Final Exam Ma 661 Spring 2014

You may use one sheet of your own notes during the exam. No other materials are permitted.

I have added brief hints for solutions.

Q.1 Let K be any field and let $R = K^{[2]}$ be a polynomial ring in two variables over K.

Find the following with proof:

- A non zero prime ideal P which is principal.

  Answer: Let $R = K[X,Y]$ be the ring. (X) is non zero, principal and $R/(X) = K[Y]$ a domain. Done!

- A non zero prime ideal Q which needs exactly two generators.

  Answer: Take the ideal $Q = (X,Y)$. $R/Q = K$, so maximal and hence prime. If Q were principal - say (f), then f is a non unit and divides both X, Y. In the U.F.D. R, X, Y are irreducible and not associates, so they cannot have such a common factor.

- A non zero ideal I which needs at least three generators.

  Answer: Take $I = (X^2, XY, Y^2)$ and let $M = (X,Y)$. Then argue that I/IM needs 3 generators and this implies, so does I. To argue, note that I/IM is annihilated by M and hence is a vector space over $K = R/M$. Prove that the three generators of I are independent in I/IM.

Q.2. Let $k \subset L$ be a Galois extension of degree $1001(13 \cdot 11 \cdot 7)$. Argue that every intermediate field $k \subset E \subset L$ is a Galois extension of k.

Extra/Hint Argue that the Galois group of L/k is abelian.

Answer: Let $H1, H2, H3$ be sylow subgroups of orders $13, 11, 7$ respectively. By Sylow theorems, they are all normal. Let a, b, c be their generators respectively. Then the commutator $[a,b] \in H_1 \bigcap H_2 = \{1\}$ and hence a, b commute. Similar deduce that all three commute and they generate the Galois group, which is then abelian. So all subgroups are normal and hence all their fixed fields are Galois.

Q.3. Consider the polynomial $f(X) = X^4 + 1$ in one variable X over the field $\mathbb{Q}$ of rational numbers. Answer the following with proof.

- What is the Galois group of the splitting field of $f(X)$ over $\mathbb{Q}$?

  Answer: Argue that it is irreducible over $\mathbb{Q}$ by checking for absence of linear and quadratic factors. Usual tests will give $V = \mathbb{Z}_2 \times \mathbb{Z}_2$ as the answer.

- What would be the Galois group of the splitting field of $f(X)$ over $\Re$, the field of reals?

  Answer: The equation has at least one (indeed all) complex root(s), so the splitting field is bigger than reals. Only such algebraic field is the complex field of degree 2, so the Galois group must be $\mathbb{Z}_2$.

- What would be the Galois group of the splitting field of $f(X)$ over $\mathbb{C}$, the field of complex numbers?

  Answer: For algebraically closed fields, it is $\{1\}$.

- What would be the Galois group of the splitting field of $f(X)$ over $\mathbb{Z}_{11}$

  Answer: The polynomial factors as two quadratics, so the splitting field has degree 2 (LCM2, 2)). Hence Galois group is $\mathbb{Z}_2$ by theory of finite fields.

Q.4. Let E/K be a Galois extension of degree 55 such that the Galois group is **not abelian.**

Answer the following with justification.

- How many intermediate fields L are there for which $[L : K] = 11$?

  Answer: Note that by Sylow theory, there is a unique (hence normal) 11-sylow group. There can be 11 or 1, 5-sylow subgroups, but the group will be abelian in the second case.

  Thus, there are 11 fixed fields of degree 11, i.e. index 5.

- How many intermediate fields F are there for which $[F : K] = 5$?

  Answer: Answer 1 by above.

- How many other intermediate fields are there?

  Answer: Only possible degrees are $1, 55$ and these give K, E resp.

Q.5. Let $f(X) = (X^5 + 8X^3 + 18)(X^5 - 1) \in \mathbb{Q}\,[X]$. Let K be its splitting field and $[K : \mathbb{Q}\,] = n$.

Prove that 20 divides n.

Answer: Argue that $p(X) = (X^5 + 8X^3 + 18$ is irreducible by Eisenstein for $p = 2$. So K contains a degree 5 subfield.

Let $q(X) = X^4 + X^3 + X^2 + X + 1$. Argue irreducible, either by cyclotomic theory or by reduction mod 2. So, K contains a field of degree 4.

Since $4, 5$ are coprime, their LCM=20 divides n.