Solutions to worksheet 4 for MA 113 - Calculus I (Spring 08)

2/27/08

1. Find all tangent lines to the ellipse $x^2 + 5y^2 = 6$ which pass through the point $(-6, 0)$. Your solution should describe how you know that you have found all of the solutions.

Solution: Let f be the function whose graph describes the given ellipse near a point (a, b) on the ellipse. Thus, f satisfies the equation

$$x^2 + 5[f(x)]^2 = 6.$$

Differentiating with respect to x we obtain

$$2x + 10f(x)f'(x) = 0. \tag{1}$$

Solving for $f'(x)$, we find that

$$f'(x) = -\frac{x}{5f(x)}.$$

Thus the equation of the tangent line through the point $(a, b) = (a, f(a))$ on the ellipse is given by

$$y - b = -\frac{a}{5b}(x - a). \tag{2}$$

This line passes through $(-6, 0)$ if and only if

$$-b = -\frac{a}{5b}(-6 - a).$$

Multiplying by $5b$ (see footnote[1] below) we get

$$a^2 + 5b^2 + 6a = 0.$$

Since the point (a, b) lies on the ellipse, we also have that $a^2 + 5b^2 = 6$. If we substitute this, the previous equation becomes

$$6a + 6 = 0.$$

Hence, we obtain $a = -1$ and $(-1)^2 + 5b^2 = 6$, thus $b^2 = 1$, hence $b = 1$ or -1. Using $(a, b) = (-1, 1)$ in Equation (2) gives the equation of the first tangent line

$$y = \frac{1}{5}x + \frac{6}{5}$$

and using $(a, b) = (-1, -1)$ provides the second tangent line

$$y = -\frac{1}{5}x - \frac{6}{5}.$$

2. A plane is flying east at 360 kilometers/hour. The plane's altitude is 4 kilometers.

(a) Find the angle of elevation 30 seconds after the plane flies directly over an observer. (You will not be able to give an exact answer to this question. Please round your answer to three decimal places.)

(b) Make a sketch showing the observer, the plane and the angle of elevation.

(c) Find the derivative of the angle of elevation 30 seconds after the plane flies directly over the observer.

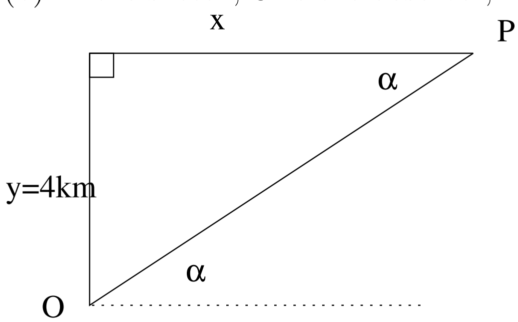
[1] Strictly speaking, we have to rule out the possibility $b = f(a) = 0$. However, if $b = 0$, then Equation (1) implies $a = 0$, but $(0, 0)$ is not a point on the ellipse. – Students are not expected to discuss this case.

(d) Find the rate of change of the angle of elevation 30 seconds after the plane flies directly over the observer. Is the angle of elevation increasing or decreasing?

(e) Find the rate at which the distance between the plane and the observer is changing 30 seconds after the plane flies over the observer. Is the distance increasing or decreasing?

Solution: (a) In 30 s $= \frac{1}{2}$ min $= \frac{1}{120}$ h the plane travels $360/120 = 3$ kilometers. As indicated in the sketch below, the plane P and the observer O give the endpoints of the hypotenuse of a right triangle. The angle of elevation α satisfies $\tan\alpha = 4/x = 4/3$. Since α is acute, it follows that $\alpha = \arctan(4/3) \approx 0.927$ radians.

(b) In the sketch, O is the observer, P is the plane, and α is the angle of elevation.



(c) Write $\alpha(t)$ and $x(t)$ for the angle α and the distance x after t hours. Using the right triangle, we see that $\tan(\alpha(t)) = \frac{4}{x(t)}$. Differentiating with respect to the time t we obtain

$$\frac{1}{\cos^2(\alpha(t))}\alpha'(t) = -\frac{4}{[x(t)]^2}x'(t) = -\frac{1440}{[x(t)]^2}$$

because we are given that the velocity is $x'(t) = 360$. Solving for the derivative $\alpha'(t)$ we get

$$\alpha'(t) = -\cos^2(\alpha(t))\frac{1440}{[x(t)]^2}.$$

Writing $c(t)$ for the hypotenuse, we have $\cos(\alpha(t)) = \frac{x(t)}{c(t)}$. Substituting this, the last equation becomes

$$\alpha'(t) = -\frac{1440}{[c(t)]^2}.$$

We observed in part (a) that $x(\frac{1}{120}) = 3$. Since $y = 4$, Pythagoras's theorem provides $c(\frac{1}{120}) = 5$. Thus, we get

$$\alpha'(\frac{1}{120}) = -\frac{1440}{5^2} = -\frac{288}{5} = -57.6.$$

Hence, the derivative of the angle after one minute is $-57.6\frac{1}{\text{h}}$. (Recall that the radian measure of an angle is a ratio of lengths and thus has no units.)

(d) This question is asking for the same information as the previous question. The angle is decreasing at $57.6\frac{1}{\text{h}}$.

(e) Pythagoras's theorem provides $[c(t)]^2 = [x(t)]^2 + 4^2$. Differentiating both sides with respect to t, we obtain

$$2c(t)c'(t) = 2x(t)x'(t).$$

Dividing both sides by $2c(t)$ and using $x(\frac{1}{120}) = 3$, $c(\frac{1}{120}) = 5$, $x'(t) = 360$, we get

$$c'(\frac{1}{120}) = \frac{x(\frac{1}{120})x'(\frac{1}{120})}{c(\frac{1}{120})} = \frac{3 \cdot 360}{5} = 216.$$

Hence after one minute the distance is increasing at $216\frac{\text{km}}{\text{h}}$.

3. (2 bonus points) When Adam gets home after a long day at the university, he decides to go for a jog. He begins jogging right outside his front door. First he runs on a level road, then he comes to a hill and runs to the top. When he gets to the top of the hill, he turns around and he runs back exactly the way he came.

Now, on level ground, Adam can run at 4 miles an hour. Uphill, 3 miles an hour. And downhill, 6 miles an hour. Upon his return home, he notices that he had run for exactly two hours.

How far did Adam run? (It appears that there is not enough information here to solve the problem, but there is.)

Solution: Let x be the distance between Adam's front door and his point of return. Denote by y the length of the uphill part of Adam's way. Since velocity is distance over time, Adam had run for $2\frac{x-y}{4}$ hours on level ground, $\frac{y}{3}$ hours uphill, and $\frac{y}{6}$ hours downhill. In total Adam had run for two hours, thus we get

$$2 = 2\frac{x-y}{4} + \frac{y}{3} + \frac{y}{6} = \frac{6(x-y) + 4y + 2y}{12} = \frac{6x}{12} = \frac{x}{2},$$

so $x = 4$. Hence has Adam ran $2 \cdot 4 = 8$ miles.

Grading guidelines:

1. (3 points) 1 point for derivative, 1 point for the two points of tangency, and 1 point for the two equations of the tangent lines.

2. (7 points)
- (a), (b), (d): each 1 point.
- (c) and (e): 1 point for expression of the derivative, 1 point for the answer.

3. (2 bonus points) 1 point for the answer, 1 point for the reasoning.

Be sure to comment favorably on papers of students who do an unusually good job.

Take the time to recognize and provide guidance to students who attempt unusual approaches.

Deductions:

- If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point if there are two such errors that are not otherwise penalized.
- Deduct one point for one of the following offences and two points for three or more offences:
  - Misuse of equality sign.
  - Omitting units from answers.
  - Unusually messy or poorly organized papers.