Extension Theorems for Sublinear Codes

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$$f(a_1, \ldots, a_n) = (u_1 a_{\pi(1)}, \ldots, u_n a_{\pi(n)})$$

for all $(a_1, \ldots, a_n) \in \mathbb{F}_q^n$.

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  for all $(a_1, \ldots, a_n) \in \mathbb{F}_q^n$.
- Linear weight preserving maps of $\mathbb{F}_q^n$ are monomial maps.
- *Theorem (MacWilliams, 1961):* If $\mathcal{C} \subseteq \mathbb{F}_q^n$ is a linear code and $f : \mathcal{C} \to \mathbb{F}_q^n$ is a linear weight preserving map, then f is a monomial map.

Generalizations

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- *Theorem (Wood, 1999):* Let R be a finite Frobenius ring. If $\mathcal{C} \subseteq R^n$ is a right linear code and $f : \mathcal{C} \to R^n$ is a linear weight preserving map, then there exist $u_i \in R^*$ and $\pi \in S_n$ such that

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  for all $(a_1, \dots, a_n) \in \mathcal{C}$.
- *Theorem (Greferath, Nechaev, Wisbauer, 2003):* Let R be a finite ring with identity and ${}_RM_R$ be a finite Frobenius bimodule. If $\mathcal{C} \subseteq M_R^n$ is a right linear code and $f : \mathcal{C} \to M_R^n$ is a right linear weight preserving map, then there exist $f_i \in Aut(M_R)$ and $\pi \in S_n$ such that

$$f(a_1, \dots, a_n) = (f_1(a_{\pi(1)}), \dots, f_n(a_{\pi(n)}))$$

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- *Definition:* For a given vector $x = (x_1, \ldots, x_n) \in R^n$, its RT-weight is defined as

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- *Theorem (Barra, Gluesing-Luerssen, 2014):* Let R be a finite Frobenius ring and $\mathcal{C} \subset R^n$ be a left code. Then, any left linear wt_{RT}-preserving map $f : \mathcal{C} \to R^n$ satisfies the MacWilliams extension theorem.

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- *Theorem (Dyshko, 2014):* Let L be a finite field, K be a proper subfield and $\mathcal{C} \subseteq L^n$ be a K linear code. Then, every $K-$linear weight-preserving map $f : \mathcal{C} \to L^n$ extends to an isometry of L^n if and only if $n \leq |K|$.

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- **Question:** Is the same theorem true for (nice) ring extensions R/S?
- **Answer:** No! For $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ and S it's diagonal subring, the theorem fails.

Back to Rosenbloom-Tsfasman Weight

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- *Proposition :* Let $f : L^n \to L^n$ be a K-linear map. Then f is $\mathrm{wt}_{\mathrm{RT}}$-isometry if and only if there exists a matrix

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

with $A_{ij} \in M_m(K), [L : K] = m > 1$ and $A_{ii} \in Gl_m(K)$, such that $f(x) = xA$ for all $x \in L^n$.

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- *Proposition:* For any two words $u, v \in L^n$ of the same RT-weight, there exists a matrix A as above such that $u = vA$.
- *Corollary:* Let $\mathcal{C} \subset L^n$ be a K-linear code and $f : \mathcal{C} \to L^n$ be an RT-isometry. Then for every $u \in \mathcal{C}$ there exists a matrix A_u as above such that $f(u) = uA_u$.

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- *Theorem:* Let L be a finite field, K be a proper subfield and $\mathcal{C} \subset L^n$ be a K-linear code. Then any K-linear RT-isometry $f : \mathcal{C} \to L^n$ extends to an RT-isometry of the entire space.

Open Problem

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- A finite Frobenius bimodule has the extension property with respect to the Rosenbloom-Tsfasman weight.

Thank You!