Solutions to worksheet 2 for MA 113 - Calculus I (Spring 08)

01/30/08

1. Let x be a positive number. Find the limit

$$\lim_{h\to 0} \frac{\sqrt{7+4(x+h)}-\sqrt{7+4x}}{h}.$$

(The result will be a function of x).

Solution: Expanding the fraction by the conjugate of the numerator and then simplifying we obtain,

$$\begin{aligned}
&\frac{\sqrt{7+4(x+h)}-\sqrt{7+4x}}{h} \\
&= \frac{[\sqrt{7+4(x+h)}+\sqrt{7+4x}\ ]\cdot[\sqrt{7+4(x+h)}-\sqrt{7+4x}\ ]}{h[\sqrt{7+4(x+h)}+\sqrt{7+4x}]} \\
&= \frac{7+4(x+h)-[7+4x]}{h[\sqrt{7+4(x+h)}+\sqrt{7+4x}\ ])} \\
&= \frac{4}{\sqrt{7+4(x+h)}+\sqrt{7+4x}}
\end{aligned}$$

Since $x>0$, the function $\sqrt{7+4(x+h)}$ is continuous at $h=0$. Hence, we conclude that

$$\lim_{h\to 0} \frac{\sqrt{7+4(x+h)}-\sqrt{7+4x}}{h} = \lim_{h\to 0} \frac{4}{\sqrt{7+4(x+h)}+\sqrt{7+4x}} = \frac{4}{2\sqrt{7+4x}} = \frac{2}{\sqrt{7+4x}}.$$

2. Consider the function $f(x)=\sqrt{9-x^2}$ and let a be a number in the open interval $(-1,1)$.

(a) Using the definition, find the slope of the tangent line to the graph of f at the point $(a, f(a))$.

(b) Determine the slope of the line through the origin and the point $(a, f(a))$. Sketch this line, the graph of f, and the tangent line you computed in (a). Interpret your result in (a) geometrically.

Solution: (a) The slope of the tangent is

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\sqrt{9-x^2}-\sqrt{9-a^2}}{x-a},$$

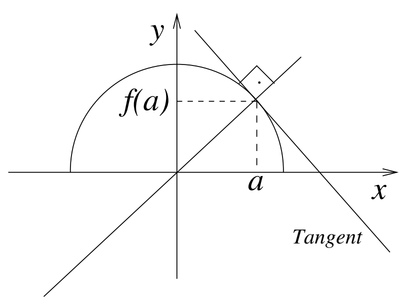
provided this limit exists. Expanding the difference quotient by the conjugate of the numerator and then simplifying we obtain,

$$\begin{aligned}\frac{\sqrt{9-x^2}-\sqrt{9-a^2}}{x-a} &= \frac{[\sqrt{9-x^2}-\sqrt{9-a^2}\ ]\cdot[\sqrt{9-x^2}+\sqrt{9-a^2}\ ]}{[x-a]\cdot[\sqrt{9-x^2}+\sqrt{9-a^2}\ ]}\\ &= \frac{9-x^2-[9-a^2]}{[x-a]\cdot[\sqrt{9-x^2}+\sqrt{9-a^2}\ ]}\\ &= \frac{-x^2+a^2}{[x-a]\cdot[\sqrt{9-x^2}+\sqrt{9-a^2}\ ]}\\ &= \frac{-[x-a][x+a]}{[x-a]\cdot[\sqrt{9-x^2}+\sqrt{9-a^2}\ ]}\\ &= \frac{-[x+a]}{\sqrt{9-x^2}+\sqrt{9-a^2}}\end{aligned}$$

Now taking the limit, we find that the slope of the tangent is

$$\begin{aligned} m=\lim_{x\to a}\frac{-[x+a]}{\sqrt{9-x^2}+\sqrt{9-a^2}} &= \frac{-2a}{2\sqrt{9-a^2}}\\ &= -\frac{a}{\sqrt{9-a^2}}.\end{aligned}$$

(b) Sketch:



If $a \neq 0$, the line L through the origin and the point $P(a, f(a))$ has slope

$$\frac{f(a)}{a}=\frac{\sqrt{9-a^2}}{a},$$

which is the negative reciprocal of the slope of the tangent to $y = f(x)$ at the point P. Hence, the line L and the tangent line at P are perpendicular, and this is also true if $a = 0$ because then the tangent is horizontal. Since the graph of f is the upper half of a circle centered at the origin, we have shown the geometric fact that the tangent to the circle at P is perpendicular to the radius from the origin of the circle to P.

3. Write the definition of the derivative of a function f at a point a.

Solution: The *derivative of f at a point a* is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

provided the limit exists.

Grading guidelines:

1. (3 points) Two points for simplifying the difference quotient, one for finding the limit.
2. (6 points) (a) One point for the correct limit to compute the slope, two points for simplifying the difference quotient, one for finding the limit.

   (b) One point for the sketch, which should in particular indicate that the graph of f is the upper half of a circle, and one point for recognizing that the radius and the tangent are perpendicular.
3. (1 point) Accept either form of the definition.

Be sure to comment favorably on papers of students who do an unusually good job.

Take the time to recognize and provide guidance to students who attempt unusual approaches.

Deductions:

- If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point if there are two such errors that are not otherwise penalized.
- Deduct one point for unusually messy or poorly organized papers. (At most one or two per section.)