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Generalized Newton-Puiseux Expansion and Abhyankar-Moh Semigroup Theorem

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1. Introduction

Let k be an algebraically closed field of characteristic zero and $k((X))$ the field of formal power series in X over k.

If $f(Y)\in k((X^n))[Y]$ is monic and irreducible of degree n then Newton's Theorem states that $f(Y)$ factors in $k((X))[Y]$ as

$$f(Y)=\prod_{\omega^n=1}(Y-\eta(\omega X)) \quad \text{where} \quad \eta(X)\in k((X)).$$

If further $f(Y)\in S[Y]$ for some subring S of $k((X^n))$, then Abhyankar and Moh discovered important structure theorems for the semigroup

$$\{\mathrm{ord}_X H(\eta(X))\,|\,0\neq H(Y)\in S[Y]\}.$$

Accounts of these results are found in the various papers of Abhyankar and others listed at the end of this introduction.

What we present here is a generalization of these results when X is replaced by an ordered p-tuple of indeterminates $(x_1,\ldots,x_p)$. Among the various treatments available we have chosen the notation and results from the lecture notes (termed ET in this paper) by B. Singh. The reason is that this treatment avoids the use of irreducibility criteria, which, even though true in our generalized situation, are somewhat lengthy.

In a sense this paper should be a one-liner: Let $X=(x_1,\ldots,x_p)$ and imitate all proofs as if it is still a single variable. We have stuck to this philosophy as far as possible. All the new definitions are given unless they are trivially routine generalizations. We have not given the details of the proofs when they are similar to the corresponding proofs in ET but we have written down the statements of the main steps of the whole proof. In short, this paper is intended to be read with a copy of ET in hand (unless the reader knows the details by heart).

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The raison de être of this paper is the Main Corollary (3.3) which is the *crucial step* in our "Polynomial rings in two variables over a d.v.r.: a criterion" (in this issue pages 159–168). That paper settles a conjecture of Dolgachev and Weisfeiler concerning affine plane bundles over d.v.r. by removing the deficiency from a previous attempt of Kambayashi ("On one parameter family of affine plane," Invent. Math. **52**, 275–281 (1979).)

Main References for the Abhyankar-Moh Theory

1. Abhyankar, S.S., Moh, T.T.: Newton-Puiseux expansion and generalized Tschirnhausen transformation. J. Reine Angew. Math. **260**, **261**, 47–83 and 29–54 (1973)
   – The original treatment of most of the important technical results; actual applications are found elsewhere
2. Abhyankar, S.S.: Expansion techniques in algebraic geometry. Lecture Notes by B. Singh, T.I.F.R. (1977)
   – One of the shortest treatment of the results, yet fully self-contained and elementary. Various partial results on the Jacobian Problem is an added bonus
3. Abhyankar, S.S.: On the semigroup of a meromorphic curve. Proc. Int. Symp. on Algebraic Geometry, Kyoto 1977, 249–414
   – The "extremely elementary" treatment with every little detail spelled out. It is written in a leisurely style of classical mathematical works; only handicap, if it should be considered as such, is the length.

2. Preliminaries

2.1. Notation and Convention

We assume all notations and conventions of ET. Here we recall a few for the convenience of the reader.

$$\mathbb{Z} = \text{the set of integers}, \quad \mathbb{Z}_+ = \{a \in \mathbb{Z} \mid a \geqq 0\}$$

$$\mathbb{Q} = \text{the set of rational numbers}$$

$$\mathbb{Q}_+ = \{a \in \mathbb{Q} \mid a \geqq 0\}.$$

A field k will be fixed throughout and will be even assumed to be algebraically closed of characteristic zero in most results.

For $0 \neq n \in \mathbb{Z}_+$ we let $\mu_n(k) = \{\omega \in k \mid \omega^n = 1\}$. Since k is usually algebraically closed in our setup we simply write μ_n.

Given a (possibly long) power series

$$a = \sum_{i \in A} a_i X^i$$

where A could be a well-ordered subset of an ordered abelian group, we write

$$\text{Supp}\, a = \{i \in A \mid a_i \neq 0\}$$

and

$$\operatorname{ord} a = \operatorname{ord}_X a = \inf\{i \mid i \in \operatorname{Supp} a\}.$$

If $\operatorname{ord} a = \alpha$ then we also define

the initial form of $a = \operatorname{info}(a) = a_\alpha X^\alpha$

and

the initial coefficient of $a = \operatorname{inco}(a) = a_\alpha$.

Finally we recall Abhyankar's generic "nonzero" θ. By the symbol θ we denote some nonzero element of an appropriate set of coefficients (usually the field k). The symbol θ can stand for several different quantities even in the same formula and $a = \theta$, $b = \theta$ should not be taken to imply $a = b$.

2.2. *Multi-Laurent Series*

Let $x = (x_1, \ldots, x_p)$ be a vector of indeterminates over a field k. By $k\langle\langle x \rangle\rangle$ we denote the field $k((x_1))((x_2))\ldots((x_p))$. In other words $k\langle\langle (x_1, \ldots, x_p)\rangle\rangle$ may be inductively defined as $k\langle\langle (x_1, \ldots, x_{p-1})\rangle\rangle((x_p))$ where for any field L and an indeterminate y, $L((y)) =$ the field of Laurent series in y over $L = \left\{ \sum_{i=s}^{\infty} a_i y^i \,\middle|\, s \in \mathbb{Z} \right.$ and $\left. a_i \in L \right\}$.

If $n = (n_1, \ldots, n_p) \in \mathbb{Q}_+^p$ is a vector of positive fractions then we can easily define $k\langle\langle x^n \rangle\rangle$ in a similar way. Let $m = (m_1, \ldots, m_p)$ be a similar vector of positive fractions. Now if $n_i = h_i m_i$ for suitable positive integers h_i then in a natural fashion we get $k\langle\langle x^n \rangle\rangle \subset k\langle\langle x^m \rangle\rangle$.

We now define the Multi-Laurent field

$$\mathcal{L}(k, p) = \bigcup \{k\langle\langle x^m \rangle\rangle \mid m \in \mathbb{Q}^p \text{ is a vector of positive fractions}\}.$$

2.3. *Generalized Newton's Lemma*

The usual Newton's Lemma ET (5.5) van be easily seen to prove a slightly stronger statement:

Any finite algebraic extension K of $k((x))$ is contained in $k'((x^{1/n}))$ for some finite algebraic extension of k and some positive integer n; provided char $k = 0$.

To prove this statement we simply note that the usual lemma says that K is contained in $k^*((x^{1/n}))$ where k^* is an algebraic closure of k and then a simple Galois theory argument shows that it is already contained in $k'((x^{1/n}))$.

Now we get a further

Generalized Newton's Lemma. *If* $\operatorname{char} k = 0$, *and* k *is algebraically closed, then* $\mathcal{L}(k, p)$ *is an algebraic closure of* $k\langle\langle (x_1, \ldots, x_p)\rangle\rangle$.

Proof. Easy induction on p.

2.4. Another Representation of $k\langle\langle x\rangle\rangle$

We fix indeterminate X over k and for $n=(n_1,\ldots,n_p)\in\mathbb{Q}_+^p$ define $X^n=x_1^{n_p}\ldots x_p^{n_p}$.

We also define the reverse lexicographic order on $\mathbb{Q}^p$ be declaring

$$n=(n_1,\ldots,n_p)<(m_1,\ldots,m_p)=m$$

if $n\neq m$ and the last $m_i\neq n_i$ satisfies $m_i>n_i$.

It is then easy to see that $k\langle\langle(x_1,\ldots,x_p)\rangle\rangle$ can be also represented as power series in X with exponents in the ordered abelian group $\mathbb{Z}^p$ such that the coefficients are in k and the support of each power series is well-ordered.

Denote by $k\langle\langle X,\mathbb{Q}^p\rangle\rangle$ the field

$$\{\sum_{\lambda\in\Lambda} a_\lambda X^\lambda \mid \Lambda\subset\mathbb{Q}^p \text{ is well-ordered and } a_\lambda\in k\}.$$

As a natural notation we set $\operatorname{ord}_X(\sum a_\lambda X^\lambda)=\min\{\lambda\in\operatorname{Supp}(\sum a_\lambda X^\lambda)\}=\min\{\lambda\mid a_\lambda\neq 0\}$.

More generally for any subsemigroup H of $\mathbb{Q}^p$ we define

$$k\langle\langle X,H\rangle\rangle=\{f\in k\langle\langle X,\mathbb{Q}^p\rangle\rangle\mid\operatorname{Supp} f\subset H\}.$$

In particular we get

$$k\langle\langle(x_1,\ldots,x_p)\rangle\rangle=k\langle\langle X,\mathbb{Z}^p\rangle\rangle \quad \text{etc.}$$

2.5. Characteristic Sequences (ET, § 6).

We fix $p\geqq 1$ and $x=(x_1,\ldots,x_p)$ as before. Let $v\neq 0\in\mathbb{Z}$ and $J\subset\mathbb{Z}^p$ be a well-ordered set. *For any $j=(j_1,\ldots,j_p)\in\mathbb{Q}^p$ we define $\bar{j}=j_1$.*

We define $m_{i+1}(v,J)$, $\bar{m}_i(v,J)$ and $d_{i+1}(v,J)$ by induction on i as in ET (6.4).

Set $\bar{m}_0(v,J)=v$, $d_1(v,J)=|v|$, $m_1(v,J)=\inf(J)$, $\bar{m}_1(v,J)=\overline{m_1(v,J)}$ and for $i\geqq 2$

$$d_i(v,J)=\text{g.c.d.}\,(d_{i-1}(v,J),\bar{m}_{i-1}(v,J)),$$
$$m_i(v,J)=\inf\{j\in J\mid\bar{j}\not\equiv 0\,(\operatorname{mod} d_i(v,J))\},$$
$$\bar{m}_i(v,J)=\overline{m_i(v,J)}.$$

As in ET (6.5) we get a unique integer h depending on v,J such that for the first time $m_{h+1}(v,J)$ is the infimum of an empty set. By general convention we take it to be ∞ (here interpreted as a terminal element to be attached to the ordered group $\mathbb{Q}^p$). We denote $\bar{m}_{h+1}(v,J)$ also by (the usual) $\infty-$ the meaning in either case will be evident from the context. $d_{h+1}(v,J)$ becomes g.c.d. $(\{\bar{j}\mid j\in J\}\cup\{v\})=d$ say. The rest of ET (6.5) generalizes easily.

From now on we make the

Hypothesis. *k is algebraically closed of characteristic zero.*

Recall that μ_n is the set off all n^{th} roots of unity. If $\omega\in\mu_n$ define a k-

automorphism $\tilde{\omega}$ of $k\langle\langle X, \mathbb{Z}^p\rangle\rangle = k\langle\langle x\rangle\rangle$ by

$$\tilde{\omega}(\sum a_j X^j) = \sum a_j \omega^{\bar{j}} X^j.$$

Note that $\tilde{\omega}$ is nothing but the unique k-automorphism setting $\tilde{\omega}(X^{(j_1,\ldots,j_p)}) = \omega^{j_1} X^{(j_1,\ldots,j_p)}$.

$$\text{Set } \tilde{\mu}_n = \{\tilde{\omega} | \omega \in \mu_n\}.$$

Let $y(x) \in k\langle\langle x\rangle\rangle$ and let

$$f = \prod_{\tilde{\omega}\in\tilde{\mu}_n} Y - \tilde{\omega}(y(x)).$$

Evidently $f \in k\langle\langle x^{\tilde{n}}\rangle\rangle[Y]$ where $\tilde{n} = (n, 1, \ldots, 1)$.

By $\operatorname{Supp} f$ we denote $\operatorname{Supp} y(x)$ and it is evident that $\operatorname{Supp} y(x) = \operatorname{Supp} \tilde{\omega}(y(x))$ for any $\tilde{\omega} \in \tilde{\mu}_n$. Suppose that $\{\bar{j} | j \in \operatorname{Supp} f\}$ is a subset of $\mathbb{Z}$ with g.c.d. 1. Then it is easy to see that $f \in k\langle\langle x^{\tilde{n}}\rangle\rangle[Y]$ is irreducible and monic of degree n.

We now define the various characteristic sequences associated with f.

Setting $J = \operatorname{Supp} f$ we get $h(v, f) = h$, say, as before. The *m-sequence associated with* f is

$$m(v, f) = (m_0(v, J), \ldots, m_{h+1}(v, J)).$$

We define $q(v, f)$, $s(v, f)$, $r(v, f)$ exactly as in ET (6.9) and in addition define $\bar{q}(v, f)$, $\bar{s}(v, f)$, $\bar{r}(v, f)$ by taking "$\bar{\ }$" (i.e. the first components) of the elements of corresponding sequences. The rest of ET (§6) generalizes easily if we use appropriate elements of m, q, s, r sequences if order comparison is involved and the corresponding "$\bar{\ }$" ed quantities if divisibility comparison is involved.

2.6. *Approximate and Pseudoapproximate Roots*

Let $y(x)$, f and $0 \neq v \in \mathbb{Z}$ be fixed as in 2.5 and such that $|v| = n = \deg_Y f$. (*Since now v, f are fixed we suppress them from the notation of characteristic sequences.*)

Let S be a ring contained in $k\langle\langle x^{\tilde{n}}\rangle\rangle$ such that $f \in S[Y]$.

Write as before $J = \operatorname{Supp} y(x)$ and *write*

$$y(x) = \sum_{j\in J} a_j X^j.$$

Define for $1 \leqq e \leqq h+1$

$$y_e^{\#} = \sum_{m_e > j \in J} a_j X^j.$$

Define G_e by the equation

$$G_e^{d_e} = \prod_{\omega\in\mu_e} Y - \tilde{\omega}(y_e^{\#}).$$

Then G_e is monic of degree n/d_e in Y and is contained in the ring $k\langle\langle x^{\tilde{n}}\rangle\rangle[Y]$. G_e is called the *pseudo-approximate* d_e^{th} *root of* f.

The *approximate* d_e^{th} *root of* f denoted by $\text{App}_Y^{d_e}(f)$ or simply $\text{App}^{d_e}(f)$ is defined to be the unique monic polynomial g_e of degree n/d_e satisfying

$$\deg(f - g_e^{d_e}) < n - n/d_e.$$

The existence and uniqueness of $\text{App}^{d_e}(f)$ and the fact that $f \in S[Y] \Rightarrow \text{App}^{d_e}(f) \in S[Y]$ is proved in ET (§4) and all the arguments are valid without a change.

Given any monic polynomial g of degree n/d_e we have the g-adic expansion of f, namely:

$$f = g^{d_e} + \sum_{i=0}^{d_e - 1} c_f^{(i)}(g) g^i$$

where $c_f^{(i)}(g)$ are polynomials of degree $< n/d_e$ in Y and are uniquely determined by f, g.

The Tschirnhausen operator

$$\tau_f(g) = g + c_f^{(d_e - 1)}(g)/d$$

changes g to $\tau_f(g)$ which is again monic of degree n/d_e. (See ET § 3 for the above.)

The main result about the operators τ_f is that if g is any polynomial of degree n/d_e as above then $\tau_f^i(g) = \text{App}_Y^{d_e}(f)$ for all $i > n/d_e$, and in particular $\tau_f^i(g) \in S[Y]$ whenever $f \in S[Y]$. (See ET (4.8).)

3. The Main Results

Now we outline the contents of ET § 7 as required for our main theorem and as before we only point out variations, if any, in the proofs. Note that all polynomials in ET have variables X, Y or t^n, Y. In adopting the proofs we just drop the reference to X or t^n as these have been absorbed into coefficients in our notation.

3.1. The Main Lemma. *For* $1 \leqq e \leqq h+1$, *set*

$$g_e = \begin{cases} Y & \text{if } e = 1 \\ \text{App}_Y^{d_e}(f) & \text{if } e > 1 \end{cases}$$

Let $y_e^* = y_e^\# + Z X^{m_e} + u$ *where* $\text{ord}_X u > m_e$ *and* y_e^* *belongs to* $k' \langle\langle X, \mathbb{Q}^p \rangle\rangle$ *for some overfield* k' *of* k. *Thus* $\text{info}(y_e^* - y_e^\#) = Z X^{m_e}$.

Then we have

$$\text{Info}(G_e(y_e^*)) = \theta Z X^{r_e}, \tag{3.1.1}$$

$$\text{Info}(G_i(y_e^*)) = \theta X^{r_i} \quad \text{if } i < e, \tag{3.1.2}$$

$$\text{Info}(f(y_e^*)) = \theta (Z^{n_e} - a_{m_e}^{n_e})^{d_{e+1}} X^{s_e}. \tag{3.1.3}$$

In particular if $n_e > 1$ *then* $\text{Info}(f(y_e^*))$ *does not contain terms with* Z*-degree* $n_e d_{e+1} - 1 = d_e - 1$.

(3.1.4) *If* $c \in k \langle\langle x^n \rangle\rangle [Y]$ *with* $\deg c < n/d_e$ *then* $\text{Inco}\, c(y_e^*) = \theta$.

(3.1.5) *If* $g \in k\langle\langle x^{\bar{n}}\rangle\rangle[Y]$ *is monic of degree* n/d_e *with* $\mathrm{Info}(g(y_e^*)) = \theta Z X^{r_e}$ *then* $\mathrm{Info}(\tau_f(g)(y_e^*)) = \theta Z X^{r_e}$.

(3.1.6) $$\mathrm{Info}(g_e(y_e^*)) = \theta Z X^{r_e}.$$

Proof. (3.1.1) to (3.1.3) are routine calculations and can be explicitly seen in ET (7.13) to (7.16). (3.1.4) follows using (3.1.2) and g-adic expansions and the proof from ET (7.17) can be used with little variation.

(3.1.5) follows immediately from ET (7.18) and (3.1.6) is obtained by observing that G_e satisfies the hypothesis for (3.1.5) and then by applying τ_f repeatedly (and using (3.1.5) repeatedly). (We simply need to know that for $e > 1$ more than n/d_e iterations of τ_f on G_e give $\mathrm{App}_Y^{d_e}(f) = g_e$, whereas for $e = 1$ there is nothing to be proved.)

3.2. The Main Theorem. *Let* $f \in k\langle\langle x^{\bar{n}}\rangle\rangle[Y]$ *be such that*

$$f(Y) = \prod_{\omega \in \mu_n} Y - \tilde{\omega}(y(x)).$$

Let S *be a subring of* $k\langle\langle x^{\bar{n}}\rangle\rangle$ *such that* $f \in S[Y]$. *Then we have the following:*

(3.2.1) *Let* $\Theta: S[Y] \to S[y(x)] = S^* \approx S[Y]/(f)$ *be the homomorphism which is identity on* S *and sends* Y *to* $y(x)$. *Let* $\mathscr{G}_i = \Theta(g_i)$ *for* $i = 1, \ldots, h$. *Then* S^* *has a free* S*-basis*

$$\left\{ \prod_{i=1}^{h} \mathscr{G}_i^{c_i} \,\middle|\, 0 \leqq c_i < n_i \right\}.$$

(3.2.2) *If* $0 \leqq p_i, q_i < n_i$ *for* $i = 1, \ldots, h$ *are two distinct sequences, then*

$$\mathrm{ord}_X \left(b \prod_1^h \mathscr{G}_i^{p_i} \right) \neq \mathrm{ord}_X \left(c \prod_1^h \mathscr{G}_i^{q_i} \right).$$

(3.2.3) *For any subset* A *of* S^* *let us define*
$\Gamma(A) = \{\mathrm{ord}_X g \mid g \in A,\ g \neq 0\}$ *and* $\bar{\Gamma}(A) = \{\bar{\lambda} \mid \lambda \in \Gamma(A)\}$.
Then $\Gamma(S^*)$ *is generated by* $\Gamma(S)$ *and* $r_1, \ldots, r_h$ *whereas* $\bar{\Gamma}(S^*)$ *is generated by* $\bar{\Gamma}(S)$ *and* $\bar{r}_1, \ldots, \bar{r}_h$ *as semigroups.*

(3.2.4) *If* $\bar{\Gamma}(S) = -n\mathbb{Z}_+$ *and* $\bar{r}_1 < 0$ *then*
$-1 \in \bar{\Gamma}(S^*) \Rightarrow -n \mid \bar{r}_1$ *or* $\bar{r}_1 \mid -n$,

Proof. (3.2.1) is simply the statement about $g = (g_1, \ldots, g_h)$-adic expansions and is found in ET (8.3), (8.4).

(3.2.2) is essentially ET (8.5(ii)). To prove it we take "$\bar{\ }$" of the assumed inequality and observe that it is enough to show that

$$\overline{\mathrm{ord}_X b} + \sum_1^h p_i \overline{\mathrm{ord}_X \mathscr{G}_i} \neq \overline{\mathrm{ord}_X c} + \sum_1^h q_i \overline{\mathrm{ord}_X \mathscr{G}_i}.$$

Note that $\overline{\mathrm{ord}_X b}$ and $\overline{\mathrm{ord}_X c}$ are both divisible by $\bar{r}_0$, since $\bar{r}_0$ is an integer with absolute value n and $S \subset k\langle\langle x^{\bar{n}}\rangle\rangle$ clearly implies $\overline{\mathrm{ord}_X u} \equiv 0(n)$ for any $u \in S$.

Thus if we define p_0 by $\operatorname{ord}_X b = p_0 \bar{r}_0$ and q_0 by $\operatorname{ord}_X c = q_0 \bar{r}_0$, then we need to show that

$$\sum_0^h p_i \bar{r}_i \neq \sum_0^h q_i \bar{r}_i$$

whenever $0 \leqq p_i, q_i < n_i$ for $i = 1, \ldots, h$ and $(p_1, \ldots, p_h) \neq (q_1, \ldots, q_h)$. This follows from ET (1.5) by purely numerical properties of $\bar{r}_i$. (This proof is adapted from ET (8.5).)

(3.2.3) follows immediately from the above two parts by observing that

$$\operatorname{ord}_X \Big(\sum_{0 \leqq p_i < n_i} b_{p_1, \ldots, p_h} \mathscr{G}_1^{p_1} \ldots \mathscr{G}_h^{p_h} \Big)$$

is then simply equal to

$$\min \{ \operatorname{ord}_X (b_{p_1, \ldots, p_h} \mathscr{G}_1^{p_1} \ldots \mathscr{G}_h^{p_h}) \}.$$

(3.2.4) is a purely numerical exercise as in ET (1.8). We simply set $\bar{r}_0 = -n$ and note that from (3.2.3) $\bar{\Gamma}(S^*)$ is seen to be strictly generated by $(\bar{r}_0, \ldots, \bar{r}_h)$.

3.3. The Main Corollary. *We now give the special application for which this paper needed to be written.*

(3.3.1) **Corollary.** *Let* $U, V \in k[W_1][\![W_2, \ldots, W_p]\!]$ *such that* $\alpha =$ *the degree of* $U(W_1, 0, \ldots, 0)$ *and* $\beta =$ *the degree of* $V(W_1, 0, \ldots, 0)$ *are nonzero integers.*

We consider $k[W_1][\![W_2, \ldots, W_p]\!]$ as a subring of $k\langle\!\langle (W_1^{-1}, W_2, \ldots, W_p) \rangle\!\rangle$ and let ord denote the induced lexicographic order as multi-Laurent series in $(W_1^{-1}, W_2, \ldots, W_p)$.

Suppose that there exists $\psi \in k[U, V][\![W_2, \ldots, W_p]\!]$ such that

$$\operatorname{ord} \psi = (-1, b_2, \ldots, b_p).$$

Then $\alpha | \beta$ or $\beta | \alpha$.

Proof. Write

$$\psi = \sum \eta_{i_2, \ldots, i_p} W_2^{i_2} \ldots W_p^{i_p}$$

where $\eta_{i_2, \ldots, i_p} \in k[U, V]$. Set $\psi^* = \sum \eta_{i_2, \ldots, i_p} W_2^{i_2} \ldots W_p^{i_p}$ where the summation is taken over $(i_2, \ldots, i_p) \leqq (b_2, \ldots, b_p)$. Then ψ^* has the property that $\operatorname{ord}(\psi - \psi^*) > \operatorname{ord} \psi = (-1, b_2, \ldots, b_p)$ and consequently $\operatorname{ord} \psi^* = (-1, b_2, \ldots, b_p)$.

Thus we may assume without loss of generality that

$$\psi = \psi^* \in k[U, V, W_2, \ldots, W_p].$$

Now U, V, ψ all are members of the ring

$$k[U][W_2, \ldots, W_p][V] \subset k[U][\![W_2, \ldots, W_p]\!][V].$$

Note that

$$\operatorname{Info}(U) = \theta W_1^{\alpha}$$

and hence we can write

$$U = \theta W_1^{\alpha}(1 + H)$$

where $\operatorname{ord} H > (0, \ldots, 0)$.

By binomial expansion we can write $U = \theta X_1^{\alpha}$ where $X_1 = W_1^{-1}(1 + H)^{-1/\alpha}$.

Note that $k\langle\langle(W_1^{-1}, W_2, \ldots, W_p)\rangle\rangle = k\langle\langle(X_1, X_2, \ldots, X_p)\rangle\rangle$ where we set $(X_2, \ldots, X_p)=(W_2, \ldots, W_p)$. For any $\phi\in k\langle\langle(W_1^{-1}, W_2, \ldots, W_p)\rangle\rangle$ if the initial form is $\theta(W_1^{-1})^{i_1}W_2^{i_2}\ldots W_p^{i_p}$ then it is easy to see that as a power series in $X_1, \ldots, X_p$ the initial form is $\theta X_1^{i_1}X_2^{i_2}\ldots X_p^{i_p}$ and thus "ord" has the same meaning as $\mathrm{ord}_{(X_1,\ldots,X_p)}$.

Set $S=k[U][\![X_2, \ldots, X_p]\!]$ and we get that $\psi\in S[V]$ and thus

$$-1\in\bar{\Gamma}(S[V])\subset -\mathbb{Z}_+ \quad \text{so that } \bar{\Gamma}(S[V])=-\mathbb{Z}_+. \tag{1}$$

Now we

Claim.
$$F=\prod_{\omega\in\mu_\alpha}(Y-\tilde{\omega}(V))\in S[Y]$$

where as before $\tilde{\omega}$ is defined by

$$\tilde{\omega}(X_1^{a_1}\ldots X_p^{a_p})=\omega^{a_1}X_1^{a_1}\ldots X_p^{a_p}.$$

If the claim is assumed for a moment then we see that the result follows from (3.2.4) by taking $n=\alpha$ and noticing that $\bar{r}_1=\mathrm{ord}_{(X_1,\ldots,X_p)}V=-\beta$.

Now to prove the claim we write

$$V=\sum a_jX^j\in k\langle\langle X, \mathbb{Z}^p\rangle\rangle.$$

We must have that g.c.d. $(\{\bar{j}|j\in\mathrm{Supp}\,V\}\cup\{n\})=d$ is necessarily 1 since otherwise $S[V]\subset k((X_1^d))\langle\langle X_2, \ldots, X_p\rangle\rangle$ and (1) is not possible.

Note that the fixed field of all automorphisms in $\tilde{\mu}_\alpha$ is exactly

$$\mathscr{L}=k((X_1^\alpha))\langle\langle X_2, \ldots, X_p\rangle\rangle$$

so that

$$F\in\mathscr{L}[Y] \text{ is the minimum polynomial of } V \text{ over } \mathscr{L}.$$

If we show that V is integral over $S\subset\mathscr{L}$ then it will follow that the minimum polynomial $F\in S[Y]$ as required.

We note that upon writing

$$U=\sum_{i_2,\ldots,i_p\geqq 0}u_{i_2,\ldots,i_p}(W_1)W_2^{i_2}\ldots W_p^{i_p}$$

and repeated substituting

$$u_{0,\ldots,0}(W_1)=U-\sum_{\substack{i_2,\ldots,i_p\geqq 0\\(i_2,\ldots,i_p)\neq(0,\ldots,0)}}u_{i_2,\ldots,i_p}(W_1)W_2^{i_2}\ldots W_p^{i_p}$$

we can write

$$U=W_1^\alpha+\zeta_1W_1^{\alpha-1}+\ldots+\zeta_\alpha$$

where $\zeta_i\in k[U][\![W_2, \ldots, W_p]\!]=S$. Thus W_1 is integral over S. Substituting U for the monic polynomial in W_1 as obtained above and applying the division algorithm to the coefficients of V we can write

$$V=\eta_1W_1^{\alpha-1}+\ldots+\eta_0 \quad \text{with } \eta_i\in S.$$

Thus V is also integral over S.

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