Homework for Ma 561 - Modern Algebra I (Fall 09)

Set 6

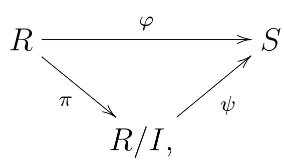
19. (7 points) Let I, J be ideals of a ring R. Show:
(a) $I + J := \{a + b \mid a \in I, b \in J\}$ is an ideal of R, and it is the smallest ideal containing both I and J.
(b) $I \cap J$ is an ideal of R.
(c) $IJ := \{\sum_{j=1}^{n} a_j b_j \mid n \in \mathbb{N}, a_j \in I, b_j \in J\}$ is an ideal of R that is contained in $I \cap J$.
(d) Give an example, where $IJ \neq I \cap J$.
(e) If R is a commutative ring and $I + J = R$, then $IJ = I \cap J$.

20. (4 points) Convince yourself that the set $R := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}$ is a commutative ring with the usual sum and product of functions. Prove:
(a) The set $I := \{f \in R \mid f(5) = f'(5) = 0\}$ is an ideal of R.
(b) $\mathbb{R}[X]/(X^2) = \{(a + bX) \bmod (X^2) \mid a, b \in \mathbb{R}\}$.
(c) The rings R/I and $\mathbb{R}[X]/(X^2)$ are isomorphic.

21. (4 points) Let I, J be ideals of a ring R, and let $\pi : R \to R/I$ be the canonical epimorphism. Show:
(a) $\pi(J)$ is an ideal in R/I. It is denoted by $(J + I)/I$.
(b) The epimorphism π induces an inclusion-preserving bijection

$$\{J \mid J \text{ is an ideal of } R \text{ such that } I \subset J\} \to \{\text{ideals of } R/I\}.$$

22. (4 points) Let $\varphi : R \to S$ be a ring homomorphism and let $I \subset R$ be an ideal such that $I \subset \ker \varphi$. Prove:
(a) There is a unique ring homomorphism $\psi : R/I \to S$ such that there is a commutative diagram



i.e. $\varphi = \psi \circ \pi$, where $\pi : R \to R/I$ is the canonical epimorphism.
(b) One has $\ker \psi = \ker \varphi / I$, thus ψ induces a ring isomorphism

$$(R/I) \,/\, (\ker \varphi / I) \cong R / \ker \varphi.$$

Due date: October 12, 2009