Homework for Ma 661 - Modern Algebra II (Spring 04)

Set 14

50. (4 points) Let H be a subgroup of the group G. Show:
(a) G acts on the set $\mathcal{P}(G)$ of all its subsets by $G \times \mathcal{P}(G) \to \mathcal{P}(G)$, $(g, M) \mapsto gMg^{-1}$. (Note that for every non-empty subset $M \subset G$, the stabilizer of M with respect to this action is the normalizer $N_G(M)$.)
(b) $H \lhd N_G(H)$.
(c) $H \lhd G$ if and only if $N_G(H) = G$.
(d) If G is finite then $[G : N_G(H)]$ is the number of subgroups of G that are conjugates of H.

51. (4 points) Let G be a group. Prove that G is abelian if $G/Z(G)$ is a cyclic group.

52. (4 points) Let p be a prime number. Show that every group of order p^2 is isomorphic to $\mathbb{Z}_{p^2}$ or to $\mathbb{Z}_p \times \mathbb{Z}_p$.

53. (4 points) Let H, N be a normal subgroups of the group G with $H < N$. Prove:
(a) $H \lhd N$ and $N/H \lhd G/H$.
(b) The groups $(G/H)/(N/H)$ and G/N are isomorphic. (Observe the similarity to Exercise 15.)

10*. (4 points extra credit) Let E be a splitting field of the irreducible and separable polynomial $f \in K[X]$ over the field K. Denote by $\alpha_1, \dots, \alpha_n \in E$ the roots of f. Assume that the Galois group $G(E, K)$ is abelian. Show that $E = K(\alpha_i)$ for all $i = 1, \dots, n$ and thus $[E : K] = \deg f$.

Due date: February 13, 2004