with $D' \in \mathcal{D}$ such that $y \mapsto (y) + D'$ gives a bijection $\mathbb{P}(H) \to \mathcal{C}$, and for which there does not exist $D_0 \in \mathcal{D}_+$ such that for all $D \in \mathcal{C}$ we have $D \geq D_0$; the second proviso means that we assume our linear systems to be devoid of fixed components. It is easily seen that the dimension of $H$ depends only on $\mathcal{C}$, and we call this dimension decreased by one to be the dimension of $\mathcal{C}$. If the dimension of $\mathcal{C}$ is one then we call $\mathcal{C}$ a pencil.

Now assume that $k$ is of characteristic zero with $n = 2$, and $E$ is an irreducible nonsingular projective algebraic surface. For any $C \in E_1$ let $\gamma(C)$ be its genus, and let us generalize this to any $D \in \mathcal{D}_+$ by putting

$$\gamma(D) = 1 + \sum_{C \in E_1} D(C)(\gamma(C) - 1).$$

For any $Q \in E_0$ the completion $\widetilde{Q}$ of $k(Q)$ is clearly isomorphic to $k[[X, Y]]$. For any $D \in \mathcal{D}_+$ we let $\chi(D; Q)$ denote the number of branches of $D$ at $Q$, i.e., upon letting $M$ stand for maximal ideal,

$$\Big(\textstyle\prod_{\{C \in E_1 : D(C) > 0 \text{ and } k(Q) \subset k(C)\}} [k(Q) \cap M(k(C))]^{D(C)}\Big)\, \widetilde{Q} = U_0 U_1 \ldots U_{\chi(D;Q)} \widetilde{Q}$$

where $U_0$ is a unit in $\widetilde{Q}$, and $U_1, \ldots, U_{\chi(D;Q)}$ are irreducible nonunits in $\widetilde{Q}^{\times}$. Let

$$\overline{\chi}(D; Q) = \chi(D; Q) - 1$$

and put

$$\overline{\chi}(D; E) = \sum_{\{Q \in E_0 : \chi(D;Q) > 1\}} \overline{\chi}(D : Q).$$

Let $\mathrm{rad}(D) \in \mathcal{D}_+$ be defined by

$$(\mathrm{rad}(D))(C) = \begin{cases} 1 & \text{if } D(C) > 0 \\ 0 & \text{if } D(C) = 0. \end{cases}$$

Also put

$$\rho_a(D) = 2\gamma(\mathrm{rad}(D)) + \overline{\chi}(\mathrm{rad}(D); E).$$

Finally define the base point set of a pencil $\mathcal{C}$ on $E$ by putting

$$B(\mathcal{C}) = \cap_{D \in \mathcal{C}} S_0(D)$$

where the curve-support and point-support of $D$ are given by

$$\begin{aligned} &S_1(D) = \{C \in E_1 : D(C) \neq 0\} \text{ and} \\ &S_0(D) = \{Q \in E_0 : k(Q) \subset \cup_{C \in S_1(D)} k(C)\}. \end{aligned}$$

If there exists a finite subset defset($\mathcal{C}$) and an integer $\rho_\pi(\mathcal{C})$ such that for any $D \in \mathcal{C}$ we have: $\rho_\pi(\mathcal{C}) = \rho_a(D) \Leftrightarrow D \in \mathcal{C}\setminus\text{defset}(\mathcal{C})$, then these two objects are clearly unique and we call them the **deficiency set** and the **pencil-rank** of the pencil $\mathcal{C}$ on $E$. When this is so,

we define the Zeuthen-Segre invariant of $\mathcal{C}$ to be the integer $\zeta(\mathcal{C})$ given by

$$\zeta(\mathcal{C}) = -|B(\mathcal{C})| - 2\rho_\pi(\mathcal{C}) + \sum_{D\in\text{defset}(\mathcal{C})} [\rho_\pi(\mathcal{C}) - \rho_a(D)].$$

**EPILOGUE.** Assume it has been shown that defset($\mathcal{C}$) and $\rho_\pi(\mathcal{C})$ exist for every pencil on $E$, and $\zeta(\mathcal{C})$ depends only on $E$ and not on $\mathcal{C}$. Let $\zeta(E)$ be the Zeuthen-Segre invariant of the surface $E$, i.e., the common value of $\zeta(\mathcal{C})$ for all pencils $\mathcal{C}$ on $E$. Taking any two distinct members $F$ and $G$ of $\mathcal{C}$, and adding $\rho_a(F)+\rho_a(G)-\zeta(E)$ to both sides of the above equation we get the jungian formula

$$\rho_a(F) + \rho_a(G) = -\zeta(E) - |B(\mathcal{C})| + \sum_{D\in\text{defset}(\mathcal{C})\setminus\{F,G\}} [\rho_\pi(\mathcal{C}) - \rho_a(D)].$$

Thinking of $F$ and $G$ as "curves on the surface" $E$, their "common points" or "set-theoretic intersection" is given by $I^*(F,G) = S_0(F) \cap S_0(G)$, and clearly we have $B(\mathcal{C}) = I^*(F,G)$. Moreover, $F$ and $G$ have no common component, i.e., there is no $C \in E_1$ with $F(C) \neq 0 \neq G(C)$. Also $\mathcal{C}$ is "generated" by $F$ and $G$; so every member of $\mathcal{C}$ can symbolically be written as $F - cG$ with $c \in k \cup \{\infty\}$ with $F - \infty G = G$. In the situation of the previous section, this symbolism becomes more real by taking $E$ to be the projective plane $\mathcal{P}$ over $k$, and taking $F = Z^d f(X/Z, Y/Z)$ and $G = Z^d w(X/Z, Y/Z)$. As a thought for the future, going back to the section on More General Pencils, $\rho_\pi$ of the pencil could be defined as $\rho_a$ of the generic member $(f,w)^\flat$ with affine coordinate ring $k(f/w)[X,Y]$ over ground field $k(f/w)$. Similar trick could be played when $E$ is any nonsingular surface.

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