Homework for Ma 561 - Modern Algebra I (Fall 03)

Set 7

23. (4 points) Consider the ring $R := \mathbb{Q}[X]/(X-1)(X+2)$.
(a) Determine all ideals of R.
(b) Find (up to isomorphism) all rings R' such that there is a surjective ring homomorphism $R \to R'$.

24. (4 points) Let $a, b \neq 0$ two elements in a factorial domain R. Show that a and b have a greatest common divisor and a least common multiple.

25. (4 points) Determine all integers x such that

$$\begin{aligned} x &\equiv 1 \bmod 5 \\ x &\equiv 2 \bmod 7 \\ x &\equiv 3 \bmod 11 \\ x &\equiv 5 \bmod 19 \end{aligned}$$

26. (6 points) Let $d \neq 0$ be an integer that is not a square in $\mathbb{Z}$. Consider the subring

$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$$

of $\mathbb{C}$. Show:
(a) Defining $N(a + b\sqrt{d}) := a^2 - b^2 d$ provides a well defined map $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$. ($N(z)$ is called the *norm* of z.)
(b) If $z, w \in \mathbb{Z}[\sqrt{d}]$ then $N(zw) = N(z)N(w)$.
(c) For the ring $\mathbb{Z}[\sqrt{-5}]$:

(i) The set of units is $\mathbb{Z}[\sqrt{-5}]^\times = \{1, -1\}$.
(ii) 2 is irreducible, but not a prime element.
(iii) The elements 6 and $2(1+\sqrt{-5})$ do not have a greatest common divisor.

(Hint: $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.)

3*. (4 points extra credit) Let R be a UFD with the property that every ideal that is generated by two elements is a principal ideal. Prove that R must be a PID.

Due date: November 10, 2003