

Solutions to worksheet 6 for MA 113 - Calculus I (Spring 08)

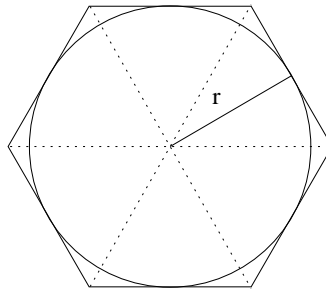
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1. Let u and v be two non-negative numbers, whose sum is 6. Find the maximum value of u^2v . Determine u and v when the maximum is attained.

Solution We are given $u + v = 6$ with $u, v \geq 0$. It follows that $v = 6 - u \geq 0$, thus u is in the interval $[0, 6]$. We want to find the maximum of $u^2v = u^2(6 - u)$. This means that we are looking for the maximum value of $f(u) = u^2(6 - u) = 6u^2 - u^3$ on the interval $[0, 6]$.

To find it we compute the derivative $f'(u) = 12u - 3u^2 = 3u(4 - u)$. Solving $f'(u) = 0$ we get that the critical number of f is $u = 4$. Since f is defined on a closed interval, it attains its maximum value at a critical number or at an endpoint. We compute $f(0) = 0$, $f(4) = 32$, and $f(6) = 0$. Since 32 is larger than 0, the maximum value occurs when $u = 4$ and $v = 6 - 4 = 2$.

2. Suppose a circle of radius r is inscribed in a regular hexagon as pictured. Find the area of the hexagon. (This formula for the area of a hexagon will be needed in the next problem.)

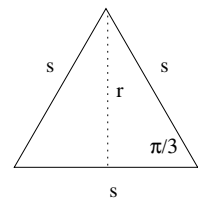


Solution:

We may divide the hexagon into six equilateral triangles with altitude of length r . If s denotes the sidelength of the equilateral triangle, then by considering the indicated triangle we get $\sin(\frac{\pi}{3}) = \frac{r}{s}$, thus

$$s = \frac{r}{\sin(\frac{\pi}{3})} = \frac{r}{\frac{\sqrt{3}}{2}} = \frac{2r}{\sqrt{3}}.$$

The area of an equilateral triangle with sidelength s and altitude of length r is $\frac{1}{2}rs = \frac{r^2}{\sqrt{3}}$. Thus the area of the hexagon is $6 \frac{r^2}{\sqrt{3}} = 2\sqrt{3}r^2$.



3. Answer parts 1–2 of the project in Stewart, pages 288–89. It will be helpful to read Example 2 on page 279–280 before beginning the project.

Solution: 1. If we cut the top and bottom from a square of sidelength $2r$, the area A of sheet metal used is

$$\begin{aligned} A &= \text{Area of side} + \text{Area of top and bottom} \\ &= 2\pi rh + 8r^2, \end{aligned}$$

where h is the height of the can. The volume of the can is fixed by $V_0 = \pi r^2 h$. Solving for h in terms of r , we obtain $h = \frac{V_0}{\pi r^2}$. Eliminating h from the expression for the area, we can write the area A as a function of r and obtain

$$A(r) = \frac{2V_0}{r} + 8r^2, \quad r > 0.$$

To find the minimum of $A(r)$, we look for its critical numbers. The derivative is

$$A'(r) = -\frac{2V_0}{r^2} + 16r.$$

Hence $A'(r) = 0$ if and only if $2V_0 = 16r^3$. It follows that $r_0 = \frac{1}{2}\sqrt[3]{V_0}$ is the only critical number of $A(r)$. Since $A'(r) < 0$ if $0 < r < \frac{1}{2}\sqrt[3]{V_0}$ and $A'(r) > 0$ if $r > \frac{1}{2}\sqrt[3]{V_0}$, we conclude that $A(r)$ assumes its absolute minimum value at $r_0 = \frac{1}{2}\sqrt[3]{V_0}$.

We are asked to find the ratio $\frac{h_0}{r_0}$ of the can that minimizes the cost of material. Since $h_0 = \frac{V_0}{\pi r_0^2}$ and $r_0^3 = \frac{V_0}{8}$, we get

$$\frac{h_0}{r_0} = \frac{V_0}{\pi r_0^3} = \frac{8}{\pi} \approx 2.55,$$

as claimed.

2. We can save material by cutting the circular lids from hexagons rather than squares. By Problem 2 above, we know that we need a hexagon of area $2\sqrt{3}r^2$ to make a circle of radius r . Thus, in this case the area A of sheet metal to produce the can is

$$A = 2\pi r h + 4\sqrt{3}r^2.$$

Eliminating h as above gives the area as function of r :

$$A(r) = \frac{2V_0}{r} + 4\sqrt{3}r^2, \quad r > 0.$$

Its derivative is

$$A'(r) = \frac{-2V_0}{r^2} + 8\sqrt{3}r.$$

It follows that the only critical number of $A(r)$ is $r_0 = \sqrt[3]{\frac{V_0}{4\sqrt{3}}}$. For the same reason as in part 1, $A(r)$ attains its absolute minimum at r_0 . Then the height is $h_0 = \frac{V_0}{\pi r_0^2}$. Hence we get for the ratio

$$\frac{h_0}{r_0} = \frac{V_0}{\pi r_0^3} = \frac{4\sqrt{3}}{\pi} \approx 2.21,$$

as claimed.

Grading guidelines:

1. (3 points) 1 point for finding the function to be maximized, 1 point for using the method for finding the maximum value on a closed interval, and 1 point for values of u and v .
2. 1 point for finding the area as a function of r .
3. (6 points) In part 1., 1 point for area of material used in terms of r ; in each part, 1 point for critical value and 1 point for ratio; 1 point for explanation as to why we have found minimum value in both parts.

Be sure to comment favorably on papers of students who do an unusually good job.

Take the time to recognize and provide guidance to students who attempt unusual approaches.

Deductions: Deduct one point for two of the following offences and two points for three or more offences:

- Not using complete sentences.
- Lack of explanations.
- Unusually messy or poorly organized papers.