Hw1: Noetherian Rings.

All rings shall be commutative with $1 \neq 0$. Unless otherwise stated, we only consider non unit ideals (i.e. ideals different from R.)

(1) Write down a proof that every ideal I in a Noetherian ring is an intersection of finitely many irreducible ideals.
(2) Prove that every irreducible ideal I is primary.
(3) Prove that the radical $\sqrt{Q}$ of a primary ideal is prime. **Note:** We say that an ideal Q is primary for P, if

$$xy \in Q \Rightarrow x \in Q \text{ or } y \in P$$

where we assume that $P \subseteq \sqrt{Q}$.

Prove that Q is primary for P iff Q is primary and $P = \sqrt{Q}$.
(4) Prove that if Q_1 and Q_2 are primary for P, then so is $Q_1 \cap Q_2$.
(5) Prove that if Q is primary for P and $x \notin Q$, then $[Q : (x)]$ is also primary for P.
(6) Prove that any ideal I in a Noetherian ring R is an intersection of finitely many primary ideals $Q_i, i = 1, \cdots, r$ such that the prime ideals $P_i = \sqrt{Q_i}$ are distinct for distinct i.

Note: We usually say that Q_i is a primary ideal belonging to P_i.
(7) **Moreover** we may arrange that the intersection is irredundent, i.e. I is not an intersection of a proper subset of the Q_i.

Suppose that $I = \bigcap_{i=1}^{r} Q_i$ is an irredundent intersection with $P_i = \sqrt{Q_i}$.

The prime ideals P_i are said to be **associated primes of I.**

Any one of these P_i is said to be **isolated** if it does not contain any P_j for $j \neq i$.

Any one of these primes P_i said to be **embedded** if it contains one of the other P_j.
(8) Prove that if P is an **isolated associated prime** of I, then there is an $x \in R$ such that $P = (I : (x))$.

Hint: Prove that there exists $x \in R$ which is not in the primary component Q corresponding to P, but is in all other primary components.

Moreover prove that there is $y \in R$ such that $Q = (I : (y))$, if Q belongs to the isolated associated prime P.

Thus the isolated prime component as well as the corresponding primary are completely determined by the ideal I alone.
(9) Consider the ideal $I = (x^2, xy)$ in $K[x, y]$, the polynomial ring in two variables over a field K. Determine a primary decomposition and the associated primes.

(10) Find two different irredundent intersections for the above ideal.
Note that the isolated component stays the same.