Homework for Ma 661 - Modern Algebra II (Spring 04)

Set 13

47. (4 points) let E/K be a finite field extension and let F be an algebraically closed extension field of E. Show:
(a) There are at most $[E : K]$ K-homomorphisms $\varphi : E \to F$.
(b) There are exactly $[E : K]$ K-homomorphisms $\varphi : E \to F$ if and only if E/K is separable.
(Hint: Consider a simple extension first.)

48. (4 points) Let E/K be a field extension where $\mathrm{char}(K) = p > 0$. Let $\alpha \in E$ be an algebraic element over K. Prove:
(a) There is an integer $e \geq 0$ such that α^{p^e} is separable over K.
(Hint: Consider the integer e such that $\mu_K(\alpha) \in K[X^{p^e}]$, but $\mu_K(\alpha) \notin K[X^{p^{e+1}}]$.)
(b) If E/K is purely inseparable then there is an integer $e \geq 0$ such that $\alpha^{p^e} \in K$.

49. (8 points) Let H, U be subgroups of the group G and let $M := \{gH \mid g \in G\}$ be the set of left cosets of G modulo H. Show:
(a) U acts on M by $U \times M \to M,\ (a, gH) \mapsto agH$.
(b) The stabilizer of gH with respect to this action is $U_{gH} = U \cap H^g$ where $H^g := gHg^{-1} := \{ghg^{-1} \mid h \in H\}$.
(c) Let $g, g' \in G$ and set $UgH := \{agh \mid a \in U, h \in H\}$. Then $UgH \neq Ug'H$ if and only if $UgH \cap Ug'H = \emptyset$.
(d) If G is finite then

$$|UgH| = \frac{|U| \cdot |H|}{|U \cap H^g|} = \frac{|U| \cdot |H|}{|H \cap U^{g^{-1}}|}.$$

(e) If G is finite then there are elements $g_1, \ldots, g_m \in G$ such that

$$|G| = \sum_{i=1}^{m} \frac{|U| \cdot |H|}{|H \cap U^{g_i}|}.$$

9*. (4 points extra credit) Let E/K be a finite field extension and let F be an algebraically closed extension field of E. Show:
(a) There are $[E : K]_s$ K-homomorphisms $\varphi : E \to F$.
(b) If E/K is also normal then $|G(E, K)| = [E : K]_s$.

Due date: February 6, 2004