Homework for Ma 561 - Modern Algebra I (Fall 03)

Set 6

19. (4 points) Let R be the ring of continuous functions $f : [a, b] \to \mathbb{R}$. Let $c \in [a, b]$.
(a) Decide whether the sets

$$I = \{f \in R \mid f(c) = 0\}$$
$$J = \{f \in R \mid f(c) = 1\}$$

are ideals of R.
(b) Show that R/I is isomorphic to a (very familiar) field.

20. (4 points) Let f, g be polynomials over the field K. Show that $g \bmod f$ has a multiplicative inverse in $K[X]/f$ if and only if f and g are coprime.

21. (4 points) Let $f, g \neq 0$ be polynomials over the field K. Let $n := \deg f$.
(a) Show that the residue class of g in $K[X]/f$ contains a unique polynomial h such that $\deg h < \deg f$; h is called the *canonical representative* of $g \bmod f$.
(b) How many elements has $K[X]/f$ if K has q elements?
(c) Let $K := \mathbb{Q}, f := X^4 + X^3 + X^2 + X - 5, g := X^2 + 5X + 6$. Compute the canonical representative of the multiplicative inverse of $g \bmod f$.

22. (4 points) The ring

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

is called the ring of Gaußian integers. Show that $\mathbb{Z}[i]$ is a Euclidean domain with respect to the norm $\nu : \mathbb{Z}[i] \to \mathbb{N}_0, z \mapsto |z|^2$.

2*. (6 points extra credit) Let a, b, c be elements of a Euclidean domain R. Let d be a greatest common divisor of a and b.
(a) Show that there are $x, y \in R$ such that $xa + yb = c$ if and only if d divides c.
(b) If $x_0, y_0 \in R$ satisfy $x_0 a + y_0 b = c$ then the set of all pairs $(x, y) \in R^2$ with $xa + yb = c$ is

$$\left\{ \left(x_0 + m\frac{b}{d}, y_0 - m\frac{a}{d} \right) \mid m \in R \right\}.$$

(c) Determine all integer solutions of $85x + 145y = 505$.

Due date: November 3, 2003