Solutions to worksheet 1 for MA 113 - Calculus I (Spring 08)

23/01/08

1. Find a formula that depends only on n to compute $\sum_{k=1}^{n}(2k-1)$ following the steps below:

(a) For $n = 1, 2, 3, 4$, compute

$$\sum_{k=1}^{n}(2k-1).$$

Make a guess for the value of this sum for $n = 1, 2, \ldots$.

(b) Use mathematical induction to prove that your guess is correct.

Solution: (a) We compute the sums

$$\sum_{k=1}^{1}(2k-1) = 1, \quad \sum_{k=1}^{2}(2k-1) = 4, \quad \sum_{k=1}^{3}(2k-1) = 9, \quad \sum_{k=1}^{4}(2k-1) = 16.$$

Recognizing that the numbers 1, 4, 9 and 16 are the squares of 1, 2, 3 and 4, respectively, we guess that

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

(b) *Base case.* The base case has been established in part (a).

Induction step. Our induction hypothesis is that, for some positive integer N,

$$\sum_{k=1}^{N}(2k-1) = N^2 \tag{1}$$

is true. We have to show that

$$\sum_{k=1}^{N+1}(2k-1) = (N+1)^2. \tag{2}$$

Writing out the last term of the sum separately, we get

$$\begin{aligned}\sum_{k=1}^{N+1}(2k-1) &= \sum_{k=1}^{N}(2k-1) + [2(N+1)-1] \\ &= \sum_{k=1}^{N}(2k-1) + 2N + 1.\end{aligned}$$

Now we apply the induction hypothesis (1) and conclude that

$$\begin{aligned}\sum_{k=1}^{N+1}(2k-1) &= N^2 + 2N + 1 \\ &= (N+1)^2,\end{aligned}$$

as claimed in (2). (Note that for the last equality we used the binomial formula $(a+b)^2 = a^2+2ab+b^2$.) Hence, the principle of mathematical induction implies that $\sum_{k=1}^{n}(2k-1) = n^2$ is true for every positive integer $n = 1, 2, 3 \ldots$.

2. Use the principle of mathematical induction to prove that, for every positive integer n,

$$\sum_{k=1}^{n} 2k^2 = \frac{n(n+1)(2n+1)}{3}.$$

Solution: *Base case.* If $n = 1$, the left-hand side becomes

$$\sum_{k=1}^{1} 2k^2 = 2$$

and the right-hand side becomes

$$\frac{1(1+1)(2\cdot 1+1)}{3} = 2.$$

Thus, both sides are equal, and we have established the base case.

Induction step. Our induction hypothesis is that, for some positive integer N,

$$\sum_{k=1}^{N} 2k^2 = \frac{N(N+1)(2N+1)}{3}. \tag{3}$$

We have to show that

$$\sum_{k=1}^{N+1} 2k^2 = \frac{(N+1)\cdot[(N+1)+1]\cdot[2(N+1)+1]}{3},$$

which simplifies to

$$\sum_{k=1}^{N+1} 2k^2 = \frac{(N+1)(N+2)(2N+3)}{3}. \tag{4}$$

Writing out the last term of the sum separately, we get

$$\sum_{k=1}^{N+1} 2k^2 = \sum_{k=1}^{N} 2k^2 \; + \; 2(N+1)^2.$$

Applying the induction hypothesis (3) and then obtaining a common denominator we deduce:

$$\begin{aligned}
\sum_{k=1}^{N+1} 2k^2 &= \frac{N(N+1)(2N+1)}{3} + 2(N+1)^2 \\
&= \frac{N(N+1)(2N+1) + 6(N+1)^2}{3} \\
&= \frac{(N+1)}{3} \cdot [(N(2N+1) + 6(N+1)] \\
&= \frac{(N+1)}{3} \cdot [2N^2 + N + 6N + 6] \\
&= \frac{(N+1)}{3} \cdot [2N^2 + 7N + 6].
\end{aligned}$$

Since $(N+2)(2N+3) = 2N^2 + 7N + 6$, we finally get

$$\sum_{k=1}^{N+1} 2k^2 = \frac{(N+1)(N+2)(2N+3)}{3},$$

as claimed in (4). Hence, the principle of mathematical induction allows us to conclude that

$$\sum_{k=1}^{n} 2k^2 = \frac{n(n+1)(2n+1)}{3}$$

is true for every positive integer $n = 1, 2, 3, \ldots$.

3. For $n = 1, 2, \ldots$, define the function f_n by $f_1(x) = x - 3$ and $f_{n+1}(x) = f_n(x) - 3$. (It is the principle of mathematical induction which tells us that these two statements suffice to define f_n for all n.) Use mathematical induction to prove that

$$f_n(x) = x - 3n.$$

Solution: *Base case.* If $n = 1$, the right-hand side becomes $x - 3 \cdot 1 = x - 3$, which is equal to f_1 by assumption. Thus, the base case has been established.

Induction step. Our induction hypothesis is that, for some positive integer N,

$$f_N(x) = x - 3N. \tag{5}$$

We have to show that

$$f_N(x) = x - 3(N+1). \tag{6}$$

To this end we use the definition of f_{N+1} and then the induction hypothesis (5). We get:

$$\begin{aligned} f_{N+1}(x) &= f_N(x) - 3 \\ &= (x - 3N) - 3 \\ &= x - 3(N+1), \end{aligned}$$

as wished. This completes the proof by induction.

Grading guidelines:

1. (3 points) 1 point for part (a). In part (b), students should make an effort to explain how the induction hypothesis is used.
2. (4 points) One point for the base case. One point for attempting to use the induction hypothesis, 2 more points for a complete proof.
3. (3 points) 1 point for the base case, 2 for the induction step.

Be sure to comment favorably on papers of students who do an unusually good job.

Take the time to recognize and provide guidance to students who attempt unusual approaches.

Deductions:

- If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point for three or more such mistakes which are not otherwise penalized.
- Deduct one point for unusually messy or poorly organized solutions. (At most one or two per paper.)