Double Negative Behavior in Metamaterials

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This talk is an overview of results presented in:

- Chen, Y. & Lipton, R.: *Double negative dispersion relation from coated plasmonic rods.* (Submitted)
Metamaterials are artificial materials designed to have electromagnetic properties not generally found in nature.

- Veselago (1968) identified novel effects associated with hypothetical materials for which both the dielectric constant and magnetic permeability are simultaneously negative.

- At the end of the last century, Pendry (1998) demonstrated that unconventional properties can be derived from subwavelength configurations of different conventional materials.

- Smith et al. (2000) experimentally demonstrated that metamaterials made from arrays of metallic posts and split ring resonators generate an effective negative refractive index at microwave frequencies.
These double negative materials are promising materials for the creation of negative index super lenses that overcome the small diffraction limit and have great potential in applications such as biomedical imaging and data storage.

Figure: Pendry, J. & Smith, D., *Reversing Light: Negative Refraction*, 2003
Figure: $H$ represents the host material, $P$ the plasmonic rod and $R$ the high dielectric rod; (a) period cell; (b) cross section of period cell.
For H-polarized Bloch-waves, the magnetic field is aligned with the cylinders and the electric field lies in the transverse plane. The direction of propagation is described by the unit vector \( \hat{\kappa} = (\kappa_1, \kappa_2) \) and \( k = 2\pi/\lambda \) is the wave number for a wave of length \( \lambda \) and the fields are of the form

\[
H_3 = H_3(x)e^{i(k\hat{\kappa} \cdot x - t\omega/c)}, \quad E_1 = E_1(x)e^{i(k\hat{\kappa} \cdot x - t\omega/c)}, \quad E_2 = E_2(x)e^{i(k\hat{\kappa} \cdot x - t\omega/c)} \quad (1)
\]

where \( H_3(x), E_1(x), \) and \( E_2(x) \) are \( d \)-periodic for \( x \) in \( \mathbb{R}^2 \).
Assume all materials are nonmagnetic and hence have magnetic permeability equal to unity. The electric field component \( \mathbf{E} = (E_1, E_2) \) of the wave is determined by

\[
\mathbf{E} = -\frac{ic}{\omega a_d} \mathbf{e}_3 \times \nabla H_3.
\]  

The oscillating dielectric permittivity for the crystal is a \( d \) periodic function in the transverse plane and is described by \( a_d = a_d(x/d) \) where \( a_d(y) \) is the unit periodic dielectric function taking the values

\[
a_d(y) = \begin{cases} 
\varepsilon_H & \text{in the host material,} \\
\varepsilon_P(\omega) & \text{in the frequency dependent “plasmonic” rod,} \\
\varepsilon_R = \varepsilon_r/d^2 & \text{in the high dielectric rod.}
\end{cases}
\]  

where \( \varepsilon_P(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \).
Setting \( h^d(x) = H_3(x)e^{i(k\hat{\kappa}\cdot x)} \), the Maxwell equations take the form of the Helmholtz equation given by

\[
-\nabla_x \cdot \left( a_d^{-1}(\frac{x}{d}) \nabla_x h^d(x) \right) = \frac{\omega^2}{c^2} h^d \quad \text{in } \mathbb{R}^2. \tag{4}
\]

Introduce some parameters:

- the dimensionless ratio \( \rho = d/\sqrt{\varepsilon_r} \)
- wave number \( \tau = \sqrt{\varepsilon_r} k \)
- square frequency \( \xi = \varepsilon_r \frac{\omega^2}{c^2} \)
- the ratio of period size to wavelength \( \eta = dk = \rho \tau \)
Set \( x = dy \) for \( y \) inside the unit period \( Y = [-0.5, 0.5]^2 \).

Write \( u(y) = H_3(dy) \) and \( u^d(y) = h^d(dy) = u(y) \exp^{i\beta \cdot y} \).

The equivalent problem over the unit period cell \( Y \) is

\[
-\nabla_y \cdot \left( a_d^{-1}(y) \nabla_y u^d \right) = \frac{d^2 \omega^2}{c^2} u^d \quad \text{in } Y. \tag{5}
\]
The variational form of the boundary value problem of the unit cell $Y$ is given by

$$
\int_Y a_d^{-1}(\nabla + i\eta \hat{k}) u \cdot (\nabla + i\eta \hat{k}) v = \int_Y \frac{\rho^2 \xi}{c^2} u \overline{v}
$$

(6)

for any $v(y) \in H^1_{\text{per}}(Y)$.

Question: existence of the solution?
Introduce the power series expansions

\[ u = \sum_{m=0}^{\infty} \eta^m u_m \quad \text{and} \quad \xi = \sum_{m=0}^{\infty} \eta^m \xi_m. \quad (7) \]

to show the existence of the solution.

Substitution of (7) into (6) and equating like powers of \( \eta \) delivers an infinite coupled system of equations that can be solved iteratively.
\[
\tau^2 B_z(\psi_m, v) + \xi_0^{-1} \epsilon_p^{-1}(\xi_0) \tau^2 \int_{Y \setminus R} \sum_{l=1}^{m-1} (-i)^l \xi_l \nabla \psi_{m-l} \cdot \nabla v + \kappa \sum_{l=0}^{m-1} (-i)^l \xi_l (\psi_{m-1-l} \nabla v - \nabla \psi_{m-1-l} v)
\]

\[
- \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} v - \xi_0^{-1} \epsilon_p^{-1}(\xi_0) \tau^2 \epsilon_r \frac{\omega_p^2}{c^2} \int_H \kappa \cdot (\psi_{m-1} \nabla v - \nabla \psi_{m-1} v) - \psi_{m-2} v
\]

\[
- \xi_0^{-1} \epsilon_p^{-1}(\xi_0) \int_R \sum_{l=0}^{m-2} (-i)^l \xi_l \nabla \psi_{m-2-l} \cdot \nabla v + \kappa \sum_{l=0}^{m-3} (-i)^l \xi_l (\psi_{m-3-l} \nabla v - \nabla \psi_{m-3-l} v)
\]

\[
- \sum_{l=0}^{m-4} (-i)^l \xi_l \psi_{m-4-l} v + \xi_0^{-1} \epsilon_p^{-1}(\xi_0) \int_R \epsilon_r \frac{\omega_p^2}{c^2} [\nabla \psi_{m-2} \cdot \nabla v + \kappa (\psi_{m-3} \nabla v - \nabla \psi_{m-3} v) + \psi_{m-4} v]
\]

\[
- \xi_0^{-1} \epsilon_p^{-1}(\xi_0) \int_Y \sum_{l=0}^{m-2} \sum_{n=0}^{l} \xi_{m-2-l} \xi_n \psi_{l-n} v - m \nabla v + \epsilon_r \frac{\omega_p^2}{c^2} \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} v
\]

\[= 0, \text{ for all } v \text{ in } H^1_{per}(Y). \]
Here \( u_m = i^m u_0 \psi_m \) and \( u_0 \) is an arbitrary constant factor. \( \psi_m = 0 \) for \( m < 0 \) and \( z = \epsilon_P^{-1} (\xi_0) = \left( 1 - \frac{\epsilon_r \omega_P^2 / c^2}{\xi_0} \right)^{-1} \).

For \( u, v \) belonging to \( H^1_{\text{per}}(Y) \), \( B_z(\psi_m, v) \) is given by the sesqulinear form

\[
B_z(u, v) = \int_H \nabla u \cdot \nabla v \, dy + \int_P z \, \nabla u \cdot \nabla v \, dy. \quad (9)
\]
There exists a convergent power series of the form

\[ \xi^n = \xi^n_0(\tau, \hat{\kappa}) + \sum_{l=1}^{\infty} (\rho \tau)^l \xi^l \]

over a countable set of intervals \( l_n \) on the positive real axis. Here the higher order terms \( \xi^l \) are real and are uniquely determined by \( \xi^n_0 \). The frequency intervals \( l_n \) associated with pass bands are shown to be governed by the poles and zeros of the effective magnetic permeability and dielectric permittivity tensors.
Theorem

There exist transverse magnetic Bloch waves given by the expansion

\[
H_3^n = u_0 \left( \psi_0^n(x/d) + \sum_{l=1}^{\infty} (\rho \tau)^l i^l \psi_l^n(x/d) \right) \exp \left\{ i \left( k_\hat{n} \cdot x - t \frac{\omega}{c} \right) \right\},
\]

provided that \( \xi_0^n(\tau, \hat{n}) \) belongs to \( I_n \) and \( \frac{\omega}{c} = \sqrt{\frac{\xi_n}{\epsilon_r}} \).
Homogenization and energy flow for double negative effective properties

For H-polarized Bloch waves, the magnetic field \( \mathbf{H}(x/d) = (0, 0, H_3(x/d)) \) and the electric field \( \mathbf{E}(x/d) = (E_1(x/d), E_2(x/d), 0) \). Therefore

\[
\mathbf{E}(x/d) = \frac{ic}{\omega a_d} \partial_{x_2} H_3(x/d) \mathbf{e}_1 - \frac{ic}{\omega a_d} \partial_{x_1} H_3(x/d) \mathbf{e}_2. \tag{11}
\]

The time average of the Poynting vector is given by

\[
P^d = \frac{1}{2} Re[\mathbf{E}(x/d) \times \overline{\mathbf{H}(x/d)}]
\]

\[
= \frac{1}{2} Re[E_2(x/d) \overline{H_3(x/d)} \mathbf{e_1} - E_1(x/d) H_3(x/d) \mathbf{e_2}]. \tag{12}
\]
Consider any fixed averaging domain $D$ transverse to the cylinders and the spatial average of the electromagnetic energy flow along the direction $\hat{\kappa}$ over this domain is written $\langle \mathbf{P} \cdot \hat{\kappa} \rangle_D$. Taking the limit of (12) as $d \to 0$ shows that the average electromagnetic energy flow along the direction $\hat{\kappa}$ is given by

$$\langle \mathbf{P} \cdot \hat{\kappa} \rangle_D = \frac{1}{2} |u_0|^2 n_{\text{eff}} \varepsilon_{\text{eff}}^{-1} \hat{\kappa} \cdot \hat{\kappa}. \quad (13)$$

In the $d \to 0$ limit, the phase velocity is along the direction $\hat{\kappa}$ and determined by

$$\mathbf{v}_p = \frac{c}{n_{\text{eff}}} \hat{\kappa}. \quad (14)$$
Equations (13) and (14) show that in the homogenization limit the energy flow and phase velocity are in opposite directions over frequency intervals where the double negative property happens, i.e., $\epsilon^{-1}_{\text{eff}} \hat{\kappa} \cdot \hat{\kappa} < 0$ and $\mu_{\text{eff}} < 0$.

These results are indicative of negative index behavior in the homogenization limit.
Related spectra problems

- The frequency dependent effective magnetic permeability $\mu_{\text{eff}}$ is determined by the Dirichlet eigenvalues and eigenfunctions associated with the cross section $R$:

$$-\Delta \phi_n = \mu_n \phi_n \quad \text{in } R, \quad \phi_n = 0 \quad \text{on } \partial R. \quad (15)$$

- The frequency dependent dielectric permittivity $\epsilon_{\text{eff}}$ is determined by the generalized electrostatic resonances and eigenfunctions associated with the exterior of $R$. 
The generalized electrostatic resonances are characterized by all eigenvalues $\lambda$ in $\mathbb{C}$ and eigenfunctions $\psi$ in $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ that solve

$$-rac{1}{2} \int_{P} \nabla \psi \cdot \nabla v \, dy + \frac{1}{2} \int_{H} \nabla \psi \cdot \nabla v \, dy = (\lambda \psi, v), \quad (16)$$

for every $v$ in $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$. Here $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ is the subspace of functions $u$ belonging to $H^1_{\text{per}}(Y \setminus R)$ with zero mean $\int_{Y \setminus R} u \, dy = 0$. This space is a Hilbert space with inner product

$$(u, v) = \int_{Y \setminus R} \nabla u \cdot \nabla v \, dy. \quad (17)$$
Introduce a suitable orthogonal decomposition of $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$:

$$H^1_{\text{per}}(Y \setminus R)/\mathbb{C} = W_1 \oplus W_2 \oplus W_3 \oplus \mathbb{C}. \quad (18)$$

- $W_1$: all periodic continuously differentiable functions with support inside $H$ and extension by zero to $Y \setminus R$.
- $W_2$: all smooth functions with compact support on $P$ and extension by zero to $Y \setminus R$.
- $W_3$: all functions $w$ in $H^1_{\text{per}}(Y \setminus R)$ for which the boundary integral $\int_{\partial P} w \, dS$ vanishes and that belong to the orthogonal complement of $W_1 \cup W_2$ with respect to the inner product (17).
The eigenvalues for (16) are real and constitute a denumerable set contained inside \([-1/2, 1/2]\) with the only accumulation point being zero. The eigenspaces associated with \(\lambda = 1/2\) and \(\lambda = -1/2\) are \(W_1\) and \(W_2\) respectively. Eigenspaces associated with distinct eigenvalues in \((-1/2, 1/2)\) are finite dimensional, pairwise orthogonal, and their union spans the subspace \(W_3\).
In $Y \setminus R$, (8) shows that $\psi_0$ is the solution of

$$B_z(\psi_0, v) = 0, \text{ for all } v \text{ in } H^1_{per}(Y \setminus R).$$  \hspace{1cm} (19)

1. $\xi_0$ satisfies $z = \epsilon_P^{-1}(\xi_0) = (\lambda_i + 1/2)/(\lambda_i - 1/2)$ and $\psi_0$ is an eigenfunction for (16).

2. $\xi_0$ satisfies $z = \epsilon_P^{-1}(\xi_0) \neq (\lambda_i + 1/2)/(\lambda_i - 1/2)$, $i = 1, 2, \ldots$ and $\psi_0 = \text{constant}$.

Here we assume the second alternative.
Restricting to test functions $v$ with support in $R$ in (8) we get

\[ \int_R (\nabla \psi_0 \cdot \nabla v - \xi_0 \psi_0 v) \, dy = 0. \tag{20} \]

From continuity we have the boundary condition for $\psi_0$ on $R$ given by $\psi_0 = const.$

Here we have the alternative:

1. If $\xi_0$ is a Dirichlet eigenvalue $\nu_i$ of $-\Delta$ in $R$ then $\psi_0(y) = 0$ for $y$ in $Y \setminus R$.

2. If $\xi_0 \neq \nu_i, \ i = 1, 2, \ldots$ then $\psi_0$ is the unique solution of the Helmholtz equation (20) and $\psi_0 = const.$ in $Y \setminus R$.

In this treatment we will choose the second alternative $\xi_0 \neq \nu_i$. Without loss of generality take the choice $\psi_0 = 1$ in $Y \setminus R$. 

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A straightforward calculation gives $\psi_0$ in $R$ in terms of the complete set of Dirichlet eigenfunctions and eigenvalues:

$$\psi_0 = \sum_{n=1}^{\infty} \frac{\mu_n < \phi_n >_R}{\mu_n - \xi_0} \phi_n, \text{ in } R. \quad (21)$$

Note that $\mu_n$ denote the Dirichlet eigenvalues of $-\Delta$ in $R$ whose eigenfunctions $\phi_n$ have nonzero mean, $< \phi_n >_R = \int_R \phi_n(y)dy \neq 0$. The Dirichlet eigenvalues associated with zero mean eigenfunctions are denoted by $\mu'_n$ and $\{\nu_n\}_{n=1}^{\infty} = \{\mu_n\}_{n=1}^{\infty} \cup \{\mu'_n\}_{n=1}^{\infty}$.
To find $\psi_1$ in $Y \setminus R$, we appeal to (8) with $\psi_0 = 1$ in $Y \setminus R$ to discover

$$B_z(\psi_1, v) = -\int_H \hat{\kappa} \cdot \nabla V - \int_P \epsilon_p^{-1}(\xi_0) \hat{\kappa} \cdot \nabla V \quad \forall \ v \in H^1_{per}(Y)$$

(22)

This problem has a unique solution subject to the mean-zero condition:

$$\int_{Y \setminus R} \psi_1 = 0$$
A straightforward calculation gives the representation for $\psi_1$ in $Y \setminus R$

$$
\psi_1 = - \sum_{-1/2 < \lambda_n < 1/2} \left( \frac{(\alpha_1^\lambda_n + \epsilon_P^{-1}(\xi_0)\alpha_2^\lambda_n)}{1 + (\epsilon_P^{-1}(\xi_0) - 1)(1 - \lambda_n)} \right) \psi_{\lambda_n} + \sum_{n=1}^{\infty} \alpha_{1,n} \psi_{1,n}^{1}, \text{ in } Y \setminus R \quad (23)
$$

with

$$
\alpha_1^\lambda_n = \hat{\kappa} \cdot \int_H \nabla \psi_{\lambda_n} \, dy, \quad \alpha_2^\lambda_n = \hat{\kappa} \cdot \int_P \nabla \psi_{\lambda_n} \, dy, \quad \text{and} \quad \alpha_{1,n} = \hat{\kappa} \cdot \int_H \nabla \psi_{1,n}^{1} \, dy.
$$

- $\{\psi_{\lambda_n}\}_{n=1}^{\infty}$: the complete set of orthonormal eigenfunctions associated with $-1/2 < \lambda_n < 1/2$ for $W_3$
- $\{\psi_{1,n}^{1}\}_{n=1}^{\infty}$: the complete orthonormal sets of functions for $W_1$.
- $\{\psi_{n}^{2}\}_{n=1}^{\infty}$: the complete orthonormal sets of functions for $W_2$. 
Setting $v = 1$ and $m = 2$ in (8) we recover the solvability condition given by

$$
\tau^2 \int_{H \cup P} \left[ -\hat{\kappa} \cdot \xi_0 \nabla \psi_1 + \xi_0 \right] - \tau^2 \epsilon_r \frac{\omega_p^2}{c^2} \int_H (-\hat{\kappa} \nabla \psi_1 + 1) = \int_Y (\xi_0^2 \psi_0 - \epsilon_r \frac{\omega_p^2}{c^2} \xi_0 \psi_0),
$$

which delivers the quasistatic dispersion relation

$$
\xi_0 = \tau^2 n_{eff}^{-2} (\xi_0),
$$

where the effective index of refraction $n_{eff}$ depends upon the direction of propagation $\hat{\kappa}$ and is written

$$
n_{eff}^2(\xi_0) = \frac{\mu_{eff}(\xi_0)}{\epsilon_{eff}^{-1}(\xi_0) \hat{\kappa} \cdot \hat{\kappa}}.
$$
The frequency dependent effective magnetic permeability $\mu_{\text{eff}}$ and effective dielectric permittivity $\varepsilon_{\text{eff}}$ are given by

$$
\mu_{\text{eff}}(\xi_0) = \int_Y \psi_0 = \theta_H + \theta_P + \sum_{n=1}^{\infty} \frac{\mu_n < \phi_n >^2_R}{\mu_n - \xi_0}
$$

(26)

and

$$
\varepsilon_{\text{eff}}^{-1}(\xi_0) = \int_H (I - P) \hat{H} \cdot \hat{K} dy + \frac{\xi_0}{\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2}} \theta_P
$$

$$
- \sum_{-1/2 < \lambda_h < 1/2} \left( \frac{\left( \xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2} \right) |\alpha^{(1)}_{\lambda_h}|^2 + 2 \frac{\varepsilon_r \omega_p^2}{c^2} |\alpha^{(1)}_{\lambda_h}|^2 \alpha^{(1)}_{\lambda_h} \lambda_h + \frac{\left( \frac{\varepsilon_r \omega_p^2}{c^2} \right)^2}{\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2}} |\alpha^{(2)}_{\lambda_h}|^2}{\xi_0 - (\lambda_h + \frac{1}{2}) \frac{\varepsilon_r \omega_p^2}{c^2}} \right),
$$

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**Figure:** the relation between $\mu_{\text{eff}}$ and $\xi_0$

$$\epsilon_{\text{eff}}^{-1} \hat{\kappa} \cdot \hat{\kappa}$$

**Figure:** the relation between $\epsilon_{\text{eff}}^{-1} \hat{\kappa} \cdot \hat{\kappa}$ and $\xi_0$
Figure: leading order behavior for pass bands associated with double negative and double positive behavior
Any element \( u \) of \( H^1_{\text{per}}(Y \setminus R)/\mathbb{C} \) can be written as

\[
    u = \mathcal{P}_1 u + \mathcal{P}_2 u + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \mathcal{P}_{\lambda_n} u + d. 
\]  

(27)

where \( d \) is chosen such that \( \int_{Y \setminus R} u \, dy = 0 \).

- \( \mathcal{P}_i \) are the orthogonal projections onto \( W_i, i = 1, 2 \).
- \( \mathcal{P}_{\lambda_n} \) are the orthogonal projections onto the finite dimensional eigenspaces associated with the eigenvalues \( \lambda_n \) in \((-1/2, 1/2)\).
Theorem

Suppose $z \neq \frac{(\lambda_i + 1/2)}{(\lambda_i - 1/2)}$ for $\lambda_i \in [-\frac{1}{2}, \frac{1}{2}]$. Then for any $F \in \left[ H^1_{\text{per}}(Y \setminus R)/\mathbb{C} \right]^*$ such that $F(v) = 0$ for constant $v$, there exists a unique solution $u \in H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ of the variational problem $B_z(u, v) = \overline{F(v)}$ for all $v \in H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$.
Proof of the existence theorem of the exterior problem

For \( u, v \) in \( H^1_{\text{per}}(Y \setminus R)/\mathbb{C} \),

\[
B_z(u, v) = \sum_{i=1}^{2} B_z(P_i u, P_i v) + \sum_{\lambda_n \in \mathbb{R} \setminus \mathbb{C}} B_z(P_{\lambda n} u, P_{\lambda n} v) \\
= (P_1 u, v) + z(P_2 u, v) + \sum_{\lambda_n \in \mathbb{R} \setminus \mathbb{C}} (1 + (z - 1)(\frac{1}{2} - \lambda_n))(P_{\lambda n} u, v) \\
= (T_z u, v). 
\]  

(28)

Here

\[
T_z = P_1 + zP_2 + \sum_{\lambda_n \in \mathbb{R} \setminus \mathbb{C}} (1 + (z - 1)(\frac{1}{2} - \lambda_n))P_{\lambda n}. 
\]  

(29)

It is evident from (29) that for \( z \neq (\lambda_i + 1/2)/(\lambda_i - 1/2) \) that \( T_z \) is a bounded one to one and onto map in \( H^1_{\text{per}}(Y \setminus R)/\mathbb{C} \).
The formula for $T_z^{-1}$ is given by

$$T_z^{-1} = P_1 + z^{-1}P_2 + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} (1 + (z - 1)(\frac{1}{2} - \lambda_n))^{-1} P_{\lambda_n}. \quad (30)$$

Taking conjugates on both sides of (28) gives $\overline{B_z(u, v)} = (v, T_z u)$ and choosing $u = T_z^{-1} q$ for $q$ in $H^1_{per}(Y \setminus R)/\mathbb{C}$ delivers the identity

$$\overline{B_z(T_z^{-1} q, v)} = (v, q). \quad (31)$$

To complete the proof consider any linear functional $F$ in $[H^1_{per}(Y \setminus R)/\mathbb{C}]^*$ with $F(v) = 0$ for $v = const$. Applying the Reisz representation theorem shows that there exists a unique solution $u$ in $H^1_{per}(Y \setminus R)/\mathbb{C}$ of

$$B_z(u, v) = \overline{F(v)}, \text{ for all } v \text{ in } H^1_{per}(Y \setminus R)/\mathbb{C}. \quad (32)$$
Comparsion of power series for dispersion relation with numerical simulation

Figure: Coated cylinder microgeometry
Calculation of the resonances for coated cylinders

We consider a metamaterial crystal characterized by a period cell containing a centered coated cylinder with plasmonic coating and high dielectric core. The core radius and the coating radius are denoted by $a$ and $b$ respectively.

![The period cell](image)

**Figure:** The period cell
The electrostatic resonances $\lambda_h$ are found by solving the following problem for the potential $u$ inside a unit cell, i.e., $d = 1$:

\[
\begin{cases}
\Delta u = 0 \quad \text{in } H, \\
\Delta u = 0 \quad \text{in } P,
\end{cases}
\]

with the boundary conditions

\[
\begin{cases}
|\partial^- u| = |\partial^- u| \quad \text{on } \partial P, \\
|\partial_r u|_{r=a} = 0 \quad \text{on } \partial R, \\
\lambda[\partial_r u]^+ = -\frac{1}{2}(\partial_r u^- + \partial_r u^+) \quad \text{on } \partial P, \\
u \text{ is } Y\text{-periodic}.
\end{cases}
\]
In polar coordinates \((r, \theta)\), the expansions of the potential \(u(r, \theta)\) are

\[
    u_p(r, \theta) = \sum_{l=1}^{\infty} (A_l r^l + B_l r^{-l}) \cos l\theta \quad \text{in } P, \tag{35}
\]

\[
    u_h(r, \theta) = \sum_{l=1}^{\infty} (C_l r^l + D_l r^{-l}) \cos l\theta \quad \text{in } H. \tag{36}
\]

Use the method of Rayleigh to calculate the generalized electrostatic resonances.
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</tbody>
</table>

**Table:** The eigenvalues corresponding to $N = 10, 15, 20$ with $a = 0.2$, $b = 0.4$. 
Then

\[ u(r, \theta) \approx \begin{cases} a^{-2} r \cos \theta + r^{-1} \cos \theta & \text{in } P, \\ \left( \frac{a^2 b^{-2}}{1-2\lambda} \right) a^{-2} r \cos \theta + \left( \frac{a^{-2} b^2}{1-2\lambda} \right) r^{-1} \cos \theta & \text{in } H. \end{cases} \]  

(37)

The solution \( u \) corresponding to the first two eigenvalues with \( a = 0.2 \) and \( b = 0.4 \) are illustrated in the following figures.
Figure: (a) the solution corresponding to the eigenvalue \( \lambda = 3.5080 \times 10^{-1} \); (b) the solution corresponding to the eigenvalue \( \lambda = 1.5379 \times 10^{-2} \).
Numerical calculation of the dispersion relation and comparison with power series

Our aim: verify that the leading order dispersion relation expressed in terms of effective properties is a good predictor of the dispersive behavior of the metamaterial for periods with finite size $d > 0$.

We fix $d = c/\omega_p$ and the dimensionless ratio $\rho = d/\sqrt{\varepsilon_r}$. With this choice of variables the frequency dependent effective magnetic permeability $\mu_{\text{eff}}$ and effective dielectric permittivity $\varepsilon_{\text{eff}}$ are written as

$$\mu_{\text{eff}}(\omega_0/\omega_p) = \int_Y \psi_0 = \theta_H + \theta_P + \sum_{n=1}^{\infty} \frac{\mu_n < \phi_n >_R^2}{\rho^{-2} \left(\mu_n \rho^2 - (\omega_0/\omega_p)^2\right)}$$

and
\[
\epsilon^{-1}_{\text{eff}}(\omega_0/\omega_p) \hat{\kappa} \cdot \hat{\kappa} = \theta_H + \frac{(\omega_0/\omega_p)^2}{(\omega_0/\omega_p)^2 - 1} \theta_p \\
- \sum_{-\frac{1}{2} < \lambda_h < \frac{1}{2}} \left( \frac{(\omega_0/\omega_p)^2 - 1}{(\omega_0/\omega_p)^2 - (\lambda_h + \frac{1}{2})} \frac{\alpha^{(1)}_h |\alpha^{(1)}_h|^2 + 2 \left( (\omega_0/\omega_p)^2 - 1 \right) \alpha^{(1)}_h \alpha^{(2)}_h + |\alpha^{(2)}_h|^2}{(\omega_0/\omega_p)^2 - 1} \right).
\]

In these variables, the leading order dispersion relation is given by

\[
(dk)^2 = (\frac{\omega_0}{\omega_p})^2 n_{\text{eff}}^2,
\]

(39)

where the effective index of diffraction \( n_{\text{eff}}^2 \) depends upon the direction of propagation \( \hat{\kappa} \) and normalized frequency \( \frac{\omega_0}{\omega_p} \) and is written

\[
n_{\text{eff}}^2 = \mu_{\text{eff}} \left( \frac{\omega_0}{\omega_p} \right) / \left( \epsilon_{\text{eff}}^{-1} \left( \frac{\omega_0}{\omega_p} \right) \hat{\kappa} \cdot \hat{\kappa} \right).
\]

(40)
The following figures show the exact numerical solutions via COMSOL and how they compare to the leading order dispersion relation.

Figure: the case of $a = 0.2d$, $b = 0.4d$ and $\epsilon_R = 285$. 
Figure: the case of $a = 0.15d$, $b = 0.4d$ and $\epsilon_R = 285$. 
Future work

- Extend the power series methodology to handle dissipative media for dielectric inclusions with dielectric function given by multiple oscillator models

\[
\epsilon_P(\omega) = 1 + \sum_{j=1}^{N} \frac{\omega_p^2}{\omega_j^2 - \omega^2 - i\gamma_j\omega}. \tag{41}
\]

- Investigate metamaterials made from periodic arrays consisting of both metallic and dielectric rods for (41).
Chen, Y. & Lipton, R., Double negative dispersion relation from coated plasmonic rods. (Submitted)

Chen, Y. & Lipton, R., Resonance and double negative behavior in metamaterials.

Chen, Y. & Lipton, R. Tunable double negative band structure from non-magnetic coated rods.


Smith, D., Padilla, W., Vier, D., Nemat-Nasser, S. & Schultz, S. Composite medium with simultaneously negative permeability and permittivity.

Veselago, V., G. The electrodynamics of substances with simultaneously negative values of $\varepsilon$ and $\mu$. 
Thank You!