Ma109 Fall 2005

Notes for Exam 2.

We discuss the topics for the second exam period as covered in the homework.

1 Coordinates

1. Use the distance formula for points in the plane to find points with desired properties.

   Example. What is the distance between $A(-1,-4)$ and $B(2,-5)$? Find the y-coordinate of a point $P(2,y)$ such that the distance between P and A is $3\sqrt{5}$.

   Answer. The distance between $P_1(a_1,b_1), P_2(a_2,b_2)$ is:

   $$d(P_1,P_2) = \sqrt{(a_2-a_1)^2+(b_2-b_1)^2}.$$

   So
   $$d(A,B) = \sqrt{(2-(-1))^2+(-5-(-4))^2} = \sqrt{9+1} = \sqrt{10}.$$

   Also we get

   $$d(A,P) = \sqrt{(2-(-1))^2+(y-(-4))^2} = 3\sqrt{5}.$$

   Squaring both sides and simplifying, we get:

   $$9+(y+4)^2 = 45 \textbf{ or } (y+4)^2 = 36.$$

   Taking square roots, we deduce that $y+4 = \pm 6$ and hence $y = 6-4$ or $y = -6-4$, i.e. $y = 2$ or $y = -10$.

2. Set up a coordinate system in the plane and calculate the coordinate of various points using their description.

   Example. A coordinate system is set up in the plane and Sally starts from the origin. She travels 2 miles East and 22 miles North. At the same time Libby travels 22 miles East and 2 miles North.

   Andy Stays at the origin.

   What are their final positions and distances?

   Answer. Sally ends at $(2,22)$ while Libby is at $(22,2)$ where the units are in miles. Andy is still at $(0,0)$.

   Then the distances are

between Sally and Libby $\sqrt{(22-2)^2+(2-22)^2} = \sqrt{400+400} = \sqrt{800}$.

between Andy and Sally $\sqrt{(22-0)^2+(2-0)^2} = \sqrt{484+4} = \sqrt{488}$.

The distance between Andy and Libby is clearly the same as $\sqrt{488}$.

3. Determine the equations of change of coordinates in a line using given information.

   Example. Assume that you are given coordinate change on a line which changes the coordinate x to a new coordinate z given by the formula $z = Ux+P$ where U, P are real numbers with $U \neq 0$.

   If the new coordinate of the point 12 becomes -51 and the new coordinate of the point -10 becomes 37 then find U, P.

   For the same transformation, if the new coordinate is -9 the what is the original coordinate?

   Answer. Using the given information, we get two equations:

   $$-51 = 12U + P \text{ and } 37 = -10U + P.$$

   We solve these for U, P.

   Subtracting the second from the first, we get

   $$-51 - 37 = (12 - (-10))U \text{ or } U = \frac{-88}{22} = -4.$$

   Using this in the first equation, we get:

   $$-51 = (-12)(-4) + P \text{ or } P = 48 - 51 = -3.$$

   Thus the transformation is $z = -4x - 3$. If $z = -9$ then we must have:

   $$-9 = -4x - 3 \text{ or } -4x = -6 \text{ or } x = \frac{3}{2}.$$

   Thus the original coordinate x is $\frac{3}{2}$.

4. Determine the equations of change of coordinates in a plane using given information.

   Example. Assume that you are given a coordinate change in the plane which changes the coordinates (x, y) to new coordinates (z, w) given by the formula

   $$z = Ux + P, w = Vy + Q$$

where U, V are ± 1 and P, Q are real numbers.If the new coordinates of $(4, 1)$ become $(13, 8)$ and the new coordinates of $(5, -4)$ become $(14, 13)$ then calculate U, V, P, Q.

Answer. We solve:

$$13 = 4U + P, 14 = 5U + P, 8 = V + Q, 13 = -4V + Q.$$

The solution of the first two gives $U = 1, P = 9$. The last two give $V = -1, Q = 9$. Thus the transformation is given by

$$(z, w) = (x + 9, -y + 9).$$

5. Construct formulas for linear expressions for one variable in terms of the other variables, using the given description.

   Example. A chemical company produces two kinds of chemicals called A and B. They cost 3 and 8 dollars respectively to produce and the resulting chemicals weigh 2.3 and 1.5 kg per gallon. The production manager records the number of gallons produced per month denoted by x, y respectively. The supply manager records the amount of money left over from a monthly allotment of $\$28,000$ and the total weight of the monthly production in kg. He uses variables z and w respectively for his records.

   Determine the relationship between z, w and x, y.

   Answer.

   Notice that the money left over, or z is equal to $\$28000-$ amount used. The expences for the production are clearly $\$3x$ for chemical A and $\$8y$ for the chemical B. Hence

$$z = 28000 - 3x - 8y.$$

   Similarly, the weight of the produced chemical A is $2.3x$ kg while that for chemical B is $1.5y$ kg so we get:

$$w = 2.3x + 1.5y.$$

2 Lines

1. Given two points A, B set up coordinates on the line. The simplest method is to let a point P with coordinate t be calculated as

   $$P(t) = (1-t)A + tB.$$

   The resulting formulas for the x, y coordinates are called the parametric equations (or parametric form) of the line AB.

   Example. Make a parametric form for the line joining points $A(2,3)$ and $B(-1,5)$.

   Choose your coordinates so that A has coordinate 0 and B has coordinate 1. What is the point with coordinate 2?

   Answer. Choose a parametrization given by

   $$P(t) = (1-t)A + tB = (1-t)(2,3) + t(-1,5).$$

   This gives the parametrization $x = 2-2t-t = 2-3t$ and $y = 3-3t+5t = 3+2t$.

   For $t = 2$ we get the point $(-4, 7)$.

   Extra. In this setup, the midpoint is always at $t = \frac{1}{2}$ and other division points are also easy to define and find.

2. Given two points A, B, calculate the equation of the line joining them. This can be done in two ways:

   Either find the slope and use a point slope formula or use a cleverly constructed formula from the two points.

   The resulting equation is the equational form of the line AB.

   Example. Given $A(-5,0), B(5,-4)$ calculate the line joining them and determine its slope and y-intercept.

   Answer. Let two given points be $P_1(a_1,b_1), P_2(a_2,b_2)$. The slope of the line joining them is:

   $$\text{slope } = m = \frac{b_2 - b_1}{a_2 - a_1}.$$

   The point-slope form of the equation of the line gives:

   $$(y - b_1) = m(x - a_1).$$

This equation can also be rewritten by the definition of the slope m as

$$(y - b_1)(a_2 - a_1) = (x - a_1)(b_2 - b_1).$$

It is possible to read off the y-intercept as :

$$\frac{b_1 a_2 - b_2 a_1}{a_2 - a_1}$$

with suitable adjustment made when $a_1 = a_2$.

In some sense, this is easier to write and lets us get all the necessary information as needed.

Thus, for our A, B we get:

$$(y - (0))(5 - (-5)) = (x - (-5))(-4 - 0) \text{ or } 10y = -4(x + 5).$$

To get the slope and intercept, rewrite this as:

$$y = -\frac{4}{10}x - \frac{20}{10} \text{ or } y = -\frac{4}{10}x - 2.$$

Thus the slope is -0.4 and y-intercept is -2.

3. Convert an equational form of the line to a parametric form and vice versa.

   Example. Convert the parametric form of a line $x = 3t + 3, y = t + 2$ to equational form. Also convert the equational form $2x - 3y = 1$ of a line to a parametric form.

   Answer. To convert from parametric to equational form make a suitable combination of the two equations to eliminate the parameter.

   Thus $x - 3y = (3t + 3) - 3(t + 2)$ produces $x - 3y = 3 - 6 = -3$ the desired answer!

   The converse problem is even easier. Put any convenient non constant expression for x and then get the expression for y using the given equation. Thus put $x = t$ and $2x - 3y = 1$ becomes $2t - 3y = 1$ or $3y = 2t - 1$ or $y = \frac{2}{3}t - \frac{1}{3}$.

   Important: This idea will fail for a vertical line like $x = 4$, but it has an obvious parametric form $x = 4, y = t$.

4. Find the intersection of two lines where one is in parametric form and the other is in equational form.

Example. Consider the line with parametric equations $x = 1 - t, y = -t + 2$ and another line defined by the equation $4y - 3x + 1 = 0$.

Answer. At a common point the equation as well as the parametric expression must hold, so just substitute the parametric expression into the equation of the line:

$4(-t+2)-3(1-t)+1 = 0$ which becomes $t(-4+3)+8-3+1 = 0$ or $-t+6 = 0$.

This gives $t = 6$ and hence the point $(1 - 6, -6 + 2) = (-5, -4)$.

5. Find the intersection of two lines given in parametric form.

   Example. Consider the parametric Line 1: $x = t + 1, y = -3t + 1$ and another Line 2: $x = -2s + 4, y = -s + 1$. Find their common point.

   Answer. One straightforward method is to write the two implied equations for the common point:

   $$t + 1 = -2s + 4 \text{ and } -3t + 1 = -s + 1.$$

   Treat these as two equations in two unknowns s, t and solve. Here we get $t = \frac{3}{7}$ and $s = \frac{9}{7}$. The point comes out to be $x = t + 1 = \frac{10}{7}$ and $y = -2(\frac{9}{7}) + 4 = \frac{-18+28}{7} = \frac{10}{7}$.

   It is also possible to shorten the work a bit by converting one line to equational form. The Line 2 for example is x-2y=2 and plugging in the parametrization of Line 1 gives:

   $$(t + 1) - 2(-3t + 1) = 2 \text{ or } 7t = 3 \text{ or } t = \frac{3}{7}.$$

6. Find the intersection of two lines in equational form.

   This is just the technique of solving two equations in two unknowns and does not need further elaboration.

7. Calculate equations of reflected lines when the equation is in a horizontal or vertical line.

   Example. A laser beam is directed from the point $A(19, 10)$ to the point $B(4, 0)$ on the x-axis. It reflects off the x-axis and hits the y-axis at some point and then reflects again from that point. What is the equation of the line followed by that final ray?

   Answer.

   It is easy to see the following general fact:

Given a line $ax + by = c$ the equation of its reflection in x-axis is simply $ax - by = c$ or the expression obtained by changing y to $-y$.

Similarly, the equation of the reflection in y axis is simply $-ax + by = c$.

As a consequence, if we reflect in both the axes, the equation becomes $-ax - by = c$ or $ax + by = -c$.

The equation of the line AB is easily found to be:

$$(y-0)(19-4) = (x-4)(10-0) \text{ or } 15y = 10x - 40.$$

Thus the reflected line has equation $-15y = 10x - 40$. It meets the y-axis at $(0, \frac{40}{15}) = (0, \frac{8}{3})$. The equation of the next reflection is then $-15y = -10x - 40$ or $15y = 10x + 40$, i.e $y = \frac{2}{3}x + \frac{8}{3}$.

Food for thought. How would you make a reflection in other horizontal or vertical lines? How about a general line?

8. Given an equational form of a line, find an equational form of a parallel line with desired properties.

   Given an equational form of a line, find an equational form of a perpendicular line with desired properties.

   Example. You are given a line L: $2x + 6y = 3$ and a point $P(-3, -3)$. Find a line M parallel to L passing thru P. Also find a line N perpendicular to L passing thru P.

   Answer. The problem can be done by calculating the slope of the given line and thereby determining the slope of the desired line. Then use the point slope form to get the answer.

   However, it is more convenient to recall that a line parallel to $ax+by = c$ is gven by $ax+by = k$ for a convenient k. Similarly, a perpendicular line to $ax+by = c$ is given by $bx - ay = k$ for a convenient k.

   Thus the line M must be $2x+6y = k$ and the value of k must be $2(-3)+6(-3) = -24$ since it passes thru P.

   Similarly, the line N must be $6x - 2y = k$ and the condition of passing thru P gives $k = 6(-3) - 2(-3) = -12$.

   Important: Remember that given one equation of a line you can multiply it by any non zero number and get another equally valid equation. Thus, you may have to multiply your final answer to match a given answer.

9. Check when a triangle is a right angle triangle by using the slope condition for perpendicular lines (either the product of the slopes is -1 or one of the lines has slope 0 and the other is vertical (slope ∞).

   Example. For which value of t does the triangle with vertices $A(-5,3), B(5,1), C(t,-1)$ have a right angle at C?

   Answer. Find the slopes of the lines AC and BC and equate the product to -1. You need to adjust the work if one of the slopes becomes zero, then the other line has to be vertical.

   Slope of AC is $\frac{-1-3}{t-(-5)} = \frac{-4}{t+5}$. The slope of BC is $\frac{-1-1}{t-5} = \frac{-2}{t-5}$.
   When we multiply them and equate to -1 we get:

   $$\frac{-4}{(t+5)}\frac{-2}{(t-5)} = -1 \text{ or } (t^2-25) = -8.$$

   This leads to $t^2 = 17$ or $t = \pm\sqrt{17}$.

10. Calculate the equation of a circle with two given points as the end of a diameter.

    Example. Find the equation of a circle with points $A(6,5)$ and $B(-4,-4)$ as the end of a diameter.

    Answer. We know that equation of a circle with $P_1(a_1,b_1), P_2(a_2,b_2)$ as the end of a diameter is given by

    $$(x-a_1)(x-a_2)+(y-b_1)(y-b_2) = 0.$$

    Thus for our problem, we get:

    $$(x-6)(x+4)+(y-5)(y+4) = 0 \text{ or } x^2+y^2-2x-y-44 = 0.$$

11. Calculate the equation of the circle containing three given points.

    Example. You are given three points: $A(2,3), B(2,-3), C(0,-3)$. Determine a point equidistant from all three. In other words, find the intersection of the perpendicular bisectors of the line segments AB, AC.

    Answer.

    It may be useful to find a formula for the perpendicular bisector of a line segment.

    Let two given points be $P_1(a_1,b_1), P_2(a_2,b_2)$.

    We know that their midpoint is $(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2})$. We need a line thru this point with correct slope, namely $-\frac{a_2-a_1}{b_2-b_1}$.

Write the equation in point slope form and simplify to:

$$(y - \frac{b_1 + b_2}{2})(b_2 - b_1) = -(x - \frac{a_1 + a_2}{2})(a_2 - a_1).$$

This can be put in an even simpler form thus:

$$(b_2 - b_1)y + (a_2 - a_1)x = \frac{b_2^2 - b_1^2 + a_2^2 - a_1^2}{2}.$$

Thus the perpendicular bisector of AB is given by:

$$(-3 - 3)y + (2 - 2)x = \frac{(-3)^2 - (3)^2 + (2)^2 - (2)^2}{2} \text{ or } -6y = 0.$$

The perpendicular bisector of AC is given by:

$$(-3 - 3)y + (0 - 2)x = \frac{(-3)^2 - (3)^2 + (0)^2 - (2)^2}{2} \text{ or } -6y - 2x = -2.$$

The common point of the two lines is clearly $x = 1, y = 0$.

The common distance of these points from each of the given points is $\sqrt{10}$ and the desired circle is simply the circle with center $(1, 0)$ and radius $\sqrt{10}$.

It equation is:

$$(x - 1)^2 + (y - 0)^2 = 10 \text{ or } x^2 + y^2 - 2x - 9 = 0.$$

It is possible to do the above problem with a different setup with the same results. Here is how you could proceed.

Suppose that the desired equation of the circle is:

$$x^2 + y^2 + ux + vy + w = 0.$$

Plug in each of the points and get three equations in three unknowns u, v, w. Then solve them. Indeed, the easiest way to eliminate w from the equations is to subtract one equation from the other two. Compare the work to see that you indeed get the equations of the perpendicular bisectors again!

We just show the first step:

$$\text{For point } A \;\; 13 + 2u + 3v + w = 0.$$

$$\text{For point } B \ 13 + 2u - 3v + w = 0.$$

$$\text{For point } C \ 9 - 3v + w = 0.$$

12. Find the intersection of a line with a circle.

    Example. The line $3y - 5x + 6 = 0$ meets the circle $x^2 + y^2 + 2x + 6y + 8 = 0$ at point $P(0, -2)$. Find the other point of intersection.

    Answer. The strategy is simple. Convert the line to a parametric form and plug in the circle. Solve for the parameter to get the points.

    Thus, the line is $x = t, y = \frac{5}{3}t - 2$. Substitution in the circle gives the equation:

    $$t^2 + (\frac{5}{3}t - 2)^2 + 2t + 6(\frac{5}{3}t - 2) + 8 = 0.$$

    This simplifies to

    $$\frac{34}{9}t^2 + \frac{16}{3}t = 0$$

    It factors as

    $$\frac{2t(17t + 24)}{9} = 0$$

    and the solutions are $t = 0, -\frac{24}{17}$.

    Clearly, $t = x = 0$ gives the original point P and $t = x = -\frac{24}{17}$ gives the other point with $y = \frac{5}{3}t - 2 = -\frac{74}{17}$.

    Observe. In general, we will always a quadratic equation, which may have two roots or a single root or no roots at all. If you know one root, then it is easy to factor the quadratic expression. This makes it easy to find the second root.

3 Circles

1. Construct the equation of a circle with given center and radius.

   Example. Find the equation of the circle with center $(-2, -3)$ and radius 6.

   Answer. Simply plug in the formula $(x-a)^2 + (y-b)^2 = r^2$ for a circle with center (a, b) and radius r.

   $$(x+2)^2 + (y+3)^2 = 36 \text{ or } x^2 + y^2 + 4x + 6y = 23.$$

2. Recognize the center and radius of a given circle from its equation.

   Example. What is the center and the radius of the circle

   $$x^2 + y^2 + 6x + 4y + 8 = 0?$$

   Answer. Complete squares in x, y separately to rewrite:

   $$\begin{aligned} x^2 + y^2 + 6x + 4y + 8 &= (x^2 + 6x) + (y^2 + 4y) + 8 \\ &= (x+3)^2 - 9 + (y+2)^2 - 4 + 8 \\ &= (x+3)^2 + (y+2)^2 - 5 \end{aligned}$$

   Thus clearly, the circle is

   $$(x+3)^2 + (y+2)^2 = 5$$

   and has center $(-3, -2)$ with radius $\sqrt{5}$.

3. Construct Pythagorean Triples using a standard formula derived from a parametrization of a circle.

   Example. Recall that a Pythagorean triple is a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$. A simple technique to create such a triple is to take

   $$a = \frac{s^2 - t^2}{2}, b = st, c = \frac{s^2 + t^2}{2}$$

   where s, t are integers which are both odd or both even.

   Verify this.

   Calculate such triples for various values of s, t.

   Answer. Note that since both s, t are odd or both are even, the expressions are actually integers. The verification that $a^2 + b^2 = c^2$ is simple algebra!

For $s = 8, t = 2$ we get:

$$a = \frac{8^2 - 2^2}{2} = 30, b = (8)(2) = 16, c = \frac{8^2 + 2^2}{2} = 34.$$

For $s = 3, t = 1$ we get:

$$a = \frac{3^2 - 1^2}{2} = 4, b = (3)(1) = 3, c = \frac{3^2 + 1^2}{2} = 5.$$

4. Find the equation of the line passing thru the two intersection points of given circles, without finding the points themselves.

   Use the line to find the intersection points of two circles.

   Example. Find the equation of the line thru the intersection of the two circles:

   $$x^2 + y^2 + 8y = 0 \text{ and } x^2 + y^2 + 8x - 2y = -1.$$

   Answer. Subtraction of one equation from the other kills the quadratic terms and hence gives us a line. Clearly, it passes thru all the common points and hence is the answer!

   We get $-8x + 10y = 1$.

   If we need the actual points of intersection, we solve this for y and plug into one of the two circle equations. This quadratic does not have nice solutions!

5. Find the equation of a circle thru three given points. **Follow the example in the section on Lines.**

6. Learn the formula for the distance from a point to a line.

   Example. The distance from $P(u, v)$ to the line $ax + by + c = 0$ is

   $$\frac{|au + bv + c|}{\sqrt{a^2 + b^2}}.$$

   Use this to find the distance from $(-2, -1)$ to $3x - 4y + 5 = 0$.

   Answer.

   $$\frac{|3(-2) - 4(-1) + 5|}{\sqrt{3^2 + 4^2}} = \frac{3}{5}.$$

7. Calculate the equation of a circle with a given center and tangent to a given line.

   Example. Find the equation of the circle centered at $P(3, -8)$ and tangent to the line $L: \quad -2x + 4y - 8 = 0$.

   Answer. The radius of the circle should be the distance from the point P to L.

   This distance is given by

$$\frac{|-2(3) + 4(-8) - 8|}{\sqrt{(-2)^2 + (4)^2}} = \frac{46}{\sqrt{20}}.$$

   Thus the equation of the circle shall be:

$$(x-3)^2 + (y+8)^2 = \frac{(46)^2}{20} = \frac{529}{5}.$$

8. Calculate angles between two lines by geometric reasoning.

   Example. What is the angle between two hands of the clock at 1 : 15 PM

   Answer. The minute hand is at 3, so it makes a $90°$ angle with the 12-o-clock position. Each successive number is separated by $\frac{360}{12} = 30$ degrees. The hour hand must have advanced a quarter of angular distance from the position of 1 so must be making an angle of $30 + \frac{30}{4}$ degrees with the 12-position.

   The desired angle must be $90 - (30 + \frac{30}{4})$ or $\frac{105}{2}$ degrees. In radians it becomes

$$\frac{\pi}{180}\frac{105}{4} = \frac{7\pi}{48}.$$

9. Understand the relation between the radius of the circle, the area of a sector of it and the angle at the vertex of the sector.

   Example. Area of a pie-slice is 24π and the measure of the angle of the slice is $60°$. What is the radius of the circle?

   Answer. Remember that the area of the circle is

$$\pi r^2 = \frac{r^2}{2} 2\pi.$$

   The whole circle corresponds to an angle of 2π radians. Hence, the formula for the area of a slice corresponding to θ radians must be $\frac{r^2}{2}\theta$.

Thus, in our problem, we have

$$24\pi = \frac{r^2}{2}\frac{\pi}{3}$$

so

$$r^2 = (24)(6) = 12^2.$$

It follows that $r = 12$ feet!

10. Calculate trigonometric functions of an angle using usual definitions (and the Pythagorian theorem as needed.)

    Example. Find the cosine of the acute angle α that the line $y = 3x$ makes with the positive x-axis.

    Answer. Choose the point $(1, 3)$ on the line and consider the right triangle $A(0, 0), B(1, 0), C(1, 3)$.

    The distance $d(A, B)$ is 1 and the distance $d(B, C)$ is 3. The distance $d(A, C)$ is $\sqrt{1 + 9} = \sqrt{10}$.

    Hence the desired $\cos(\alpha) = \frac{d(A,C)}{d(A,B)} = \frac{1}{\sqrt{10}}$ Also similarly, $\sin(\alpha) = \frac{d(B,C)}{d(A,B)} = \frac{3}{\sqrt{10}}$.