Homework for Ma 561 - Modern Algebra I (Fall 03)

Set 5

16. (4 points) Let E/K be a field extension and let $\alpha \in E$ be algebraic over K with minimal polynomial $p := \mu_K(\alpha)$. Consider the multiplication map $\varphi : K(\alpha) \to K(\alpha),\ y \mapsto \alpha y$. Show that:
(a) φ is K-linear.
(b) The minimal polynomial of the endomorphism φ is p (i.e. the minimal polynomials of α and φ agree).

17. (4 points) Let I be an ideal in the ring R and let $\varphi : R \to R'$ be a ring homomorphism. Prove that φ factors through the canonical projection $\pi : R \to R/I$, i.e. there is a ring homomorphism $\psi : R/I \to R'$ such that $\varphi = \psi \circ \pi$, if and only if $I \subset \ker \varphi$.

18. (4 points) Let $f \in K[X]$ be a polynomial of degree $n > 0$ over the field K. Use Kronecker's theorem to prove that there is an extension field E of K such that f splits completely into linear factors in $E[X]$, i.e. there are $\lambda, \alpha_1, \ldots, \alpha_n \in E$ such that

$$f = \lambda(X - \alpha_1) \cdot \ldots \cdot (X - \alpha_n).$$

Furthermore, show that $[K(\alpha_1, \ldots, \alpha_n) : K] \leq n!$.

1*. (4 points extra credit) Let E/K be a finite field extension of degree n. Show for the function fields over E and K that $[E(X) : K(X)] = n$.

Due date: October 24, 2003