Solutions to worksheet 5 for MA 113 - Calculus I (Spring 08)

03/26/08

1. Carry out the following steps to sketch the graph of

$$f(x) = \frac{x}{1+x^2}.$$

(a) Find the local maxima and minima of f. Compute the local maximum and minimum values. Give the intervals of increase and decrease.
(b) Find the inflection points of f. Give the intervals where f is concave upward and where f is concave downward.
(c) Determine if f is even or odd.
(d) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
(e) Make a careful sketch of the graph of f that reflects the above information.

Solution: (a) Using the quotient rule we get for the derivative of $f(x)$:

$$f'(x) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

We have $f'(x) = 0$ if and only if $1 - x^2 = 0$, which is equivalent to $x = 1$ or -1. Since f is differentiable on $\mathbb{R}$, the critical numbers of f are -1 and 1. Furthermore, since f' is continuous on $\mathbb{R}$, it has constant sign on each of the intervals $(-\infty,-1), (-1,1)$, and $(1,\infty)$. To determine the sign, we use that $(1+x)^2$ is always positive and that $(1-x^2)$ is positive if and only if $|x| < 1$. Hence, we get

intervall	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
sign of $f'(x)$	$-$	$+$	$-$
I/D	decreasing	increasing	decreasing

Hence, f is decreasing on the intervals $(-\infty,-1)$ and $(1,\infty)$, and f is increasing on the interval $(-1,1)$.

Using the first derivative test for local extrema, we see that f has a local minimum at $x=-1$ and a local maximum at $x=1$. The local minimum value at -1 is $f(-1) = -1/2$ and the local maximum value at 1 is $f(1) = 1/2$.

(Alternative method to find intervals of increase and decrease: Since we have found the intervals where f' has constant sign, we can determine this sign by computing the value of f at any number in the interval. Choosing as test points $-2, 0$, and 2, we get

intervall	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Test point	-2	0	2
$f'(x)$	$-3/25$	1	$-3/25$
I/D	decreasing	increasing	decreasing

,

which leads, of course, to the same conclusion as above.)

(Some papers may use the second derivative test to classify local extrema. Give credit. However, encourage students to learn how to use the first derivative to classify local extrema.)

(b) Using the quotient rule, canceling (x^2+1), and simplifying, we get for the second derivative

$$\begin{aligned} f''(x) &= \frac{-2x\cdot(1+x^2)^2-(1-x^2)\cdot 2(1+x^2)2x}{(1+x^2)^4} = \frac{-2x\cdot(1+x^2)-4x(1-x^2)}{(1+x^2)^3} \\ &= \frac{2x(x^2-3)}{(1+x^2)^3}. \end{aligned}$$

The solutions of $f''(x)=0$ are $x=0,\pm\sqrt{3}$. Using $(1+x^2)^3>0$ and $x^2-3>0$ if and only if $|x|>\sqrt{3}$, we get the following table:

intervall	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
sign of x	$-$	$-$	$+$	$+$
sign of (x^2-3)	$+$	$-$	$-$	$+$
sign of $f''(x)$	$-$	$+$	$-$	$+$
concavity	down	up	down	up

Hence, the second derivative test for concavity yields that f is concave up on the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$ and that f is concave down on the intervals $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$.

Since f changes concavity precisely at $x=0,\sqrt{3}$, and $-\sqrt{3}$, we conclude that the inflection points of f are $(-\sqrt{3},-\sqrt{3}/4)$, $(0,0)$ and $(\sqrt{3},\sqrt{3}/4)$.

(Alternative method to decide concavity: Since f'' is continuous on $\mathbb{R}$ and only zero at $-\sqrt{3},0$, and $\sqrt{3}$, the sign of f'' is constant on each interval $(-\infty,-\sqrt{3})$, $(-\sqrt{3},0)$, $(0,\sqrt{3})$. Hence we can determine the sign by choosing appropriate test points. Taking as such $-2,-1,1$, and 2, we get the table

	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
Test point	-2	-1	1	2
$f''(x)$	$-4/125$	$1/2$	$-1/2$	$4/125$
concave	down	up	down	up

and then conclude as above.)

(c) The function f is odd since $f(-x)=\dfrac{-x}{1+x^2}=-f(x)$.

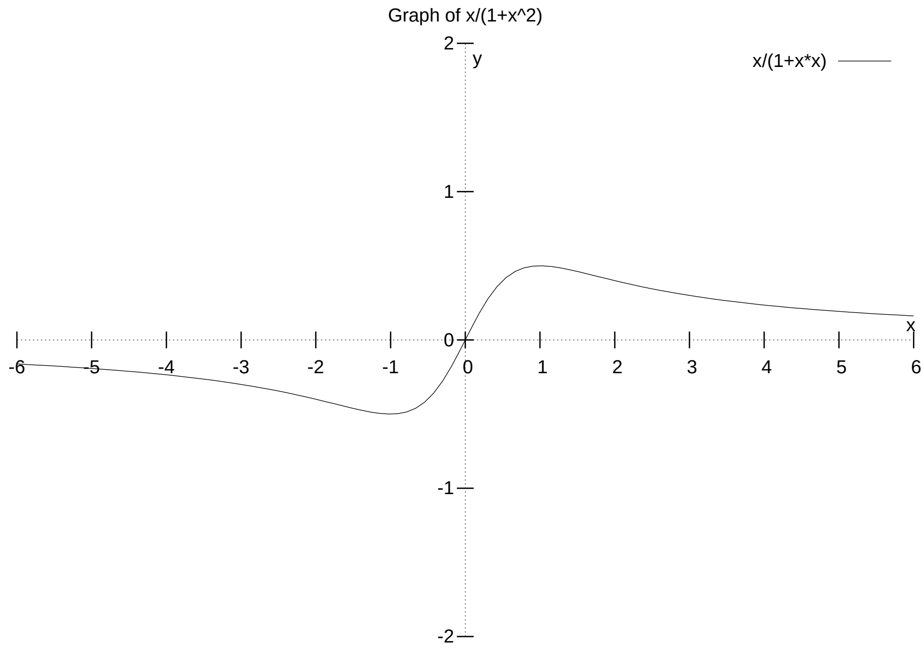
(d) Dividing numerator and denominator by x^2, we get

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x^2}+1} = \frac{0}{0+1} = 0$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{\frac{1}{x}}{\frac{1}{x^2}+1} = \frac{0}{0+1} = 0.$$

e) The graph of f is sketched below.



2. Let t be a real number and consider the cubic polynomial $f(x) = x^3 + tx^2 + 3x$.

(a) Find $f'(x)$.
(b) Find the value(s) of t for which f has exactly one critical number.
(c) Find the value(s) of t for which f has two critical numbers.
(d) Find the value(s) of t for which f has no critical numbers.
(e) Draw a sketch of the graph of the polynomial when $t = 3$ and verify that your sketch agrees with your answers to (b) - (d).

Solution: (a) $f'(x) = 3x^2 + 2tx + 3$.

(b) - (d) Since f is differentiable on $\mathbb{R}$, the critical numbers of f are exactly the solutions of $f'(x) = 0$, that is, of $3x^2 + 2tx + 3 = 0$. These solutions are

$$x = \frac{-2t \pm \sqrt{4t^2 - 36}}{6}.$$

Now we look at the discriminant $4t^2 - 36$. it equals zero if and only if $4t^2 = 36$ or, equivalently, if t is -3 or 3.

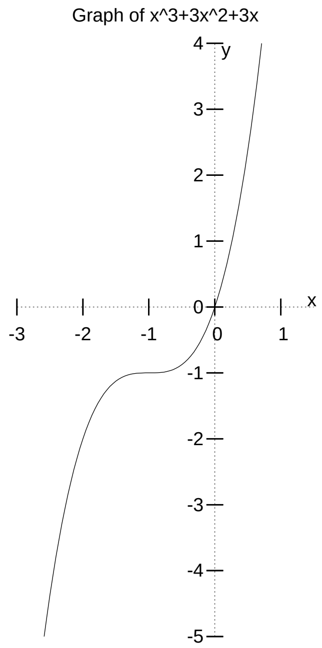
The discriminant $4t^2 - 36$ is positive if and only if $t^2 > 9$, that is, if $|x| > 3$.

The discriminant $4t^2 - 36$ is negative if and only if $t^2 < 9$, that is, if $|x| < 3$. Hence, we conclude

- f has exactly one critical number if t is -3 or 3.
- f has exactly two critical number if t is in $(-\infty, -3) \cup (3, \infty)$.
- f has no critical number[1] if t is in $(-3, 3)$.

(e) When $t = 3$, we are in the case discussed in part (b). There is exactly one critical number, namely -1. The graph of f is sketched below.

[1] We do not consider complex solutions in this course.



Grading guidelines:

1. (7 points)
   - (a): 1 point for intervals, 1 point for extrema.
   - (b): 1 point for intervals, 1 point for inflection points.
   - (c) - (e): each 1 point.
2. (3 points) Critical numbers (1 point), cases (1 point),) sketch (1 point)

Be sure to comment favorably on papers of students who do an unusually good job.

Take the time to recognize and provide guidance to students who attempt unusual approaches.

Deductions: Deduct one point for two of the following offences and two points for three or more offences:

- Not using complete sentences.
- Lack of explanations.
- Misuse of equality sign.
- Unusually messy or poorly organized papers.