Marie Meyer

(1) Let G be a finite group and let P be a normal p-subgroup of G. Show that P is contained in every Sylow p-subgroup of G.

(2) Determine all groups of order 21 up to isomorphism.

(3) Let P be s Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

(4) Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

Fouché F. Smith

(1) Scavenger Hunt 1 : Algebra Prelim June 2004
Let $(G, \cdot)$ be a group with identity element e. Suppose that $a \neq e$ is an element of G such that $a^6 = a^1 0 = e$. Determine the order of a.
(2) Scavenger Hunt 2 : J. Fraleigh Section 6 Exercise 44
Let G be a cyclic group with generator a, and let G' be a group isomorphic to G. If $\phi : G \to G$ is an isomorphism, show that, for every $x \in G$, $\phi(x)$ is completely determined by the value $\phi(a)$. That is, if $\phi : G \to G'$ and $\psi : G \to G'$ are two isomorphisms such that $\phi(a) = \psi(a)$, then $\phi(x) = \psi(x)$ for all $x \in G$.
(3) Scavenger Hunt : D. Dummit Section 1.6 Exercise 22
Let A be an abelian group and fix some $k \in \mathbb{Z}$. Prove that the map $a \mapsto a^k$ is a homomorphism from A to itself. If $k = -1$ prove that this homomorphism is an isomorphism (i.e, is an automorphism of A.)
(4) Scavenger Hunt : D. Dummit Section 3.2 Exercise 31
Let $N \leq G$ and N is a normal subgroup of H, then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of G in which N is normal(i.e., is the join of all subgroups H for which $N \lhd H$)

Sarah Orchard

(1) 1. Let G be a finite group and let H be a normal Sylow p-subgroup of G. Show that $\alpha(H) = H$ for all authomorphisms α of G.

(2) 2. Suppose that G is a group of order p^n, where p is prime, and G has exactly one subgroup for each divisor of p^n. Show that G is cyclic.
(3) 3. Let H be a Sylow p-subgroup of G. Prove that H is the only Sylow p-subgroup of G contained in $N(H)$.
(4) 4. Show that if G is a group of order 168 that has a normal subgroup of order 4, then G has a normal subgroup of order 28.

Florian Kohl

(1) Prove that there are 45 elements of order 2 in A_6.
(2) Let G be an abelian group, K a group and $f\colon G \to K$ a group homomorphism. Prove that $f(G) \subset K$ is an abelian subgroup of K.
(3) Prove that G is abelian if and only if the map $f\colon G \to G$ by $f(g) = g^2$ is a group homomorphism.
(4) Prove that $(\mathbb{Q} \setminus 0, \cdot)$ is not a cyclic group.

George Lytle

(1) Let K be a Sylow p-subgroup of G and N a normal subgroup of G. Prove that $K \cap N$ is a Sylow p-subgroup of N.
(2) Prove that there are no simple subgroups of order 30.
(3) Let K be a p-Sylow subgroup of G and N a normal subgroup of G. If K is a normal subgroup of N, prove that K is normal in G.
(4) If K is a p-Sylow subgroup of G and H is a subgroup that contains $N(K)$, prove that $[G : H] \equiv 1 \mod p$. [1]

Lola Davidson

(1) How many elements of order 5 does a non-cyclic group of order 55 have?
(2) If P is a Sylow p-subgroup of G, prove that P is the only Sylow p-subgroup of $N(P)$.
(3) Let G be a group of order 105. Show that G has a subgroup of order 35.
(4) If $|G| = pqr$ with $p \leq q \leq r$ primes, prove that G is not simple.

Ian Barnett

(1) Prove that every non-abelian group of order 6 has a non-normal subgroup of order 2, and in fact there is only one such group.
(2) Prove that there are 28 homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to D_4.
(3) Prove that for every integer $1 \leq n \leq 59$ there are no non-abelian simple groups of order n.

[1] All problems are from *Abstract Algebra: an Introduction* Second Edition by Thomas Hungerford

(4) An abelian group has 8192 has elements of the following orders:

order	1	2	4	8	16	32
# of elements	1	31	224	1792	2048	4096

Determine the isomorphism type of the group.

Robert Cass

1.: Show that the multiplicative group $(\mathbb{Z}/2^l\mathbb{Z})^*$ for $l \geq 3$ is a direct product of a cyclic group of order 2 and another cyclic group of order 2^{l-2}.

To do this, it will help to show that $\{(-1)^a 5^b | a = 0, 1 \text{ and } 0 \leq b < 2^{l-2}\}$ is a reduced residue system mod 2^l. You may also use the fact that the order of 5 modulo 2^l is 2^{l-2}.

Source: Ireland and Rosen, A Classical Introduction to Modern Number Theory, Second Edition.

2.: Let H be a proper subgroup of a finite group G. Prove the group G is not the union of the conjugate subgroups of H.

Source: Artin, Algebra, Second Edition.

3.: Prove that any group of order 1365 is not simple.

Source: Jim Brown, homework problem from MA 851 Fall 2010 at Clemson University

4.: Show that there are two isomorphism classes of groups of order 6, the class of the cyclic group with six elements and the class of the symmetric group S_3.

Source: Artin, Algebra, Second Edition.

Cyrus Hettle

- **1.** Count and give a combinatorial interpretation of the number of abelian groups of order 2^n for $n \in \mathbb{N}$. Give a geometric interpretation of the abelian groups of order 8.
- **2.** Suppose a group G has elements u and v such that $u^m = e, uvu^{-1} = v^k$, where $k > 1, m > 0$. Prove that $|v|$ is finite.
- **3.** Let G be a group, and let $f : G \to G$ be defined by $f(g) = g^2$. Give necessary and sufficient conditions for f to be an automorphism.
- **4.** Let G be a finite group and let P be a normal$p$-subrgroup of G. Show that P is contained in every Sylow p-subgroup of G.

Eric Kaper

(1) Show that $\mathbb{A}_5$ has no subgroup of order 15.
(2) Show that $\mathbb{A}_5$ has no subgroup of order 30. (One possible approach to this is showing that every group of order 30 has a subgroup of order 15).

(3) Show that the number of conjugacy classes in S_n is $p(n)$ where $p(n)$ is the number of ways, neglecting the order of the summands, that n can be expressed as a sum of positive integers. The number $p(n)$ is the number of partitions of n.
(4) Show that the number of conjugacy classes in S_n is also the number of different abelian groups (up to isomorphism) of order p^n, where p is a prime number.
(5) Let H be a normal subgroup of order p^k of a finite group G. Show that H is contained in every p-Sylow subgroup of G.
(6) Let G be a finite group with the property that for each positive integer n, the equation $x^n = 1$ has at most n solutions in the group. Prove that G is cyclic.
(7) Show that any finite p-group G is isomorphic to a group of upper triangular matrices with ones on the diagonal (unitriangular matrices) over $\mathbb{F}_p$.
A possible approach to this problem follows:
    - Take $n \in \mathbb{N}$ to be given. Use a counting argument to show that the unitriangular group (group of all $n \times n$ unitriangular matrices) is a p-Sylow subgroup of the general linear group (group of all invertible $n \times n$ matrices) over $\mathbb{F}_p$.
    - Note that the symmetric group embeds in the general linear group using permutation matrices.
    - Note that G is isomorphic to a subgroup of a symmetric group.
    - Apply the fact that any two p-Sylow subgroups are conjugate.

Chase Russell

(1) Let G be a group, and let $\mathrm{Aut}(G)$ be the group of all automorphisms of G together with the operation of function composition. Suppose that G is non-Abelian. Show that $\mathrm{Aut}(G)$ is not cyclic.
(2) Let G be a group and p be a prime. Suppose that $H = \{g^p | g \in G\}$. Show that H is a normal subgroup of G and that every nonidentity element of G/H has order p.
(3) Let G be an Abelian group. Determine all homomorphisms from S_3 to G.
(4) Let G be an Abelian group and let n be a positive integer. Let $G_n = \{g \in G | g^n = e\}$ and $G^n = \{g^n | g \in G\}$. Prove that G/G_n is isomorphic to G^n.

Olsen McCabe

(1) How many elements of order 5 does a non-cyclic group of order 55 have?
(2) Prove that there are no simple groups of order 120.
(3) Show that every group of order 56 has a proper normal subgroup.
(4) If $|G| = pqr$, with $p < q < r$ primes, the G is not simple.
(5) If $G/Z(G)$ is cyclic, prove that G is abelian.
(6) Prove that a non-cyclic group of order 21 must have 14 elements of order 3.