Handout on induction for MA 113 - Calculus I (Spring 08)

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Motivation for studying mathematical induction. For many students, mathematical induction is probably an unfamiliar topic. Nonetheless, this is an important topic and useful in the study of calculus. Calculus involves many new ideas. To study derivatives, we have to look at the slope between pairs of points that are arbitrarily close together. To define the integral, we have to subdivide an interval into n sub-intervals for infinitely many values of n. To fully understand these operations, we have to see why infinitely many statements are true. Mathematical induction is one way to see that infinitely many statements are true.

In mathematics, we engage in deductive reasoning. We make assumptions and deduce conclusions from these assumptions. The induction step in a proof by mathematical induction provides practice in this type of reasoning.

Finally, mathematical induction provides a framework which allows us to understand why many important results in calculus, such as the rule for the derivative of a power, are true.

Summation notation. First, we explain use of $\sum$ for summation or repeated addition. The notation

$$\sum_{k=1}^{n} f(k)$$

means to evaluate the function $f(k)$ at $k = 1, 2, \ldots, n$ and add up the results. In other words:

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

For example:

$$\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30,$$

$$\sum_{k=1}^{n} (2k - 1) = 1 + 3 + 5 + \cdots + (2n - 1),$$

and

$$\sum_{k=3}^{2n} 1 = 2n - 2.$$

Often the following identity will be useful

$$\sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n+1).$$

The principle of mathematical induction is used to establish the truth of a sequence of statements or formula which depend on a natural number, $n = 1, 2, \ldots$. We will use P_n to stand for a statement which depends on n. For example, P_n might stand for the statement "The number $2n - 1$ is odd." These statements are true for $n = 1, 2, \ldots$

The principle of mathematical induction is:

Principle of mathematical induction. Suppose that P_n is a sequence of statements depending on a natural number $n = 1, 2, \ldots$. If we show that:

- (Base case) P_1 is true
- (Induction step) For each natural number N: If P_N is true, then P_{N+1} is true.

Then, we may conclude that all the statements P_n are true for all positive integers $n = 1, 2, \ldots$.

To see why this principle makes sense, suppose that we know P_1 is true, then the second step allows us to conclude P_2 is true. Now that we know P_2 is true, the second step allows us to conclude P_3 is true. If we repeat this $n-1$ times, we conclude that P_n is true.

This principle is useful because it allows us to prove an infinite number of statements are true in just two easy steps! The statement P_N that we assume to hold is called the *induction hypothesis*. The key point in the induction step is to use the induction hypothesis, P_N, in order to deduce P_{N+1}.

Below are several examples to illustrate how to use this principle.

Example 1. Show that for $n = 1, 2, 3, \ldots$, the number $n^2 - n$ is even.

Solution. Base case. If $n = 1$, then $n^2 - n = 1^2 - 1 = 0$ and 0 is even.

Induction step. Our induction hypothesis is that $N^2 - N$ is even and we want to use this assumption to show that $(N+1)^2 - (N+1)$ is even. We write $(N+1)^2 - (N+1) = N^2 + 2N + 1 - N - 1 = N^2 - N + 2N$. Now $2N$ is even when N is an integer and $N^2 - N$ is even by our induction hypothesis. As the sum of two even numbers is again even, we conclude that $(N+1)^2 - (N+1)$ is even. $\square$

Example 2. Show that for $n = 1, 2, \ldots$, we have

$$\sum_{j=1}^{n} 2j = n(n+1).$$

Solution. Base case. If $n = 1$, then the right-hand side of the claimed formula becomes $n(n+1) = 1 \cdot 2 = 2$. The left-hand side is

$$\sum_{j=1}^{1} 2j = 2.$$

Thus both sides are equal if $n = 1$, as claimed.

Induction step. Our induction hypothesis is that the formula $\sum_{j=1}^{N} 2j = N(N+1)$ is true. We have to show that $\sum_{j=1}^{N+1} 2j = (N+1)[(N+1)+1]$, which we rewrite more simply as

$$\sum_{j=1}^{N+1} 2j = (N+1)(N+2).$$

To prove this we write the last term in the sum on left-hand side separately and obtain

$$\sum_{j=1}^{N+1} 2j = \sum_{j=1}^{N} 2j + 2(N+1).$$

Now we use our induction hypothesis saying that $\sum_{j=1}^{N} 2j = N(N+1)$ to conclude that

$$\sum_{j=1}^{N+1} 2j = N(N+1) + 2(N+1).$$

Simplifying the right-hand side provides

$$N(N+1) + 2(N+1) = N^2 + N + 2N + 2 = N^2 + 3N + 2 = (N+1)(N+2).$$

(Alternatively, we could factor out $(N+1)$ to obtain the same result:
$N(N+1) + 2(N+1) = (N+2)(N+1)$.)
Thus, we have shown that the formula

$$\sum_{j=1}^{N+1} 2j = (N+1)(N+2).$$

is true. This completes the induction step and thus the proof by induction. □

Example 3. All horses are the same color.

Solution. We will show by induction that any set of N horses consists of horses of the same color.

The base case is easy. If we have a set with one horse, then all horses in the set are the same color.

We assume as our induction hypothesis that any set of N horses consists of horses of the same color. We take a set of $N+1$ horses, and put one of the horses in the barn for a moment. By our induction hypothesis, the remaining N horses are all of the same color. Now, we put a different horse in the barn. Again, the remaining N horses are all the same color. It follows that the set of $N+1$ horses are all the same color. □

Warning: This example is to illustrate that you should be careful in writing your solutions. Obviously, the above statement is wrong, so the above argument is **not correct**. Try to locate the error!