Homework for Ma 561 - Modern Algebra I (Fall 03)

Set 9

31. (4 points) Determine the automorphism group of the following fields: $\mathbb{Q}$, $\mathbb{Q}[\sqrt{3}]$, $\mathbb{Q}[\sqrt[3]{3}]$, $\mathbb{Q}[\sqrt[4]{3}]$.

32. (4 points) (a) Let $(R, +, \cdot)$ be ring. Show that the set $R^\times$ of units of R forms a group under multiplication.
(b) Let $n := p_1^{e_1} \dots p_s^{e_s}$ where $p_1, \dots, p_s$ are distinct prime numbers and $e_1, \dots, e_s$ are positive integers. Show that $(\mathbb{Z}/n\mathbb{Z})^\times$ has $\varphi(n) := (p_1 - 1)p_1^{e_1-1} \dots (p_s - 1)p_s^{e_s-1}$ elements. (This function $\varphi : \mathbb{N} \to \mathbb{N}$ is called *Euler's φ-function*.)

33. (4 points) Let G be a group and let $a \in G$ be an element of finite order n. Show that for every integer $m \neq 0$ the order of a^m is $\frac{n}{d}$ where d is the positive greatest common divisor of m and n.

34. (4 points) Prove that every group with 4 elements is either isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

5*. (4 points extra credit) Let $d \geq 2$ be a square free integer. Let $\alpha \in \mathbb{R}$ be an element such that $\alpha^n = d$ for some $n \in \mathbb{N}$. Show:
(a) The rings

$$\mathbb{Z}[\alpha] := \{b_{n-1}\alpha^{n-1} + \dots b_1\alpha + b_0 \mid b_{n-1}, \dots, b_0 \in \mathbb{Z}\}$$

and $\mathbb{Z}[X]/(X^n - d)$ are isomorphic.
(b) $\mathbb{Z}[\alpha]$ is an integral domain with quotient field $\mathbb{Q}[\alpha]$.

Due date: December 3, 2003