

Homework for Ma 764 - Algebraic Geometry (Fall 02)

Set 4

17. Let L be an extension field of the field K . Let $V \subset K^n$ be a K -vector space and let $W \subset L^n$ be the L -vector space generated by V . Show that the vector space dimensions of V and W agree.

(This exercise justifies that for computations of a Hilbert function we may assume the ground field K to be algebraically closed.)

18. Prove the uniqueness of d -th Macaulay presentations.

19. Compute the Hilbert function and the Hilbert polynomial of the following rings: $K[x, y]/(x^3, y)$, $K[x, y]/(x^2, xy, y^2)$, $K[x, y, z]/(x^3, y)$, $K[x, y, z]/(x^2, xy, y^2)$.

20. Consider the subschemes $X, Y \subset \mathbb{P}^3$ defined by the following ideals:

$I_X = (Z_0, Z_1)^2 \cap (Z_2, Z_3)$, $I_Y = (Z_0, Z_1)^2 \cap (Z_1, Z_2^2, Z_3)$. Sketch their reduced schemes and compute the degree and dimension of X, Y, X_{red}, Y_{red} .

21. Let $\mathfrak{p}, \mathfrak{p}' \subset R$ be homogeneous prime ideals such that $\mathfrak{p} \subset \mathfrak{p}'$. Show that if $\dim R/\mathfrak{p} = \dim R/\mathfrak{p}'$ then $\mathfrak{p} = \mathfrak{p}'$.

(Notice the similarity to the situation for vector spaces.)

22. Let $\mathfrak{a}, \mathfrak{b} \subset R$ be homogeneous ideals and consider for any $j \in \mathbb{Z}$ the following maps:

$$\varphi : [\mathfrak{a}]_j \oplus [\mathfrak{b}]_j \rightarrow [\mathfrak{a} + \mathfrak{b}]_j, (f, g) \mapsto f - g,$$

$$\psi : [\mathfrak{a} \cap \mathfrak{b}]_j \rightarrow [\mathfrak{a}]_j \oplus [\mathfrak{b}]_j, f \mapsto (f, f).$$

Show that φ is surjective, ψ is injective, and $\text{im } \psi = \ker \varphi$.

Due date: November 6, 2002 (Wednesday)