On the Free Resolution of n+1 General Forms

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Introduction

Notation: K denotes a field and $R = K[x_1, \ldots, x_n]$ a polynomial ring in n variables. $*$ denotes the R-dual and $\vee$ the K-dual.

The Problem: Given an ideal $I = \langle G_1, \ldots, G_h \rangle$ generated by h general forms, what can we say about its minimal free resolution (MFR) if we only know the degrees of the generators?

If $h \leq n$ then I is a complete intersection and so its MFR is given by the Koszul complex.
Assume $h \geq n + 1$.

In some cases we know the Hilbert function

When $n = 2$ this was solved by R. Fröberg (1985).
When $n = 3$ it was solved by D. Anick (1986).

The Minimal Resolution Conjecture

A generically chosen set of points in $\mathbb{P}^{n-1}$ is a set of points whose Hilbert function depends only on the number of points.
The Minimal Resolution Conjecture (MRC) says that the graded Betti numbers are the smallest that the Hilbert function allows.

If the free resolution has a term like:

$$\begin{array}{ccccccc} & & F_i & \longrightarrow & F_{i-1} & & \\ \cdots & \longrightarrow & \bigoplus & & \bigoplus & \longrightarrow & \cdots \\ & & R(-j) & \overset{\text{id}}{\dashrightarrow} & R(-j) & & \end{array}$$

then the $R(-j)$ terms are not seen by the Hilbert function.

The MRC says such terms are not allowed.

If the the map is the identity when restricted to $R(-j)$ then we say the $R(-j)$ terms "split off".

Such terms can be removed and still leave a free resolution behind.

If the $R(-j)$ do not split of then, since they are still invisible to the Hilbert function, we them summands "ghost terms".

Eisenbud and Popescu showed the MRC is false (1999); ghost terms can occur.

There is a counterexample with 11 points in $\mathbb{P}^6$ having a $R(-5)$ ghost in terms 3 and 4 (found via computer by Schreyer).

The Thin Resolution Conjecture

$I = \langle G_1, \ldots, G_k \rangle \subseteq R$ (G_i general) has two generator of degree 4 and one of degree 8.

Then the first term in the MFR will have a $R(-8)$ summand corresponding to the generator.

The second will also have a $R(-8)$ summand from the Koszul syzygy.

This is a ghost term which we can never avoid.

The Thin Resolution Conjecture says that these are the only ghost terms. Due to A. Iarrobino, 1997.

This conjecture is false. In $K[x, y, z]$ consider the ideal I generated by general forms of degrees 4,4,4,8. The MFR is :

$$0 \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-11)^2 \end{matrix} \to \begin{matrix} R(-8)^3 \\ \bigoplus \\ R(-9)^2 \\ \bigoplus \\ R(-10) \end{matrix} \to \begin{matrix} R(-4)^3 \\ \bigoplus \\ R(-8) \end{matrix} \to R \to R/I \to 0$$

The $R(-8)$ term is expect from Koszul.

The $R(-10)$ terms do no split off and do not come from a Koszul syzygy.

This is due to a theorem we prove later.

The MFR of n+1 general forms

We restrict ourselves to
$I = \langle G_1, \ldots, G_n, G_{n+1} \rangle \subseteq R = K[x_1, \ldots, x_n]$
where the G_i are chosen generally and
$d_i = \deg G_i$.

We also assume that $J = \langle G_1 \ldots, G_n \rangle$ is a complete intersection. Let $D = d_1 + d_2 + \cdots + d_n$

Note that $\dim R/I = \dim R/J = 0$.

By local duality and the fact that J is Cohen-Macaulay,

$$R/J \cong (R/J)^{\vee}(D - n).$$

So the Hilbert function of R/J ends in degree $D - n$.

Thus, if $d_{n+1} > D - n$, $G_{n+1} \in J$. So I is a complete intersection and we know its MFR. Thanks Koszul!

Assume $d_{n+1} \leq D - n$.

Lemma 1. *The degree of the last non-zero component of R/I is*

$$\left\lfloor \frac{D + d_{n+1} - n - 1}{2} \right\rfloor.$$

Proof. Stanley and Watanabe showed the R/J has the strong Lefschetz property.
This means the map $[R/J]_i \to [R/J]_{i+d_{n+1}}$ given by multiplication by G_{n+1} has maximal rank.
We now know the Hilbert function of R/I,

$$h_{R/I}(t) = \max\{h_{R/J}(t) - h_{R/J}(t - d_{n+1}), 0\}.$$

We want the smallest t such that $\dim[R/J]_{t-d_{n+1}} > \dim[R/J]_t$.
The Hilbert function of R/J ends in degree $D - n$ and is symmetric.
We simply have to check the 4 different parity cases to verify the result. QED

Let $G = J\colon I$ be the ideal linked to I by the complete intersection J.

J has a Koszul resolution:

$$0 \to K_n \to K_{n-1} \to \cdots \to K_1 \to R \to R/J \to 0$$

and G has a MFR:

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to R \to R/G \to 0.$$

$K_n = R(-D)$

$K_{n-1} = \bigoplus_{i=1}^{n} R(D - d_i)$

$K_1 = \bigoplus_{i=1}^{n} R(-d_i)$

The mapping cone and standard exact sequence tell us that the MFRs of J and G have the same length.

Note that the canonical module of R/I is

$$\omega_{R/I} \cong \mathrm{Ext}_R^n(R/I, R)(-n)$$

Behold! the mapping cone!

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & G & \longrightarrow & \operatorname{Ext}^n(R/I,R)(-D) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K_1 & \dashrightarrow & F_1 & & F_1 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K_{n-1} & \dashrightarrow & F_{n-1} & & F_{n-1} \oplus K_{n-2} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K_n & \dashrightarrow & F_n & & F_n \oplus K_{n-1} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & K_n \cong R(-D) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$

By taking the R-dual we get a free resolution for R/I.

$$
0 \to F_1^*(-D) \to \begin{array}{c} F_2^*(-D) \\ \bigoplus \\ K_1^*(-D) \end{array} \to \cdots \to \begin{array}{c} F_n^*(-D) \\ \bigoplus \\ K_{n-1}^*(-D) \end{array} \to
$$

$$
\to R \to R/I \to 0
$$

The rank of F_n must be 1; we then call G *Gorenstein.*
$F_n = R(-D + d_{n+1})$, by the next lemma.

This resolution has the right length, so if we split off all the terms we can we will have the MFR.

Question: Where is the splitting?

¿From the exact sequence

$$0 \to \operatorname{Ext}^n(R/I, R)(-D) \to R/J \to R/G \to 0$$

we get

$$\begin{aligned} h_{R/G}(t) &= h_{R/J}(t) - h_{R/I}(D-n-t) \\ &= h_{R/J}(D-n-t) - h_{R/I}(D-n-t) \end{aligned}$$

We know $h_{R/J}$ by the Koszul resolution and $h_{R/I}$ by the strong Lefshetz property.
We also know that the Hilbert function of R/G is symmetric.

Lemma 2. *Let G and J be as above.*

(a) $h_{R/G}$ *ends in degree* $s = D - d_{n+1} - n$*.*

(b) $h_{R/G}$ *is unimodal with one peak if s is even and two if s is odd.*

(c) $\forall t \leq \frac{1}{2}s$, $h_{R/G}(t) = h_{R/J}(t)$

(d) $h_{R/G}(t) = \binom{t+n-1}{n-1}$ *for all* $t \leq \frac{1}{2}s$ *if* $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$

Proof: Clearly R/J and R/I are identical up to degree d_{n+1}.

$$h_{R/G}(s+k) = h_{R/J}(d_{n+1} - k) - h_{R/I}(d_{n+1} - k)$$

For positive k this is 0 and for $k = 0$ it is non-zero. (a) follows.

Since $D - d_{n+1}$ and $D + d_{n+1}$ have the same parity, (b) follows from (a) once we know $h_{R/G}$ is unimodal.

There are two, almost identical cases.

Assume s is even.

Since G is Gorenstein we know it is symmetric across $\frac{1}{2}s$. If $t \geq 0$ then

$$h_{R/G}\left(\frac{D - d_{n+1} - n}{2} - t\right) = h_{R/J}\left(\frac{D + d_{n+1} - n}{2} + t\right)$$

The right hand side is decreasing and so, by symmetry of $h_{R/G}$ we get unimodality.

(c) follows from $h_{R/G} = h_{R/J}(t) + h_{R/I}(D - n - t)$ and the Hilbert function calculation for R/I.

In (d) note that our condition shows that $d_1 > \frac{1}{2}s$. R/J has no generators until degree d_1 and so R/G is identical to the polynomial ring until after degree $\frac{1}{2}s$ by (c). QED

The resolution of R/I now looks like:

$$0 \to F_1^*(-D) \to \begin{array}{c} F_2^*(-D) \\ \bigoplus \\ K_1^*(-D) \end{array} \to \cdots \to \bigoplus_{i=1}^{n+1} R(-d_i) \to$$

$$\to R \to R/I \to 0$$

We can compute the MFR of G when
$D + d_{n+1} - n$ is even,
$d_2 + \cdots + d_n < d_1 + d_{n+1} + n$ and $2 \leq d_1$.

Also when $D + d_{n+1} - n$ is odd,
$d_2 + \cdots + d_n < d_1 + d_{n+1} + n$, $2 \leq d_1$ and n is even.

In both cases we can also determine the splitting.

In other cases we have bounds on the graded Betti numbers.

n=3

When $n = 3$ we have a complete result.
Our resolution is

$$0 \to F_1^*(-D) \to \begin{array}{c} \oplus_{i=1}^3 R(d_i - D) \\ \bigoplus \\ F_2^*(-D) \end{array} \to \bigoplus_{i=1}^{4} R(-d_i) \to$$

$$R \to R/I \to 0$$

Where the F_i come from the MFR of G

$$0 \to R(-D + d_{n+1}) \to F_2 \to F_1 \to R \to R/G \to 0$$

If we knew the MFR of G then we would know the MFR of I

Let G' be a general Gorenstein ideal whose Hilbert function is given by Lemma 2.

In particular the Hilbert function should be unimodal and end in degree $s = d_1 + d_2 + d_3 - d_4 - 3$ and be equal to $h_{R/J}$ for the first half.

Let $l = \lfloor s/2 \rfloor$.
Note that $d_2 > l + 1$

There are 4 cases

Case I: $d_2 + d_3 < d_1 + d_4 + 3$ and $D + d_{n+1}$ is odd

$$0 \to R(-2l-3) \to R(-l-2)^{2l+3} \to R(-l-2)^{2l+3} \to R \to R/G' \to 0$$

Case I $d_2 + d_3 < d_1 + d_4 + 3$ and $D + d_{n+1}$ is even

$$0 \to R(-2l-4) \to \begin{matrix} R(-l-3)^{l+2} \\ \bigoplus \\ R(-l-2)^{\delta} \end{matrix} \to \begin{matrix} R(-l-1)^{l+2} \\ \bigoplus \\ R(-l-2)^{\delta} \end{matrix} \to R \to R/G' \to 0$$

Where $\delta = 1$ if l is even and 0 otherwise.

Case III: $d_2 + d_3 \geq d_1 + d_4 + 3$ and $D + d_{n+1}$ is odd

$$0 \to R(-2l-3) \to \begin{array}{c} R(-2l-3+d_1) \\ \bigoplus \\ R(-l-2)^{2d_1} \end{array} \to \begin{array}{c} R(-d_1) \\ \bigoplus \\ R(-l-1)^{2d_1} \end{array}$$

$$\to R \to R/G' \to 0$$

Case IV: $d_2 + d_3 \geq d_1 + d_4 + 3$ and $D + d_{n+1}$ is even

$$0 \to R(-2l-4) \to \begin{array}{c} R(-2l-4+d_1) \\ \bigoplus \\ R(-l-3)^{d_1} \\ \bigoplus \\ R(-l-2)^{\delta} \end{array} \to \begin{array}{c} R(-d_1) \\ \bigoplus \\ R(-l-1)^{d_1} \\ \bigoplus \\ R(-l-2)^{\delta} \end{array}$$

$$\to R \to R/G' \to 0$$

Where $\delta = 1$ if d_1 is odd and 0 otherwise.

Consider case III.

We first compute the Hilbert function of R/G'.

R/G' and R/J are equal up to degree l.

Since $d_2 > l+1$, G has only one generator (of degree d_1) up to degree l.

By semicontinuity this also holds for G'

We know $h_{R/G'}$ up to degree l and so we know it everywhere by symmetry.

Until degree d_1 R/G' looks like our polynomial ring. ¿From degree d_1 to l it looks like a principle ideal.

$$h_{R/G'}(t) = \begin{cases} \binom{t+2}{2} & \text{if } 0 \leq t < d_1 \\ \binom{t+2}{2} - \binom{t+2-d_1}{2} & \text{if } d_1 \leq t \leq l \\ \text{symmetric} & \text{otherwise} \end{cases}$$

It is known that the number of minimal generators of G' is bounded from below by $-\Delta^3 h_{R/G'}$

We compute:

$$-\Delta^3 h_{R/G'}(t) = \begin{cases} 0 & \text{if } 0 < t < d_1 \\ 1 & \text{if } t = d_1 \\ 2d_1 & \text{if } t = l+1 \\ \text{negative} & \text{otherwise} \end{cases}$$

If s is even (it is here) then there is a G' with precisely these generators.

If s is odd there is one more generator of degree $l+2$.

We know the generators so we can compute the MFR. This is then the MFR for every G' by semicontinuity.

The other cases are similar.

Now that we know the MFR of G we can compute a free resolution for I.

We also know all the splitting that will occur in this free resolution.

So we can find the MFR for I

Of interest is the MFR when all the generators of I have the same degree a. Case II applies

$$0 \to R(-2a-1)^a \to \begin{matrix} R(-2a)^3 \\ \bigoplus \\ R(-2a+1)^a \end{matrix} \to R(-a)^4 \to R \to R/I \to 0$$

Although the MFR of G varies with the parity of a the MFR of I does not.

This MFR has no ghost terms.

It is conjectured that this is true whenever $a > n$.