

Homework for Ma 764 - Algebraic Geometry (Fall 02)

Set 3

11. Let $\mathfrak{a} \subset \mathfrak{m}$ be a homogeneous ideal of $R := K[Z_0, \dots, Z_n]$. Show

- (a) $\mathfrak{a}^{sat} = \{f \in R \mid \text{for each } i = 0, \dots, n \text{ there is a positive integer } e_i \text{ such that } fZ_i^{e_i} \in \mathfrak{a}\}$
(b) As a K -vector space $\mathfrak{a}^{sat}/\mathfrak{a}$ has finite dimension.

12. Let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{m}$ be homogeneous ideals. Prove that \mathfrak{a} and \mathfrak{b} have the same saturation if and only if $[\mathfrak{a}]_j = [\mathfrak{b}]_j$ for all $j \gg 0$.

13. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subset R$ be ideals. Define the colon ideal

$$\mathfrak{a} : \mathfrak{b} := \{f \in R \mid f \cdot \mathfrak{b} \subset \mathfrak{a}\}.$$

Show the following properties:

- (a) $(\mathfrak{a} \cap \mathfrak{b}) : \mathfrak{c} = (\mathfrak{a} : \mathfrak{c}) \cap (\mathfrak{b} : \mathfrak{c})$
(b) $\mathfrak{a} : (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}) \cap (\mathfrak{a} : \mathfrak{c})$.

14. Let $\mathfrak{a} \subset \mathfrak{m}$ be a homogeneous ideal of $R := K[Z_0, \dots, Z_n]$. Show that

$$\mathfrak{a}^{sat} = \bigcup_{k \geq 1} \mathfrak{a} : \mathfrak{m}^k.$$

15. Let \mathfrak{q} be an \mathfrak{p} -primary ideal. Prove that for any element $f \in R$ we have

$$\mathfrak{q} : f := \mathfrak{q} : (f) = \begin{cases} \mathfrak{q} & \text{if } f \notin \mathfrak{p} \\ R & \text{if } f \in \mathfrak{q} \\ \text{a } \mathfrak{p}\text{-primary ideal properly containing } \mathfrak{q} & \text{if } f \in \mathfrak{p} \setminus \mathfrak{q}. \end{cases}$$

16. Let $\mathfrak{a} \subset R$ be an ideal. Prove that the residue class of an element $f \in R$ is not a zero-divisor of R/\mathfrak{a} if and only if f is not contained in any of the associated prime ideals of \mathfrak{a} .

Due date: October 23, 2002 (Wednesday)