

## Homework for Ma 561 - Modern Algebra I (Fall 09)

### Set 8

**27.** (4 points) Let  $p$  be a fixed prime number and consider the set

$$R := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ are coprime, } p \nmid b \right\}.$$

- (a) Show that  $R$  is a subring of  $\mathbb{Q}$  and not a field.
- (b) Determine the set of units  $R^\times$ .
- (c) Show that each non-zero ideal of  $R$  is principal and of the form  $(p^e)$  for some  $e \in \mathbb{N}_0$ .
- (d) Argue that  $R$  has a quotient field  $K$  that is a subfield of  $\mathbb{Q}$  and determine  $K$ .

**28.** (4 points) Let  $f, g$  be polynomials over the field  $K$ . Show that  $g \bmod f$  has a multiplicative inverse in  $K[X]/f$  if and only if  $f$  and  $g$  are coprime.

**29.** (4 points) Consider the polynomials

$$f := X^4 + X^3 + X^2 + X - 5, g := X^2 + 5X + 6 \in \mathbb{Q}[X].$$

Find a polynomial  $h \in \mathbb{Q}[X]$  whose degree is at most 3 such that  $h \bmod f$  is the multiplicative inverse of  $g \bmod f$  in  $\mathbb{Q}[X]/f$ . Argue that the polynomial  $h$  is unique.

**30.** (4 points) Show that the ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

with the norm  $\nu : \mathbb{Z}[i] \rightarrow \mathbb{N}_0, z \mapsto |z|^2$ , is a Euclidean domain.

**2\*.** (6 points extra credit) Let  $a, b, c$  be elements of a Euclidean domain  $R$  such that  $(a, b) \neq (0, 0)$ . Let  $d$  be a greatest common divisor of  $a$  and  $b$ .

- (a) Show that there are  $x, y \in R$  such that  $xa + yb = c$  if and only if  $d$  divides  $c$ .
- (b) If  $x_0, y_0 \in R$  satisfy  $x_0a + y_0b = c$  then the set of all pairs  $(x, y) \in R^2$  with  $xa + yb = c$  is

$$\left\{ \left( x_0 + m \frac{b}{d}, y_0 - m \frac{a}{d} \right) \mid m \in R \right\}.$$

- (c) Determine all integer solutions of  $85x + 145y = 505$ .

**Due date:** October 30, 2009