Dispersion of Rayleigh Waves in Weakly Anisotropic Media with Vertically-Inhomogeneous Initial Stress

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Working within a special context in linear elasticity with initial stress, we present a procedure by which a high-frequency asymptotic formula can be derived for dispersion relations of Rayleigh waves that propagate in various directions along the free surface of a vertically-inhomogeneous, prestressed, and generally anisotropic half-space. The special context in question is defined by three assumptions, namely: (i) the incremental elasticity tensor of the material half-space can be written as the sum of a homogeneous isotropic part $C^{iso}$ and a depth-dependent perturbative part $A$; (ii) at the free surface both the initial stress and $A$ are small as compared with $C^{iso}$; (iii) the mass density, the initial stress, and $A$ are smooth functions of depth from the free surface. We derive formulas and Lyapunov-type equations that can iteratively deliver each term of an asymptotic expansion of the surface impedance matrix, which leads to the aforementioned high-frequency asymptotic formula for Rayleigh-wave dispersion. As illustration we consider a thick-plate sample of AA 7075-T651 aluminum alloy, which has one face treated by low plasticity burnishing that induced a (depth-dependent) prestress at and immediately beneath the treated surface. We model the sample as a prestressed, weakly-textured orthorhombic aggregate of cubic crystallites and work out explicitly, up to the third order, the dispersion relations that pertain to Rayleigh waves propagating in several directions along the treated face of the sample.

Keywords: Rayleigh waves, vertical inhomogeneity, dispersion, initial stress, weakly anisotropic media, surface impedance matrix, direct problem

1. Introduction

Recently Man et al. (2013) developed a general procedure, under the framework of linear elasticity with initial stress (Biot (1965); Hoger (1986); Man and Lu (1987); Man and Carlson (1994)), for obtaining a high-frequency asymptotic formula for the dispersion of the phase velocity of Rayleigh waves propagating in a vertically-inhomogeneous, prestressed and anisotropic medium. That work was motivated by an industrial need to develop a nondestructive measurement technique for monitoring the retention of protective surface and subsurface compressive stress induced by surface treatments to improve the fatigue life of metal parts such as critical components of aircraft engines. The theory in Man et al. (2013) does not consider the effects of surface roughness and will apply to surface treatments such as low plasticity burnishing (LPB) which leave a mirror-smooth surface finish. On the other hand, the theory is developed in the setting that the constitutive equation in linear elasticity with initial stress is
put in its most general form, which makes the derivation of explicit dispersion relations difficult.

In this paper we adapt the general procedure in Man et al. (2013) to the special case where the incremental elasticity tensor $L$ can be written as the sum of an isotropic part $C_{\text{Iso}}$ and a perturbative part $\Delta$. Under a Cartesian coordinate system where the material medium occupies the half-space $x_3 \leq 0$, the perturbative part $\Delta$, the initial stress $\sigma^{\circ}$, and the mass density $\rho$ are assumed to be smooth functions of $x_3$. Moreover, at the free surface of the material medium the perturbative part $\Delta(0)$ and the initial stress $\sigma^{\circ}(0)$ are assumed to be sufficiently small as compared with $C_{\text{Iso}}$ that for all expressions and formulas which depend on $\Delta(0)$ and $\sigma^{\circ}(0)$ it suffices to keep only those terms linear in the components of these tensors. Under this setting, after outlining some preliminaries in Section 2, we derive in Sections 3–5 and Appendix A specific formulas with which the procedure presented in Man et al. (2013) can be implemented.

In Section 6, we further specialize the formulation to an illustrative example where the material medium in question is a real-world sample of an AA 7075-T651 aluminum alloy that carries a prestress induced by prior LPB-treatment. The sample is modeled as a weakly-textured orthorhombic aggregate of cubic crystallites. The prestress in the sample was ascertained by destructive means (X-ray diffraction and hole-drilling), and so was the crystallographic texture (X-ray diffraction). To describe the elastic response of the sample, we adopt the constitutive equation given in Man (1999) and Tanuma and Man (2002), where the perturbative part $\Delta$ of the incremental elasticity tensor is given explicitly in terms of the prestress and texture. We use the information given in the ASM Handbook for the 7075-T651 alloy to get the Lamé constants of $C_{\text{Iso}}$ and, for the 10 material parameters pertaining to $\Delta$, we adopt the values predicted by the Man-Paroni model (Man and Paroni (1996); Paroni and Man (2000)) from single-crystal elastic constants of aluminum (Thomas (1968); Sarma and Reddy (1972)). With the prestress, texture, and material parameters of the sample specified, we implement the procedure given in Sections 3–5 to solve the direct problem of deriving high-frequency asymptotic formulas for dispersion relations that pertain to Rayleigh waves with various propagation directions. To shed light on how crystallographic texture would affect the dispersion relations, we prescribe two other textures to the sample and repeat the calculations with the prestress and material parameters unchanged.

2. Preliminaries

In a Cartesian coordinate system let $(x_1, x_2, x_3)$ be the Cartesian coordinates of place $x$, and let $u = u(x) = (u_1, u_2, u_3)$ be the displacement at $x$ pertaining to the superimposed small elastic motion. We work in the theoretical context of linear elasticity with initial stress, under which the constitutive equation can be put in the form (cf. Man and Lu (1987); Man and Carlson (1994))

$$S = \bar{T} + \bar{H} \bar{T} + L[E];$$

(2.1)

here $S = (S_{ij})$ is the first Piola-Kirchhoff stress, $\bar{T} = (\bar{T}_{ij})$ the initial stress, $H = (\partial u_i/\partial x_j)$ the displacement gradient pertaining to the superimposed small elastic motion, and $E = (H + H^T)/2$ the corresponding infinitesimal strain, where the superscript $T$ denotes transposition; $L$ is the incremental elasticity tensor which, when regarded as a fourth-order tensor on symmetric tensors, has its components $L_{ijkl}$ ($i, j, k, l = 1, 2, 3$) satisfying the minor symmetries

$$L_{ijkl} = L_{ijlk} = L_{jikl}, \quad i, j, k, l = 1, 2, 3.$$  
(2.2)
We adopt the assumption that there exists a stored energy function for deformations from the initial configuration, which dictates that the components of \( L \) enjoy also the major symmetry
\[
L_{ijkl} = L_{klij}, \quad i,j,k,l = 1,2,3.
\] (2.3)

We choose the Cartesian coordinate system so that the material half-space occupies the region \( x_3 \leq 0 \) whereas the 1- and 2- axes are chosen arbitrarily. In this paper we assume that the initial stress \( \mathbf{T} = \mathbf{T}(x_3) \), the incremental elasticity tensor \( \mathbf{L} = \mathbf{L}(x_3) \), and the mass density \( \rho = \rho(x_3) \) are smooth functions of the coordinate \( x_3 \) \((x_3 \leq 0)\). Here and hereafter we use the term “smooth function” to denote an infinitely differentiable function all of whose derivatives are bounded and continuous. We assume that the initial stress \( \mathbf{T} \) satisfies the equation of equilibrium
\[
\sum_{j=1}^{3} \frac{\partial \mathbf{T}_{ij}}{\partial x_j} = 0, \quad (i = 1,2,3)
\]
and that the surface \( x_3 = 0 \) of the half-space is free of traction, which implies that the components \( \mathbf{T}_{ij}(x_3) \) \((i = 1,2,3)\) of \( \mathbf{T} \) vanish at the surface \( x_3 = 0 \). We call \( x_3 \geq 0 \) the depth of place \( x \) beneath the free surface \( x_3 = 0 \).

In what follows we suppose that \( \mathbf{L} \) can be written as a sum of two terms: a principal part \( \mathbf{C}^{\text{Iso}} \) which is homogeneous and isotropic, and a perturbative part \( \mathbf{A} = \mathbf{A}(x_3) \) which is a smooth function of \( x_3 \) \((x_3 \leq 0)\) and is generally anisotropic. Then \( \mathbf{L} \) can be expressed as a fourth-order tensor on symmetric tensors \( \mathbf{E} \) in the form
\[
\mathbf{L}[\mathbf{E}] = \mathbf{C}^{\text{Iso}}[\mathbf{E}] + \mathbf{A}[\mathbf{E}],
\] (2.4)
where both \( \mathbf{C}^{\text{Iso}} \) and \( \mathbf{A} \) are 4th-order tensors that enjoy the major and minor symmetries, \( \mathbf{C}^{\text{Iso}} \) is given by
\[
\mathbf{C}^{\text{Iso}}[\mathbf{E}] = \lambda (\text{tr} \mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}
\] (2.5)
with the identity matrix \( \mathbf{I} \) and Lamé constants \( \lambda \) and \( \mu \), and \( \mathbf{A} \) can be written componentwise in terms of a \( 6 \times 6 \) matrix under the Voigt notation as
\[
\mathbf{A} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{33} & a_{34} & a_{35} & a_{36} \\
a_{44} & a_{45} & a_{46} \\
a_{55} & a_{56} \\
a_{66}
\end{pmatrix}, \quad \text{Sym.}
\] (2.6)
with the 21 components in the upper triangular part of matrix (2.6) being smooth in \( x_3 \) \((x_3 \leq 0)\) and being generally all independent.

We suppose that at the free surface \( x_3 = 0 \), the perturbative part \( \mathbf{A} \) of \( \mathbf{L} \) and the initial stress \( \mathbf{T} \) are sufficiently small as compared with the isotropic part \( \mathbf{C}^{\text{Iso}} \) of \( \mathbf{L} \), by which we mean the Euclidean norm \( \| \cdot \| \) of \( \mathbf{C}^{\text{Iso}} \), of \( \mathbf{T}(0) \) and of \( \mathbf{A}(0) \) satisfy
\[
\| \mathbf{T}(0) \| \ll \| \mathbf{C}^{\text{Iso}} \|, \quad \| \mathbf{A}(0) \| \ll \| \mathbf{C}^{\text{Iso}} \|
\] (2.7)
so that for all expressions and formulas which depend on \( \mathbf{A}(0) \) and \( \mathbf{T}(0) \) it suffices to keep only those terms linear in the components of these tensors.
Substituting the componentwise expression of (2.1), i.e.

\[ S_{ij} = T_{ij}^\circ + \sum_{k,l=1}^3 (T_{jl}^\circ \delta_k + L_{ijkl}) \frac{\partial u_k}{\partial x_l}, \]  

(2.8)

where \( \delta_k \) is the Kronecker delta, into the equations of motion with zero body force, namely

\[ \rho \frac{\partial^2}{\partial t^2} u_i = \sum_{j=1}^3 \frac{\partial}{\partial x_j} S_{ij}, \quad i = 1, 2, 3 \]  

(2.9)

where \( t \) denotes the time, we obtain elastic wave equations of the form

\[ \rho \frac{\partial^2}{\partial t^2} u_i = \sum_{j,k,l=1}^3 \frac{\partial}{\partial x_j} (B_{ijkl} \frac{\partial u_k}{\partial x_l}), \quad i = 1, 2, 3, \]  

(2.10)

where

\[ B_{ijkl} = B_{ijkl}(x_3) = \delta_{ik} T_{jl}^\circ(x_3) + L_{ijkl}(x_3) \]  

(2.11)

are the effective elastic coefficients.

We consider Rayleigh waves propagating in a given direction along the traction-free surface of the aforementioned vertically-inhomogeneous, anisotropic and prestressed half-space \( x_3 \leq 0 \). These waves are described as a surface-wave solution to (2.10) in \( x_3 \leq 0 \) which is time-harmonic, has the form

\[ u(x, t) = (u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3)) = e^{-i k (x_1 \eta_1 + x_2 \eta_2 - \nu t)} a(x, \eta_1, \eta_2, \nu, k), \]  

(2.12)

and satisfies the traction-free boundary condition

\[ s_n(u) \big|_{x_3=0} = 0, \]  

(2.13)

where

\[ s_n(u) = \left( \sum_{j,k,l=1}^3 B_{ijkl} \frac{\partial u_k}{\partial x_l} \right)_{i=1,2,3}. \]  

(2.14)

Here \( i = \sqrt{-1} k \) is the wave number, \( \eta = (\eta_1, \eta_2, 0) \) is the direction of wave propagation on the surface, \( \nu \) is the phase velocity in the subsonic range to be determined, and \( a(x, \eta_1, \eta_2, \nu, k) \) is the complex-valued polarization vector which decays exponentially as \( x_3 \to -\infty \).

Under the assumption on existence of Rayleigh waves Man \textit{et al.} (2013) recently derived a high-frequency asymptotic formula

\[ v_R = v_R(k) = v_0 + v_1 k^{-1} + v_2 k^{-2} + v_3 k^{-3} + \cdots \]  

(2.15)

which, for a large wave number \( k \), expresses the phase velocity \( v_R \) of the Rayleigh waves in question in terms of \( \mathbb{L}(x_3), \mathbb{T}(x_3), \rho(x_3) \) at \( x_3 = 0 \) and their first and higher-order \( x_3 \)-derivatives at \( x_3 = 0 \). They developed a procedure which can deliver an expression for each term \( v_i (i = 0, 1, 2, \cdots) \). The asymptotic formula for \( v_R \) thus gives a characterization of the frequency-dependence of the Rayleigh-wave velocity, i.e., the dispersion of Rayleigh waves, as caused by the vertical inhomogeneity of the medium.
The surface impedance matrix \( Z(v) = Z(v, \eta, k) \) is a 3 \( \times \) 3 matrix that expresses a linear relationship between the displacements at the surface on which surface waves propagate with phase velocity \( v \) and the surface tractions needed to sustain them;

\[
s_n(u)\big|_{\kappa_3=0} = Z(v) \left( u \big|_{\kappa_3=0} \right),
\]

where \( u \) is the solution (2.12). (Strictly speaking, we need to consider surface displacements supported in a bounded portion of the surface so that we can construct in an appropriate Sobolev space an asymptotic solution to (2.10) and (2.13) which describes the Rayleigh waves. See Sections 3 and 4 of Man et al. (2013).) By Section 5 of Man et al. (2013), \( Z(v) \) admits an asymptotic expansion

\[
Z(v) = kZ_0(v) + Z_1(v) + k^{-1}Z_2(v) + k^{-2}Z_3(v) + \cdots;
\]

where \( kZ_0(v) \) is nothing but the surface impedance matrix of the comparative homogeneous elastic half-space which has its incremental elasticity tensor, mass density, and initial stress equal to \( L(0), \rho(0), \) and \( \hat{T}(0) \), respectively. Note that in the literature on the Stroh formalism for homogeneous elastic media it is \( Z_0(v) \) which is usually called the “surface impedance matrix”; see, for example, Lothe and Barnett (1976), Chapter 7 of Chadwick and Smith (1977), Chapter 12 of Ting (1996) and Definition 4.3 of Tanuma et al. (2013). It is proved in Man et al. (2013) that each \( Z_n(v) \) \((n = 0, 1, 2, \cdots)\) is Hermitian, i.e., \( Z_n(v) = Z_n(v)^\ast \), where the overbar denotes complex conjugation.

The surface impedance matrix plays a crucial role on the procedure which delivers each term of the asymptotic expansion (2.15). It follows from (2.13) and (2.16) that the matrix \( Z(v) \) has a non-trivial null space in a three-dimensional complex linear space at the phase velocity of the Rayleigh waves \( v_R \). This leads us to the asymptotic representation of a secular equation for \( v_R \)

\[
det \left[ Z_0(v) + Z_1(v)k^{-1} + Z_2(v)k^{-2} + Z_3(v)k^{-3} + \cdots \right] = 0,
\]

from which the high-frequency asymptotic formula (2.15) can be derived by a simple routine through the implicit function theorem (cf. Section 6 of Man et al. (2013)).

In this paper, under the assumption (2.7) we develop a perturbation method for determining each term \( v_i \) \((i = 0, 1, 2, \cdots)\) in (2.15). For this purpose, we first of all give a formula for \( Z_0(v) \) written in terms of \( L(0), \hat{T}(0) \) and \( \lambda(0) \), which is correct to within terms linear in \( \hat{T}(0) \) and \( \lambda(0) \), in the form

\[
Z_0(v) = Z_0^{\text{iso}}(v) + Z_0^{\text{ph}(v)},
\]

where \( Z_0^{\text{iso}}(v) \) is of zeroth order in \( \hat{T}(0) \) and \( \lambda(0) \), and \( Z_0^{\text{ph}(v)} \) is of first order in \( \hat{T}(0) \) and \( \lambda(0) \). A formula for \( Z_0^{\text{iso}}(v) \) is well known (see Proposition 3.1), which pertains to surface waves propagating along the surface of a comparative homogeneous, isotropic unstressed elastic half-space. In contrast with \( Z_0^{\text{iso}}(v) \), the formula for \( Z_0^{\text{ph}(v)} \) is complicated. Each entry of \( Z_0^{\text{ph}(v)} \) is a linear combination of the components of \( \hat{T}(0) \) and \( \lambda(0) \). In Proposition 3.2 and Appendix A we will write each coefficient of these linear combinations explicitly in terms of \( V : = \rho(0)v^2 \) and the Lamé constants \( \lambda \) and \( \mu \) of \( C^{\text{iso}} \).

These formulas show explicitly how each entry of \( \hat{T}(0) \) and \( \lambda(0) \) affects each entry of \( Z_0^{\text{ph}(v)} \).

Again under the assumption (2.7), we derive equations which determine each of \( Z_n(v) \) \((n = 1, 2, 3, \cdots)\) in the lower-order terms of (2.17), to within terms linear in \( \hat{T}(0) \) and \( \lambda(0) \). These matrix equations are of Lyapunov-type, which can be reduced to systems of linear equations whose coefficients matrices are given explicitly in terms of \( \lambda, \mu, \hat{T}(0) \) and \( \lambda(0) \), and whose inhomogeneous terms are written in terms
of $\lambda, \mu, \hat{T}(0), A(0)$ and the first and higher-order $x_3$-derivatives of $A(x_3)$ and of $\hat{T}(x_3)$ at $x_3 = 0$. We can then solve those linear systems iteratively to obtain the terms in $Z_n(v)(n = 1, 2, 3, \cdots)$ up to those linear in $\hat{T}(0)$ and $A(0)$. The explicit form of $Z_{n}^{\Phi}(v)$ is useful when we calculate $Z_n(v)(n = 1, 2, 3, \cdots)$ both symbolically and numerically.

Throughout this paper, we do not put any condition on the first and higher-order $x_3$-derivatives of $A(x_3)$ and of $\hat{T}(x_3)$ at $x_3 = 0$. This corresponds to the physical situation that the initial stress is small at the surface $x_3 = 0$ along which Rayleigh waves propagate, whereas its $x_3$-derivatives near $x_3 = 0$ need not be negligible, which is typical of initial stresses induced by surface conditioning for the purpose of lifetime enhancement of metal parts.

Henceforth, we consider Rayleigh waves which propagate along the surface of the prestressed half-space $x_3 \leq 0$ in the direction of the 2-axis ($\eta = (0, 1, 0)$). Doing so does not lose any generality of the setting, because under the rotation of the coordinate system around the 3-axis the form of the isotropic part (2.5) is invariant while the perturbative part (2.6) and initial stress $\hat{T}(x_3)$ have been assumed to appear in their most general forms.

3. Surface impedance matrix of weakly-anisotropic homogeneous elastic half-space

In what follows, by the comparative homogeneous elastic half-space $x_3 \leq 0$ we mean that which has the incremental elasticity tensor $\mathbb{L}$, mass density $\rho$, and initial stress $\hat{T}$ given by

$$L = \mathbb{L}(0) = C^{\text{iso}} + A(0),$$

$$\rho = \rho(0),$$

and $\hat{T} = \hat{T}(0)$, respectively. The constitutive equation in this homogeneous half-space is then

$$S = \hat{T}(0) + \mathbf{H} \hat{T}(0) + C^{\text{iso}}[\mathbf{E}] + A(0)[\mathbf{E}].$$

We recall that the components $\hat{T}_{i3}(x_3)$ ($i = 1, 2, 3$) of $\hat{T}(x_3)$ vanish at the traction-free surface $x_3 = 0$, which implies that $\hat{T}(0)$ is represented by a $3 \times 3$ matrix as

$$\hat{T}(0) = \begin{pmatrix} \hat{T}_{11}(0) & \hat{T}_{12}(0) & 0 \\ \hat{T}_{12}(0) & \hat{T}_{22}(0) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ (3.3)

In this section we give a formula for $Z_0(v)$ which appears as the dominant term in the asymptotic expansion (2.17), i.e., a formula for the surface impedance matrix that pertains to surface waves which propagate in the direction of the 2-axis along the surface of the comparative homogeneous elastic half-space $x_3 \leq 0$. Under assumption (2.7), we are concerned only with the terms in $Z_0(v)$ up to those linear in $\hat{T}(0)$ and $A(0)$, which leads us to write

$$Z_0(v) \approx Z_{0}^{\text{iso}}(v) + Z_{0}^{\text{ph}}(v).$$ (3.4)

Here and hereafter we use the notation $\approx$ to indicate that we are retaining terms up to those linear in $A(0)$ and $\hat{T}(0)$ and that we are neglecting the higher order terms. $Z_{0}^{\text{iso}}(v)$ is of zeroth order in $\hat{T}(0)$ and $A(0)$, whereas $Z_{0}^{\text{ph}}(v)$ is of first order in $\hat{T}(0)$ and $A(0)$. Note that $kZ_{0}^{\text{iso}}(v)$ is the surface impedance matrix pertaining to a homogeneous isotropic elastic half-space with constitutive equation $S = C^{\text{iso}}[\mathbf{E}]$ and with density $\rho = \rho(0)$.
The Hermitian matrix $Z_0^{\text{iso}}(\nu)$ has a well-known formula (cf. for example, Sec. 12–10 of Ting (1996)), which is given by

**Proposition 3.1**

$$Z_0^{\text{iso}}(\nu) = Z_0^{\text{iso}}(\nu)^T = \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & -i s_{23} \\ 0 & i s_{23} & s_{33} \end{pmatrix},$$  \hspace{1cm} (3.5)

where

$$s_{11} = \sqrt{\mu (\mu - V)}, \quad s_{22} = \frac{\sqrt{\mu (\lambda + 2 \mu - V)} \left( \sqrt{\mu (\lambda + 2 \mu) + \sqrt{(\mu - V)(\lambda + 2 \mu - V)}} \right)}{\lambda + 3 \mu - V},$$

$$s_{33} = \frac{\sqrt{(\lambda + 2 \mu)(\mu - V)} \left( \sqrt{\mu (\lambda + 2 \mu) + \sqrt{(\mu - V)(\lambda + 2 \mu - V)}} \right)}{\lambda + 3 \mu - V},$$

$$s_{23} = \frac{1}{\lambda + 3 \mu - V} \left( \mu (\lambda + 4 \mu - 2 V) - \sqrt{\mu (\lambda + 2 \mu)(\mu - V)(\lambda + 2 \mu - V)} \right),$$ \hspace{1cm} (3.6)

and $\lambda$ and $\mu$ are the Lamé constants in (2.5).

Now we give an explicit formula for $Z_0^{\text{ph}}(\nu)$. All the components of $\tilde{T}$ and $\hat{V}$ in the formula denote their values at $x_3 = 0$.

**Proposition 3.2**

$$Z_0^{\text{ph}}(\nu) = Z_0^{\text{ph}}(\nu)^T,$$  \hspace{1cm} (3.7)

$$= \begin{bmatrix} \ell_{11}(a_{05},a_{06},\hat{F}_{22}) & \ell_{12}(a_{026},a_{035},a_{044}) + i \ell_{13}(a_{025},a_{035},a_{044}) & \ell_{14}(a_{024},a_{035},a_{044}) + i \ell_{13}(a_{025},a_{035},a_{044}) \\ \ast & \ell_{22}(a_{022},a_{033},a_{044},\hat{F}_{22}) & \ast \\ \ast & \ast & \ell_{33}(a_{022},a_{033},a_{044},\hat{F}_{22}) \end{bmatrix},$$

where the diagonal components $\ell_{ii}$ ($i = 1, 2, 3$) and $\ell_{jj}$, $\ell_{ij}$ ($i, j = (1, 2), (1, 3), (2, 3)$) in the off-diagonal components are real-valued linear functions of their arguments whose coefficients are given explicitly in terms of the Lamé constants $\lambda$, $\mu$ and $V = \rho(0) v^2$, and the label “$\ast$” in the $(j,i)$ component of the matrix denotes the complex conjugate of the $(i,j)$ component of the matrix ($i, j = (1, 2), (1, 3), (2, 3)$). The formulas for the aforementioned linear functions are given in Appendix A.

**Outline of derivation of Proposition 3.2.** We use the integral representation of $Z_0(\nu)$, which will be given in Lemma 3.1. Let us take orthogonal unit vectors $\mathbf{m} = (0, 1, 0)$, which is the propagation direction of the surface waves in question and $\mathbf{n} = (0, 0, 1)$, which is the unit outward normal of the boundary $x_3 = 0$ of the material half-space. Let $\mathbf{m} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ and $\mathbf{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ be orthogonal unit vectors which are obtained by rotating the orthogonal unit vectors $\mathbf{m}$ and $\mathbf{n}$ around their vector.
product \( \mathbf{m} \times \mathbf{n} \) by an angle \( \phi \) \((-\pi \leq \phi < \pi)\) so that
\[
\mathbf{m} = \mathbf{m}(\phi) = \mathbf{m} \cos \phi + \mathbf{n} \sin \phi = (0, \cos \phi, \sin \phi),
\]
\[
\mathbf{n} = \mathbf{n}(\phi) = -\mathbf{m} \sin \phi + \mathbf{n} \cos \phi = (0, -\sin \phi, \cos \phi).
\] (3.8)

Let \( \mathbf{R}_0(\phi) \) and \( \mathbf{T}_0(\phi) \) be the 3 \times 3 real matrices given by
\[
\mathbf{R}_0(\phi) = \left( \sum_{j,l=2} B_{ijkl}(0) \tilde{m}_j \tilde{m}_l \right) + \rho(0) v^2 \cos \phi \sin \phi \mathbf{I},
\]
\[
\mathbf{T}_0(\phi) = \left( \sum_{j,l=2} B_{ijkl}(0) \tilde{n}_j \tilde{n}_l \right) - \rho(0) v^2 \sin^2 \phi \mathbf{I},
\] (3.9)

where \( B_{ijkl}(0) \) are the effective elastic coefficients at \( x_3 = 0 \) and are written with the help of (2.4), (2.5) and (2.11) as
\[
B_{ijkl}(0) = \delta_{ik} \tilde{T}_{jl}(0) + L_{ijkl}(0) = \delta_{ik} \tilde{T}_{jl}(0) + \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + a_{ijkl}(0),
\] (3.10)

and \( \mathbf{I} \) is the 3 \times 3 identity matrix.

Let \( v_L \) be the lowest velocity for which the matrix \( \mathbf{T}_0(\phi) \) becomes singular for some angle \( \phi \):
\[
v_L = \inf\{ v > 0 \mid \exists \phi \text{ such that } \det \mathbf{T}_0(\phi) = 0 \}.
\] (3.11)

The velocity \( v_L \) and the interval \( 0 < v < v_L \) are called the limiting velocity and the subsonic range, respectively.

For \( 0 \leq v < v_L \), we define the 3 \times 3 real matrices \( \mathbf{S}_1 = \mathbf{S}_1(v) \) and \( \mathbf{S}_2 = \mathbf{S}_2(v) \) by
\[
\mathbf{S}_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_0(\phi)^{-1} R_0(\phi) \mathbf{T} \, d\phi, \quad \mathbf{S}_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_0(\phi)^{-1} \, d\phi,
\] (3.12)

where \( \mathbf{R}_0(\phi) \) and \( \mathbf{T}_0(\phi) \) are given by (3.9).

The integral representation of \( \mathbf{Z}_0(v) \) is given by (cf. Lothe and Barnett (1976); Chadwick and Smith (1977))

**Lemma 3.1** For \( 0 \leq v < v_L \),
\[
\mathbf{Z}_0(v) = \mathbf{S}_2^{-1} + i \mathbf{S}_2^{-1} \mathbf{S}_1,
\] (3.13)

where the matrices \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) are given by (3.12).

By (3.3), (3.8) and (3.10), we can write \( \mathbf{R}_0(\phi) \) and \( \mathbf{T}_0(\phi) \) in (3.9) as
\[
\mathbf{R}_0(\phi) = \mathbf{R}^{\text{iso}}_v(\phi) + \mathbf{R}^{\text{ph}}_0(\phi), \quad \mathbf{T}_0(\phi) = \mathbf{T}^{\text{iso}}_v(\phi) + \mathbf{T}^{\text{ph}}_0(\phi),
\]
where
\[
\mathbf{R}^{\text{iso}}_v(\phi) = (\lambda \tilde{m} \tilde{m}_k + \mu \tilde{n} \tilde{n}_k) + \rho(0) v^2 \cos \phi \sin \phi \mathbf{I},
\] (3.14)
\[
\mathbf{T}^{\text{iso}}_v(\phi) = (\lambda + \mu) \tilde{n} \tilde{n}_k + (\mu - \rho(0) v^2 \sin^2 \phi) \mathbf{I}
\] (3.15)
are of zeroth order in $\tilde{T}(0)$ and $A(0)$, and

$$R_{0}^{ph}(\phi) = \tilde{T}_{22}(0) \tilde{m}_2 \tilde{n}_2 I + \left( \sum_{j,l=2}^{3} a_{ijkl}(0) \tilde{m}_j \tilde{n}_l \right),$$  \hspace{1cm} (3.16)$$

$$T_{0}^{ph}(\phi) = \tilde{T}_{22}(0) \tilde{n}_2 \tilde{m}_2 I + \left( \sum_{j,l=2}^{3} a_{ijkl}(0) \tilde{n}_j \tilde{m}_l \right)$$  \hspace{1cm} (3.17)$$

are of first order in $\tilde{T}(0)$ and $A(0)$. Then the argument which leads to (42) and (43) in Tanuma et al. (2013) can be applied in a parallel way to deduce from (3.13) that

$$Z_{iso}^{0}(v) = \left( S_{iso}^{0} \right)^{-1} + i \left( S_{iso}^{0} \right)^{-1} S_{iso}^{0},$$  \hspace{1cm} (3.18)$$

$$Z_{Ptb}^{0}(v) = \left( S_{Ptb}^{0} \right)^{-1} + i \left( S_{Ptb}^{0} \right)^{-1} S_{Ptb}^{0};$$  \hspace{1cm} (3.19)$$

are the terms of zeroth order in $\tilde{T}(0)$ and $A(0)$ included in $S_{1}$ and $S_{2}$ respectively, and $S_{Ptb}^{1} = S_{Ptb}^{1}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{iso}^{0}(\phi)^{-1} R_{v}^{iso}(\phi) I \ d\phi$ (3.20)$$

and

$$S_{2}^{iso} = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{iso}^{0}(\phi)^{-1} d\phi$$  \hspace{1cm} (3.21)$$

are the terms of first order in $\tilde{T}(0)$ and $A(0)$ included in $S_{1}$ and $S_{2}$ respectively.

The explicit forms of $S_{1}^{iso}$ and $S_{2}^{iso}$ are computed from (3.14), (3.15), (3.20) and (3.21) as (cf. (6.55), (6.56) in Chadwick and Smith (1977) and (37) in Tanuma and Man (2008))

$$S_{1}^{iso} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -S(v) \\ 0 \sqrt{\frac{\mu + 2\nu}{\mu + \nu}} & \sqrt{\frac{\lambda + 2\mu - \nu}{\lambda + 2\mu}} & S(v) \end{pmatrix}$$  \hspace{1cm} (3.24)$$

and

$$S_{2}^{iso} = \begin{pmatrix} (s_{11})^{-1} & 0 & 0 \\ 0 & (s_{22})^{-1} & 0 \\ 0 & 0 & (s_{33})^{-1} \end{pmatrix}$$  \hspace{1cm} (3.25)$$

where

$$S_{iso}^{1} = S_{iso}^{1}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{iso}^{0}(\phi)^{-1} R_{v}^{iso}(\phi) I \ d\phi$$  \hspace{1cm} (3.22)$$

$$S_{iso}^{2} = S_{iso}^{2}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{iso}^{0}(\phi)^{-1} d\phi$$  \hspace{1cm} (3.23)$$

are the terms of zeroth order in $\tilde{T}(0)$ and $A(0)$ included in $S_{1}$ and $S_{2}$ respectively, and $S_{Ptb}^{1} = S_{Ptb}^{1}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{iso}^{0}(\phi)^{-1} R_{v}^{iso}(\phi) I \ d\phi$ (3.20)$$

and

$$S_{Ptb}^{2} = S_{Ptb}^{2}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{iso}^{0}(\phi)^{-1} T_{v}^{iso}(\phi)^{-1} R_{v}^{iso}(\phi) I \ d\phi$$  \hspace{1cm} (3.23)$$

are the terms of first order in $\tilde{T}(0)$ and $A(0)$ included in $S_{1}$ and $S_{2}$ respectively.
where
\[ S(v) = \frac{1}{v} \left( \frac{\lambda + 2\mu}{\lambda + 2\mu} - V \right) - \frac{\mu - V}{\mu} 2\mu \] (3.26)
and \(s_{11}, s_{22}\) and \(s_{33}\) are given by (3.6). Note that integral representation (3.18), combined with (3.24) and (3.25), recovers formula (3.5). Let us put
\[ S_{1}^{a b} = (\alpha_{ij}), \quad S_{2}^{a b} = (\beta_{ij}). \] (3.27)
From (3.5) and (3.19) we get
\[ Z_{0}^{a b}(v) = Z_{0}^{a b}(v)^{T} \] (3.28)
\[ = \begin{bmatrix}
- s_{11}^{2} \beta_{11} & - s_{11} s_{22} \beta_{12} - i s_{22} \alpha_{21} & - s_{11} s_{33} \beta_{13} - i s_{33} \alpha_{31} \\
* & - s_{22}^{2} \beta_{22} & - s_{22} s_{33} \beta_{23} + i s_{33} (s_{23} \beta_{22} + \alpha_{23}) \\
* & * & - s_{33}^{2} \beta_{33}
\end{bmatrix}, \]
where the label "*" at the \((j, i)\) component of the matrix denotes the complex conjugate of the \((i, j)\) component of the matrix \(((i, j) = (1, 2), (1, 3), (2, 3))\).
To obtain (3.7), we need formulas for the relevant entries of (3.27), each of which is a linear combination of the components of \(\hat{T}(0)\) and \(\hat{\lambda}(0).\) The coefficients of the linear combinations in question can be written explicitly in terms of \(\lambda\) and \(\mu\) of \(C^{\text{iso}}\) and \(V.\) In what follows we give a brief sketch on the derivation of the formulas for the entries \(\alpha_{21}, \alpha_{31}\) and \(\alpha_{23}\) of \(S_{1}^{a b}\) that are used in (3.28), which follows arguments parallel to those described in Section 6 of Tanuma and Man (2002) and Section 7 of Tanuma and Man (2008).
Using (3.8) and (3.14)–(3.17), we compute the integrands of \(S_{1}^{a b}\) and \(S_{2}^{a b}\), which are, by (3.22) and (3.23),
\[ -T_{v}^{\text{iso}}(\phi)^{-1}R_{0}^{a b}(\phi)^{T} + T_{v}^{\text{iso}}(\phi)^{-1}T_{0}^{a b}(\phi)T_{v}^{\text{iso}}(\phi)^{-1}R_{v}^{a b}(\phi)^{T} \] (3.29)
and
\[ T_{v}^{\text{iso}}(\phi)^{-1}T_{0}^{a b}(\phi)T_{v}^{\text{iso}}(\phi)^{-1}, \] (3.30)
respectively. Note that these integrands are linear functions of \(\hat{T}(0)\) and \(\hat{\lambda}(0).\) By computations, we observe that the coefficients of \(a_{25}, a_{35}\) and \(a_{46}\) in the \((2, 1)\) component of the integrand (3.29) and the coefficients of \(a_{26}, a_{36}\) and \(a_{45}\) in the \((3, 1)\) component of (3.29) have the form
\[ P_{1} = \frac{1}{(\lambda + 2\mu - V \sin^{2} \phi)(\mu - V \sin^{2} \phi)^{2}}, \] (3.31)
with \(P_{1}\) being a polynomial of \(\lambda, \mu, \cos \phi, \sin \phi\) and \(V,\) whereas the coefficients of \(a_{22}, a_{23}, a_{33}, a_{44},\) and \(\hat{T}_{22}\) in the \((2, 3)\) component of (3.29) have the form
\[ P_{2} = \frac{1}{(\lambda + 2\mu - V \sin^{2} \phi)^{2}(\mu - V \sin^{2} \phi)^{2}}, \] (3.32)
with \(P_{2}\) being a polynomial of \(\lambda, \mu, \cos \phi, \sin \phi\) and \(V.\) The coefficients of the other \(a_{ij}\) and \(\hat{T}_{ij}\) in the aforementioned components of (3.29) either vanish or are odd functions in \(\phi.\) This assertion easily
follows without going into the computation that leads to the forms (3.31) and (3.32); see the method in Lemma 6.2 and Lemma 6.3 of Tanuma and Man (2008). Hence those coefficients are zero when (3.29) is integrated over \([-\pi, \pi]\) with respect to \(\phi\).

We decompose each of the coefficients of the forms (3.31) and (3.32) into partial fractions of the four types

\[
\frac{C_1}{(\lambda + 2\mu - V\sin^2\phi)}, \quad \frac{C_2}{(\lambda + 2\mu - V\sin^2\phi)^2}, \quad \frac{C_3}{\mu - V\sin^2\phi}, \quad \frac{C_4}{(\mu - V\sin^2\phi)^2},
\]

where \(C_i (i = 1, \ldots, 4)\) are functions of \(\lambda, \mu\) and \(V\). Note that when we decompose (3.31), fraction of the second type does not appear. Integration of the preceding partial fractions over \([-\pi, \pi]\) with respect to \(\phi\) yields

\[
\begin{align*}
\frac{2\pi C_1}{\sqrt{(\lambda + 2\mu)(\lambda + 2\mu - V)}}, & \quad \frac{\pi C_2}{\sqrt{(\lambda + 2\mu)(\lambda + 2\mu - V)}}, & \quad \frac{2(\lambda + 2\mu) - V}{(\lambda + 2\mu - V)}, \\
\frac{2\pi C_3}{\sqrt{\mu(\mu - V)}}, & \quad \frac{\pi C_4}{\sqrt{\mu(\mu - V)}}, & \quad \frac{2(\mu - V)}{\mu(\mu - V)},
\end{align*}
\]

respectively. Summing up these terms, we obtain formulas for the \(\alpha_{ij}\)'s in question. A parallel argument can be applied to integrate (3.29) to get formulas for \(\beta_{ij}\).

\[\square\]

4. Lower-order terms of the asymptotic expansion of surface impedance matrix

Now we turn to study the surface impedance matrix \(Z(v)\) that pertains to surface waves which propagate in the direction of the 2-axis along the surface of the vertically-inhomogeneous, anisotropic and prestressed elastic half-space \(x_3 \leq 0\). Recall that the constitutive equation is expressed by (2.1), where the incremental elasticity tensor has the form (2.4), and the mass density is given by \(\rho(x_3)\). Let \(Q, R\) and \(T\) be \(3 \times 3\) real matrices given by

\[
Q = Q(x_3, v) = (B_{2222}(x_3) - \rho(x_3)v^2\delta_{11}), \quad (4.1)
\]

\[
R = R(x_3) = (B_{2223}(x_3)), \quad T = T(x_3) = (B_{3233}(x_3)),
\]

where \(B_{ijkl}(x_3)\) are the effective elastic coefficients (2.11). Let \(Q_n, R_n, T_n\) and \(S_n\) \((n = 0, 1, 2, \cdots)\) be the coefficients in the Taylor expansions of \(Q, R, T\) and \(T^{-1}\) at \(x_3 = 0\), respectively.

According to the arguments in Section 3 and 5 of Man et al. (2013), each lower-order term in (2.17), i.e., each of the \(3 \times 3\) matrices \(Z_n(v) (n = 1, 2, 3, \cdots)\), is obtained by solving some systems of Lyapunov-type equations. From equations (58) and (101) of Man et al. (2013) we get

\[
Z_n(v) = iG_0^{(-n)} (n = 1, 2, 3, \cdots), \quad (4.2)
\]

where the \(3 \times 3\) matrix \(G_0^{(-n)}\) is the last term of a sequence of \(3 \times 3\) matrices

\[
\{ G_n^{(-n)}, G_{n-1}^{(-n)}, \cdots, G_1^{(-n)}, G_0^{(-n)} \}
\]

whose elements are obtained inductively by solving Lyapunov-type equations; see (79), the first equation of (80), (81), the first equation of (82), (83) and (87) in Man et al. (2013).
In what follows under the assumption (2.7) we shall apply a perturbation argument to the aforementioned equations in Man et al. (2013) to derive equations for the matrices which approximate \( Z_n(v) \) \((n = 1, 2, 3, \ldots)\) to within terms linear in \( T^{\circ}(0) \) and \( A^0(0) \). We recall that the effective elastic coefficients (2.11) can be written through (2.4) and (2.5) as

\[
B_{ijkl} = \delta_{ik} \tilde{T}_{jl}(x_3) + \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + a_{ijkl}(x_3)
\]  

Then (3.3) and (4.1) imply that

\[
T_0 = T_0^{\text{Iso}} + T_0^{\text{Pb}}, \quad R_0 = R_0^{\text{Iso}} + R_0^{\text{Pb}},
\]

where

\[
T_0^{\text{Iso}} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda + 2\mu \end{pmatrix}, \quad R_0^{\text{Iso}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix}
\]

are of zeroth order in \( \tilde{T}(0) \) and \( \tilde{A}(0) \), and

\[
R_0^{\text{Pb}} = \begin{pmatrix} a_{56}(0) & a_{46}(0) & a_{36}(0) \\ a_{25}(0) & a_{24}(0) & a_{23}(0) \\ a_{45}(0) & a_{44}(0) & a_{34}(0) \end{pmatrix}, \quad T_0^{\text{Pb}} = \begin{pmatrix} a_{55}(0) & a_{45}(0) & a_{35}(0) \\ a_{45}(0) & a_{44}(0) & a_{34}(0) \\ a_{35}(0) & a_{34}(0) & a_{33}(0) \end{pmatrix}
\]

are of first order in \( \tilde{T}(0) \) and \( \tilde{A}(0) \). For

\[
K_0 = T_0^{-1/2}R_0^{\text{Pb}} - iT_0^{-1}Z_0,
\]

which appears in all the left hand sides of the aforementioned equations in Man et al. (2013), we can write

\[
K_0 \approx K_0^{\text{Iso}} + K_0^{\text{Pb}},
\]

where \( K_0^{\text{Iso}} \) is of zeroth order in \( \tilde{T}(0) \) and \( \tilde{A}(0) \), and \( K_0^{\text{Pb}} \) is of first order in \( \tilde{T}(0) \) and \( \tilde{A}(0) \). It then follows that

\[
K_0^{\text{Iso}} = (T_0^{\text{Iso}})^{-1} \left( (R_0^{\text{Iso}})^T - iZ_0^{\text{Iso}} \right),
\]

where \( Z_0^{\text{Iso}} \) is given by (3.5), which implies that

\[
K_0^{\text{Iso}} = \begin{pmatrix} -ik_{11} & 0 & 0 \\ 0 & -ik_{22} & k_{32} \\ 0 & k_{23} & -ik_{33} \end{pmatrix},
\]
where
\[ k_{11} = \sqrt{\frac{\mu - V}{\mu}}, \quad k_{22} = \sqrt{\frac{\lambda + 2\mu - V}{\mu}} \left( \frac{\sqrt{\mu(\lambda + 2\mu)} + \sqrt{(\mu - V)(\lambda + 2\mu)}}{\lambda + 3\mu - V} \right), \]
\[ k_{33} = \sqrt{\frac{\mu - V}{\lambda + 2\mu}} \left( \frac{\sqrt{\mu(\lambda + 2\mu)} + \sqrt{(\mu - V)(\lambda + 2\mu)}}{\lambda + 3\mu - V} \right), \]
\[ k_{23} = \sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu}} \left( \frac{\sqrt{(\lambda + 2\mu)(\lambda + 2\mu - V)} - \sqrt{\mu(\mu - V)}}{\lambda + 3\mu - V} \right), \]
\[ k_{32} = \sqrt{\frac{\mu - V}{\lambda + 2\mu}} \left( \frac{\sqrt{(\lambda + 2\mu)(\lambda + 2\mu - V)} - \sqrt{\mu(\mu - V)}}{\lambda + 3\mu - V} \right). \]

It also follows that
\[
K_0^{Ph} = (T_0^{iso})^{-1} (R_0^{Ph})^{T} - iZ_0^{iso} - (T_0^{iso})^{-1} T_0^{iso} (T_0^{iso})^{-1} (R_0^{iso})^{T} - iZ_0^{iso},
\]
where \(Z_0^{Ph}\) is given by (3.7). Here we have used the approximation (3.4) and
\[
(T_0)^{-1} \approx (T_0^{iso})^{-1} - (T_0^{iso})^{-1} T_0^{Ph} (T_0^{iso})^{-1}.
\]

Since \(Z_n(v)\) \((n = 1, 2, 3, \cdots)\) are related to \(G_n^{(-n)}\) by (4.2), it is sufficient to give equations which determine \(G_n^{(-n)}\) \((l = 0, 1, 2, \cdots, n)\) to within terms linear in \(\bar{k}(0)\) and \(\bar{T}(0)\). Hereafter we use the symbol “\(G_n^{(-n)}\)” to denote a matrix which approximates \(G_n^{(-n)}\) up to terms linear in \(\bar{k}(0)\) and \(\bar{T}(0)\) \((n = 1, 2, 3, \cdots; l = 0, 1, 2, \cdots, n)\).

From (4.1) and (4.3) it follows that
\[
Q_n = \frac{1}{n!} \begin{pmatrix}
\frac{\partial^n}{\partial \xi_3^n} (a_{66} + \bar{T}_{22} - \rho v^2) & \frac{\partial^n}{\partial \xi_3^n} a_{26} & \frac{\partial^n}{\partial \xi_3^n} a_{46} \\
\frac{\partial^n}{\partial \xi_3^n} a_{26} & \frac{\partial^n}{\partial \xi_3^n} (a_{22} + \bar{T}_{22} - \rho v^2) & \frac{\partial^n}{\partial \xi_3^n} a_{24} \\
\frac{\partial^n}{\partial \xi_3^n} a_{46} & \frac{\partial^n}{\partial \xi_3^n} a_{24} & \frac{\partial^n}{\partial \xi_3^n} (a_{44} + \bar{T}_{22} - \rho v^2)
\end{pmatrix}
\]
\[
R_n = \frac{1}{n!} \begin{pmatrix}
\frac{\partial^n}{\partial \xi_3^n} (a_{56} + \bar{T}_{21}) & \frac{\partial^n}{\partial \xi_3^n} a_{56} & \frac{\partial^n}{\partial \xi_3^n} a_{56} & \frac{\partial^n}{\partial \xi_3^n} a_{56} \\
\frac{\partial^n}{\partial \xi_3^n} a_{56} & \frac{\partial^n}{\partial \xi_3^n} (a_{22} + \bar{T}_{22} - \rho v^2) & \frac{\partial^n}{\partial \xi_3^n} a_{24} & \frac{\partial^n}{\partial \xi_3^n} a_{24} \\
\frac{\partial^n}{\partial \xi_3^n} a_{56} & \frac{\partial^n}{\partial \xi_3^n} a_{24} & \frac{\partial^n}{\partial \xi_3^n} (a_{44} + \bar{T}_{22} - \rho v^2) & \frac{\partial^n}{\partial \xi_3^n} a_{44} \\
\frac{\partial^n}{\partial \xi_3^n} a_{56} & \frac{\partial^n}{\partial \xi_3^n} a_{24} & \frac{\partial^n}{\partial \xi_3^n} a_{44} & \frac{\partial^n}{\partial \xi_3^n} (a_{44} + \bar{T}_{22} - \rho v^2)
\end{pmatrix},
\]
Equation (4.12) is of Lyapunov type. All the eigenvalues of $K$ can be solved for Proposition 1 in Section 3 of Man et al. (2013), we see that $(4.6)$ have negative imaginary parts (cf. Proposition 1 in Section 3 of Man et al. (2013)), which implies under the assumption (2.7) that (4.12) can be solved for $\hat{G}_i^{(-1)}$ uniquely in terms of the right-hand side (cf. Section 8.3 of Gantmacher (1960) or Chapter 12 of Bellman (1997)).

Once we have obtained $\hat{G}_i^{(-1)}$, putting $m = 1$ in (81) and in the first equation of (80) in Man et al. (2013), we see that $\hat{G}_0^{(-1)}$ is obtained as the solution to

$$
(\text{K}_0^{(lo)} + \text{K}_0^{(ph)})^\dagger \hat{G}_0^{(-1)} - \hat{G}_0^{(-1)} (\text{K}_0^{(lo)} + \text{K}_0^{(ph)}) = Q_1 - (R_1 + R_1^T) (\text{K}_0^{(lo)} + \text{K}_0^{(ph)}) + (\text{K}_0^{(lo)})^* T_1 \text{K}_0^{(lo)} + 2 \mathcal{H} (\text{K}_0^{(lo)})^* T_1 \text{K}_0^{(ph)},
$$

where $\text{M}^*$ denotes the adjoint of the matrix $\text{M}$ (i.e., $\text{M}^* = \text{M}^\top$) and $\mathcal{H}(\text{M})$ denotes the Hermitian part of $\text{M}$, i.e.,

$$\mathcal{H}(\text{M}) = \frac{1}{2} (\text{M} + \text{M}^*).$$

Equation (4.12) is of Lyapunov type. All the eigenvalues of $\text{K}_0$ (4.5) have negative imaginary parts (cf. Proposition 1 in Section 3 of Man et al. (2013)), which implies under the assumption (2.7) that (4.12) can be solved for $\hat{G}_i^{(-1)}$ uniquely in terms of the right-hand side (cf. Section 8.3 of Gantmacher (1960) or Chapter 12 of Bellman (1997)).

From Leibniz’ rule it follows that

$$S_0 T_0 = T_0 S_0 = I, \quad S_n T_0 = - \sum_{l=1}^n S_{n-l} T_l \quad (n = 1, 2, 3, \cdots),$$

from which by induction we obtain

$$S_n T_0 = \sum_{k=1}^n (-1)^k \sum_{l_1 + l_2 + \cdots + l_k = n} (S_0 T_{l_1}) (S_0 T_{l_2}) \cdots (S_0 T_{l_k}) \quad (n = 1, 2, 3, \cdots).$$
Put
\[
\begin{align*}
\widetilde{S}_0 &= (T_0^{\text{iso}})^{-1} - (T_0^{\text{iso}})^{-1} T_0^{\text{Pb}} (T_0^{\text{iso}})^{-1}, \\
\widetilde{S}_m &= \sum_{k=1}^{n} (-1)^k \sum_{\ell_1, \ell_2, \ldots, \ell_k = n} (\widetilde{S}_0 T_{\ell_1}) (\widetilde{S}_0 T_{\ell_2}) \cdots (\widetilde{S}_0 T_{\ell_k}) \widetilde{S}_0 
\end{align*}
\]
\((n = 1, 2, 3, \cdots).\)

Then it follows from (4.10) that
\[
S_0 = (T_0)^{-1} \approx \widetilde{S}_0, \quad S_n \approx \widetilde{S}_n \quad (n = 1, 2, 3, \cdots). \tag{4.14}
\]

For \(m = 2, 3, 4, \cdots,\) suppose that we have obtained the sequences of matrices
\[
\{ \widetilde{G}_1^{(-1)}, \widetilde{G}_0^{(-1)} \}, \quad \{ \widetilde{G}_2^{(-2)}, \widetilde{G}_1^{(-2)}, \widetilde{G}_0^{(-2)} \}, \cdots , \tag{4.15}
\]
\[
\{ \widetilde{G}_{m-1}^{(-m-1)}, \widetilde{G}_{m-2}^{(-m-1)}, \cdots, \widetilde{G}_1^{(-m)}, \widetilde{G}_0^{(-m)} \}.\]

Then the sequence \(\{ \widetilde{G}_m^{(-m)}, \widetilde{G}_{m-1}^{(-m)}, \cdots, \widetilde{G}_1^{(-m)}, \widetilde{G}_0^{(-m)} \}\) is determined as follows: Equations (79) and (87) in Man et al. (2013), combined with (4.6), (4.14) and \(G_0^{(0)} = T_0^{k_0},\) imply that \(\widetilde{G}_m^{(-m)}\) can be obtained as the solution to
\[
\left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right)^* \widetilde{G}_m^{(-m)} - \widetilde{G}_m^{(-m)} \left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right) = Q_m - (R_m + R_m^T) \left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right) - \left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right)^* \left( \widetilde{S}_0 \right)^{-1} \widetilde{S}_m \left( \widetilde{S}_0 \right)^{-1} \left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right) - \mathcal{F}_1^{m}, \tag{4.16}
\]

where
\[
\mathcal{F}_1^{m} = \sum_{\alpha = 1}^{m-1} \left( \widetilde{G}_\alpha^{(-\alpha)} \right)^* \widetilde{S}_0 \widetilde{G}_{m-\alpha}^{(-m-\alpha)} + 2 \mathcal{H} \left( \sum_{n=1}^{m-1} \left( \widetilde{G}_{m-n}^{(-m-n)} \right)^* \widetilde{S}_n \left( \widetilde{S}_0 \right)^{-1} \left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right) \right) \sum_{n=1}^{m-2} \sum_{\alpha = 1}^{m-n} \left( \widetilde{G}_\alpha^{(-\alpha)} \right)^* \widetilde{S}_n \widetilde{G}_{m-\alpha-n}^{(-m-\alpha-n)},
\]

and the last term on the right hand side of the preceding equation drops out when \(m = 2.\) Note that \(\mathcal{F}_1^{m}\) is determined from the entries in (4.15). Hence we can solve (4.16) for \(\widetilde{G}_m^{(-m)}\) uniquely.

Once we have obtained \(\widetilde{G}_m^{(-m)}\), from (81) and the first equation of (80) in Man et al. (2013) we see that \(\widetilde{G}_{m-1}^{(-m)}\) can be obtained as the solution to
\[
\left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right)^* \widetilde{G}_{m-1}^{(-m)} - \widetilde{G}_{m-1}^{(-m)} \left( \widetilde{K}_0^{\text{iso}} + \widetilde{K}_0^{\text{Pb}} \right) = -\mathcal{F}_2^{m} + i m (R_m^T \widetilde{G}_{m}^{(-m)}), \tag{4.17}
\]
where
\[
\mathcal{F}_2^m = 2\mathcal{H} \left( (G_0^{-1})^* S_0 G_{m-1}^{(m-1)})^* + \sum_{n=1}^{m-1} (G_{m-1-n}^{(-m-n)})^* S_n (S_0)^{-1} \left( K_0^{iso} + K_0^{ph} \right) \right)
+ \sum_{\alpha=1}^{m-2} \frac{\alpha+1}{\alpha} \left( G_0^{(-j)} \right)^* S_0 G_{m-1-\alpha}^{(-m-j)} + 2\mathcal{H} \left( \sum_{n=1}^{m-2} (G_{m-1-n}^{(-m-n)})^* S_n (S_0)^{-1} \right)
+ \sum_{n=1}^{m-3} \sum_{\alpha=1}^{m-2-n} \frac{\alpha+1}{\alpha} \left( G_0^{(-j)} \right)^* S_0 G_{m-1-\alpha-n}^{(-m-j-n)}
\]
and the second and the third terms on the right hand side of the preceding equation drop out when \(m = 2\) and the last term disappears when \(m = 2\) and \(m = 3\). Again, we see that \(\mathcal{F}_2^m\) is determined from the entries in (4.15). Hence we can solve (4.17) for \(G_{m-1}^{(-m)}\) uniquely.

For \(l = 0, 1, \ldots, m-2\), suppose that we already know the matrix \(G_{l+1}^{(-m)}\). It then follows from (83) and the first equation of (82) in Man et al. (2013) that \(G_l^{(-m)}\) can be obtained as the solution to
\[
\left( K_0^{iso} + K_0^{ph} \right)^* G_l^{(-m)} - G_l^{(-m)} \left( K_0^{iso} + K_0^{ph} \right) = -\mathcal{F}_3^{m-l} - i(l+1) G_{l+1}^{(-m)}
\]
where
\[
\mathcal{F}_3^{m-l} = 2\mathcal{H} \left( \sum_{j=1}^{m-l} (G_0^{(-j)} \right)^* S_0 G_{l-j}^{(-j)} + \sum_{n=1}^{l} \frac{\alpha+1}{\alpha} (G_{l-n}^{(-m-n)})^* S_n (S_0)^{-1} \left( K_0^{iso} + K_0^{ph} \right) \right)
+ \sum_{j=1}^{l-1} (G_0^{(-j)} \right)^* S_0 G_{l-j}^{(-j)} + \sum_{n=1}^{l-1} \sum_{\alpha=1}^{l-1-n} \frac{\alpha+1}{\alpha} (G_0^{(-j)} \right)^* S_0 G_{l-j-\alpha}^{(-j)}
\]
\[
+ 2\mathcal{H} \left( \sum_{n=1}^{l-2} \sum_{j=1-n} G_0^{(-j)} S_0 G_{l-n-j}^{(-j)} \right) + \sum_{n=1}^{l-2} \sum_{\alpha=1}^{l-2-n} \sum_{j=1-n} G_0^{(-j)} S_0 G_{l-j-\alpha-n}^{(-j-n)}
\]
for \(l \geq 1\), and
\[
\mathcal{F}_3^{m-l} = \sum_{j=1}^{m-l} (G_0^{(-j)} \right)^* S_0 G_{l-j}^{(-j)}
\]
for \(l = 0\), and the third and the forth terms on the right hand side of (4.19) are not there when \(l = 1\) and the last term drops out when \(l = 1\) and \(l = 2\). We see that \(\mathcal{F}_3^{m-l}\) is determined from the entries in (4.15). Hence we can solve (4.18) for \(G_l^{(-m)}\) uniquely. Then
\[
Z_m \approx i G_0^{(-m)}
\]
Finally, we comment on how to solve equations (4.12), (4.13), (4.16) to (4.18). These equations can all be expressed in the form of a Lyapunov-type equation for \(G\), namely:
\[
L' G - GL = B,
\]
(4.20)
where \( \mathbf{L} = \mathbf{K}_{\text{iso}}^0 + \mathbf{K}_{\text{bph}}^0 \) and \( \mathbf{B} \) denotes the right hand sides of the respective equations. The three-dimensional matrix equation (4.20) can be recast into a nine-dimensional linear system. In fact, under the component-wise expressions of the \( 3 \times 3 \) matrices

\[
\mathbf{L} = (l_{ij}), \quad \mathbf{G} = (g_{ij}), \quad \mathbf{B} = (b_{ij}),
\]

(4.20) is equivalent to

\[
\mathcal{L} \mathbf{g} = \mathbf{b},
\]

where \( \mathcal{L} \) is the \( 9 \times 9 \) matrix given by

\[
\begin{pmatrix}
-t_{11} & -t_{12} & -t_{13} & t_{21} & t_{22} & t_{23} & 0 & 0 & 0 \\
-t_{12} & t_{11} - t_{21} & -t_{22} & 0 & 0 & t_{31} & 0 & t_{31} & 0 \\
-t_{13} & t_{13} & t_{13} - t_{33} & 0 & 0 & t_{21} & 0 & t_{21} & 0 \\
\end{pmatrix}
\]

and \( \mathbf{g} \) and \( \mathbf{b} \) are nine-dimensional column vectors defined by

\[
\mathbf{g} = (g_{11} \ g_{12} \ g_{13} \ g_{21} \ g_{22} \ g_{23} \ g_{31} \ g_{32} \ g_{33})^T, \quad \mathbf{b} = (b_{11} \ b_{12} \ b_{13} \ b_{21} \ b_{22} \ b_{23} \ b_{31} \ b_{32} \ b_{33})^T.
\]

Later in the numerical implementations we shall solve (4.12), (4.13), and the series of equations (4.16) to (4.18) for \( m = 2 \) and \( m = 3 \) by appealing to the reduction of them to a nine-dimensional linear system of the form (4.21) in order to compute the first several terms of asymptotic expansion (2.17).

5. Asymptotic formula for phase velocity of Rayleigh waves

We apply the implicit function theorem to the asymptotic representation (2.18) of the secular equation to obtain the asymptotic formula (2.15) of \( v_0 \) for a large wave number \( k \). This procedure is a simple routine and is outlined in Section 6 of Man et al. (2013). Here we note that the first term of (2.15), namely \( v_0 \), solves \( \det \mathbf{Z}_0(v) = 0 \), i.e., \( v_0 \) is the phase velocity of Rayleigh waves propagating along the surface of the comparative homogeneous elastic half-space \( \chi_3 = 0 \) whose incremental elasticity tensor and initial stress have the forms (3.1) and (3.3), respectively, and whose density is equal to \( \rho(0) \). By (3.4), \( v_0 \) is written as

\[
v_0 \approx v_0^{\text{iso}} + v_0^{\text{bph}}.
\]

where \( v_0^{\text{iso}} \) is of zeroth order in \( \hat{T}(0) \) and \( \hat{\chi}(0) \) and \( v_0^{\text{bph}} \) is of first order in \( \hat{T}(0) \) and \( \hat{\chi}(0) \). The term \( v_0^{\text{iso}} \) is the velocity of Rayleigh waves in the comparative isotropic medium defined by \( \lambda = \mathbb{1}(0) = \mathbb{C}^{\text{iso}}, \chi(0) = 0 \). and \( \hat{T}(0) = 0 \). The formula for \( v_0^{\text{bph}} \), where each coefficient of \( \hat{T}(0) \) and \( \hat{\chi}(0) \) is written explicitly in terms of \( \lambda \) and \( \mu \) of \( \mathbb{C}^{\text{iso}} \), is given in formula (12) of Tanuma and Man (2006).

To obtain \( v_i \) \( (i = 1, 2, 3, \cdots) \) in the lower-order terms of (2.15), we can use the procedure in Section 6 of Man et al. (2013) with \( v_0 \) there replaced by \( v_0^{\text{iso}} + v_0^{\text{bph}} \).
Finally, let us comment on how each component of $\hat{T}$ and $\hat{\Lambda}$ under their respective most general form would affect each term of $v_i$ ($i = 1, 2, 3, \cdots$). We can deduce the following two assertions from (2.15), (2.18), Proposition 3.2, (4.11) and Lyapunov-type equations (4.12), (4.13) and (4.16)–(4.18). Recall that the Rayleigh waves in question propagate in the direction of the $2$-axis along the surface of the vertically-inhomogeneous, prestressed, anisotropic elastic half-space $x_3 \leq 0$, where the constitutive equation is expressed by (2.1), the incremental elasticity tensor has the form (2.4), and the mass density is given by $\rho(x_3)$.

1. To obtain the terms in (2.15) up to those of $v_n$, it suffices to know $\lambda, \mu, \hat{T}(0), \hat{\Lambda}(0), \rho(0)$ and the $x_3$-derivatives of $\hat{\Lambda}(x_3), \hat{T}(x_3), \rho(x_3)$ at $x_3 = 0$ up to those of order $n$.

2. None of the components of $\hat{\Lambda}(x_3)$ which have the subscript “1” in the Voigt notation and none of the components of $\hat{T}(x_3)$ which have the subscript “1” can affect the dispersion of Rayleigh waves.

6. An Illustrative Example

Among engineering material systems covered by the theory developed in this paper are metal structural parts surface-treated by low plasticity burnishing (LPB), which leaves a mirror-smooth surface finish and creates a thin layer of compressive residual stress that improves the fatigue life of the parts so treated. There the splitting (2.4) of the incremental elasticity tensor $L$ is valid, where the perturbative part $\hat{\Lambda}$ is originated from the presence of crystallographic texture and of the prestress $\hat{T}$. Moreover, the assumptions on the sizes of $\|\hat{T}(0)\|$ and $\|\hat{\Lambda}(0)\|$ in (2.7) are satisfied. The problem at issue, which is of considerable engineering interest, is to investigate the possibility of using Rayleigh waves to monitor the retention of the protective prestress $\hat{T}$ during the lifetime of a structural component. We shall study this inverse problem in another paper. Our solution of the inverse problem, however, is based on what we have done in the preceding sections on the following direct problem, namely, to determine dispersion curves for Rayleigh waves propagating in various directions when the material parameters, texture coefficients, and initial stresses are given. In this section we illustrate our solution of this direct problem by a concrete example.

A 10 cm $\times$ 10 cm $\times$ 2 cm sample was cut from an AA 7075-T651 aluminum plate. One face of the sample was surface treated with low plasticity burnishing, which introduced in the sample depth-dependent compressive stresses to a depth of about 1 mm from the treated surface (cf. Moreau and Man (2006) for more details on sample preparation). Henceforth we fix a spatial coordinate system $OXYZ$ and model the prestressed sample as a half space that occupies the region $x_3 \leq 0$, while the 1- and 2-axis are chosen arbitrarily. By the “depth” of a point in the material half-space is meant the value of $-x_3$ (in mm), where $x_3$ is the 3-coordinate of the given point. We consider only Rayleigh waves propagating in the direction of the 2-axis. X-ray diffraction measurements indicated that there were three distinguished mutually-orthogonal directions for material points at the LPB-treated surface, namely the 3-direction (normal to the free surface), the direction of LPB-rolling (which apparently was the same as the original rolling direction of the manufacturing process), and the direction transverse to the two. We define another Cartesian coordinate system $OXYZ'$ which has the 1'-, 2'-, and 3'-axis agree with the aforementioned rolling, transverse, and normal direction (i.e., the 3-direction) of the sample, respectively. We call $OXYZ'$ the material coordinate system, because it is attached to the sample. Let $\theta$ be the angle of rotation about the 3-axis that will bring the 2-axis to the 2'-axis. Different propagation directions in the sample are obtained by rotating the material half-space about the 3-axis, i.e., by varying
\[ T_{ij}(x_3) = \begin{pmatrix} \tilde{T}_{11}(x_3) & \tilde{T}_{12}(x_3) & 0 \\ \tilde{T}_{12}(x_3) & \tilde{T}_{22}(x_3) & 0 \\ 0 & 0 & 0 \end{pmatrix} \] 

(6.1)

under the \( OXYZ \) coordinate system, was measured by X-ray diffraction (and supplemented by information gathered from hole-drilling) up to a depth of 1.25 mm from the treated surface. Let \( e_1(x_3) \) and \( e_2(x_3) \) be the principal directions of the stress that are perpendicular to the 3-axis, and \( \sigma_1(x_3) \) and \( \sigma_2(x_3) \) be the corresponding principal stresses. Let \( \zeta(x_3) \) be the angle between \( e_2(x_3) \) and the \( 2' \)-axis. Then \( \varphi(x_3) = \theta + \zeta(x_3) \) is the angle of rotation about the 3-axis that will bring the direction of the 2-axis to \( e_2(x_3) \); see Fig. 1. It follows that \( \tilde{T}_{ij}(x_3) \) in (6.1) can be written as

\[ \tilde{T}_{11} = \tilde{T}_m - \tilde{T}_d \cos 2\varphi, \quad \tilde{T}_{22} = \tilde{T}_m + \tilde{T}_d \cos 2\varphi, \quad \tilde{T}_{12} = -\tilde{T}_d \sin 2\varphi, \] 

(6.2)

\( \theta \). Henceforth we call \( \theta \) the propagation direction of the Rayleigh wave (relative to the \( 2' \)-direction of the material half-space).

The surface and near-surface crystallographic texture of the sample at the LPB-treated face were measured by X-ray diffraction up to a depth of 0.225 mm. The texture was found to be essentially constant with depth and was orthorhombic with respect to the \( O'X'Y'Z' \) coordinate system. The values of those texture coefficients relevant to this study, namely \( W_{400}', W_{420}', W_{440}', W_{600}', W_{620}', W_{640}' \) and \( W_{660}' \), are given in Appendix B. Regrettably, for the 7075-T651 sample, texture measurement was not made at depths that exceed 0.225 mm, as the material within a surface layer of about 1 mm thick would be relevant to the present study. For the present purpose of working out an illustrative example, we will simply take the crystallographic texture to be constant at all depths in our model of the sample.
The measured data-points of the principal stresses $s_1$ and $s_2$ are shown in Fig. 2, where adjacent data-points are joined by straight-line segments. The top (red) curve and the bottom (blue) curve give the principal stresses $s_1$ and $s_2$, respectively. In our model we fit the data points with cubic polynomials. The cubic fitting curves for the principal stresses $s_1$ and $s_2$ are shown in Fig. 3 and Fig. 4, respectively.

The corresponding equations of the principal stresses are given by

$$s_1(x_3) = 1.438710^3 x_3^3 + 3.409010^3 x_3^2 + 1.798110^3 x_3 - 2.03510^2,$$

(6.4)

$$s_2(x_3) = 1.194310^3 x_3^3 + 2.729110^3 x_3^2 + 1.163610^3 x_3 - 4.12510^2,$$

(6.5)

where $s_1, s_2$ are in MPa and $x_3$ is in mm.

In the stress measurements it was found that $\zeta(x_3) \approx 10^{6}$ for $0 \geq x_3 \geq -0.5$ mm. As shown in Fig. 2, $s_1(x_3) \approx s_2(x_3)$ for $x_3 \leq -0.5$ mm. Hence we may take $\zeta(x_3) \approx 10^{6}$ for $x_3 \leq -0.5$ mm, as $\zeta(x_3)$ is, to within experimental error, arbitrary there. In our example, we will simply put $\zeta(x_3) = 10^{6}$ for all $x_3 \leq 0$ in our model of the sample. Later in our computations, we shall have to make use of the components $\hat{T}_{ij}(x_3)$ of the prestress under the $OXYZ$ coordinate system. The non-trivial components $\hat{T}_{ij}(x_3)$ that pertain to (6.4) and (6.5) for the principal stresses, $\zeta(x_3) = 10^{6}$, and various propagation direction $\theta$ can be obtained from (6.2) and (6.3), where $\varphi = \theta + \zeta(x_3)$. For instance, for $\theta = 0^{\circ}$ we have:

$$\hat{T}_{11}(x_3) = 1.431310^3 x_3^3 + 3.388510^3 x_3^2 + 1.778910^3 x_3 - 2.09810^2,$$

(6.6)

$$\hat{T}_{22}(x_3) = 1.201610^3 x_3^3 + 2.749610^3 x_3^2 + 1.182810^3 x_3 - 4.06210^2,$$

(6.7)

$$\hat{T}_{12}(x_3) = 0.041810^3 x_3^3 + 0.116310^3 x_3^2 + 0.108510^3 x_3 + 0.35710^2,$$

(6.8)
The texture of the sample has coefficients

\[ P_{ij} \]s are in MPa and \( x_3 \) is in mm.

We treat the material points of the prestressed 7075-T651 aluminum sample as weakly-textured orthorhombic aggregates of cubic crystallites and, in our model, adopt what follows as their constitutive equation (cf. Man (1999); Tanuma and Man (2002)):

\[
S = \tilde{T} + H\tilde{T} + L_1[E] = \tilde{T} + H\tilde{T} + C^{iso}[E] + A[E]
\]

\[
= \tilde{T} + H\tilde{T} + \lambda (uE)I + 2\mu E + \alpha \Phi(W'_{400}, W'_{420}, W'_{440})[E]
\]

\[
+ \beta_1(u\tilde{T})(uE)I + \beta_2(u\tilde{T})E + \beta_3((trE)\tilde{T} + tr(\tilde{E}E)I) + \beta_4(\tilde{E}E + \tilde{T}E)
\]

\[
+ \sum_{i=1}^{4} b_i \Psi^{(i)}(W'_{400}, W'_{420}, W'_{440})[\tilde{T}, E] + \alpha \Theta(W'_{600}, W'_{620}, W'_{640}, W'_{660})[\tilde{T}, E].
\]  

(6.9)

Here \( \Phi \) is a fourth-order tensor and \( \Psi^{(i)} (i = 1, \ldots, 4) \) are sixth-order tensors defined in terms of the texture coefficients \( W'_{400}, W'_{420}, W'_{440} \), and \( \Theta \) a sixth-order tensor defined in terms of \( W'_{600}, W'_{620}, W'_{640} \) and \( W'_{660} \); the components of these tensors in the \( OXYZ \) coordinate system are given explicitly in Appendix B. Note that the values of texture coefficients \( W'_{iso} \) of a polycrystalline material depend on the coordinate system. Here we use a superimposed “prime” to indicate that the texture coefficients \( W'_{iso} \) etc. refer to the material coordinate system \( O'X'Y'Z' \). Because of the orthorhombic texture symmetry and the cubic crystal symmetry, all the texture coefficients \( W'_{iso} \) are real. Constitutive equation (6.9) has 12 material parameters, namely \( \lambda, \mu, \alpha, \beta_i (i = 1, \ldots, 4), b_j (j = 1, \ldots, 4), \) and \( a \), which should be determined empirically. For the 7075-T651 aluminum alloy, however, only the Lamé constants \( \lambda \) and \( \mu \) are available in the literature. As for the other material parameters, we will adopt for our illustrative example the values of these parameters as estimated by the Man-Paroni model (Man and Paroni (1996); Paroni and Man (2000)) from elastic constants of single-crystal pure aluminum. The values of all the 12 material parameters are given in Appendix B.

Let \( \rho_0 \) be the density of the aluminum alloy in question when it is stress free. The presence of vertically-inhomogeneous residual stress \( \tilde{T}(x_3) \) will change the density of the material point from \( \rho_0 \) to \( \rho(x_3) \), which is related to \( \rho_0 \) and \( \tilde{T}(x_3) \) by the formula

\[
\rho(x_3) = \rho_0 (1 - uE), \quad \text{where} \quad E = (C^{iso} + \alpha \Phi)^{-1}[\tilde{T}],
\]  

(6.10)

In our example we take \( \rho_0 = 2.81 \times 10^3 \text{ kg/m}^3 \), which is the (nominal) density of AA7075 alloy as computed from those of its alloying elements and their concentrations (Aluminum Standards and Data (2000), pp. 2–14).

In this paper we want to examine also how the texture would affect the dispersion relations. Hence, for comparison purposes, we consider in addition two hypothetical but possible scenarios in the texture of the 7075-T651 sample to yield three cases as follows:

- The sample has its actual texture, the relevant coefficients of which are given in Appendix B. We call this case Texture (I).

- The sample of the texture has coefficients

\[
W'_{400} = 0.00159, W'_{420} = -0.00368, W'_{440} = 0.00175, W'_{600} = -0.00529, W'_{620} = 0.00348, \\
W'_{640} = -0.00299, \text{ and } W'_{660} = 0.00197.
\]

These coefficients are those that pertain to the surface texture of a 6061-T6 aluminum alloy plate (Man et al. (1999)). We refer to this case as Texture (II).
The 7075-T651 sample has no texture, i.e., $W'_{lm0} = 0$.

From (2.15) and (5.1), the dispersion relation can be written in the lower-order terms of the asymptotic expansion as

$$v_R \approx v_0 + v_1 \varepsilon + v_2 \varepsilon^2 + v_3 \varepsilon^3 = v_1^{\text{iso}} + v_0^{\text{Ph}} + v_1 \varepsilon + v_2 \varepsilon^2 + v_3 \varepsilon^3,$$  \hspace{1cm} (6.11)

where $\varepsilon = k^{-1}$ and $k$ denotes the wave number. In (6.11) $v_0 = v_0^{\text{iso}} + v_0^{\text{Ph}}$ is the zeroth-order term. As shown by Tanuma and Man (2002), $v_0$ can be estimated by the formula

$$v_0 = v_0^{\text{iso}} - \frac{1}{2\rho(0)v_0^{\text{iso}}} \left( A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta + (B_0 + B_2 \cos 2\theta + B_4 \cos 4\theta) \hat{T}_m(0) \right) \right. + 
(C_0 + C_2 \cos 2\theta + C_4 \cos 4\theta + C_6 \cos 6\theta) \hat{T}_d(0) \cos 2(\theta + \zeta(0)) 

+ \left. (D_2 \sin 2\theta + D_4 \sin 4\theta + D_6 \sin 6\theta) \hat{T}_d(0) \sin 2(\theta + \zeta(0)) \right),$$ \hspace{1cm} (6.12)

where $v_0^{\text{iso}}$ is the phase velocity of Rayleigh waves in the isotropic base material; $\rho(0)$, $\hat{T}_m(0)$, $\hat{T}_d(0)$, and $\zeta(0)$ are the values of the density $\rho$, stress parameters $\hat{T}_m$, $\hat{T}_d$, and $\zeta$ at the free surface $x_3 = 0$, respectively. Explicit formulas that relate the parameters $A_i$ ($i = 0, 2, 4$), $B_i$ ($i = 0, 2, 4$), $C_i$ ($i = 0, 2, 4, 6$), and $D_i$ ($i = 2, 4, 6$) to material parameters and texture coefficients are given in Tanuma and Man (2002).

Appealing to (6.3), and using $\zeta(0) = 10^\circ$ and the values of $\sigma_1(0)$ and $\sigma_2(0)$ given in (6.4) and (6.5), respectively, we compute the velocities $v_0$ of the sample in question for the aforementioned three cases of texture by formula (6.12) for $\theta = 0^\circ$, $45^\circ$, $90^\circ$, and $135^\circ$. The results are shown in Table 1.

To compute $v_i$ ($i = 1, 2, 3$), we start from the following truncated form of the asymptotic expansion (2.17) of the surface impedance $Z$:

$$Z = Z_0^{\text{iso}} + Z_0^{\text{Ph}} + Z_1 \varepsilon + Z_2 \varepsilon^2 + Z_3 \varepsilon^3,$$ \hspace{1cm} (6.13)

where $Z_0^{\text{iso}}$ is the surface impedance matrix of the homogeneous isotropic base material with $\mathbb{L} = \mathbb{C}^{\text{iso}}$, $A = 0$, $\hat{T} = \mathbf{0}$ and $\rho = \rho(0)$, $Z_0^{\text{Ph}} = Z_0 - Z_0^{\text{iso}}$, and $\varepsilon = k^{-1}$. We set the approximate secular equation as

$$R(v, \varepsilon) = \det \left[ Z_0^{\text{iso}} + Z_0^{\text{Ph}} + Z_1 \varepsilon + Z_2 \varepsilon^2 + Z_3 \varepsilon^3 \right] = 0.$$ \hspace{1cm} (6.14)

By the implicit functions theorem, we obtain from (6.14) the formulas

$$v_1 = -\frac{N_1}{D}, \quad v_2 = -\frac{N_2}{2D}, \quad v_3 = \frac{N_3}{6D}$$ \hspace{1cm} (6.15)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\theta$ (degree) & 0$^\circ$ & 45$^\circ$ & 90$^\circ$ & 135$^\circ$ \\
\hline
Texture (I) & 2882.7 & 2873.1 & 2873.2 & 2877.7 \\
\hline
Texture (II) & 2876.3 & 2874.2 & 2881.4 & 2878.3 \\
\hline
No texture & 2876.4 & 2876.8 & 2863.8 & 2872.4 \\
\hline
\end{tabular}
\caption{Zeroth-order velocity $v_0$ (in m/s) for Rayleigh waves with different propagation directions $\theta$ in sample prestressed half-space with three different textures.}
\end{table}
where

\[ D = \frac{\partial R}{\partial v} \bigg|_{v=v_0, \varepsilon=0} \]
\[ N_1 = \frac{\partial R}{\partial \varepsilon} \bigg|_{v=v_0, \varepsilon=0} \]
\[ N_2 = \frac{\partial^2 R}{\partial v^2} \bigg|_{v=v_0, \varepsilon=0} + 2v_1 \frac{\partial^2 R}{\partial \varepsilon \partial v} \bigg|_{v=v_0, \varepsilon=0} + \left(v_1\right)^2 \frac{\partial^3 R}{\partial v^3} \bigg|_{v=v_0, \varepsilon=0}, \]
\[ N_3 = \frac{\partial^3 R}{\partial v^3} \bigg|_{v=v_0, \varepsilon=0} + 3v_1 \frac{\partial^3 R}{\partial \varepsilon^2 \partial v} \bigg|_{v=v_0, \varepsilon=0} + 3 \left(v_1\right)^2 \frac{\partial^4 R}{\partial \varepsilon \partial v^2} \bigg|_{v=v_0, \varepsilon=0} + \left(v_1\right)^3 \frac{\partial^5 R}{\partial v^5} \bigg|_{v=v_0, \varepsilon=0} + 6v_2 \frac{\partial^2 R}{\partial \varepsilon \partial v} \bigg|_{v=v_0, \varepsilon=0} + 6v_1^2 v_2 \frac{\partial^3 R}{\partial \varepsilon^2 \partial v^2} \bigg|_{v=v_0, \varepsilon=0}, \]

and \( v_0 \) is estimated by using (6.12) above.

By (6.14) and (6.15) we take the following steps to obtain \( v_1, v_2, \) and \( v_3 \) in the dispersion relation (6.11):

**Step 1** Determine \( Z_1^{iso} \) and \( Z_2^{iso} \) by using the well-known formula reproduced in Proposition 3.1 and the formulas given in Proposition 3.2 and Appendix A, respectively.

**Step 2** Determine \( Z_1 \) by the formula \( Z_1 = \frac{1}{i} G_0^{(-1)} \), where \( G_0^{(-1)} \) is calculated inductively by solving the Lyapunov-type equations (4.12) and (4.13).

**Step 3** Determine \( Z_2 \) by the formula \( Z_2 = \frac{1}{i} G_0^{(-2)} \), where \( G_0^{(-2)} \) is obtained inductively by solving Lyapunov-type equations as follows:

- Solve for \( G_0^{(-2)} \) from equation (4.16) by setting \( m = 2 \);
- Solve for \( G_0^{(-2)} \) from equation (4.17) by setting \( m = 2 \);
- Solve for \( G_0^{(-2)} \) from equation (4.18) by setting \( m = 2 \) and \( l = 0 \).

**Step 4** Determine \( Z_3 \) by the formula \( Z_3 = \frac{1}{i} G_0^{(-3)} \), where \( G_0^{(-3)} \) is computed inductively by solving Lyapunov-type equations as follows:

- Solve for \( G_0^{(-3)} \) from equation (4.16) by setting \( m = 3 \);
- Solve for \( G_0^{(-3)} \) from equation (4.17) by setting \( m = 3 \);
- Solve for \( G_0^{(-3)} \) from equation (4.18) by setting \( m = 3 \) and \( l = 1 \);
- Solve for \( G_0^{(-3)} \) from equation (4.18) by setting \( m = 3 \) and \( l = 0 \).

**Step 5** Compute \( v_1, v_2, \) and \( v_3 \) from (6.15).

**Step 6** Find the dispersion relation. In practice the limit in accuracy of measurement of \( v_R \) is about 0.1\%. Hence for the truncated dispersion relation (6.11), the approximation in replacing \( v_R \) by \( v_0 \) in the formula \( \varepsilon = v_R/(2\pi f) \) will be acceptable if \( v_R = v_0 \) and the correction terms \( v_1/k, v_2/k^2, v_3/k^3 \) are all within 1\% of \( v_0 \). Substitution of \( \varepsilon \approx v_R/(2\pi f) \) in the approximate formula (6.11) for the phase velocity \( v_R \) of the Rayleigh waves leads to the dispersion relation between \( v_R \) and the frequency \( f \) that we seek.
See Remark 6.2 for further discussions.

**Remark 6.1** We use MAPLE to carry out the above steps. In the program we apply the central finite difference with a fourth-order accuracy to approximate the derivatives with respect to \( v \). The formulas of the finite difference for the first, second and third derivatives are given by

\[
\frac{\partial g}{\partial v} = \frac{g(v - 2h) - 8g(v - h) + 8g(v + h) - g(v + 2h)}{12h} + O(h^4),
\]

\[
\frac{\partial^2 g}{\partial v^2} = \frac{-g(v - 2h) + 16g(v - h) - 30g(v) + 16g(v + h) - g(v + 2h)}{12h^2} + O(h^4),
\]

\[
\frac{\partial^3 g}{\partial v^3} = \frac{g(v - 3h) - 8g(v - 2h) + 13g(v - h) - 13g(v + h) + 8g(v + 2h) - g(v + 3h)}{8h^3} + O(h^4).
\]

Here \( g \) is a given function of \( v \) and the step size \( h \) is taken to be 0.5 m/s.

The computation results are shown below.

For each of the specified propagation directions, the dispersion relations between Rayleigh-wave velocity \( v_R \) and frequency \( f \) for the sample half-space endowed with the specified prestress and three different textures are as follows:

**Case 1** \( \theta = 0^\circ \)

For Texture (I)

\[
v_R = 2882.7 - \frac{4.248 \times 10^7}{\pi f} - \frac{5.244 \times 10^{14}}{\pi^2 f^2} + \frac{4.757 \times 10^{21}}{\pi^3 f^3}; \quad (6.16)
\]

for Texture (II),

\[
v_R = 2867.3 - \frac{4.437 \times 10^7}{\pi f} - \frac{4.952 \times 10^{14}}{\pi^2 f^2} + \frac{4.605 \times 10^{21}}{\pi^3 f^3}; \quad (6.17)
\]

for the case of no texture,

\[
v_R = 2876.4 - \frac{2.319 \times 10^7}{\pi f} - \frac{7.029 \times 10^{14}}{\pi^2 f^2} + \frac{5.432 \times 10^{21}}{\pi^3 f^3}; \quad (6.18)
\]

The corresponding dispersion curves are shown in Fig. 5.

**Case 2** \( \theta = 45^\circ \)

For Texture (I)

\[
v_R = 2873.1 + \frac{1.879 \times 10^8}{\pi f} - \frac{2.586 \times 10^{15}}{\pi^2 f^2} + \frac{1.748 \times 10^{22}}{\pi^3 f^3}; \quad (6.19)
\]

for Texture (II),

\[
v_R = 2874.2 + \frac{1.849 \times 10^8}{\pi f} - \frac{2.591 \times 10^{15}}{\pi^2 f^2} + \frac{1.763 \times 10^{22}}{\pi^3 f^3}; \quad (6.20)
\]
Fig. 5. Dispersion curves for propagation direction $\theta = 0^\circ$.

Fig. 6. Dispersion curves for propagation direction $\theta = 45^\circ$.

for the case of no texture,

$$v_R = 2867.8 + \frac{8.946 \times 10^7}{\pi f} - \frac{1.561 \times 10^{15}}{\pi^2 f^2} + \frac{1.035 \times 10^{22}}{\pi^3 f^3}.$$  \hspace{1cm} (6.21)

The corresponding dispersion curves are shown in Fig. 6.

**Case 3  $\theta = 90^\circ$**

For Texture (I)

$$v_R = 2873.2 + \frac{3.154 \times 10^8}{\pi f} - \frac{3.714 \times 10^{15}}{\pi^2 f^2} + \frac{2.619 \times 10^{22}}{\pi^3 f^3};$$  \hspace{1cm} (6.22)

for Texture (II),

$$v_R = 2881.4 + \frac{3.118 \times 10^8}{\pi f} - \frac{3.681 \times 10^{15}}{\pi^2 f^2} + \frac{2.500 \times 10^{22}}{\pi^3 f^3};$$  \hspace{1cm} (6.23)

for the case of no texture,

$$v_R = 2863.8 + \frac{1.408 \times 10^8}{\pi f} - \frac{1.920 \times 10^{15}}{\pi^2 f^2} + \frac{1.172 \times 10^{22}}{\pi^3 f^3}.$$  \hspace{1cm} (6.24)

The corresponding dispersion curves are shown in Fig. 7.

**Case 4  $\theta = 135^\circ$**

For Texture (I)

$$v_R = 2877.7 + \frac{1.239 \times 10^8}{\pi f} - \frac{2.093 \times 10^{15}}{\pi^2 f^2} + \frac{1.414 \times 10^{22}}{\pi^3 f^3};$$  \hspace{1cm} (6.25)
for Texture (II),

\[
v_R = 2878.3 + \frac{1.198 \times 10^8}{\pi f} - \frac{2.072 \times 10^{15}}{\pi^2 f^2} + \frac{1.384 \times 10^{22}}{\pi^3 f^3}; \tag{6.26}
\]

for the case of no texture

\[
v_R = 2872.4 + \frac{2.982 \times 10^7}{\pi f} - \frac{1.120 \times 10^{15}}{\pi^2 f^2} + \frac{8.103 \times 10^{21}}{\pi^3 f^3}. \tag{6.27}
\]

The corresponding dispersion curves are shown in Fig. 8.

To get a more quantitative grasp on the size of dispersion, which is important when we look into the possibility of using Rayleigh-wave dispersion for nondestructive evaluation of stress, we further display in tabular form relations (6.16), (6.19), (6.22), and (6.25) between \(v_R\) and \(f\) for the 7075-T651 sample (i.e. Texture (I)). See Table 2.

**Remark 6.2** Dispersion relations (6.16)–(6.27) are third-order high-frequency asymptotic formulas. Moreover, in obtaining these dispersion relations, we have replaced \(v_R\) by \(v_0\) in the formula \(\varepsilon := 1/k = \nu/(2\pi f)\); cf. the discussion under Step 6 just before Remark 6.1. Table 3 displays the values of the first-, second-, and third-order terms in dispersion relation (6.22), which pertains to Rayleigh waves propagating at \(\theta = 90^\circ\) along the treated surface of the 7075-T651 sample. This example is singled out for illustration, partly because it concerns the real-world sample of our primary interest, and partly because it shows the largest dispersion among the cases considered. Note that for \(f = 1\) MHz and 2 MHz, we have

\[
|v_1|/k \ll |v_2|/k^2 \ll |v_3|/k^3,
\]

which suggests that these frequencies are too low for the high-frequency asymptotic formula (6.22) to be applicable; moreover, the magnitudes of the correction terms in question render the approximation
Texture (I)

<table>
<thead>
<tr>
<th>Rayleigh-wave Velocity (m/s)</th>
<th>Texture (I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (MHz)</td>
<td>θ = 0°</td>
</tr>
<tr>
<td>1</td>
<td>2969.7</td>
</tr>
<tr>
<td>2</td>
<td>2881.9</td>
</tr>
<tr>
<td>3</td>
<td>2878.0</td>
</tr>
<tr>
<td>4</td>
<td>2878.4</td>
</tr>
<tr>
<td>5</td>
<td>2879.1</td>
</tr>
<tr>
<td>6</td>
<td>2879.7</td>
</tr>
<tr>
<td>8</td>
<td>2880.5</td>
</tr>
<tr>
<td>10</td>
<td>2881.0</td>
</tr>
<tr>
<td>20</td>
<td>2881.9</td>
</tr>
<tr>
<td>30</td>
<td>2882.2</td>
</tr>
<tr>
<td>40</td>
<td>2882.3</td>
</tr>
<tr>
<td>50</td>
<td>2882.4</td>
</tr>
<tr>
<td>60</td>
<td>2882.4</td>
</tr>
<tr>
<td>70</td>
<td>2882.5</td>
</tr>
<tr>
<td>80</td>
<td>2882.5</td>
</tr>
<tr>
<td>90</td>
<td>2882.5</td>
</tr>
<tr>
<td>100</td>
<td>2882.5</td>
</tr>
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<td>110</td>
<td>2882.6</td>
</tr>
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<td>120</td>
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<td>2882.6</td>
</tr>
<tr>
<td>140</td>
<td>2882.6</td>
</tr>
<tr>
<td>150</td>
<td>2882.6</td>
</tr>
</tbody>
</table>

Table 2. Rayleigh-wave velocity $v_R$ vs. frequency $f$ for the case of Texture (I).

<table>
<thead>
<tr>
<th>Texture (I), θ = 90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (MHz)</td>
</tr>
<tr>
<td>$v_1/k$ (m/s)</td>
</tr>
<tr>
<td>$v_2/k^2$ (m/s)</td>
</tr>
<tr>
<td>$v_3/k^3$ (m/s)</td>
</tr>
<tr>
<td>$v_R - v_0$ (m/s)</td>
</tr>
</tbody>
</table>

Table 3. Comparison of first-, second-, and third-order terms in high-frequency asymptotic formula for $v_R$ in 7075-T651 sample.

of replacing $v_R$ by $v_0$ in the formula $\varepsilon = v/(2\pi f)$ problematic. On the other hand, nothing looks fishy for $f \geq 5$ MHz. Albeit somewhat doubtful the dispersion formula may still work for $f = 4$ MHz, while its validity for $f = 3$ MHz is much more questionable. Parallel studies on the first-, second-, and third-order terms of the other cases show the same pattern. In practical applications of the dispersion relations delivered by the method presented above, e.g., in using them in nondestructive evaluation of stress, it will be of paramount importance to determine the window of frequencies for which the high-frequency asymptotic formulas would be applicable, a task that can be achieved by comparing some of
the predicted dispersion relations with the corresponding experimentally-determined dispersion curves.

**Remark 6.3** The frequency window within which an \( n \)-th order dispersion relation is applicable will depend on the order \( n \). In this section, all dispersion relations are computed to the third order. We could push our computations to the fourth order if desired. In fact, that should be pursued in applications where extending the lower end of the frequency window, say to include \( f = 4 \) MHz as an applicable frequency, is beneficial.

**Remark 6.4** In the example above, the texture is assumed to be homogeneous in all three instances considered and the vertical inhomogeneity of the half space is due to that of the principal stresses alone. Even so, Figs. 5–8 show that dispersion is influenced by homogeneous texture in the presence of inhomogeneous stress. It can be expected that inhomogeneities in texture will strongly affect Rayleigh-wave dispersion. For the direct problem, where the relevant texture coefficients are known functions of depth, this is not an issue, as we may follow exactly the same procedure as what we did in this section to derive dispersion relations.

## 7. Conclusion

In this paper we consider the direct problem of deriving dispersion relations for Rayleigh waves propagating in various directions along the surface of a vertically-inhomogeneous prestressed anisotropic medium when all details about the prestress and the constitutive equation of the medium are given. For the case where the incremental elasticity tensor can be written as an isotropic part and a perturbative part, we solve the aforementioned problem by deriving necessary formulas to implement the general procedure recently developed by Man *et al.* (2013) to obtain a high-frequency asymptotic formula for each dispersion relation. We illustrate our solution of the direct problem by deriving dispersion relations for a 7076-T651 aluminum alloy sample with a prestress induced by low plasticity burnishing.

The formulas and procedure presented in this paper can serve as the mathematical foundation of a nondestructive technique that uses the dispersion of Rayleigh waves to monitor changes in the protective prestress placed on metal parts by surface treatments such as low plasticity burnishing.

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### References


**Appendix**

A. Formulas for $\ell_{ij}$ in $Z^\text{Ph}_0(v)$

Put

\[
P = \frac{1}{\sqrt{(\lambda + 2\mu)(\lambda + 2\mu - V)}}, \quad S = \frac{1}{\sqrt{\mu(\mu - V)}}, \quad R = \frac{1}{\sqrt{\mu(\lambda + 2\mu)(\mu - V)(\lambda + 2\mu - V)}},
\]

where $\lambda$ and $\mu$ are the Lamé constants in (2.5) and $V = \rho(0)v^2$. Then

\[
\ell_{11}(a_{55}, a_{66}, \hat{\rho}_{22}) = \frac{1}{2} \left( (\mu - V)a_{55} + \mu(a_{66} + \hat{\rho}_{22}) \right) S,
\]
\[
\ell_{22}(a_{22}, a_{23}, a_{33}, a_{44}, T_{22}) = \frac{\mu}{2(\lambda + \mu)(\lambda + 3\mu - V)^2V} \\
\times \left[ (\lambda + 2\mu) \left( (\lambda + 2\mu)(\lambda + 3\mu)^2 - (3\lambda^2 + 15\lambda\mu + 20\mu^2)V + 2(\lambda + 3\mu)V^2 \right) P \\
-(\lambda - V) \left( (\lambda + 2\mu)(\lambda + 3\mu)^2 - (2\lambda^2 + 11\lambda\mu + 13\mu^2)V + (\lambda + \mu)V^2 \right) S \right] a_{22} \\
+ \frac{\mu(\lambda + 2\mu - V)}{(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ (\lambda + 2\mu)(\lambda + 3\mu - 2V)P + (\lambda + 3\mu)(\mu - V)S \right] a_{23} \\
+ \frac{\mu(\lambda + 2\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)^2V} \left[ (\lambda + 2\mu - V) \left( (\lambda + 3\mu)^2 - (3\lambda + 7\mu)V \right) P \\
-(\lambda - V) \left( (\lambda + 3\mu)^2 - (\lambda + 5\mu)V \right) S \right] a_{33} + \frac{\lambda + 2\mu - V}{2(\lambda + \mu)(\lambda + 3\mu - V)^2V} \\
\times \left[ -2(\lambda + 2\mu) \left( 2\mu(\lambda + 3\mu)^2 - (\lambda^2 + 9\lambda\mu + 20\mu^2)V + (\lambda + 5\mu)V^2 \right) P \\
+ \left( 4\mu^2(\lambda + 3\mu)^2 - 2\mu(\lambda^2 + 11\lambda\mu + 22\mu^2)V - (\lambda^2 - 9\mu^2)V^2 + (\lambda + \mu)V^2 \right) S \right] a_{44} \\
+ \frac{\mu}{2(\lambda + 3\mu - V)^2V} \left[ -(\lambda + 2\mu)(\lambda + \mu - V)P + \left( \lambda^2 + 5\lambda\mu + 8\mu^2 - (2\lambda + 7\mu)V + V^2 \right) S \right] T_{22},
\]

\[
\ell_{33}(a_{22}, a_{23}, a_{33}, a_{44}, T_{22}) = \frac{(\lambda + 2\mu)(\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)^2V} \\
\times \left[ -(\lambda + 2\mu) \left( (\lambda + 3\mu)^2 - (\lambda + 5\mu)V \right) P + \mu \left( (\lambda + 3\mu)^2 - (3\lambda + 7\mu)V \right) S \right] a_{22} \\
+ \frac{(\lambda + 2\mu)(\mu - V)}{(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ (\lambda + 3\mu)(\lambda + 2\mu - V)P - \mu(\lambda + 3\mu - 2V)S \right] a_{23} \\
+ \frac{\mu - V}{2(\lambda + \mu)(\lambda + 3\mu - V)^2V} \\
\times \left[ -(\lambda + 2\mu - V) \left( (\lambda + 2\mu)(\lambda + 3\mu)^2 - (2\lambda^2 + 11\lambda\mu + 13\mu^2)V + (\lambda + \mu)V^2 \right) P \\
+ \mu \left( (\lambda + 2\mu)(\lambda + 3\mu)^2 - (3\lambda^2 + 15\lambda\mu + 20\mu^2)V + 2(\lambda + 3\mu)V^2 \right) S \right] a_{33} \\
+ \frac{\lambda + 2\mu}{2(\lambda + \mu)(\lambda + 3\mu - V)^2V}
\]
\[
\left[ 2(\lambda + 2\mu - V)(2\mu(\lambda + 3\mu)^2 - (\lambda^2 + 11\lambda\mu + 22\mu^2)V + 2(\lambda + 3\mu)V^2)P \\
- \left(4\mu^2(\lambda + 3\mu)^2 - 2\mu(3\lambda^2 + 21\lambda\mu + 38\mu^2)V + (\lambda^2 + 16\lambda\mu + 47\mu^2)V^2 - (\lambda + 9\mu)V^3 \right)S \right] a_{44}
+ \frac{\lambda + 2\mu}{2(\lambda + 3\mu - V)^2} \left[ (2\lambda^2 + 9\lambda\mu + 11\mu^2 - (3\lambda + 8\mu)V + V^2)P + \mu(\lambda + \mu + V)S \right] \hat{P}_{22},
\]

\[
\ell_{12}^R(a_{26, a_{36}, a_{45}}) = \frac{\mu}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ (\lambda + 2\mu)(\lambda + 3\mu)V - (\lambda + \mu)V \right] a_{26}
+ \frac{\mu(\lambda + 2\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ -(\lambda + 2\mu)(\lambda + 3\mu - 2V)V + (\lambda + 3\mu)(\mu - V)S \right] a_{36}
+ \frac{\lambda + 2\mu - V}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ -(\lambda + 2\mu)(2\mu(\lambda + 3\mu) - (\lambda + 5\mu)V) \right] P
+ \mu V \left( 2\mu(\lambda + 3\mu) + (\lambda + \mu) \right) a_{45},
\]

\[
\ell_{12}^I(a_{25, a_{35}, a_{46}}) = \frac{\mu - V}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ (\lambda + 2\mu)(\lambda + 3\mu - V) - \mu(\lambda + \mu)V \right] a_{25}
+ \frac{(\mu - V)(\lambda + 2\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ -(\lambda + 3\mu) + \mu(\lambda + 2\mu)(\lambda + 3\mu - 2V) \right] a_{35}
+ \frac{\lambda + 2\mu - V}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ 2\mu(\lambda + 3\mu) + (\lambda + \mu) \right] V
- \mu(\lambda + 2\mu)(2\mu(\lambda + 3\mu) - (\lambda + 5\mu)V) a_{46},
\]

\[
\ell_{13}^R(a_{25, a_{35}, a_{46}}) = \frac{(\lambda + 2\mu)(\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ (\lambda + 3\mu)(\lambda + 2\mu - V)P - \mu(\lambda + 3\mu - 2V)S \right] a_{25}
+ \frac{\mu - V}{2(\lambda + \mu)(\lambda + 3\mu - V)V} \left[ -(\lambda + 2\mu)(\lambda + 2\mu - V)(\lambda + 3\mu - 2V) \right] P
+ \mu \left( (\lambda + 2\mu)(\lambda + 3\mu) - 2(\lambda + 3\mu)V + 2V^2 \right) a_{35}
\]
\[
\ell_{13}^{1}(a_{26},a_{36},a_{45}) = \frac{\lambda + 2\mu}{2(\lambda + \mu)(\lambda + 3\mu - V)} \left[ (\lambda + 2\mu - V) \left( 2\mu(\lambda + 3\mu) - (\lambda + 5\mu)V \right) P \right. \\
\left. - \mu \left( 2\mu(\lambda + 3\mu) - (3\lambda + 11\mu)V + 4V^2 \right) S \right] a_{46}
\]

\[
\ell_{23}^{1}(a_{24},a_{34}) = \frac{\lambda + 2\mu}{2(\lambda + \mu)(\lambda + 3\mu - V)} \left[ (\lambda + 2\mu - V) \left( 2\mu(\lambda + 3\mu) - (\lambda + 4\mu)V \right) P \right. \\
\left. - \mu(\mu - V) \left( 2(\lambda + 3\mu) - 3V \right) S \right] a_{24}
\]

\[
\ell_{33}^{1}(a_{22},a_{32},a_{44},\circ_{22}) = \frac{\mu(\lambda + 2\mu)(\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)^2} \left[ 2 - \left( \lambda^2 + 4\lambda\mu + 5\mu^2 - (\lambda + 3\mu)V \right) R \right] a_{22}
\]

\[
+ \frac{\mu(\mu - V)}{(\lambda + \mu)(\lambda + 3\mu - V)^2} \left[ -1 + (\lambda + 2\mu)(\lambda + 2\mu - V) R \right] a_{23}
\]

\[
+ \frac{\mu(\mu - V)(\lambda + 2\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)^2} \left[ 2 - \left( \lambda^2 + 4\lambda\mu + 5\mu^2 - (\lambda + 3\mu)V \right) R \right] a_{33}
\]

\[
+ \frac{\lambda + 2\mu)(\lambda + 2\mu - V)}{2(\lambda + \mu)(\lambda + 3\mu - V)^2} \left[ -2(\lambda + 5\mu - 2V) \right]
\]
\[ + \left( 2\mu (\lambda^2 + 5\lambda \mu + 8\mu^2) - (\lambda^2 + 8\lambda \mu + 19\mu^2)V + (\lambda + 5\mu)V^2 \right) R \right] a_{44} \]

\[ + \frac{\mu(\lambda + 2\mu)}{2(\lambda + 3\mu - V)^2} \left[ -2 + \left( \frac{\lambda^2 + 4\lambda \mu + 5\mu^2}{(\lambda + 3\mu)V} \right) R \right] \tau_{22}. \]

We have used MATHEMATICA to carry out the computations of some expressions above.

B. Details on Constitutive Equation of 7075-T651 Aluminum Sample

In this appendix we provide the details that complete the constitutive equation (6.9) of the 7075-T651 sample studied in Section 6.

Material parameters

In our computations we take \( \lambda = 60.79 \) GPa and \( \mu = 26.9 \) GPa, which correspond to the mean values of \( \mu \) and Young’s modulus \( E = 71.43 \) GPa obtained by Radovic et al. (2004) in their RUS (resonant ultrasound spectroscopy) measurements on sixteen 7075-T651 samples. As for the other 10 parameters, we are not aware of any experimentally determined value reported in the literature. In our illustrative example, we adopt the values predicted by the Man-Paroni model (Man and Paroni (1996); Man (1999); Paroni and Man (2000)) from second-order and third-order elastic constants of single-crystal pure aluminum reported by Thomas (1968) and Sarma and Reddy (1972), respectively: \( \alpha = -16.49 \) GPa, \( \beta_1 = 0.89, \beta_2 = 0.96, \beta_3 = -2.63, \beta_4 = -4.54, b_1 = -3.32, b_2 = -0.61, b_3 = 0.14, b_4 = 1.54 \) and \( \alpha = 12.10 \).

Texture coefficients

\( W_{400} = 0.00393, W_{240} = -0.00083, W_{440} = -0.00233, W_{600} = 0.00025, W_{620} = -0.0004, W_{640} = -0.0033, \) and \( W_{660} = 0.00035. \)

Components of tensors \( \Phi, \Theta, \) and \( \Psi \)

All components of tensors below refer to the coordinate system \( OXYZ \) defined in Section 6.

An \( r \)-th order tensor \( H \) is said to be harmonic if it is totally symmetric and traceless, i.e., its components \( H_{\pi^{i_1}...i_r}^{\pi^{j_1}...j_r} \) satisfy \( H_{\pi^{i_1}...i_r}^{\pi^{j_1}...j_r} = H_{\pi^{j_1}...j_r}^{\pi^{i_1}...i_r} \) for any permutation \( \tau \) of \( \{1, 2, ..., r\} \), and \( \text{tr}_{j<k}H = 0 \) for any pair of distinct indices \( j \) and \( k \). For example, for \( r = 3 \) we have \( H_{112} = H_{121} = H_{211} \), etc. from total symmetry, and \( H_{111} + H_{212} + H_{313} = 0 \), etc. from the traceless condition.

The fourth-order tensor \( \Phi \) and the sixth-order tensor \( \Theta \) are harmonic. All the non-trivial components of \( \Phi \) can be obtained from the five following harmonic components of the total symmetry of and the traceless condition on the harmonic tensor \( \Phi \):

\[ \Phi_{1122} = W_{400}^\prime - \sqrt{70} W_{440}^\prime \cos 4\theta, \quad \Phi_{1133} = -4W_{400}^\prime + 2\sqrt{70} W_{420}^\prime \cos 2\theta, \]
\[ \Phi_{2233} = -4W_{400}^\prime - 2\sqrt{70} W_{420}^\prime \cos 2\theta, \quad \Phi_{1112} = -\sqrt{70} W_{420}^\prime \sin 2\theta + \sqrt{70} W_{440}^\prime \sin 4\theta, \]
\[ \Phi_{2212} = -\sqrt{70} W_{420}^\prime \sin 2\theta - \sqrt{70} W_{440}^\prime \sin 4\theta. \]
The non-trivial components of $\Theta$ can be obtained from the following seven by using the total symmetry of and the traceless condition on $\Theta$:

\[
\Theta_{22211} = -W'_{600} - \frac{\sqrt{105}}{15} W_{620} \cos 2\theta + \sqrt{14} W_{640} \cos 4\theta + \sqrt{231} W_{660} \cos 6\theta,
\]

\[
\Theta_{22233} = 6W_{600} + \frac{16\sqrt{105}}{15} W_{620} \cos 2\theta + 2\sqrt{14} W_{640} \cos 4\theta,
\]

\[
\Theta_{33311} = -8W'_{600} + \frac{16\sqrt{105}}{15} W_{620} \cos 2\theta,
\]

\[
\Theta_{33322} = -8W'_{600} - \frac{16\sqrt{105}}{15} W_{620} \cos 2\theta,
\]

\[
\Theta_{12222} = \frac{\sqrt{105}}{3} W_{620} \sin 2\theta + 2\sqrt{14} W_{640} \sin 4\theta + \sqrt{231} W_{660} \sin 6\theta,
\]

\[
\Theta_{12233} = -\frac{8\sqrt{105}}{15} W_{620} \sin 2\theta - 2\sqrt{14} W_{640} \sin 4\theta.
\]

The components of the sixth-order tensors $\Psi^{(i)}$ are given in terms of those of the harmonic tensor $\Phi$ by the following formulae:

\[
\Psi_{ijklmn}^{(1)} = \Phi_{ijkl} \delta_{mn}, \quad \Psi_{ijklmn}^{(2)} = \Phi_{klmn} \delta_{ij} + \Phi_{ijmn} \delta_{kl},
\]

\[
\Psi_{ijklmn}^{(3)} = \delta_{ik} \Phi_{jlmn} + \delta_{jk} \Phi_{ilmn} + \delta_{kl} \Phi_{jmn} + \delta_{mj} \Phi_{ikmn} + \delta_{nk} \Phi_{lijm} + \delta_{ln} \Phi_{mij} + \delta_{im} \Phi_{knj} + \delta_{jm} \Phi_{lni} + \delta_{kn} \Phi_{lij} + \delta_{lj} \Phi_{ikm} + \delta_{il} \Phi_{knj} + \delta_{km} \Phi_{lij} + \delta_{jm} \Phi_{lni} + \delta_{ln} \Phi_{mij} + \delta_{ij} \Phi_{knm},
\]

where $\delta_{ij}$ is the Kronecker delta.