Homework for Ma 661 - Modern Algebra II (Spring 04)

Set 16

58. (4 points) Let $f \in K[X]$ be an irreducible polynomial. Assume there is an extension field E of K such that E/K is a radical extension and E contains one root of f. Show that f is solvable by radicals.

59. (4 points) Let $f \in K[X]$ be a polynomial of degree n and let M be the set of roots of f (in some splitting field of E) that are not in K. Show:
(a) There is an injective group homomorphism $\Psi : G(f, K) \to S(M)$ into the symmetric group $S(M)$.
(b) If $G(f, K)$ has order $n!$ then f is irreducible and separable.

60. (4 points) Let $f := X^4 + 1 \in \mathbb{Q}[X]$. Compute the Galois group $G(f, \mathbb{Q})$ and all its subgroups.

61. (4 points) Let $f \in K[X]$ be a polynomial of degree n and let E be a splitting field of f over K. Prove that $[E : K]$ divides $n!$.
(This improves Exercise 18.
Hint: If s is an integer such that $0 < s < n$ argue that $s! \cdot (n - s)!$ divides $n!$.)

12*. (4 points extra credit) Let E/K be a finite field extension and let F be an extension field of E that is algebraically closed. Show that there is a unique smallest subfield L of F that contains E such that L/K is normal. L is called a *normal hull* of E/K. Furthermore argue that if L, L' are two normal hulls of E/K then there is a K isomorphism $L \to L'$.

Due date: February 27, 2004