GRÖBNER FLAGS

AND

KOSZUL ALGEBRAS

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1 Introduction

Notation:

K = field, algebraically closed, $char(K) = 0$.

$R = K[x_1, \ldots, x_n]$.

A = a standard graded algebra.

(A graded commutative noetherian K-algebra $A = \bigoplus_{i \in \mathbb{N}} [A]_i$ is *standard graded* if $[A]_0 = K$ and A is generated (as a K-algebra) by elements of degree 1. Such an algebra can be presented as a quotiend of R by a homogeneous ideal I, i.e. we can write $A \simeq R/I$.)

1.1 Preliminaries

A *flag* of a finite-dimensional vector space V is a filtration of subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$. A flag is *complete* if $\dim V_i = i$ for each i.

A *basis of the flag* is a list of vectors $v_1, \ldots, v_{d_n}$, $dim V_k = d_k$, such that the first d_1 vectors are a basis of V_1, first d_2 a basis of V_2, etc.

EXAMPLE: $V = \mathbb{R}^n$, then for $V_k = span\{e_1, \ldots, e_k\}$, $V_1 \subseteq \cdots \subseteq V_n$ is a complete flag with basis $e_1, \ldots, e_n$.

1.2 Gröbner Basics

Definition 1 *A monomial order on R is a total order $>$ on the monomials of R such that if $r, s, t \in R$ are monomials with $r \neq 1$, then $s > t$ implies $rs > rt > t$.*

Let $a = (a_1, \ldots, a_n)$. We define

$$x^a = x_1^{a_1} \ldots x_n^{a_n}.$$

EXAMPLES.

- If $R = K[x_1]$, then the requirement that $\forall t \in R$ we must have $rt > t$ implies that the unique order on R is the ordering by degree.

- Reverse Lexicographic (revlex) order:

  Assume $x_1 > \cdots > x_n$.

  $x^a >_{rlex} x^b$ if

  $\deg x^a > \deg x^b$, or $\deg x^a = \deg x^b$ and and $a_i < b_i$ for the *last* index i such that $a_i \neq b_i$.

Definition 2 *Fix any* $>$. *The initial term of* f, *denoted* $in_>(f)$ *or just* $in(f)$, *is the largest term with respect to* $>$. *If* $I \subseteq R$ *ideal, then* $in_>(I) = in(I) = (in_>(f) | f \in I)$ *is the initial ideal of* I.

Definition 3 *Fix a monomial order* $>$. *The collection* $\{g_1, \ldots, g_k\}$ *is called a Gröbner Basis for an ideal* I *of* R *with respect to* $>$ *if*

$$in_>(I) = (in_>(g_1), \ldots, in_>(g_k)).$$

Monomials not in $in_>(I)$ *are called standard monomials (of* R/I*).*

NOTE: A Gröbner basis of I does form a generating set for I.

2 G-quadraticity and Koszulness

- Properties of A: G-quadratic, Koszul, quadratic
- Tools for detecting them: Koszul filtration, Gröbner flag

Definition 4 *A is Koszul if K has a linear resolution as a graded A-module:*

$$\cdots \to A(-2)^{\beta_2} \to A(-1)^{\beta_1} \to A \to K \to 0.$$

Ideal $I \subset R$ is quadratic if it is generated by quadrics (homog. elements of degree 2). A is quadratic if its defining ideal is quadratic.

A is G-quadratic if $A \simeq R/I$ where there exists a set of coordinates (linear combination of the $x_1, \ldots, x_n$) and a monomial order with respect to which I has a quadratic Gröbner basis.

IMPLICATIONS: G-quadratic $\Rightarrow$ Koszul $\Rightarrow$ quadratic.

EXAMPLES:

1. A polynomial ring $R = K[x]$ is a Koszul algebra, since we have

$$0 \to R(-1) \to R \to K = R/xR \to 0.$$

2. For $A = R/(x^2)$, we get

$$\ldots A(-2) \to A(-1) \to A \to K \to 0.$$

3. The coordinate ring of a rational normal curve is G-quadratic; namely, for $A = K[s^n, s^{n-1}t, \ldots, st^{n-1}, t^n] = R/I_C$, the $2x2$ minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{bmatrix}$$

give a Gröbner basis for I_C.

Definition 5 *A family of ideals $\mathcal{F}$ of A is a Koszul filtration of A if:*

1. *every ideal $I \in \mathcal{F}$ is generated by linear forms*

2. $(0) \in \mathcal{F}$ *and* $\mathfrak{m} \in \mathcal{F}$

3. *for all $I \in \mathcal{F}$, $I \neq 0$, $\exists J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic, and $J : I \in \mathcal{F}$.*

For example, take $R = K[x_1, \ldots, x_n]$. Then the ideals $I_j = (x_1, \ldots, x_j)$ for $1 \leq j \leq n$ and $I_0 = (0)$ form a Koszul filtration of R:

- I_j/I_{j-1} is clearly cyclic

- $I_{j-1} : I_j = I_{j-1}$, since the ideals are prime.

REMARK: If A has a Koszul filtration, then A is Koszul.

Definition 6 *A Gröbner flag of A is a complete flag of* $[A]_1$*, say*

$$F : V_0 = 0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = [A]_1,$$

where V_i *is a space of dimension* i*, such that the ideals* (V_i) *form a Koszul filtration of* A*; i.e. for each* i*,* $1 \leq i \leq n$*,* $\exists j_i$ *with*

$$(V_{i-1}) : (V_i) = (V_{j_i}).$$

We also define $j(F) = \{j_1, j_2, \ldots, j_n\}$*.*

REMARK: Equivalently, there is an ordered system of generators $l_1, \ldots, l_n$ of $[A]_1$, which are a basis of the flag, such that for each i we have

$$(l_1, \ldots, l_{i-1}) : (l_1, \ldots, l_i)$$

$$= \cap_{h=1}^{i}((l_1, \ldots, l_{i-1}) : l_h) = R \cap ((l_1, \ldots, l_{i-1}) : l_i)$$

$$= (l_1, \ldots, l_{i-1}) : l_i = (l_1, l_2, \ldots, l_{j_i}).$$

COMMENT1: Roughly, a Koszul filtration of A is a family of ideals of generated by linear forms such that any ideal of the family can be filtered in such a way that all successive colon ideals belong to the family.

COMMENT2: Roughly, a Gröbner flag is just a Koszul filtration supported on a single complete flag of linear forms.

Theorem 1 *If A has a Gröbner flag, then A is G-quadratic.*

PROOF: Let $F : V_0 = 0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = [A]_1$ be a Gröbner flag of A with $j(F) = j_1, j_2, \ldots, j_n$, and let $l_1, \ldots, l_n$ be a basis of the flag.

Consider the presentation of A given by

$$K[x_1, \ldots, x_n]/I \simeq A,$$

$$x_i \mapsto l_i.$$

(By assumption, the linear forms l_i generate $[A]_1$, and thus generate A as a K-algebra.)

From the Gröbner flag, we have for each i that

$$(l_1, \ldots, l_{i-1}) : l_i = (l_1, l_2, \ldots, l_{j_i}).$$

Hence for each k, $i \leq k \leq j_i$, we get a relation

$$l_k l_i = \sum_{h=1}^{i-1} l_h L_{k,i,h},$$

where $L_{k,i,h} \in [A]_1$ by degree considerations.

(**REMARK:** $j_i \geq i-1$. The case when $j_i = i-1$ for each i will be discussed at the end of the proof.)

Now look for the preimage of $l_k l_i - \sum l_h L_{k,i,h}$: for each i and k as above, there exists a polynomial of degree 2 of the form

$$Q_{i,k} = x_i x_k - \sum_{h=1}^{i-1} x_h L_{k,i,h},$$

with $L_{k,i,h}$ a linear form.

Fix a term order τ on R such that $in_\tau(Q_{i,k}) = x_i x_k$. (e.g. take revlex order induced by $x_n > \cdots > x_1$. In our case, $x_n^{a_n} \ldots x_1^{a_1} >_{rlex} x_n^{b_n} \ldots x_1^{b_1}$ if $a_i < b_i$ for the last i with $a_i \neq b_i$.

Now, each $x_h L_{k,i,h}$ contains a power of some x_h for $h \leq i-1$, and the power of each such x_h is zero in the monomial $x_i x_k$, because of the restrictions on k. So revlex will pick $x_i x_k$ as the largest term.)

Claim: these $Q_{i,k}$ are a Gröbner basis for I with respect to τ. To prove the claim, we need to show that the initial terms of the $Q_{i,k}$ generate the initial ideal $in(I)$; so let

$$J = (x_i x_k : 1 \leq i \leq n, i \leq k \leq j_i).$$

We need that $in(J) = in(I)$, or equivalently

that the standard monomials of R/J and R/I are the same.

But:

- the standard monomials form bases of R/J and R/I,
- $in(J) \subseteq in(I)$, so we get the opposite inclusion for the standard monomials.

Thus it suffices to show that the bases are of same cardinality; i.e. that R/J and R/I have the same Hilbert series. For this we use the following

Lemma 1 *The Hilbert series of a standard graded algebra A with a Gröbner flag F depends only on $j(F)$.*

By the Lemma, it suffices to show that there exists a Gröbner flag G of R/J with $j(G) = j(F)$.

Claim:

$$G : 0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n,$$

where W_i is generated by residue classes of $x_1, \ldots, x_i$, is a Gröbner flag of R/J with $j(G) = j(F)$.

Proof: $(\bar{x}_1, \ldots, \bar{x}_{i-1}) :_{R/J} \bar{x}_i = (\bar{x}_1, \ldots, \bar{x}_{j_1})$ is true iff

$$(J + (x_1, \ldots, x_{i-1})) :_R (J + (x_i)) = J + (x_1, \ldots, x_{j_i}) \text{ iff}$$

$$((J + (x_1, \ldots, x_{i-1})) : J) \cap ((J + (x_1, \ldots, x_{i-1})) : x_i)$$

$$= (J + (x_1, \ldots, x_{i-1})) : x_i = J + (x_1, \ldots, x_{j_i}).$$

One inclusion is clear ($\supseteq$); the other becomes apparent from the details of the proof of the next Theorem.

Finally, we return to the case $j_i = i - 1$ for all i. In this case, the ideal J above is (0). Hence by the Lemma, $A = R/I$ has the same Hilbert function as $R/J = R$, thus $I = 0$. $\square$

It remains to prove the Lemma: For each i, $1 \le i \le n$, we have a short exact sequence:

$$0 \to R/(V_{j_i})(-1) \to R/(V_{i-1}) \to R/(V_i) \to 0.$$

Of course, $R/(V_n) = K$ so that $h_{R/(V_n)}(j) = 1$ if $j = 0$ and is zero otherwise. Also, we have $n \ge j_i \ge i-1$ for each i.

To calculate $h_{R/(V_{n-1})}(j)$, there could be two cases:

- if $j_n = n$, then we use the exact sequence

$$0 \to K(-1) \to R/V_{n-1} \to K \to 0$$

- and if $j_n = n-1$, then we use

$$0 \to R/V_{n-1}(-1) \to R/V_{n-1} \to K \to 0.$$

The behavior of vector space dimensions along short exact sequences provides a formula for the desired Hilbert function.

To obtain $h_{R/(V_i)}(j)$ for other i, we proceed the similar way.

□

Theorem 2 (A characterization of algebras which have a Gröbner flag) *The following are equivalent:*

1. *A has a Gröbner flag.*

2. *There exists a presentation $A \simeq R/I$ such that if τ is the degree revlex order induced by $x_n > \cdots > x_1$, then:*

   - *$in_\tau(I)$ is generated by monomials of degree 2,*

   - *if $x_a x_b \in in_\tau(I)$ with $a \leq b$, then $x_a x_c \in in_\tau(I)$ for each c with $a \leq c \leq b$.*

PROOF that (2)⇒(1): set $S_i = \{s | i \leq s \leq n, x_i x_s \in in_\tau(I)\}$, and

$$j_i = \max_i S_i, \text{ if } S_i \neq \varnothing,$$

$$j_i = i - 1 \text{ otherwise.}$$

It suffices to show that

$$(I + (x_1, \ldots, x_{i-1})) : x_i = I + (x_1, \ldots, x_{j_i}),$$

the flag will be given by residue classes of the variables.

NOTE: This will also prove the opposite inclusion in proof of Theorem 1.

$\supseteq$: If $j_i = i - 1$, nothing to prove. Assume $j_i \geq i$. Take any k with $i \leq k \leq j_i$. Then $x_i x_k \in in_\tau(I)$, by assumption. Hence $\exists f \in I$ with $in_\tau(f) = x_i x_k$. Wolog, assume no other term of f is in $in_\tau(I)$. Then

$$f = \lambda x_i x_k + g \text{ with } \lambda \in K^* \text{ and } g \in (x_1, \ldots, x_{i-1}).$$

Therefore,

$$x_k \in (I + (x_1, \ldots, x_{i-1})) : x_i.$$

The claim follows as this is valid for all k.

$\subseteq$: Let $f \in (I+(x_1,\dots,x_{i-1})) : x_i$ and suppose $f \notin (x_1,\dots,x_{j_1})+I$.

We may assume that no term of f is in $in_\tau(I) \cup (x_1,\dots,x_{j_1})$. Then

$$fx_i \in I + (x_1,\dots,x_{i-1}) \Rightarrow fx_i = g + h,$$

with $g \in I, h \in (x_1,\dots,x_{i-1})$. Note that since variables of f have indices $> j_i \geq i-1$,

$$in_\tau(f)x_i = in_\tau(g) \in in_\tau(I).$$

But as $in_\tau(f) \notin in_\tau(I)$, x_i(one variable of $in_\tau(f)$) $\in in_\tau(I)$; a contradiction since variables of $in_\tau(f)$ have indices $> j_i$.

$\square$

3 Quadratic Gröbner Bases for Ideals of Points

We are interested in the properties of the coordinate ring A of a set of m points in $\mathbb{P}^n$. Some history:

- Kempf: if $m \leq 2n$, then A is Koszul.
- Conca, Trung, Valla: if $m \leq 2n$, then A has a Koszul filtration.
- This Koszul filtration is not supported on a flag, but Conca, Rossi, Valla modify the argument and show that A actually has a Gröbner flag. We show an outline of this proof.

Let X be a set of distinct points $P_1, \ldots, P_m \in \mathbb{P}^n$, and I be the defining ideal of X in $R = K[x_0, \ldots, x_n]$. We can write

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$$

where $\mathfrak{p}_i$ is the prime ideal corresponding to P_i.

Recall: the points are in linearly general position if at most $d+1$ of them lie in a any d-dimensional linear subspace.

Theorem 3 *Let X be a set of m distinct points in linearly general position in $\mathbb{P}^n$. If $m \leq 2n$, then the coordinate ring A of X has a Gröbner flag.*

PROOF: Suppose $m \geq n+1$ (the other case is not interesting). Set $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ and $J = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. By the linearly general position property, there is a linear form $L \in R$ such that the hyperplane $L = 0$ conatins the points $P_1, \ldots, P_n$ and avoids the remaining points. Similarly, there is a linear form $M \in R$ such that $M = 0$ contains the points $P_{n+1}, \ldots, P_m$, and avoids the others. One checks that $L + M$ is regular on R/I and R/J.

Next, set $S = R/I + (L+M) = A + (L+M)$, and let $l = \bar{L}$. Then pne also verifies that l satisfies $l^2 = 0$ and $l[S]_1 = [S]_2$.

Then, S has a Gröbner flag, computed as follows: since $0 : l \supseteq (l) \supset [S]_2$, $0 : l$ has no generators in degree > 1, thus we have $0 : l = (l, \tilde{l}_2, \ldots, \tilde{l}_r)$ for suitable independent $\tilde{l}_2, \ldots, \tilde{l}_r \in [S]_1$. Completing this list to an (ordered) basis $\tilde{l}_2, \ldots, \tilde{l}_n$ of $[S]_1$ provides the basis for a Gröbner flag of S.

Finally, since $L + M$ is regular on A, the Gröbner flag of $S = A/(L+M)A$ can be lifted to a Gröbner flag of A, whose basis will be $(L+M), l_2, \ldots, l_n$ where l_i is the preimage of $\tilde{l}_i$.

$\square$

3.1 Generalizations?

- There exists an example of $9 = 2 * 4 + 1$ points in $\mathbb{P}^4$ in general linear position that are not even Koszul!
- ...but, under certain assumptions, we get Koszulness; namely, the coordinate ring of m points in $\mathbb{P}^n$ is Koszul iff:
  - the points have "generic coordinates", which means that the coordinates are algebraically independent over $\mathbb{Q}$, and
  - $m \leq 1 + n + n^2/4$.

  So, 9 points with generic coordinates in $\mathbb{P}^4$ **are** Koszul.
- With 3 additional assumptions on $2n + 2$ points in $\mathbb{P}^n$, for $n \geq 3$, the cooridanate ring A has a Gröbner flag.

MAIN REFERENCE: A. Conca, M.E. Rossi, G. Valla: Gröbner flags and Gorenstein Algebras