Worksheet 1 for MA 113 - Calculus I (Spring 08)

01/09/08

Work the following three problems related to mathematical induction and hand in your solutions. You will have time some time in recitation to begin working on these problems. Write up your solutions neatly, carefully and in complete sentences.

1. Find a formula that depends only on n to compute $\sum_{k=1}^{n}(2k-1)$ following the steps below:

(a) For $n = 1, 2, 3, 4$, compute

$$\sum_{k=1}^{n}(2k-1).$$

Make a guess for the value of this sum for $n = 1, 2, \ldots$.

(b) Use mathematical induction to prove that your guess is correct.

2. Use the principle of mathematical induction to prove that, for every positive integer n,

$$\sum_{k=1}^{n} 2k^2 = \frac{n(n+1)(2n+1)}{3}.$$

3. For $n = 1, 2, \ldots$, define the function f_n by $f_1(x) = x - 3$ and $f_{n+1}(x) = f_n(x) - 3$. (It is the principle of mathematical induction which tells us that these two statements suffice to define f_n for all n.) Use mathematical induction to prove that

$$f_n(x) = x - 3n.$$

Additional problems. Below are some additional exercises for you to consider. You will not be able to solve all of these problems at this time, so you are encouraged to come back to these problems later on.

All these additional problems will not be collected.

(1) Find the flaw in the proof that all horses are the same color.

(2) Let $f_1(x) = x - 2$ and then define f_n for $n = 1, 2, \ldots$ by $f_{n+1}(x) = f_1(f_n(x))$. (It is the principle of mathematical induction which tells us that these two statements suffice to define f_n for all $n = 1, 2, 3, \ldots$.) Use mathematical induction to prove that

$$f_n(x) = x - 2n.$$

(3) Show that if $r \neq 1$, we have

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

(4) Let P_n be the statement: $n^2 - n$ is an odd integer.

(a) Show that if P_n is true, then P_{n+1} is true.

(b) Is P_1 true?

(c) Is P_n true for any n?

(5) Let $f(x) = \sin(2x)$. Prove that for $n = 1, 2, \ldots,$

$$\frac{d^{2n}}{dx^{2n}} f(x) = (-4)^n \sin(2x).$$

(6) Let $f(x) = xe^x$. Compute f', f'', and f'''. Guess a formula for the nth derivative,

$$\frac{d^n}{dx^n} f(x).$$

Prove that your guess is right.

(7) Argue that

$$\frac{d}{dx} x^n = nx^{n-1}, \qquad n = 1, 2 \ldots.$$

Hint: For the base case $n = 1$, use the definition of the derivative. For the induction step write $x^{n+1} = x \cdot x^n$ and use the product rule.

(8) Prove that

$$\frac{d}{dx} \frac{1}{x^n} = \frac{-n}{x^{n+1}}, \qquad n = 1, 2 \ldots.$$

(9) Prove that

$$\frac{d^n}{dx^n} x^n = n!, \qquad n = 0, 1, \ldots$$

(10) (a) Find a simple formula for

$$\sum_{k=1}^{n} ((k+1)^2 - k^2) = 2^2 - 1 + (3^2 - 2^2) + \cdots + n^2 - (n-1)^2 + (n+1)^2 - n^2.$$

(b) Using your answer to part (a), find a simple expression for

$$\sum_{k=1}^{n} (2k - 1).$$

To do this you should simplify each summand on the left.

(11) Use mathematical induction to prove that

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$