Homework for Ma 561 - Modern Algebra I (Fall 03)

Set 10

35. (4 points) Let M be a subset of the group $G = (G, \cdot)$. Define $\langle M \rangle$ as the intersection of all the subgroups of G that contain M, i.e.

$$\langle M \rangle := \bigcap_{H \leq G, M \subset H} H.$$

It is called the *subgroup of G generated by M*. Show:
(a) $\langle M \rangle$ is the smallest subgroup of G that contains M.
(b) $\langle M \rangle = \{g_1^{e_1} \dots g_n^{e_n} \mid n \in \mathbb{N}_0, g_1, \dots, g_n \in M, e_1, \dots, e_n \in \{-1, 1\}\}$ where $\langle \emptyset \rangle := \{e_G\}$.

36. (4 points) Let P be a regular n-gon. Let P_0 be one of its vertices and let τ be the reflection at the line through P_0 such that $\tau(P) = P$. Let σ be the rotation of $\frac{2\pi}{n}$ such that $\sigma(P) = P$. Let D_{2n} be the subgroup of the symmetric group of P generated by $\{\sigma, \tau\}$. Prove:
(a) $\operatorname{ord} \tau = 2$, $\operatorname{ord} \sigma = n$, and $\tau \circ \sigma = \sigma^{n-1} \circ \tau$.
(b) D_{2n} has $2n$ elements, namely $\sigma, \sigma^2, \dots, \sigma^n, \tau \circ \sigma, \dots, \tau \circ \sigma^n$. It is called the *dihedral group of order* $2n$.
(Hint: Label the vertices by $1, \dots, n$ and consider the image of the vertices 1 and 2 under an element of D_{2n}.)

37. (4 points) Determine all subgroups and all normal subgroups of D_8.

6*. (6 points extra credit) Let $K(X)$ be the function field in one variable over the field K. Let $T := \frac{f}{g} \in K(X)$ where $f, g \in K[X]$ are coprime polynomials. Show
(a) $[K(X) : K(T)] = \max\{\deg f, \deg g\}$.
(b) $G(K(X), K) \cong PGL_2(K)$ where $PGL_2(K) := GL_2(K)/H$ is the projective general group and $H = \{aI_2 \mid a \in K^\times\}$.
(Hint: Show that every $\varphi \in G(K(X), K)$ is determined by $\varphi(X)$ and that $\varphi(X) = \frac{aX+b}{cX+d}$ where $a, b, c, d \in K$ and $ad - bc \neq 0$.

Due date: December 10, 2003