Homework for Ma 661 - Modern Algebra II (Spring 04)

Set 15

54. (4 points) Let $G \neq \{e_G\}$ be a group. Show that the order of G is prime if and only if G has no non-trivial subgroups, i.e. the only subgroups of G are $\{e_G\}$ and G.

55. (4 points) Let $G \neq \{e_G\}$ be a group. Prove that G is cyclic and its order is a power of a prime if and only if G has a largest proper subgroup, i.e. there is a subgroup $H \neq G$ that contains all subgroups of G except G.

56. (4 points) Let H be a subgroup of the group G and let M be the set of left cosets of G modulo H.
(a) Show that G acts on M by $G \times M \to M,\ (g, aH) \mapsto gaH$, and that the induced group homomorphism $T : G \to S(M)$ has kernel

$$\ker T = \bigcap_{g \in G} gHg^{-1}.$$

(b) Now let G be a finite group and let p be the smallest prime that divides the order of G. Prove that H must be a normal subgroup of G if its index is p. (This is a generalization of Exercise 39 in case of a finite group.)
(Hint: Argue that $[H : \ker T]$ divides $|G/\ker T|$ which itself divides $(p-1)!$.)

57. (4 points) Let E be the splitting field of the irreducible polynomial $f \in \mathbb{Q}[X]$ of degree 4 over $\mathbb{Q}$ and let $z \in \mathbb{C}$ be a root of f. Show that $z \notin \measuredangle\mathbb{Q}$ if $|G(E, \mathbb{Q})| > 8$ (though $[\mathbb{Q}(z) : \mathbb{Q}] = 2^2$).

11*. (4 points extra credit) Consider the irreducible polynomial $f := X^4 + X + 1 \in \mathbb{Q}[X]$. Show that there are $z, w \in \mathbb{C}$ such that $f = (X - z)(X - \bar{z})(X - w)(X - \bar{w})$ and that $|z|^2 + |w|^2$ is a root of $g := X^3 - 4X - 1$. Conclude that every root $\alpha \in \mathbb{C}$ of f is not in $\measuredangle\mathbb{Q}$ though $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^2$.

Due date: February 20, 2004