Homework for Ma 561 - Modern Algebra I (Fall 03)

Set 2

4. (2 points) Let E/L and L/K be field extensions such that $[E : K]$ is a prime number. Show that:
(a) $L = K$ or $L = E$.
(b) If $\alpha \in E$ does not belong to K then $K(\alpha) = E$.

5. (2 points) Determine $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}]$.

6. (8 points) Let E/K be a field extension. Prove that:
(a) If $A, B \subset E$ then

$$K(A \cup B) = K(A)(B).$$

(b) For $A \subset E$ define

$$K[A] := \text{intersection of all subrings of } E \text{ containing } K \cup A.$$

Then $K[A]$ is a subring of E. More precisely, $K[A]$ is the smallest subring of E that contains $K \cup A$.
(c) For every $\alpha \in E$ define $K[\alpha] := K[A]$ where $A := \{\alpha\}$. Then

$$K[\alpha] = \{f(\alpha) \mid f \in K[X]\}.$$

(d) For every $\alpha \in E$ we have

$$K(\alpha) = \left\{ \frac{f(\alpha)}{g(\alpha)} \mid f, g \in K[X], g(\alpha) \neq 0 \right\}.$$

7. (4 points) Let $(\alpha_n)_{n \in \mathbb{N}}$ be the sequence of real numbers such that $\alpha_1 = 2$ and $\alpha_{n+1} = \sqrt{\alpha_n}$ for all $n \in \mathbb{N}$.
(a) Prove that for every $n \in \mathbb{N}$ we have

$$[\mathbb{Q}(\alpha_{n+1}) : \mathbb{Q}] = 2^n.$$

(b) Deduce that $\measuredangle\{0, 1\}/\mathbb{Q}$ is not a finite field extension.

Due date: September 17, 2003