Minimal Curves

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- An example
- How is the new algorithm an improvement?

Some Notation

- We will be working in $\mathbb{P}^3$.
- K will denote an algebraically closed field.
- $R = K[X, Y, Z, T]$ will be a polynomial ring.
- We also note that a curve $C \subseteq \mathbb{P}^3$ is a closed subscheme of dimension 1 that is taken to be locally Cohen-Macaulay (in particular, no embedded points).
- Let $C \subset \mathbb{P}^3$ be a curve. Then the Hartshorne-Rao module of C is:

$$M(C) = \mathrm{H}^1_{\mathfrak{m}}\left({}^{R}/_{I_C}\right)$$

(f.g. since the curve is locally Cohen-Macaulay)

What is a minimal curve?

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Proof: We will prove that there is a minimal left shift of M. Let M be a f.g. unmixed module associated to a curve. Then our claim is that $\mathrm{H}^1_{\mathfrak{m}}\left({}^{R}/_{I_C}\right) \cong M(e)$ is not possible for $e >> 0$.

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Proof: We will prove that there is a minimal left shift of M. Let M be a f.g. unmixed module associated to a curve. Then our claim is that $\mathrm{H}^1_{\mathfrak{m}}\left({}^{R}/_{I_C}\right) \cong M(e)$ is not possible for $e >> 0$. We will proceed by contradiction.

What is a minimal curve?

Assume that $\mathrm{H}^1_{\mathfrak{m}}\left({}^{R}\!/_{I_C}\right) \cong M(e) \neq 0$ such that

$$\left[\mathrm{H}^1_{\mathfrak{m}}\left({}^{R}\!/_{I_C}\right)\right]_j = [M]_{e+j} = 0, \qquad \text{for all } j \geq 0$$

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Take $\ell \in [R]_1$ general. Then we have the following exact sequence:

$$0 \longrightarrow A(-1) \xrightarrow{\ell} A \longrightarrow {}^{A}\!/_{\ell A} \longrightarrow 0$$

where $A = {}^{R}\!/_{I_C}$.

What is a minimal curve?

This induces the long exact cohomology sequence:

$$\cdots \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}\left({}^{A}/_{\ell A}\right) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A)(-1) \xrightarrow{\ell} \mathrm{H}^1_{\mathfrak{m}}(A) \longrightarrow \cdots$$

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But, $\mathrm{H}^0_{\mathfrak{m}}(A) = 0$. Hence, we have:

$$0 \rightarrow \mathrm{H}^0_{\mathfrak{m}}\left({}^{A}/_{\ell A}\right) \longrightarrow M(e-1) \longrightarrow M(e)$$

since $\mathrm{H}^1_{\mathfrak{m}}(A) \cong M(e)$.

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since $\mathrm{H}^1_{\mathfrak{m}}(A) \cong M(e)$. Now, we look in degree $a+1$:

$$0 \longrightarrow \left[\mathrm{H}^0_{\mathfrak{m}}\left({}^{A}/_{\ell A}\right)\right]_{a+1} \longrightarrow [M(e-1)]_{a+1} \longrightarrow [M(e)]_{a+1}$$

where $a = \max\left\{ j \,\middle|\, [M(e)]_j \neq 0 \right\}$.

What is a minimal curve?

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But, $\left[\mathrm{H}^0_{\mathfrak{m}} \left({}^{A}/_{\ell A} \right) \right]_{a-1} \neq 0$ where $a < 0$. A contradiction.

Therefore, there is a left-most shift.$_{QED}$

What is a minimal curve?

- Definition of a Minimal Curve: Let $\mathcal{L}$ be a biliaison class. A curve $C \in \mathcal{L}$ whose Hartshorn-Rao module corresponds to the left-most possible shift is called a minimal curve.

Why are minimal curves important?

- Lazarsfeld-Rao Property: Let $\mathcal{L}$ be a biliaison class of curves in $\mathbb{P}^3$, then we say that $\mathcal{L}$ has the Lazarsfeld-Rao Property if the following conditions hold:
  1. If C_1 and C_2 are minimal curves, then there is a deformation from one to the other through subschemes which are all minimal curves.
  2. Let C_0 be a minimal curve. Given $C \in \mathcal{L}$, then there exists a sequence of subschemes $C_0, C_1, ..., C_t$ such that for all i, $1 \leq i \leq t$, C_i is a basic double link of C_{i-1}, and C is a deformation of C_t.

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- Theorem: Every biliaison class of non-arithmetically Cohen-Macaulay curves of $\mathbb{P}^3$ has the Lazarsfeld-Rao Property.

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- Theorem: Every biliaison class of non-arithmetically Cohen-Macaulay curves of $\mathbb{P}^3$ has the Lazarsfeld-Rao Property.

- So, if we were able to find a minimal curve in the biliaison class of a given module, we could recover any other curve in the biliaison class by a sequence of basic double links and possibly a deformation.

The original algorithm

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$$\begin{array}{ccccccccccccc} & & & & & & & & Y & & & & \\ & & & & & & & \nearrow & & \searrow & & & \\ 0 & \longrightarrow & F_4 & \longrightarrow & F_3 & \longrightarrow & & F_2 & \xrightarrow{\sigma_2} & F_1 & \longrightarrow & F_0 & \longrightarrow \; 0 \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & L & & & & & & \end{array}$$

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\end{array}$$

Set $r := \text{rank}(F_1) - \text{rank}(F_0)$.

- We want to find a free R-module F of rank $r-1$ s.t. there is an exact sequence

$$0 \longrightarrow F \xrightarrow{\varphi} L \longrightarrow I(s) \longrightarrow 0$$

where I is an ideal of height two.

The original algorithm

We note that the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow {}^{R}\!/_{I} \longrightarrow 0$$

induces the long exact cohomology sequence:

$$\cdots \longrightarrow \mathsf{H}^1_{\mathfrak{m}}(R) \longrightarrow \mathsf{H}^1_{\mathfrak{m}}\left({}^{R}\!/_{I}\right) \longrightarrow \mathsf{H}^2_{\mathfrak{m}}(I) \longrightarrow \mathsf{H}^2_{\mathfrak{m}}(R) \longrightarrow \cdots$$

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But, R is free, so $\mathsf{H}^1_{\mathfrak{m}}(R) = \mathsf{H}^2_{\mathfrak{m}}(R) = 0$. Hence, we get the isomorphism

$$\mathsf{H}^1_{\mathfrak{m}}\left({}^{R}\!/_{I}\right) \cong \mathsf{H}^2_{\mathfrak{m}}(I)$$

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Using this same idea, we can get the following isomorphisms:

$$\mathsf{H}^1_{\mathfrak{m}}\left({}^{R}\!/_{I}\right) \cong \mathsf{H}^2_{\mathfrak{m}}(I) \cong \mathsf{H}^2_{\mathfrak{m}}(L)(-s) \cong \mathsf{H}^1_{\mathfrak{m}}(Y)(-s) \cong M(-s)$$

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However, for I to define a curve, we need to check that I is saturated. We know that I is saturated iff $\mathsf{H}^0_{\mathfrak{m}}\left({}^{R}/_{I}\right) = 0$. But

$$\mathsf{H}^0_{\mathfrak{m}}\left({}^{R}/_{I}\right) \cong \mathsf{H}^1_{\mathfrak{m}}(I) \cong \mathsf{H}^1_{\mathfrak{m}}(L)(-s) = 0$$

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We have the following exact sequence:

$$0 \longrightarrow F \xrightarrow{\varphi} L \longrightarrow I(s) \longrightarrow 0$$

where $F = \bigoplus_{i=1}^{r-1} R(-d_i)$ with $d_1 \leq d_2 \leq \ldots \leq d_{r-1}$.

The original algorithm

Since we have free resolutions for F and L, we can use the mapping cone to get a free resolution for $I(s)$ (not necessarily minimal).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\varphi} & L & \longrightarrow & I(s) & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & F & & F_2 & & F_2 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & F_3 & & F_3 \oplus F & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & F_4 & & F_4 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
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If we considered σ_2 as a matrix, then the columns are the minimal generators of L.

Then the matrix defining φ consists of $r-1$ of these columns where we take into consideration the degrees among the generators, i.e. the d_i's.

The original algorithm

Moreover, coker$(\varphi) \cong I$ iff ht$(I_{r-1}) = 2$.

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Moreover, coker$(\varphi) \cong I$ iff ht$(I_{r-1}) = 2$. It ends up that the saturation of the ideal generated by the $(r-1) \times (r-1)$ minors of the matrix corresponding to φ is the defining ideal of the minimal curve.

Improvements of the new algorithm

- The newer algorithm is done in $K[u]$ which is a much simpler polynomial ring.

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- In $K[u]$, the problem is reduced to applying the Smith Normal Form algorithm to the new matrix.

References

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